$\mathbb{A}^1$-homotopy theory of log schemes

Motivic geometry conference, Oslo, August 11, 2022.

Why log geometry + motivic homotopy theory?

Log geometry is useful for understanding compactification and degeneration.

Compactification: Already used in $\overline{\square}$-homotopy theory [Binda-P-Østvær].
In many cases (e.g. logTHH, logTC, Hodge cohomology ...), cohomology behaves better when compactified.

Today's talk will be focused on degeneration.

§ Log geometry

Def A pre-log ring (A, M) consists of a ring A, monoid M, and a homomorphism $\theta : M \to (A, \times)$.
It is a log ring if $\theta^{-1}(A^*) \to A^*$ is an isomorphism.
M is called fine saturated (or simply fs) if M is finitely generated, $M \to M^{gp}$ is injective, and $x \in M^{gp}$, $n \in \mathbb{N}^+$, $nx \in M \Rightarrow x \in M$.

By gluing, we can define the notion of fs log schemes
$(A, M) \rightsquigarrow \mathrm{Spec}(A, M)$.

Notation For an fs log scheme X,
$\underline{X}$:= underlying scheme of X,
$\mathcal{M}_X$:= structure sheaf of monoids
$X - \partial X := \{x \in X : \mathcal{M}_{X,x} \text{ is a group}\}$, open subscheme of X.
$\partial X := X - (X - \partial X)$, reduced scheme structure.

Def For an fs monoid P such that P^{gp} is torsion free, let $\mathrm{Spec}(P)$ be the fan associated with the dual monoid $P^\vee$.

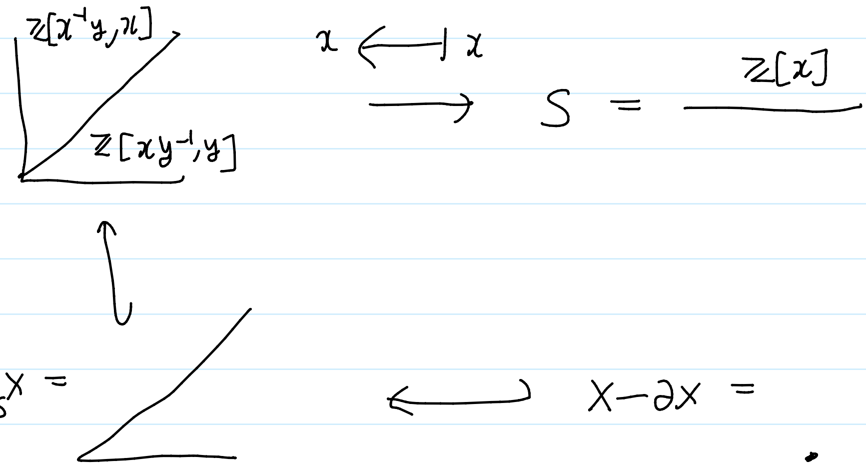
Def A morphism of fans $f : \Delta \to \Sigma$ is vertical if $f(\delta) \neq 0$ for every $\delta \in \Delta - \{0\}$.

Notation Suppose $f : X \to S$ morphism of fs log schs.
$X - \partial_S X$ is the set of points $x \in X$ s.t.
$\mathcal{M}_{S, f(x)} \to \mathcal{M}_{X, x}$ is vertical.
Regard $X - \partial_S X$ as an open subscheme of X.
$\partial_S X := X - (X - \partial_S X)$ with reduced sch structure.

Ex

$X =$ $\longrightarrow$ $S =$

$X - \partial_S X =$ $\longleftarrow$ $X - \partial X =$



Def
$$Q = \begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$ a square of fs log schs

Q is a <u>strict Nisnevich dist sq</u> if $Q \simeq S \times_{\underline{S}} \underline{Q}$ and $\underline{Q}$ is a Nisnevich dist sq.
Q is a <u>dividing dist sq</u> if $X' = S' = \emptyset$ and $X \to S$ is a surj proper log étale monomorphism.
$\rightsquigarrow$ We get the dividing Nisnevich topology.

§ Construction of SH.

Suppose S fs log sch.
lSm/S category of fs log sch log smooth over S (due to Kato)

Begin with $Shv_{dNis}(lSm/S, Spc_*)$
Invert $X \times \mathbb{A}^1 \to X$ and $X - \partial_S X \hookrightarrow X$ for all $X \in lSm/S$.
$\otimes$-invert $\mathbb{P}^1$.
$\rightsquigarrow$ We get $SH(lSm/S)$

Thm [P] If S has the trivial log structure, then

$$SH(S) \simeq SH(lSm/S).$$

Pf $\lambda : Sm/S \rightleftarrows lSm/S : \omega$ adjoint pair

$\lambda(X) := X$, $\omega(X) := X - \partial X$.

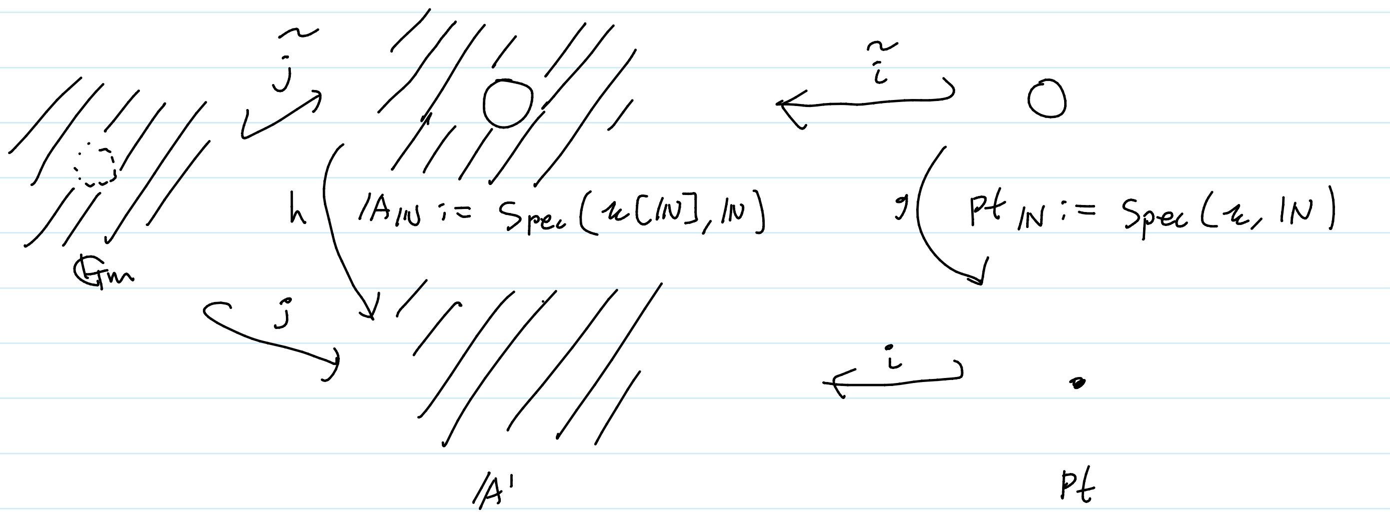
$\lambda_\# : SH(S) \rightleftarrows SH(lSm/S) : \omega_\#$ adjoint pair

$\omega_\# \lambda_\# \simeq id$ $\qquad \lambda_\# \omega_\# \Sigma^\infty X_+ \simeq \Sigma^\infty (X - \partial X)_+ \overset{\text{by ver-eq}}{\simeq} \Sigma^\infty X_+$

for $X \in lSm/S$. $\square$

Def $SH(S) := SH(lSm/S)$ for all fs log scheme S.

§ Usage of log geometry



$i^* j_* =$ "Punctured tubular neighborhood" considered by Levine over pt.

$\tilde{i}^* \tilde{j}_* =$ "Punctured tubular neighborhood" over $pt_{\mathbb{N}}$.

$i^* j_* \mathbb{1} \simeq \mathbb{1} \oplus \Sigma^{-1,-1} \mathbb{1}$.

$\tilde{i}^* \tilde{j}_* \mathbb{1} \simeq \mathbb{1}$

Expectation $g_* \tilde{i}^* \tilde{j}_* \simeq i^* h_* \tilde{j}_* \simeq i^* j_*$.

§ Cohomology theory

Def Suppose X fs log sch, $p : X \to Spec(\mathbb{Z})$ structure morphism

$\mathbb{E} \in SH(\mathbb{Z})$, $p, q \in \mathbb{Z}$.

$\mathbb{E}^{p,q}(X) := Hom_{SH(X)}(\Sigma^\infty X_+, \Sigma^{p,q} p^* \mathbb{E})$.

If $\mathbb{E} = M\Lambda$, KGL, and $MGL \in SH(\mathbb{Z})$ (Λ is a ring) we obtain motivic cohomology, homotopy K-theory, and algebraic cobordism of fs log schs.

§ Computation

<u>Thm</u> [P] Suppose $B \in Sch$, $Y \in lSm/B$.
There exists a long exact seq

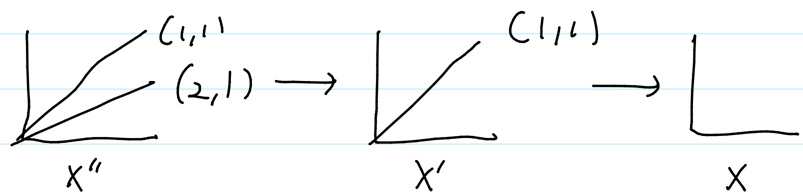
$$\cdots \to \mathbb{E}^{p,q}(\partial Y) \to \mathbb{E}^{p,q}(Y - \partial Y) \oplus \mathbb{E}^{p,q}(\underline{\partial Y}) \to \mathbb{E}^{p,q}(\underline{Y}) \to \mathbb{E}^{p+1,q}(\partial Y) \to \cdots.$$

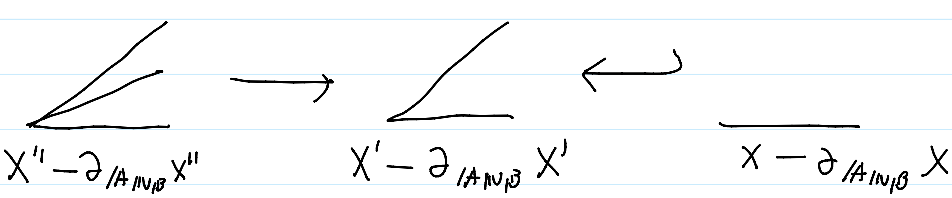
<u>Pf</u> Let us omit p,q.
The main part of the proof is to compute

$$\mathrm{Hom}_{SH(\mathbb{A}_{\mathbb{N},B})}(\Sigma^\infty X_+, \Sigma^\infty \mathbb{G}_{m+})$$

for $X \in lSm/\mathbb{A}_{\mathbb{N},B}$ in terms of cohomology in SH of schs.

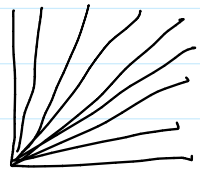
We need to compute $L_{ver \cup \mathbb{A}^1 \cup div} \Sigma^\infty \mathbb{G}_{m+}$.
However, L_{ver} has no "explicit" description since $X \mapsto X - \partial_{\mathbb{A}_{\mathbb{N},B}} X$ is <u>not</u> functorial.



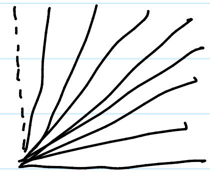


$$L_{div} \mathcal{F}(X) := \operatorname*{colim}_{X' \in X_{div}} \mathcal{F}(X').$$

$$L_{dver} \mathcal{F}(X) := \lim_{X' \in X_{div}^{op}} \mathcal{F}(X' - \partial_{/\mathbb{A}_{\mathbb{N},B}} X')$$ for div-local $\mathcal{F}$.



$\lim_{X' \in X_{div}} X'$



"dividing verticalization"

$\Phi(X) := \mathrm{Hom}_{SH(\underline{X})}(\Sigma^\infty \underline{X}_+, \Sigma^\infty \mathbb{G}_m \times_{\mathbb{A}_{\mathbb{N}}} X_+)$ for $X \in l\mathrm{Sm}/\mathbb{A}_{\mathbb{N},B}$.
Φ covariant for dividing covers, contravariant for open immersions.

Lemma. Suppose $f : X' \to \mathbb{A}_{\mathbb{N},B}$ is log smooth, $X' \to X$ div cover and X admits a fan chart.

(1) $\Phi(X') \simeq \Phi(X)$ if f is vertical

(2) $\Phi(X' - \partial_{\mathbb{A}_{\mathbb{N},B}} X') \simeq \Phi(X - \partial_{\mathbb{A}_{\mathbb{N},B}} X)$

Proven by six functors formalism for schs.

$$L_{dver} L_{div} L_{\mathbb{A}^1} L_{sNis} \Sigma^\infty \mathbb{G}_{m+}(X)$$
$$\simeq \lim_{X' \in X_{div}^{op}} L_{div} L_{\mathbb{A}^1} L_{sNis} \Sigma^\infty \mathbb{G}_{m+}(X' - \partial_{\mathbb{A}_{\mathbb{N},B}} X')$$
$$\simeq \lim_{X' \in X_{div}^{op}} \Phi(X' - \partial_{\mathbb{A}_{\mathbb{N},B}} X')$$ by Lemma 1 since $X' - \partial_{\mathbb{A}_{\mathbb{N},B}} X'$ is vertical over $\mathbb{A}_{\mathbb{N},B}$
$$\simeq \Phi(X - \partial_{\mathbb{A}_{\mathbb{N},B}} X)$$ by Lemma 2

$$L_{dver} L_{div} L_{\mathbb{A}^1} L_{sNis} \Sigma^\infty \mathbb{G}_{m+}(X - \partial_{\mathbb{A}_{\mathbb{N},B}} X) \simeq \Phi(X - \partial_{\mathbb{A}_{\mathbb{N},B}} X).$$

Hence $L_{dver} L_{div} L_{\mathbb{A}^1} L_{sNis} \Sigma^\infty \mathbb{G}_{m+} \simeq L_{ver \cup \mathbb{A}^1 \cup dNis} \Sigma^\infty \mathbb{G}_{m+}$. $\square$

For $X \in l\mathrm{Sm}/B$ s.t. $\underline{X} \to B$ proper,

$$\mathrm{Hom}_{SH(\partial X)}(\Sigma^\infty \partial X_+, \mathbb{E}) \simeq \mathrm{Hom}_{SH(B)}(\Pi_B^\infty(X - \partial X), \mathbb{E})$$

$\Pi_B^\infty(X - \partial X)$: stable motivic homotopy type at ∞ defined by Dubouloz – Déglise – Østvær

k field

$H^1_M(\partial \mathbb{A}_{\mathbb{N},k}, \mathbb{Z}(1)) \simeq k^* \oplus \mathbb{Z}$

$H^1_{log-M}(\partial \mathbb{A}_{\mathbb{N},k}, \mathbb{Z}(1)) \simeq k^*$ Gregory – Langer log motivic cohomology