Triangulated categories of log motives over a field (I)

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08.09.2020

Cohomology theories

There are lots of cohomology theories for schemes.
Many fundamental cohomology theories are not $\mathbb{A}^1$-invariant.

Type I: $\mathbb{A}^1$-invariant cohomology theories
$H^n(X) \simeq H^n(X \times \mathbb{A}^1)$ for all schemes X (including singular schemes) and integer n.

Example

- ℓ-adic cohomology $H^n_{\acute{e}t}(X, \mathbb{Z}_\ell)$, where ℓ is invertible in $\mathcal{O}_X$.
- Betti cohomology $H^n_{Betti}(X, \mathbb{Z})$ if X is a variety over $\mathbb{C}$.

Cohomology theories

Type II: $H^n(X) \simeq H^n(X \times \mathbb{A}^1)$ if X is regular but $H^n(X) \not\simeq H^n(X \times \mathbb{A}^1)$ if X is not regular in general.

Example

- (constructed) K-theory $K_n(X)$
- (unconstructed) Motivic cohomology $H^n_{\mathcal{M}}(X, \mathbb{Z}(d))$
  Wanted properties:
  1. $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) \simeq \mathrm{Pic}(X)$.
  2. $H^n_{\mathcal{M}}(X, \mathbb{Z}(n)) \simeq K^M_n(X)$ if X is the spectrum of a local ring.
  3. Motivic spectral sequence

$$E_2^{pq} := H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$

Cohomology theories

Type III: $H^n(X) \not\simeq H^n(X \times \mathbb{A}^1)$ even for regular schemes X

Example

- p-adic cohomology $H^n_{\acute{e}t}(X, \mathbb{Z}_p)$, where p is not invertible in $\mathcal{O}_X$.
- Hodge cohomology $H^n_{Zar}(X, \Omega^j)$
- Hochschild homology $HH_n(X)$ [Hochschild]
  Homology of a certain complex $\cdots \to R \otimes R \to R \to 0$ (if $X = \mathrm{Spec}(R)$).
  $HH_n(R) \simeq \Omega^n_{R/k}$ (if R is smooth) [Hochschild-Kostant-Rosenberg]
- Cyclic homology $HC_n(X)$ [Tsygan, Connes]
- Topological Hochschild homology $THH_n(X)$ [Bökstedt]
- Topolocial cyclic homology $TC_n(X)$ [Bökstedt-Hsiang-Madsen]

See *Topological Hochschild homology and integral p-adic Hodge theory* by Bhatt-Morrow-Scholze and *On topological cyclic homology* by Nikolaus-Scholze for some recent developments.

Motivation for our project

Is there a motivic homotopy theory of schemes that incorporates these cohomology theories?

Type II cohomology theories are not representable in $\mathrm{SH}(X)$ if X is singular.
Type III cohomology theories are not representable in $\mathrm{SH}(X)$ even if X is regular.
More precisely, there does not exist $\mathcal{E} \in \mathrm{SH}(X)$ such that

$$H^n(Y) \simeq \mathrm{Hom}_{\mathrm{SH}(X)}(Y_+, \Sigma^n \mathcal{E})$$

for all $Y \in Sm/X$.

Goal of our project

1. Construct a non $\mathbb{A}^1$-homotopic enlargement of $\mathrm{DM}(k)$ (done if k is a perfect field with resolution of singularities)
2. Construct a non $\mathbb{A}^1$-homotopic enlargement of $\mathrm{SH}(X)$ (work in progress)
3. Represent all cohomology theories in the previous slides (work in progress)

Recollection of Voevodsky's construction of $\mathrm{DM}(k)$

There are four main ingredients in Voevodsky's construction.

1. Invert $\mathbb{A}^1 \to \mathrm{pt}$
2. Nisnevich topology
3. Finite correspondences
4. $\mathbb{P}^1$-stabilization

The construction is as follows.

1. Use finite correspondences to form the category of presheaves with transfers.
2. Form the category of (Nisnevich) sheaves with transfers.
3. Invert $\mathbb{A}^1 \to \mathrm{pt}$ in the derived category of sheaves with transfers to obtain $\mathrm{DM}^{\mathrm{eff}}(k)$.
4. Apply $\mathbb{P}^1$-stabilization to obtain $\mathrm{DM}(k)$.

Axioms for schemes

We definitely need to replace $\mathbb{A}^1$. Here is a set of axioms that is helpful to determine the candidate.

- ($\mathbb{A}^1$-rigidity) $i_0^*, i_1^* \colon H^n(X \times \mathbb{A}^1) \to H^n(X)$ are equal.
- ($\mathbb{A}^1$-invariance) $p^* \colon H^n(X) \to H^n(X \times \mathbb{A}^1)$ is an isomorphism.
- ($\mathbb{P}^1$-rigidity) $i_0^*, i_1^* \colon H^n(X \times \mathbb{P}^1) \to H^n(X)$ are equal.
- ($\mathbb{P}^1$-invariance) $p^* \colon H^n(X) \to H^n(X \times \mathbb{P}^1)$ is an isomorphism.

($\mathbb{A}^1$-invariance) $\Leftrightarrow$ ($\mathbb{A}^1$-rigidity) $\Rightarrow$ ($\mathbb{P}^1$-rigidity) $\Leftarrow$ ($\mathbb{P}^1$-invariance)

Fortunately, $\mathbb{P}^1$ is not an inverval object. Hence ($\mathbb{P}^1$-rigidity) $\not\Rightarrow$ ($\mathbb{P}^1$-invariance).
If $\mathbb{P}^1$ were an interval object, then we could not consider $\mathbb{P}^1$-rigidity as one of the properties of non $\mathbb{A}^1$-invariant cohomology theories.

Many cohomology theories are $\mathbb{P}^1$-rigid, and most cohomology theories are not $\mathbb{P}^1$-invariant.
For $\mathbb{P}^1$-localization of cohomology theories see $\mathbb{P}^1$*-Localisation et une classe de Kodaira-Spencer arithmétique* by Joseph Ayoub.

Replacement of $\mathbb{A}^1$

Even though $\mathbb{A}^1$ resembles the interval $[0,1]$ in topology, the key difference property is that $\mathbb{A}^1$ is not compact but $[0,1]$ is compact.

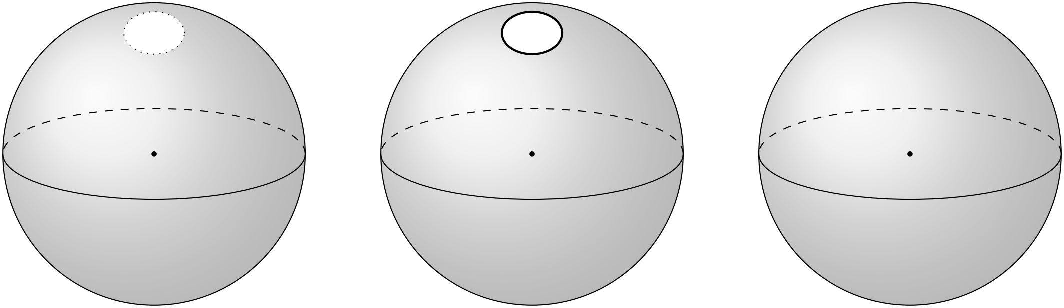
An idea is to compactify $\mathbb{A}^1$ in a suitable sense.
We make a gadget beyond the category of schemes to compactify $\mathbb{A}^1$ as follows.

Definition

$$\overline{\square} := (\mathbb{P}^1, \infty)$$

We regard ∞ as the boundary of $\overline{\square}$. This is a model for a compactification of $\mathbb{A}^1$.
This is a compromise notion between non-compact $\mathbb{A}^1$ and non-contractible $\mathbb{P}^1$.

In algebraic topology adding an infinitesimal boundary at ∞ can be described by the following figure:



$\mathbb{A}^1$ $\qquad$ $\overline{\square}$ $\qquad$ $\mathbb{P}^1$

The gadget $\overline{\square}$ is contractible as $\mathbb{A}^1$ and compact as $\mathbb{P}^1$.

To make a category of such compactifications, we can introduce the following.

Definition

$SmlSm/S$ is the category whose objects are pairs (X, D) where $X \in Sm/S$ and D is a strict normal crossing divisor, and whose morphisms are $f\colon (Y, E) \to (X, D)$ such that $f(Y - E) \subset X - D$.

We do not consider multiplicities on D, i.e., D is reduced.
This category is actually a full subcategory of the following.

Definition

lSm/S is the category of log smooth fs log schemes (with Zariski log structures) over k.

The functor $SmlSm/S \to lSm/S$ sends (X, D) to the fs log scheme whose underlying scheme is X and whose log structure is the compactifying log structure associated with the open immersion $X - D \to X$.

Replacement of $\mathbb{A}^1$

The foundations of logarithmic geometry was developed by Deligne, Faltings, Fontaine, Illusie, and K. Kato.
The compactification gadget can be efficiently studied in this well-developed geometry.
This is the reason why we chose logarithmic geometry to develop the theory of non $\mathbb{A}^1$-motivic homotopy theory.

Replacement of $\mathbb{G}_m$

What is the replacement of the Tate twist? What is the replacement of $B\mathbb{G}_m$?
We can view $(\mathbb{P}^1, 0+\infty)$ as a compactification of $\mathbb{G}_m$. One problem is that the multiplication morphism $m\colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ does not extend to a morphism

$$(\mathbb{P}^1, 0+\infty) \times (\mathbb{P}^1, 0+\infty) \to (\mathbb{P}^1, 0+\infty).$$

To remedy this, we can consider the blow-up at $(0,\infty)+(\infty,0)$ so that we have a morphism

$$(\mathrm{Bl}_{(0,\infty)+(\infty,0)}(\mathbb{P}^1 \times \mathbb{P}^1), H_1 + H_2 + H_3 + H_4 + E_1 + E_2) \to (\mathbb{P}^1, 0+\infty).$$

To view this as a multiplication, we identify this blow-up with $(\mathbb{P}^1, 0+\infty) \times (\mathbb{P}^1, 0+\infty)$.
To achieve this, we introduce a topology as follows.

Replacement of topology

Definition

A morphism $f\colon (Y,E) \to (X,D)$ is a *dividing cover* if it is a proper surjective log étale monomorphism.

It turns out that this is equivalent to the notion of log modifications defined by F. Kato. The blow-up in the previous slide is an example of a dividing cover.

The hypercover descent condition is equivalent to an invariant condition, i.e., $\mathcal{F}$ satisfies dividing descent if and only if $\mathcal{F}(X,D) \to \mathcal{F}(Y,E)$ is a weak equivalence for all dividing covers $(Y,E) \to (X,D)$. One philosophical reason why this is true is that $(Y,E) \to (X,D)$ is a monomorphism.

Hence if we use the dividing topology, then we automatically invert dividing covers.

Replacement of topology

Definition

A morphism $f\colon (Y, E) \to (X, D)$ is *strict* if $E \simeq Y \times_X D$.

To generalize the Nisnevich topology to log geometry, we use the following.

Definition

A morphism $f\colon (Y, E) \to (X, D)$ is a *strict Nisnevich cover* if f is strict and $Y \to X$ is a Nisnevich cover.

The associated topology is the *strict Nisnevich topology*.

Definition

dividing Nisnevich topology $=$ strict Nisnevich topology $+$ dividing topology

Replacement of topology

Definition

The *strict Nisnevich cd-structure* is the collection of squares of the form

$$\begin{array}{ccc} (Y',E') & \to & (Y,E) \\ \downarrow & & \downarrow f \\ (X',D') & \to & (X,D) \end{array}$$

such that all the morphisms in the square are strict and the square

$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow f \\ X' & \to & X \end{array}$$

is a Nisnevich distinguished square.

Replacement of topology

Definition

The *dividing cd-structure* is the collection of squares of the form

$$\begin{array}{ccc} \emptyset & \to & (Y,E) \\ \downarrow & & \downarrow f \\ \emptyset & \to & (X,D) \end{array}$$

such that f is a dividing cover.

Theorem

The dividing Nisnevich cd-structure is complete, quasi-bounded, and regular.

The notion of quasi-boundness is introduced because the dividing Nisnevich topology is not bounded.

The Brown-Gersten property is still satisfied by every complete, quasi-bounded, and regular cd-structure.

Replacement of the Tate twist

We do not replace $\mathbb{P}^1$.
There are two equivalent definitions of Tate twist in $\mathrm{DM}(k)$:

$$\mathbb{Z}(1) := M(\mathrm{pt} \to \mathbb{P}^1)[-2].$$
$$\mathbb{Z}(1) := M(\mathrm{pt} \to \mathbb{G}_m)[-1].$$

These can be replaced by the following:

$$\mathbb{Z}(1) := M(\mathrm{pt} \to \mathbb{P}^1)[-2].$$
$$\mathbb{Z}(1) := M(\mathrm{pt} \to (\mathbb{P}^1, 0 + \infty))[-1].$$

We also have

$$\Gamma((X, D), a_{dNis}(\mathbb{P}^1, 0 + \infty)) \simeq \Gamma(X - D, \mathcal{O}^*_{X-D}),$$

which is not true without dividing Nisnevich sheafification.
In our motivic homotopy categories using $\overline{\square}$-invariance and Mayer-Vietoris we can deduce

$$\mathbb{P}^1 \simeq S^1 \wedge (\mathbb{P}^1, 0 + \infty).$$

Finite log correspondences

We need to extend finite correspondences to lSm/k. The definition of one possible extension is as follows.

Definition

For $(X, D), (Y, E) \in SmlSm/k$ an *elementary log correspondence* from (X, D) to (Y, E) consists of

1. an integral subscheme Z of $X \times Y$ that is finite and surjective over X,
2. a morphism $(Z^N, Z^N \times_X D) \to (Y, E)$ of fs log schemes, where Z^N is the normalization of Z
   (not just a morphism $g\colon Z^N \to Y$ such that $g(Z^N - (Z^N \times_X D)) \subset Y - E$).

A *finite log correspondence* is a formal sum of elementary log correspondences.
$\mathrm{lCor}((X, D), (Y, E))$ is the free abelian group of finite log correspondences.

This can be extended to lSm/k.
We have $\mathrm{lCor}((X, \emptyset), (Y, \emptyset)) = \mathrm{Cor}(X, Y)$.
The part 2 allows us to make elementary log correspondences composable.

Construction of $\mathrm{logDM}^{\mathrm{eff}}(k)$

k is a field (with trivial log structure).
There are three main ingredients.

1. Invert $\overline{\square} \to \mathrm{pt}$.
2. Dividing Nisnevich topology.
3. Finite log correspondences.

The construction is as follows.

1. Use finite log correspondences to form the category of presheaves with log transfers.
2. Form the category of dividing Nisnevich sheaves with log transfers.
3. Invert $\overline{\square} \to \mathrm{pt}$ in the derived category of dividing Nisnevich sheaves with log transfers to obtain $\mathrm{logDM}^{\mathrm{eff}}(k)$.

If we apply $\mathbb{P}^1$-stabilization, then we can obtain $\mathrm{logDM}(k)$. Do we have the cancellation theorem?

Alternative approach: *Motives with modulus* by Kahn-Miyazaki-Saito-Yamazaki using modulus pairs where divisors can have multiplicity.

Construction of $\mathrm{logSH}(S)$

S is any fs log scheme.
There are three main ingredients.

1. Invert $\overline{\square} \to \mathrm{pt}$.
2. Dividing Nisnevich topology.
3. $\mathbb{P}^1$-stabilization.

The construction is as follows.

1. Form the category of dividing Nisnevich sheaves.
2. Invert $\overline{\square} \to \mathrm{pt}$ in the category of pointed simplicial sheaves to obtain $\mathrm{logH}(S)$.
3. Apply $\mathbb{P}^1$-stabilization to obtain $\mathrm{logSH}(S)$.

Representability

To represent a cohomology theory of schemes in $\mathrm{logH}(S)$ or $\mathrm{logSH}(S)$, we may take the following steps.

1. Extend the cohomology theory to fs log schemes
2. Prove $\overline{\square}$-invariance

Examples:

1. $\Omega^j_{(X,D)/k} := \Omega^j_{X/k}(\log(D))$ $((X,D) \in SmlSm/k)$
2. $\mathrm{logHH}_n(S)$, $\mathrm{logHC}_n(S)$ (S is a log smooth fs log scheme over a field of characteristic 0)

Work in progress / Open questions

- Can we represent
  1. $\mathrm{logHH}_n(R)$, $\mathrm{logTHH}_n(R)$ [Rognes] (R is a pre-log ring),
  2. $\mathrm{logHH}_n(S)$, $\mathrm{logHC}_n(S)$ [Olsson] (S is an fs log scheme)?
- We can define $K_n^{log}(S) := \mathrm{Hom}_{\mathrm{logSH}(S)}(\Sigma^n S_+, \mathrm{KGL})$ (S is an fs log scheme). The T-spectrum KGL is defined by $(\mathbb{L}_{\overline{\square}}(\mathbb{Z} \times \mathrm{Gr}), \mathbb{L}_{\overline{\square}}(\mathbb{Z} \times \mathrm{Gr}), \cdots)$ with a certain bonding map $T \wedge \mathbb{L}_{\overline{\square}}(\mathbb{Z} \times \mathrm{Gr}) \to \mathbb{L}_{\overline{\square}}(\mathbb{Z} \times \mathrm{Gr})$.
  Is $K_n(X) \simeq K_n^{log}(X)$ if X is a scheme (with trivial log structure)?

Proving $\overline{\square}$-invariance may (or may not) require some extra work. From our experience we expect that the projective bundle formula helps to prove this.

Work in progress / Open questions

- What is $\mathrm{Hom}_{\mathrm{logDM}^{\mathrm{eff}}(k)}(\mathbb{Z}[n], M(\mathbb{A}^1))$?
  This is a (non $\mathbb{A}^1$-invariant) singular homology of $\mathbb{A}^1$, which should not be isomorphic to the Suslin singular homology.
  We expect that there is a relation between this group and big de Rham-Witt vectors.
- Cancellation: $\mathrm{Hom}_{\mathrm{logDM}^{\mathrm{eff}}(k)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{logDM}^{\mathrm{eff}}(k)}(\mathcal{F}(1), \mathcal{G}(1))$?

Work in progress / Open questions

- Can we list axioms for cohomology theories of schemes such that any cohomology theory with these axioms can be canonically extended to $\mathrm{logSH}(S)$?
  Here are some axioms that should be satisfied:
  1. $\mathbb{P}^1$-rigidity.
  2. Nisnevich descent.
  3. Descent for cartesian squares of the form

$$\begin{array}{ccc} Z' & \hookrightarrow & \mathrm{Bl}_Z X \\ \downarrow & & \downarrow \\ Z & \overset{i}{\hookrightarrow} & X \end{array}$$

     such that i is a regular embedding.

  Are these enough?

Alternative construction

Now let S be a scheme (without log structure).
Recall that there is a fully faithful functor

$$SmlSm/S \hookrightarrow lSm/S.$$

It may be easier to define first a cohomology theory for all fs log schemes in $SmlSm/S$ and extend to for all log schemes in lSm/S by some machine.
For this purpose, we can use the following.

Theorem

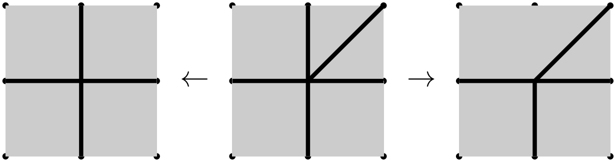
$\overline{\square}$-invariance + dNis-descent in lSm/S
$\Leftrightarrow (\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance for all n + sNis-descent in $SmlSm/S$

Here, we regard $\mathbb{P}^{n-1}$ as the hyperplane of $\mathbb{P}^n$ at ∞.
Observe that both $(\mathbb{P}^n, \mathbb{P}^{n-1})$ and $\overline{\square}^n$ are compactifications of $\mathbb{A}^n$.

Explanation of the proof

The proof is combinatorial.

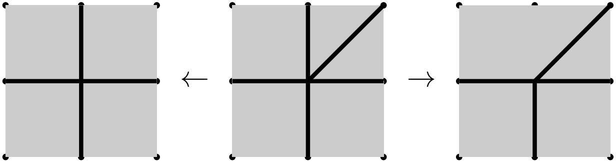
Toric picture:



$(\mathbb{P}^1, \infty) \times (\mathbb{P}^1, 0) \qquad\qquad \simeq (\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2, H_2 + H_3)$

The first and third fs log schemes are $\overline{\square}$-bundles over $\overline{\square}$, so their motives are isomorphic to the motive of pt.
Hence the left morphism is an isomorphism of motives if and only if the right morphism is an isomorphism of motives.

Explanation of the proof



$(\mathbb{P}^1, \infty) \times (\mathbb{P}^1, 0)$ $\simeq (\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2, H_2 + H_3)$

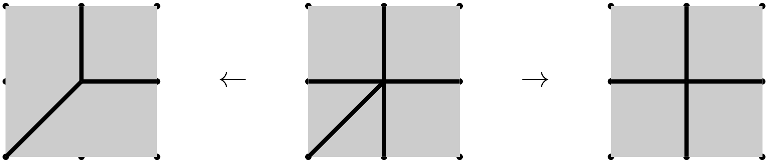
Use Mayer-Vietoris to deduce that

$$M(\mathrm{Bl}_{(0,0)}(\mathbb{A}^2), H_1 + H_2 + E) \to M(\mathbb{A}^2, H_1 + H_2) \qquad (*)$$

is an isomorphism if and only if

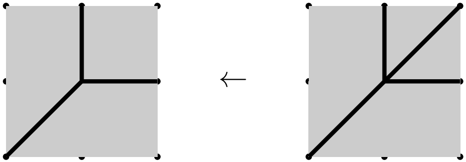
$$M(\mathrm{Bl}_{(0,0)}(\mathbb{A}^2), H_2 + E) \to M(\mathbb{A}^2, H_2) \qquad (**)$$

is an isomorphism.



$(\mathbb{P}^2, H_3)$ $\qquad$ $\overline{\square}^2$

Suppose we have $\overline{\square}$-invariance and dividing invariance.
By the previous slide and using Mayer-Vietoris the two morphisms are isomorphisms of motives.
Since $M(\overline{\square}^2) \simeq M(\mathrm{pt})$, this shows that $M(\mathbb{P}^2, \mathbb{P}^1) \simeq M(\mathrm{pt})$.



$(\mathbb{P}^2, H_1)$ $\leftarrow$ $(\mathrm{Bl}_{[0:0:1]}(\mathbb{P}^2), H_1 + E)$

Suppose we have $\overline{\square}$-invariance and $(\mathbb{P}^2, \mathbb{P}^1)$-invariance.
Then the morphism becomes an isomorphism of motives since the right fs log scheme is a $\overline{\square}$-bundle over $\overline{\square}$.
Use Mayer-Vietoris to show that the morphism

$$M(\mathrm{Bl}_{(0,0)}(\mathbb{A}^2), H_1 + E) \to M(\mathbb{A}^2, H_1) \qquad (**)$$

is an isomorphism of motives.

Explanation of the proof

Hence assuming $\overline{\square}$-invariance,

$$M(\mathrm{Bl}_{(0,0)}(\mathbb{A}^2), H_1 + H_2 + E) \simeq M(\mathbb{A}^2, H_1 + H_2) \qquad (*)$$

and

$$M(\mathbb{P}^2, \mathbb{P}^1) \simeq M(\mathrm{pt})$$

are equivalent.

Generalize this argument to arbitrary n by induction to conclude that

Theorem

$\overline{\square}$-invariance + dNis-descent in lSm/S
$\Leftrightarrow$ $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance for all n + sNis-descent in lSm/S (not SmlSm/S)

Explanation of the proof

To extend a theory $\mathcal{F}$ on *SmlSm*/*S* to *lSm*/*S*, we consider

$$\iota_\sharp \mathcal{F}(X) := \operatorname*{colim}_{Y \to X} \mathcal{F}(Y),$$

where Y runs over all dividing covers of X such that $Y \in SmlSm/S$.
We can show that $\mathcal{F}$ satisfies $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance for all n + *sNis*-descent if and only if $\iota_\sharp \mathcal{F}(X)$ satisfies $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance for all n + *sNis*-descent.
From this we can obtain

Theorem

$\overline{\square}$-invariance + dNis-descent in lSm/S
$\Leftrightarrow$ $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance for all n + sNis-descent in SmlSm/S