Triangulated Categories of Motives over fs Log Schemes

by

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Abstract

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In this thesis, we construct triangulated categories of motives over fs log schemes with rational coefficients and formulate its six operations formalism. For these, we introduce pw-topology and *log*-weak equivalences to study the homotopy equivalences of fs log schemes. We also introduce equivariant cd-structures to deal with descent theory of motives more systematically.

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Introduction

0.1. This thesis is devoted to constructing the triangulated categories of motives over fs log schemes with rational coefficients and their six operation formalism. Throughout the introduction, let Λ be a fixed ring. For simplicity, assume also that every log scheme we deal with in the introduction is a noetherian fs log schemes over the spectrum of a fixed prime field or Dedekind domain.

Construction

0.2. As illustrated in [CD12, 16.2.18], $\mathbb{A}^1$-weak equivalences and the étale topology "generate" the right homotopy equivalences needed to produce the motivic cohomology. However, in the category of fs log schemes, we may need more homotopy equivalences. For example, consider morphisms

$$Y \xrightarrow{g} X \xrightarrow{f} S$$

of fs log schemes satisfying one of the following conditions:

(a) f is exact log smooth, and g is the verticalization $X^{\mathrm{ver}} \to X$ of f.

(b) f is the identity, and g is a pullback of $\mathbb{A}_u : \mathbb{A}_M \to \mathbb{A}_P$ where $u : M \to \operatorname{spec} P$ is a proper birational morphism of monoschemes.

(c) f is the identity, the morphism $\underline{g} : \underline{Y} \to \underline{X}$ of underlying schemes is an isomorphism, and the homomorphism

$$\overline{\mathcal{M}}^{\mathrm{gp}}_{Y,\overline{y}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\overline{g(y)}}$$

of groups is an isomorphism for any point y of Y.

(d) f is the projection $S \times \mathbb{A}_{\mathbb{N}} \to S$, and g is the 0-section $S \times \mathrm{pt}_{\mathbb{N}} \to S \times \mathbb{A}_{\mathbb{N}}$ where $\mathrm{pt}_{\mathbb{N}}$ denotes the reduced strict closed subscheme of $\mathbb{A}_{\mathbb{N}}$ whose image is the origin.

For each type (a)–(d), we should expect that $g : Y \to X$ is *homotopy equivalent* over S in some sense because the Betti realization of g seems to be homotopy equivalent over the Betti realization of S. It is not clear that $\mathbb{A}^1$-weak equivalences and the étale topology can make such morphisms of the types (a)–(d) as homotopy equivalences.

0.3. Thus we decided to introduce new topologies and new weak equivalences.

(1) The *piercing topology* on the category of fs log schemes is the Grothendieck topology generated by the morphism

$$\operatorname{Spec}\mathbb{Z}\coprod\mathbb{A}_{\mathbb{N}}\to\mathbb{A}^1$$

where the morphisms $\operatorname{Spec}\mathbb{Z}\to\mathbb{A}^1$ and $\mathbb{A}_{\mathbb{N}}\to\mathbb{A}^1$ used above are the 0-section and the morphism removing the log structure respectively.

(2) The *winding topology* on the category of fs log schemes is the Grothendieck topology generated by the morphisms

$$\mathbb{A}_\theta:\mathbb{A}_Q\to\mathbb{A}_P$$

where $\theta:P\to Q$ is a Kummer homomorphism of fs monoids.

(3) The *pw-topology* on the category of fs log schemes is the minimal Grothendieck topology generated by strict étale, piercing, and winding covers.

We choose these for technical reasons. The new weak equivalences are *log*-weak equivalences, and see (1.7.2) for the description.

0.4. Our construction of the triangulated category of motives over fs log schemes using the above notions is roughly as follows. Let S be a fs log scheme, and let ft/S denote the category of fs log schemes of finite type over S. The starting category is

$$\mathrm{D}_{\mathbb{A}^1}(\mathrm{Sh}_{pw}(ft/S,\Lambda))$$

(see [CD12, 5.3.22, 5.1.4] for the definitions). We invert all the *log*-weak equivalences in this category. Then we consider the localizing subcategory of this generated by twists and motives of the form $M_S(X)$ for *exact* log smooth morphisms $X\to S$. The resulting category is denoted by

$$\mathrm{D}_{log,pw}(S,\Lambda).$$

We do not attempt to write this as $\mathrm{DM}(S,\Lambda)$ because it is not clear whether they are equivalent or not when S is a usual scheme.

Six operations

0.5. Our next goal is to develop the Grothendieck six operations formalism. Let $\mathscr{T}$ be a triangulated, fibered over the category of fs log schemes. The formalism should contain the following information.

(1) There exists 3 pairs of adjoint functors as follows:

$$\begin{aligned} f^*:\mathscr{T}(S)\rightleftarrows\mathscr{T}(X):f_*, f:X\to S &\text{ any morphism},\\ f_!:\mathscr{T}(X)\rightleftarrows\mathscr{T}(S):f^!, f:X\to S &\text{ any separated morphism of finite type},\\ (\otimes, Hom), &\text{ symmetric closed monoidal structure on } \mathscr{T}(X).\end{aligned}$$

(2) There exists a structure of a covariant (resp. contravariant) 2-functors on $f\mapsto f_*$, $f\mapsto f_!$ (resp. $f\mapsto f^*$, $f\mapsto f^!$).

(3) There exists a natural transformation

$$\alpha_f : f_! \to f_*$$

which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.

(4) For any separated morphism of finite type $f : X \to S$, there exist natural transformations

$$f_!K \otimes_S L \xrightarrow{\sim} f_!(K \otimes_X f^*L),$$
$$Hom_S(f_!K, L) \xrightarrow{\sim} f_*Hom_X(K, f^!L),$$
$$f^!Hom_S(L, M) \xrightarrow{\sim} Hom_X(f^*L, f^!M).$$

(5) *Localization property.* For any strict closed immersion $i : Z \to S$ with complementary open immersion j, there exists a distinguished triangle of natural transformations as follows:

$$j_!j^! \xrightarrow{ad'} \mathrm{id} \xrightarrow{ad} i_*i^* \xrightarrow{\partial_i} j_!j^![1]$$

where ad' (resp. ad) denotes the counit (resp. unit) of the relevant adjunction.

(6) *Base change.* Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of fs log schemes. Assume that one of the following conditions is satisfied: f is strict, f is exact log smooth, g is strict, or g is exact log smooth. Then there exists a natural isomorphism

$$g^*f_! \longrightarrow f'_!g'^*.$$

(7) *Lefschetz duality.* Let $f : X \to S$ be an exact log smooth morphism of fs log schemes of relative dimension d, and let $j : X^{\mathrm{ver/f}} \to X$ denote its verticalization of X via f. Then there exist natural isomorphisms

$$j_*j^*f^!(-d)[-2d] \xrightarrow{\sim} f^*,$$
$$f^! \xrightarrow{\sim} j_\sharp j^*f^*(d)[2d].$$

Here, the formulations (1)–(5) are extracted from [CD12, Introduction A.5.1]. In (2.9.1), borrowing a terminology from [CD12, 2.4.45], we introduce the notion of *log motivic triangulated category.* The following is our first main theorem.

Theorem 0.6 (2.9.3 in the text)**.** A log motivic triangulated category satisfies the properties (1)–(6) in (0.5), the homotopy properties (Htp–5), (Htp–6), and (Htp–7), and the purity.

0.7. We do not prove (7) in (0.5) for log motivic triangulated categories. In [Nak97, 5.1], the proper base change theorem is proved in the context of the derived category of Kummer log étale sheaves with a more general condition than that of our formalism (6), but we do not know that such a generalization is possible to our situation.

Verification of the axioms

0.8. Our second main theorem is as follows.

Theorem 0.9 (2.9.4 in the text)**.** Assume that Λ is a $\mathbb{Q}$-algebra. Then the category $\mathrm{D}_{log,pw}(-,\Lambda)$ is a log motivic triangulated category..

0.10. With (0.6), we see that $\mathrm{D}_{log,pw}(-,\Lambda)$ satisfies the properties (1)–(6) in (0.5), the homotopy properties (Htp–5), (Htp–6), and (Htp–7), and the purity.

Poincaré duality

0.11. The following is a weaker version of (7) in (0.5).

(7)' *Poincaré duality.* Let $f : X \to S$ be a vertical exact log smooth morphism of fs log schemes of relative dimension d. Then there exist a natural isomorphism

$$f^!(-d)[-2d] \xrightarrow{\sim} f^*.$$

One of the main obstacles is not only to prove that it is an isomorphism but also to construct it. Since the construction exists locally, to circumvent this obstacle, we extend log motivic triangulated categories to diagrams of fs log schemes. Here, a diagram of fs log schemes means a functor from a small category to the category of fs log schemes. For this, we adopt Ayoub's algebraic derivator in [Ayo07]. Our third main theorem is as follows.

Theorem 0.12 (10.7.2 in the text)**.** A log motivic triangulated category satisfies (7)' in (0.11) if

(i) it can be extended to diagrams of schemes,

(ii) it satisfies the axioms of (9.1.2) and strict étale descent.

0.13. By (9.5.3), $\mathrm{D}_{log,pw}(-,\Lambda)$ can be extended to diagrams of fs log schemes, and it satisfies (9.1.2) and strict étale descent. Thus it satisfies (7)' in (0.11).

Organization

0.14. *Construction part.* In Chapter 1, we first review the notion of premotivic triangulated categories. We develop an equivariant version of cd-structures, and we discuss descent and compactness using this. After dicussing localizing subcategories and Bousfield localizations for premotivic triangulated categories, we construct the category $\mathrm{D}_{log,pw}(S,\Lambda)$ as explained in (0.4).

0.15. *Six operations part.* In Chapter 2, we review properties of morphisms in [Ayo07] and [CD12]. Many properties of morphisms in them are trivially generalized to properties for *strict* morphisms. We end this chapter by introducing the notion of log motivic triangulated categories.

In Chapter 3, we discuss results on log schemes and motives that will be needed in the later chapters.

In Chapter 4, we construct purity transformations. Let $f : X \to S$ be a vertical exact log smooth morphism of fs log schemes of relative dimension d. Unlike the case of usual schemes, the diagonal morphism

$$\underline{X} \to \underline{X} \times_{\underline{S}} \underline{X}$$

of underlying schemes is *not* a regular embedding in general. Hence we cannot apply the theorem of Morel and Voevodsky [CD12, 2.4.35]. To resolve this obstacle, we assume that the diagonal morphism $X \to X \times_S X$ has a compactified version of an exactification

$$X \xrightarrow{c} E \to X \times_S X$$

in some sense. Then c becomes a strict regular embedding, so when f is a proper exact log smooth morphism, we can apply [loc. cit] to construct the purity transformation

$$f_\sharp \longrightarrow f_*(d)[2d].$$

We discuss this construction even if the exactification does not exist in Chapter 10.

In Chapter 5, we introduce the notions of the semi-universal and universal support properties, which are generalizations of the support property for *non proper* morphisms. Then we prove that Kummer log smooth morphisms satisfy the semi-universal support property. We next prove that morphisms satisfying the semi-universal support property enjoy some good properties. Then we prove the semi-universal support property for $\mathbb{A}_\theta : \mathbb{A}_{\mathbb{N}^2} \to \mathbb{A}_{\mathbb{N}}$ where $\theta : \mathbb{N} \to \mathbb{N} \oplus \mathbb{N}$ denotes the diagonal morphism and the projection $\mathbb{A}_{\mathbb{N}} \times \mathrm{pt}_{\mathbb{N}} \to \mathrm{pt}_{\mathbb{N}}$. We end this chapter by proving the support property under the additional axiom (ii) of (2.9.1).

In Chapter 6, we develop various homotopy properties. In particular, the proof of the assertion that for morphisms of types (c) and (d) in (0.2), the morphisms

$$M_S(Y) \to M_S(X)$$

in $\mathrm{D}_{log,pw}(S, \Lambda)$ are *log*-weak equivalences is given.

0.16. *Verification of the axioms part.* In Chapter 7, we prove the localization property for various premotivic triangulated categories in the order of structural complexity. In the course of proof, we introduce the dimensional density structure, which is applied to showing that the union of dividing and Zariski cd-structure is reducing with respect to the dimensional density structure. We also introduce log'''-weak equivalences for future usage.

In Chapter 8, consider the projection $g : S \times \mathrm{pt}_{\mathbb{N}} \to S$ where S is a fs log scheme with a fs chart $\mathbb{N}$. The main purpose of the first two sections is to construct the functor

$$g_\sharp : \mathrm{D}_{log''',pw}(eSm/(Y \times \mathrm{pt}_{\mathbb{N}}), \Lambda) \to \mathrm{D}_{log''',pw}(eSm/S, \Lambda),$$

which is the left adjoint of g^*. To show this, we show that various morphisms are isomorphisms or log'''-weak equivalences. This enable us to show that g^* is conservative, and then we reduce the axiom (ii) of (2.9.1) to (5.5.5). We also discuss (Htp–1), (Htp–2), (Htp–3), and (Htp–4) for $\mathrm{D}_{log,pw}(-,\Lambda)$. This completes the proof that $\mathrm{D}_{log,pw}(-,\Lambda)$ is a log motivic triangulated categories.

0.17. *Poincaré duality part.* In Chapter 9, we select axioms of algebraic derivators to define the notion of premotivic triangulated prederivators. We prove several consequences of the axioms. Then as in Chapter 1, we discuss localizing subcategories and Bousfield localizations. We end this chapter by showing that $\mathrm{D}_{log,pw}(-,\Lambda)$ can be extended to eSm-premotivic triangulated prederivators.

In Chapter 10, we introduce the notion of compactified exactifications. Applying these to various transformations defined in Chapter 4, we construct the Poincaré duality for vertical exact log smooth separated morphism $f : X \to S$ with a fs chart having some conditions, and we show the purity. Then we collect the local constructions of purity transformations using the notion of premotivic triangulated prederivators, and we discuss its canonical version.

Terminology and conventions

0.18. *General terminology and conventions.*

(1) Let Λ be a ring throughout this thesis. We often assume that Λ is a $\mathbb{Q}$-algebra.

(2) When S is an object of a full subcategory $\mathscr{S}$ of the category of fs log schemes, we say that S is an $\mathscr{S}$-scheme.

(3) When f is a morphism in a class $\mathscr{P}$ of morphisms of a category, we say that f is a $\mathscr{P}$-morphism.

(4) If we have an adjunction $\alpha : \mathcal{C} \leftrightarrows \mathcal{D} : \beta$ of categories, then the unit $\mathrm{id} \to \beta\alpha$ is denoted by ad, and the counit $\alpha\beta \to \mathrm{id}$ is denoted by ad'.

(5) We mainly deal with fs log schemes. The fiber products of fs log schemes and fiber coproducts of fs monoids are computed in the category of fs log schemes and fs monoids respectively unless otherwise stated.

(6) An abbreviation of the strict étale topology is sét.

0.19. *Terminology and conventions for monoids.*

(1) For a monoid P, we denote by $\operatorname{Spec} P$ the set of prime ideals of P. Note that $K \mapsto (P - K)$ for ideals K of P gives one-to-one correspondence between $\operatorname{Spec} P$ and the set of faces of P.

(2) A homomorphism $\theta : P \to Q$ of monoids is said to be *strict* if $\overline{\theta} : \overline{P} \to \overline{Q}$ is an isomorphism.

(3) A homomorphism $\theta : P \to Q$ of monoids is said to be *locally exact* if for any face G of Q, the induced homomorphism $P_{\theta^{-1}(G)} \to Q_G$ is exact.

(4) Let $\theta : P \to Q$ be a homomorphism of monoids. A face G of Q is said to be *θ-critical* if $\theta^{-1}(G) = \theta^{-1}(Q^*)$. Such a face G is said to be *maximal θ-critical* if G is maximal among θ-critical faces.

(5) A homomorphism $\theta : P \to Q$ of monoids is said to be *vertical* if the cokernel of θ computed in the category of integral monoids is a group. Equivalently, θ is vertical if $\theta(P)$ is not contained in any proper face of Q.

0.20. *Terminology and conventions for log schemes.*

(1) For a monoid P with an ideal K, we denote by $\mathbb{A}_{(P,K)}$ the closed subscheme of $\mathbb{A}_P$ whose underlying scheme is $\operatorname{Spec} \mathbb{Z}[P]/\mathbb{Z}[K]$.

(2) For a sharp monoid P, we denote by pt_P the log scheme $\mathbb{A}_{(P,P^+)}$.

(3) For a log scheme S, we denote by $\underline{S}$ the underlying scheme of S, and we denote by $\mathcal{M}_S$ the étale sheaf of monoids on $\underline{S}$ given by S.

(4) For a morphism $f : X \to S$ of log schemes, $\underline{f}$ denotes the morphism $\underline{X} \to \underline{S}$ of underlying schemes.

(5) For a morphism $f : X \to S$ of fs log schemes, we say that f is a monomorphism if it is a monomorphism in the category of fs log schemes. Equivalently, f is a monomorphism if and only if the diagonal morphism $X \to X \times_S X$ is an isomorphism.

(6) For a morphism $f : X \to S$ of fine log schemes and a point $x \in X$, we say that f is *vertical at x* if the induced homomorphism

$$\overline{\mathcal{M}}_{S,f(x)} \to \overline{\mathcal{M}}_{X,x}$$

is vertical. Then the set

$$X^{\mathrm{ver}/f} := \{x \in X : f \text{ is vertical at } x\}$$

is an open subset of X, and we regard it as an open subscheme of X. The induced morphism $X^{\mathrm{ver}/f} \to S$ is said to be the *verticalization* of f, and the induced morphism $X^{\mathrm{ver}/f} \to X$ is said to be the *verticalization* of X via f.

0.21. *Terminology and conventions for monoschemes.*

(1) For a monoid P, we denote by $\operatorname{spec} P$ the monoscheme associated to P defined in [Ogu14, II.1.2.1].

(2) For a monoscheme M (see [Ogu14, II.1.2.3] for the definition of monoschemes), we denote by $\mathbb{A}_M$ the log scheme associated to M defined in [Ogu14, Section III.1.2].

Chapter 1

Construction

1.1 Premotivic categories

1.1.1. Through this section, we fix a category $\mathscr{S}$ with fiber products and a class of morphisms $\mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks.

1.1.2. In this section, we will review $\mathscr{P}$-premotivic triangulated categories and exchange structures formulated in [CD12, Section 1]. First recall from [CD13, A.1] the definition of $\mathscr{P}$-primotivic triangulated categories as follows.

Definition 1.1.3. A *$\mathscr{P}$-premotivic triangulated category* $\mathscr{T}$ over $\mathscr{S}$ is a fibered category over $\mathscr{S}$ satisfying the following properties:

(PM–1) For any object S in $\mathscr{S}$, $\mathscr{T}(S)$ is a symmetric closed monoidal triangulated category.

(PM–2) For any morphism $f : X \to S$ in $\mathscr{S}$, the functor f^* is monoidal and triangulated, and admits a right adjoint denoted by f_*.

(PM–3) For any $\mathscr{P}$-morphism $f : X \to S$, the functor f^* admits a left adjoint denoted by $f_\sharp$.

($\mathscr{P}$-BC) $\mathscr{P}$-base change: For any Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in $\mathscr{S}$ with $f \in \mathscr{P}$, the exchange transformation defined by

$$Ex : f'_\sharp g'^* \xrightarrow{ad} f'_\sharp g'^* f^* f_\sharp \xrightarrow{\sim} f'_\sharp f'^* g^* f_\sharp \xrightarrow{ad'} g^* f_\sharp$$

is an isomorphism.

($\mathscr{P}$-PF) $\mathscr{P}$-projection formula: For any $\mathscr{P}$-morphism $f : X \to S$, and any objects K in $\mathscr{T}(X)$ and L in $\mathscr{T}(S)$, the exchange transformation defined by

$$Ex : f_\sharp(K \otimes_X f^* L) \xrightarrow{ad} f_\sharp(f^* f_\sharp K \otimes_X f^* L) \xrightarrow{\sim} f_\sharp f^*(f_\sharp K \otimes_S L) \xrightarrow{ad'} f_\sharp K \otimes_S L$$

is an isomorphism.

We denote by Hom_S the internal Hom in $\mathscr{T}(S)$.

Remark 1.1.4. Note that the axiom (PM–2) implies

(1) for any morphism $f : X \to S$ in $\mathscr{S}$ and objects K and L of $\mathscr{T}(S)$, we have the natural transformation
$$f^*(K) \otimes_X f^*(L) \xrightarrow{\sim} f^*(K \otimes_S L) \tag{1.1.4.1}$$
with the coherence conditions given in [Ayo07, 2.1.85, 2.1.86].

(2) for any morphism $f : X \to S$ in $\mathscr{S}$, we have the natural transformation
$$f^*(1_S) \xrightarrow{\sim} 1_X$$
with the coherence conditions given in [Ayo07, 2.1.85].

Definition 1.1.5. Let $\mathscr{T}$ be a $\mathscr{P}$-premotivic triangulated category.

(1) Let $f : X \to S$ be a $\mathscr{P}$-morphism in $\mathscr{S}$. Then we put $M_S(X) = f_\sharp 1_X$ in $\mathscr{T}(S)$. It is called the *motive* over S represented by X.

(2) A *cartesian section* of $\mathscr{T}$ is the data of an object A_S of $\mathscr{T}(S)$ for each object S of $\mathscr{S}$ and of isomorphisms
$$f^*(A_S) \xrightarrow{\sim} A_X$$
for each morphism $f : X \to S$ in $\mathscr{S}$, subject to following coherence conditions:

  (i) the morphism $\mathrm{id}^*(A_S)^* \xrightarrow{\sim} A_S$ is the identity morphism,

  (ii) if $g : Y \to X$ is another morphisms in $\mathscr{S}$, then the diagram
$$\begin{array}{ccccc} g^*f^*(A_S) & \xrightarrow{\sim} & g^*A_X & \xrightarrow{\sim} & A_Y \\ \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ (gf)^*(A_S) & & \xrightarrow{\qquad\sim\qquad} & & A_Y \end{array}$$
  in $\mathscr{T}(Y)$ commutes.

The tensor product of two cartesian sections is defined termwise.

(3) A set of *twists* τ for $\mathscr{T}$ is a set of Cartesian sections of $\mathscr{T}$ stable by tensor product. For short, we say also that $\mathscr{T}$ is τ-twisted .

1.1.6. Let i be an object of τ. Then it defines a section i_S for each object S of $\mathscr{S}$, and for an object K of $\mathscr{T}(S)$, we simply put
$$K\{i\} = K \otimes_S i_S.$$
Then when $i, j \in \tau$, we have
$$K\{i + j\} = (K\{i\})\{j\}.$$

Note also that by (1.1.4(1)), for a morphism $f : X \to S$ in $\mathscr{S}$, we have the natural isomorphism

$$f^*(K\{i\}) \xrightarrow{\sim} (f^*K)\{i\}.$$

1.1.7. Let $\mathscr{T}$ be a $\mathscr{P}$-premotivic triangulated category. Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in $\mathscr{S}$. We associates several exchange transformations as follows.

(1) We obtain the exchange transformation

$$f^*g_* \xrightarrow{Ex} g'_*f'^*$$

by the adjunction of the exchange transformation

$$f'_\sharp g'^* \xrightarrow{Ex} g^*f_\sharp.$$

Note that it is an isomorphism when f is a $\mathscr{P}$-morphism by ($\mathscr{P}$-BC).

(2) Assume that f is a $\mathscr{P}$-morphism. Then we obtain the exchange transformation

$$Ex : f_\sharp g_* \xrightarrow{ad} f_\sharp g_* f'^* f'_\sharp \xrightarrow{Ex^{-1}} f_\sharp f^* g'_* f'_\sharp \xrightarrow{ad'} g'_* f'_\sharp.$$

(3) Assume that g_* and g'_* have right adjoints, denoted by $g^!$ and $g'^!$ respectively. If the exchange transformation

$$f^*g_* \xrightarrow{Ex} g'_*f'^*$$

is an isomorphism, then we obtain the exchange transformation

$$Ex : f'^*g'^! \xrightarrow{ad} g^!g_*f'^*g'^! \xrightarrow{Ex^{-1}} g^!f^*g'_*g'^! \xrightarrow{ad'} g^!f^*.$$

(4) For objects K of $\mathscr{T}(X)$ and L of $\mathscr{T}(S)$, we obtain the exchange transformation

$$Ex : f_*K \otimes_S L \xrightarrow{ad} f_*f^*(f_*K \otimes_S L) \xrightarrow{\sim} f_*(f^*f_*K \otimes_X f^*L) \xrightarrow{ad'} f_*(K \otimes_X f^*L).$$

(5) For objects K of $\mathscr{T}(S)$ and L of $\mathscr{T}(X)$, we obtain the natural isomorphism

$$Ex : Hom_S(K, f_*L) \longrightarrow f_*Hom_T(f^*K, L)$$

by the adjunction of (1.1.4.1).

(6) Assume that f is a $\mathscr{P}$-morphism. For objects K and L of $\mathscr{T}(S)$ and K' of $\mathscr{T}(X)$, we obtain the exchange transformations

$$f^*Hom_S(K,L) \xrightarrow{Ex} Hom_X(f^*K, f^*L),$$

$$Hom_S(f_\sharp K', L) \xrightarrow{Ex} f_*Hom_X(K', f^*L)$$

by the adjunction of the $\mathscr{P}$-projection formula.

(7) Assume that f is a $\mathscr{P}$-morphism and that the diagram is Cartesian. Then we obtain the exchange transformation

$$Ex : M_{S'}(X') = f'_\sharp 1_{X'} \xrightarrow{\sim} f'_\sharp g'^* 1_X \xrightarrow{Ex} g^* f_\sharp 1_X = g^* M_S(X).$$

Note that it is an isomorphism.

(8) Assume that f and g are $\mathscr{P}$-morphisms and the the diagram is Cartesian. Then we obtain the exchange transformation

$$\begin{aligned} Ex : M_S(X \times_S S') = f_\sharp g'_\sharp f'^* 1'_S &\xrightarrow{Ex} f_\sharp f^* g_\sharp 1_{S'} \\ &\xrightarrow{\sim} f_\sharp(1_X \otimes_X f^* g_\sharp 1_{S'}) \xrightarrow{Ex} f_\sharp 1_X \otimes_S g_\sharp 1_{S'} = M_S(X) \otimes M_S(S') \end{aligned}.$$

Note that it is an isomorphism.

(9) Assume that $\mathscr{T}$ is τ-twisted and that f is a $\mathscr{P}$-morphism. For $i \in \tau$ and an object K of $\mathscr{T}(X)$, we obtain the exchange transformation

$$Ex : f_\sharp(K\{i\}) \xrightarrow{\sim} f_\sharp(K \otimes_X f^* 1_S\{i\}) \xrightarrow{Ex} (f_\sharp K)\{i\}.$$

Note that it is an isomorphism.

(10) Assume that $\mathscr{T}$ is τ-twisted. For $i \in \tau$ and an object K of $\mathscr{T}(X)$, we obtain the exchange transformation

$$Ex : (f_*K)\{i\} \xrightarrow{\sim} f_*K \otimes_S 1_S\{i\} \xrightarrow{Ex} f_*(K \otimes_X f^* 1_S\{i\}) = f_*(K\{i\}).$$

If twists are $\otimes$-invertible, then it is an isomorphism since its right adjoint is the natural isomorphism

$$f^*(L\{-i\}) \xrightarrow{\sim} (f^*L)\{-i\}$$

where L is an object of $\mathscr{T}(S)$.

1.2 Equivariant cd-structures

1.2.1. Through this section, we fix a category $\mathscr{S}$. We also assume that Λ is a $\mathbb{Q}$-algebra.

Definition 1.2.2. Let S be an $\mathscr{S}$-scheme, let $\mathscr{P}$ be a class of morphisms of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks, and let t be a topology on $\mathscr{S}$ such that every t-covering consists of $\mathscr{P}$-morphisms.

(1) We denote by
$$\mathrm{PSh}(\mathscr{P}/S, \Lambda)$$
the category of presheaves of Λ-modules on the category of $\mathscr{P}/S$-schemes.

(2) We denote by
$$\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$$
the category of t-sheaves of Λ-modules on the category of $\mathscr{P}/S$-schemes.

Definition 1.2.3. (1) Let A be a set with a left action of a group G. Then we denote by A^G the subset of A fixed by G.

(2) Let F be a sheaf on a site $\mathcal{C}$ with a left action of a group G. Then we denote by F/G the colimit of the diagram induced by the G-action
$$F \times G \rightrightarrows F$$
in the category of sheaves on $\mathcal{C}$. Note that for any sheaf F' on $\mathcal{C}$, we have
$$\mathrm{Hom}_{\mathcal{C}}(F/G, F') \cong (\mathrm{Hom}_{\mathcal{C}}(F, F'))^G.$$
Here, the right action of G on $\mathrm{Hom}_{\mathcal{C}}(F, F')$ comes from the left action of G on F.

(3) As in [CD12, 3.3.21], for any object K of $\mathrm{C}(\mathrm{PSh}(\mathscr{P}/S, \Lambda))$ or $\mathrm{C}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda))$ with a left G-action where t is a topology on $\mathscr{S}$, we denote by K^G the complex $\mathrm{im}\, p_K$ where $p_K : K \to K$ denotes the morphism defined by the formula
$$p(x) = \frac{1}{\#G} \sum_{g \in G} g \cdot x.$$
Then we get the morphisms
$$K \xrightarrow{q_K} K^G \xrightarrow{i_K} K$$
whose composition is p_K.

Definition 1.2.4. Recall from [Ayo07, Section 4.5.3] that the $t_\emptyset$-topology on $\mathscr{S}$ is the minimal Grothendieck topology such that the empty sieve is a covering sieve for the initial object $\emptyset$. Note that a presheaf F on $\mathscr{S}$ is a $t_\emptyset$-sheaf if and only if $F(\emptyset) = *$.

Definition 1.2.5. We will introduce an equivariant version of cd-structures in [Voe10a] as follows. An *equivariant cd-structure* (or ecd-structure for abbreviation) P on $\mathscr{S}$ is a collection of pairs (G, C) where G is a group and C is a commutative diagram
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$
of $\mathscr{S}$-schemes with G-actions on X over S and on X' over S' such that

(i) g' is G-equivariant over g,

(ii) if $G \cong G'$ and $C \cong C'$, then $(G, C) \in P$ if and only if $(G', C') \in P$.

For a pair $(G, C) \in P$, C is called a P-distinguished square of group G. The t_P-topology is the Grothendieck topology generated by $t_\emptyset$-topology and morphisms of the form

$$X \coprod S' \to S \tag{1.2.5.1}$$

for $(G, C) \in P$. If G is trivial for any element of P, then P is a cd-structure defined in [Voe10a, 2.1].

1.2.6. In [Voe10a] and [Voe10b], analogous results of the Brown-Gersten theorem ([BG73]) for Nisnevich topology and cdh-topology are studied by introducing cd-structures. For instance, if we take P as the collection given in (1.2.8(4)), then we recover the Nisnevich cd-structure. In [CD12, §3.3], it is applied to study descents in triangulated categories of motives over usual schemes.

However, there is a topology like the étale topology that cannot be obtained by any cd-strucutres. In [loc. cit], descent theory for the étale topology (and more generally the h-topology) is discussed with equivarient versions of distinguished squares but without cd-structures. The reason why we introduce ecd-structures here is to study descent theory for such a topology more systematically.

1.2.7. From now on, in this section, fix a fs log scheme $\mathcal{S}$. Then we assume that $\mathscr{S}$ is a full subcategory of the category of noetherian fs log schemes over $\mathcal{S}$ such that

(i) $\mathscr{S}$ is closed under finite sums and pullbacks via morphisms of finite type,

(ii) if S belongs to $\mathscr{S}$ and $X \to S$ is strict quasi-projective, then X belongs to $\mathscr{S}$,

(iii) if S belongs to $\mathscr{S}$, then $S \times \mathbb{A}_M$ belongs to $\mathscr{S}$ for every fs monoscheme M,

(iv) If S belongs to $\mathscr{S}$, then $\underline{S}$ is belongs to $\mathscr{S}$.

Definition 1.2.8. Consider a Cartesian diagram

$$C = \begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes and a group G acting on X over S. We have several ecd-structures as follows.

(1) Recall from [Voe10b] that C is called an *additive* distinguished square (with trivial G) if $X' = \emptyset$ and $S = X \amalg S'$.

(2) Recall from [Voe10b] that C is called a *plain lower* distinguished square (with trivial G) if f and g are strict closed immersions and $S = f(X) \cup g(S')$.

(3) Recall from [Voe10b] that C is called a *Zariski* distinguished square (with trivial G) if f and g are open immersions and $S = f(X) \cup g(S')$,

(4) Recall from [Voe10b] that C is called a *strict Nisnevich* distinguished square (with trivial G) if f is strict étale, g is an open immersion, and the morphism $f^{-1}(S - g(S')) \to S - g(S')$ is an isomorphism. Here, $S - g(S')$ is considered with the reduced scheme structure.

(5) C is called a *Galois* distinguished square of group G if $X' = S' = \emptyset$, f is Galois, and G is the Galois group of f.

(6) C is called a *dividing* distinguished square (with trivial G) if $X' = S' = \emptyset$ and f is a surjective proper log étale monomorphism.

(7) C is called a *piercing* distinguished square (with trivial G) if C is a pullback of the Cartesian diagram

$$\begin{array}{ccc} \mathrm{pt}_{\mathbb{N}} & \longrightarrow & \mathbb{A}_{\mathbb{N}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Z} & \longrightarrow & \mathbb{A}^1 \end{array} \tag{1.2.8.1}$$

of $\mathscr{S}$-schemes where the lower horizontal arrow is the 0-section and the right vertical arrow is the morphism removing the log structure.

(8) C is called a *quasi-piercing* distinguished square (with trivial G) if C is a plain lower distinguished or C has a decomposition

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

such that the upper square is a plain lower distinguished square and that the lower square is a piercing distinguished square or a pullback of the Cartesian diagram

$$\begin{array}{ccc} \mathrm{pt}_{\mathbb{N}} & \longrightarrow & \mathbb{A}_{\mathbb{N}} \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathbb{N}^2} & \longrightarrow & \mathbb{A}_{\mathbb{N}} \times_{\mathbb{A}^1} \mathbb{A}_{\mathbb{N}} \end{array} \tag{1.2.8.2}$$

where the lower horizontal arrow is the 0-section and the right vertical arrow is the diagonal morphism of $\mathbb{A}_{\mathbb{N}} \to \mathbb{A}^1$ removing the log structure.

(9) For $n \in \mathbb{N}^+$, let μ_n be an n-th root of unity. Then C is called a *winding* distinguished square of group G if $X' = S' = \emptyset$, f is a pullback of the composition

$$\mathbb{A}_Q \times \operatorname{Spec}\mathbb{Z}[\mu_n] \to \mathbb{A}_Q \stackrel{\mathbb{A}_\theta}{\to} \mathbb{A}_P$$

where the first arrow is the projection, $n \in \mathbb{N}^+$, and $\theta : P \to Q$ is a Kummer homomorphism of fs monoids such that the Galois group of $\mathbb{A}_Q \times \operatorname{Spec} \mathbb{Q}[\mu_n]$ over $\mathbb{A}_P \times \operatorname{Spec} \mathbb{Q}$ exists, and G is the Galois group.

By (1.2.5), we obtain the additive, plain lower, Zariski, strict Nisnevich, dividing, piercing, quasi-piercing, Galois, and winding ecd-structures and topologies.

Definition 1.2.9. Let P be an ecd-structure on $\mathscr{S}$. As in [Voe10a], we introduce the notions of *complete, regular,* and *bounded* ecd-structures as follows.

(1) A P-simple covering is a covering that can be obtained by iterating coverings of the form (1.2.5.1).

(2) P is called *complete* if any covering sieve of an object $X \neq \emptyset$ of $\mathscr{S}$ contains a sieve generated by a P-simple covering.

(3) P is called *regular* if for any $(G, C) \in P$, C is Cartesian, $S' \to S$ is a monomorphism, and the induced morphism of t_P-sheaves

$$(\rho(X') \times_{\rho(S')} (\rho(X')/G)) \coprod \rho(X) \to \rho(X) \times_{\rho(S)} (\rho(X)/G)$$

is surjective where $\rho(S)$ denotes the representable t_P-sheaf of sets of S.

(4) Recall from [Voe10a, 2.20] that a density structure on $\mathscr{S}$ is a function which assign to any object S of $\mathscr{S}$ a sequence $D_0(S),\ D_1(S) \ldots$ of family of morphisms to S with the following conditions:

   (i) $(\emptyset \to S) \in D_0(S)$ for all S,
   (ii) isomorphisms belong to D_i for all i,
   (iii) $D_{i+1} \subset D_i$,
   (iv) if $g : Y \to X$ is in $D_i(X)$ and $f : X \to S$ is in $D_i(S)$, then $gf : Y \to S$ is in $D_i(S)$.

(5) Let $D_*(-)$ be a density structure. Then $(G, C) \in P$ is called *reducing* (with respect to D_*) if for any $i \geq 0$, and any $X_0' \in D_i(X')$, $S_0' \in D_{i+1}(S')$, $X_0 \in D_{i+1}(X)$, there exist $X_1 \in D_{i+1}(X)$, a distinguished square of G

$$C_1 = \begin{array}{ccc} X_1' & \xrightarrow{g'} & X_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S_1' & \xrightarrow{g} & S_1 \end{array}$$

of $\mathscr{S}$-schemes over S, and a G-equivariant morphism $C_1 \to C$ which coincides with the morphism $S_1 \to S$ on the right corner and whose other respective components factor through X_0', S_0', X_0.

(6) A G-equivariant morphism $(G', C') \to (G, C)$ of P-distinguished squares is called a refinement if the morphism is the identity on G and the identity on the right corner.

(7) Let $D_*(-)$ be a density structure. Then P is called *bounded* by $D_*(-)$ if every element of P is reducing with respect to $D_*(-)$ and that for any object X of $\mathscr{S}$, there exists n such that any element of $D_n(X)$ is an isomorphism.

Definition 1.2.10. For a noetherian scheme S, recall from [Voe10b] the *standard density structure* $D_d(S)$ as follows. An open immersion $U \to S$ is in $D_d(S)$ if for any irreducible component Z_i of $S - U$, there is an irreducible component S_i of S containing Z_i such that $\dim S_i \geq d + \dim Z_i$.

Then for an $\mathscr{S}$-scheme X, we denote by $D_d(X)$ the family $D_d(\underline{X})$. It is again called the *standard density structure.*

1.2.11. The notions of *complete, regular,* and *bounded* ecd-structures will be used in (1.3.6). Hence let us study these notions for the ecd-structures defined in (1.2.8).

Proposition 1.2.12. *The additive, plain lower, Zariski, and strict Nisnevich ecd-structures are complete, regular, and bounded by the standard density structure.*

Proof. It follows from [Voe10b, 2.2]. □

Proposition 1.2.13. *The piercing, quasi-piercing, Galois, and winding ecd-structures are complete.*

Proof. It follows from [Voe10b, 2.5]. □

Proposition 1.2.14. *The quasi-piercing cd-structure is regular.*

Proof. Consider a commutative diagram

$$C = \begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. If it is a plain lower distinguished square, we are done by (1.2.12). Hence we may assume that C has a decomposition

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{g''} & Y \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ S' & \xrightarrow{g} & S \end{array}$$

such that the upper square is a plain lower distinguished square and the lower square is a pullback of (1.2.8.1) or (1.2.8.2). We want to show that the induced Cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle d} \\ X' \times_{S'} X' & \xrightarrow{g' \times_g g'} & X \times_S X \end{array} \qquad (1.2.14.1)$$

of $\mathscr{S}$-schemes where d denotes the diagonal morphism is again a quasi-piercing distinguished square.

Since p is a strict closed immersion, we have

$$X \cong (X \times_S X) \times_{Y \times_S Y, d'} Y,$$

$$X' \times_{S'} X' \cong (Y' \times_{S'} Y') \times_{Y \times_S Y} (X \times_S X).$$

where $d' : Y \to Y \times_S Y$ denotes the diagonal morphism. Thus (1.2.14.1) is a pullback of the Cartesian diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow d' \\ Y' \times'_S Y' & \xrightarrow{g'' \times_g g''} & Y \times_S Y \end{array} \tag{1.2.14.2}$$

of $\mathscr{S}$-schemes via $p \times p : X \times_S X \to Y \times_S Y$. Then the remaining is to show that (1.2.14.2) is a quasi-piercing distinguished square. By definition, (1.2.14.2) is a pullback of (1.2.8.2) or the Cartesian diagram

$$\begin{array}{ccc} \mathrm{pt}_{\mathbb{N}} & \longrightarrow & \mathbb{A}_{\mathbb{N}} \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ \mathrm{pt}_{\mathbb{N}} & \longrightarrow & \mathbb{A}_{\mathbb{N}} \end{array} \tag{1.2.14.3}$$

where the horizontal arrows are the 0-section. The square (1.2.8.2) is a quasi-piercing distinguished square by definition, and the square (1.2.14.3) is a plain lower distinguished, which is a quasi-piercing distinguished square. Thus (1.2.14.2) is a quasi-piercing distinguished square. □

Proposition 1.2.15. *The quasi-piercing cd-structures is bounded by the standard density structure.*

Proof. Consider a quasi-piercing distinguished square

$$C = \begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. As in the proof of [Voe10b, 2.11], if we replace X by the scheme-theoretic closure of the open subscheme $f^{-1}(S' - S)$, we get another quasi-piercing distinguished square which is a refinement of the original one. Then the same proof of [Voe10b, 2.12] can be applied to our situation. □

Proposition 1.2.16. *The union of the additive and Galois ecd-structures is regular.*

Proof. The additive structure is regular by [Voe10b, 2.2]. Thus as in the proof of [CD12, 3.3.19], the question is equivalent to the assertion that for any additive and Galois sheaf of sets F and any Galois cover $f : X \to S$, the function

$$F(S) \to F(X)^G$$

induced by $f^* : F(X) \to F(Y)$ is a bijection. This follows from the fact that the cokernel of the induced functions

$$F(X) \rightrightarrows F(X \times_S X) = F(X \times G) \cong F(X) \times G$$

is exactly $F(X)^G$. □

Proposition 1.2.17. *The union of the plain lower and winding ecd-structures is regular.*

Proof. Let $f : X \to S$ be a winding cover, which is a pullback of the composition

$$\mathbb{A}_Q \times \operatorname{Spec} \mathbb{Z}[\mu_n] \to \mathbb{A}_Q \xrightarrow{\mathbb{A}_\theta} \mathbb{A}_P$$

where the first arrow is the projection, $n \in \mathbb{N}^+$, and $\theta : P \to Q$ is a Kummer homomorphism of fs monoids such that the Galois group G of $\mathbb{A}_Q \times \operatorname{Spec} \mathbb{Q}[\mu_n]$ over $\mathbb{A}_P \times \operatorname{Spec} \mathbb{Q}$ exists. We denote by

$$\varphi(g) : Q \oplus \mathbb{Z}/(n) \to \varphi : Q \oplus \mathbb{Z}/(n)$$

the homomorphism induced by g. We have

$$X \times_S X = \bigcup_{g \in G} X_g$$

where X_g denotes the graph of the automorphism $X \to X$ induced by $g \in G$.

We will show that X_g is a closed subscheme of $X \times_S X$. We put $Q' = Q \oplus \mathbb{Z}/(n)$. It suffices to show that for any $g \in G$, the homomorphism

$$Q' \oplus_P Q' \to Q', \quad (a, b) \mapsto a + \varphi(g)(b)$$

is strict. Composing with the isomorphism

$$Q' \oplus_P Q' \to Q' \oplus Q', \quad (a, b) \mapsto (a, \varphi(g^{-1})(b)),$$

it suffices to show that the summation homomorphism

$$Q' \oplus_P Q' \to Q'$$

is strict. It follows from (1.2.18) below.

Then as in the proof of [CD12, 3.3.19], the question is equivalent to the assertion that for any plain lower and winding sheaf of sets F and any winding cover $f : X \to S$, the function

$$F(S) \to F(X)^G$$

induced by $f^* : F(X) \to F(Y)$ is a bijection. The function

$$F(X \times_S X) \to \coprod_{g \in G} F(X_g)$$

is injective since F is a plain lower sheaf, so the conclusion follows from the fact that the cokernel of the compositions

$$F(X) \rightrightarrows F(X \times_S X) \to \coprod_{g \in G} F(X_g)$$

is exactly $F(X)^G$. □

Lemma 1.2.18. *Let $\theta : P \to Q$ be a Kummer homomorphism of fs monoids. Then the summation homomorphism $\eta : Q \oplus_P Q \to Q$ is strict.*

Proof. The homomorphism $\overline{\eta} : \overline{Q \oplus_P Q} \oplus \overline{Q}$ is surjective, so the remaining is to show that $\overline{d}$ is injective. Choose $n \in \mathbb{N}^+$ such that $nq \subset \theta(P)$. For any $q \in Q$, $n(q, -q) = (nq, 0) + (0, -nq) = 0$ because $nq \in \theta(P)$. Thus $(q, -q) \in (Q \oplus_P Q)^*$ since $Q \oplus_P Q$ is saturated. Let Q' denote the submonoid of $Q \oplus_P Q$ generated by elements of the form $(q, -q)$ for $q \in Q^{\text{gp}}$. Then $Q' \subset (Q \oplus_P Q)^*$, and $Q/Q' \cong Q$. The injectivity follows from this. □

Proposition 1.2.19. *The Galois and winding ecd-structures are bounded by the standard density structure.*

Proof. If follows from [Voe10b, 2.9]. □

Theorem 1.2.20. *Any combination of unions of the additive, plain lower, Zariski, strict Nisnevich, quasi-piercing, additive+Galois, and plain lower+winding ecd-structures is complete, regular, and bounded by the standard density structure.*

Proof. It follows from (1.2.12), (1.2.13), (1.2.14), (1.2.15), (1.2.16), (1.2.17), (1.2.19), and [Voe10a, 2.6, 2.12, 2.24]. □

Definition 1.2.21. The Grothendieck topology on $\mathscr{S}$ generated by the strict étale, piercing, and winding topologies is called the pw-topology , and the Grothendieck topology on $\mathscr{S}$ generated by the strict étale, quasi-piercing, and winding topologies is called qw-topology .

1.2.22. By [CD12, 3.3.26], the strict étale topology is the minimal Grothendieck topology generated by the strict Nisnevich and Galois topologies, and the additive topology is coarser than the strict étale topology. Thus the strict étale topology and qw-topology are unions of topologies in (1.2.20).

1.3 Descents

1.3.1. Through this section, we fix a full subcategory $\mathscr{S}$ of the category of noetherian fs log schemes satisfying the conditions of (1.2.7). We also assume that Λ is a $\mathbb{Q}$-algebra.

Definition 1.3.2. Let S be an $\mathscr{S}$-scheme, let $\mathscr{P}$ be a class of morphisms of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks, and let t be a topology on $\mathscr{S}$ such that every t-covering consists of $\mathscr{P}$-morphisms.

(1) eSm/S denotes the category of $\mathscr{S}$-schemes exact log smooth over S. The class of exact log smooth morphisms in $\mathscr{S}$ is denoted by eSm.

(2) lSm/S denotes the category of $\mathscr{S}$-schemes log smooth over S. The class of exact log smooth morphisms in $\mathscr{S}$ is denoted by lSm.

(3) ft/S denotes the category of $\mathscr{S}$-schemes of finite type over S. The class of exact log smooth morphisms in $\mathscr{S}$ is denoted by ft.

(4) For any presheaf F on $\mathscr{P}/S$, we denote by $\Lambda_S(F)$ the Λ-free presheaf

$$(X \in ft/S) \mapsto \Lambda^{F(S)}.$$

Then we denote by $\Lambda_S^t(F)$ its associated t-sheaf.

(5) For any $\mathscr{P}$-morphism $X \to S$, we denote by $\Lambda_S^t(X)$ the free sheaf in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ represented by $X \to S$.

(6) We denote by $\mathrm{C}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda))$ the category of unbounded complexes in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$. An object C of this category is called a *complex* in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$.

(7) We denote by $\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda))$ the category of unbounded complexes in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ modulo the chain homotopy equivalences.

(8) If $\mathscr{X} = (\mathscr{X}_i)$ be a simplicial $\mathscr{S}$-scheme over S, then we denote by $\Lambda_S^t(\mathscr{X})$ the associated complex

$$\cdots \to \Lambda_S^t(\mathscr{X}_i) \to \cdots \to \Lambda_S^t(\mathscr{X}_0) \to 0 \to \cdots .$$

(9) Recall from [CD12, 5.1.9] that a complex C in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ is said to be *t-local* if for any $\mathscr{P}$-morphism $X \to S$ and any $n \in \mathbb{Z}$ the induced homomorphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda_S^t(X)[n], C) \to \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda_S^t(X)[n], C)$$

(10) Recall from [CD12, 5.1.9] that a complex C in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ is said to be *t-flasque* if for any $\mathscr{P}$-morphism $X \to S$, any t-hypercover $\mathscr{X} \to X$, and any $n \in \mathbb{Z}$ the induced homomorphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda_S^t(X)[n], C) \to \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda_S^t(\mathscr{X})[n], C)$$

is an isomorphism. Note that by [CD12, 5.1.13], C is t-local if and only if C is t-flasque.

1.3.3. We refer to [CD12, 3.2.5] for the definition of t-descent . For example, a complex K in $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ satisfies t-descent if and only if K is t-flasque by definition.

Definition 1.3.4. Let S be an $\mathscr{S}$-scheme, let $\mathscr{P}$ be a class of morphisms of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks, and let P be a ecd-structure on $\mathscr{S}/S$. We put $t = t_P$ for brevity. We denote by BC_P the union of the family of bounded complexes of the form

$$\Lambda_S^t(X')^G[n] \to \Lambda_S^t(X)^G[n] \oplus \Lambda_S^t(T')[n] \to \Lambda_S^t(T)[n]$$

for P-distinguished squares of group G

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of $\mathscr{S}$-schemes over S and $n \in \mathbb{Z}$ and the family of bounded complexes of the form

$$M_S(\emptyset)[n]$$

for $n \in \mathbb{Z}$. A complex C in $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ is said to be BC_P-*local* if

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(D, C) = 0$$

for any object D of BC_P.

1.3.5. Many results in [Voe10a] can be trivially generalized to ecd-structures and complexes of presheaves of Λ-modules. The following theorem is such an example.

Theorem 1.3.6. *Let S be an $\mathscr{S}$-scheme, and let K be a presheaf of complexes of Λ-modules on $\mathscr{S}/S$, and let P be a ecd-structure on $\mathscr{S}/S$. Consider the following conditions.*

(i) $K(\emptyset) = 0$, and for any P-distinguished square of G

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

of $\mathscr{S}$-schemes over S, the diagram

$$\begin{array}{ccc} K(T) & \longrightarrow & K(T') \\ \downarrow & & \downarrow \\ K(X)^G & \longrightarrow & K(X')^G \end{array}$$

is homotopy Cartesian in the derived category of Λ-modules.

(ii) The image of K in $\mathrm{D}(\mathrm{PSh}(\mathscr{P}/S,\Lambda))$ *is BC_P-local.*

(iii) The image of K in $\mathrm{D}(\mathrm{PSh}(\mathscr{P}/S,\Lambda))$ *is t_P-local.*

Then we have the implication (i) $\Leftrightarrow$ (ii). When P is a complete, regular, and bounded, we also have the implication (ii) $\Leftrightarrow$ (iii).

Proof. The equivalence of (i) and (ii) follows from the point 1 of [Voe10a, 3.8] with the generalization (1.3.5). When P is complete, regular, and bounded, the equivalence of (ii) and (iii) follows from the points 2 and 3 of [Voe10a, 3.8] with the generalization (1.3.5). □

Corollary 1.3.7. *Let S be an $\mathscr{S}$-scheme, and let K be an object of* $\mathrm{D}(\mathrm{PSh}(\mathscr{P}/S,\Lambda))$. *If P is a complete, regular, and bounded ecd-structure on $\mathscr{S}$, then the following conditions are equivalent.*

(i) For any morphism $p : T \to S$ of $\mathscr{S}$-schemes, and for any P-distinguished square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of $\mathscr{S}$-schemes with a G-action, the commutative diagram

$$\begin{array}{ccc} p_*p^*K & \xrightarrow{ad} & p_*g_*g^*p^*K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ (p_*f_*f^*p^*K)^G & \xrightarrow{ad} & (p_*h_*h^*p^*K)^G \end{array}$$

is homotopy Cartesian where $h = fg'$.

(ii) K satisfies t_P-descent.

Proof. By definition, the condition (i) is equivalent to the condition that and for any objects and E of $\mathrm{D}(\mathrm{PSh}(\mathscr{P}/S,\Lambda))$, the presheaf of complexes of Λ-modules

$$\mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(-,K))$$

on $\mathscr{S}/S$ (see [CD12, 3.2.11.3, 3.2.15] for the definitions) satisfies the condition (i) of (1.3.6). Then by the implication (i) $\Leftrightarrow$ (iv) of (loc. cit), it is equivalent to the condition that

$$\mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(-,K))$$

satisfies t_P-descent over $\mathscr{S}/S$. Finally, it is equivalent to the condition (ii) by [CD12, 3.2.18]. □

Corollary 1.3.8. *Let S be an $\mathscr{S}$-scheme, let K be an object of* $\mathrm{D}(\mathrm{PSh}(\mathscr{P}/S,\Lambda))$, *and let P be a union of the additive, plain lower, Zariski, strict Nisnevich, quasi-piercing, additive+Galois, and plain lower+winding ecd-structures. Then the conclusion of (1.3.7) is satisfied.*

Proof. It follows from (1.3.7) and (1.2.20). □

1.4 Compactness

1.4.1. Through this section, we fix a category $\mathscr{S}$ and a class of morphisms $\mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks.

Definition 1.4.2. Let $\mathscr{T}$ be triangulated category which admits small sums. Recall from [CD12, 1.3.15] the following definitions.

(1) An object X of $\mathscr{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathscr{T}}(X,-)$ commutes with small sums.

(2) A class $\mathcal{G}$ of objects of $\mathscr{T}$ is called *generating* if the family of functors

$$\mathrm{Hom}_{\mathscr{T}}(X[n],-)$$

for $X \in \mathcal{G}$ and $n \in \mathbb{Z}$ is conservative.

(3) $\mathscr{T}$ is called *compactly generated* if there exists a generating set $\mathcal{G}$ of compact objects of $\mathscr{T}$.

Definition 1.4.3. Let $\mathscr{T}$ be a $\mathscr{P}$-premotivic triangulated category over $\mathscr{S}$.

(1) We say that $\mathscr{T}$ is *generated by* $\mathscr{P}$ *and* τ if for any object S of $\mathscr{S}$, the family of objects of the form

$$M_S(X)\{i\}$$

for a $\mathscr{P}$-morphism $X \to S$ and $i \in \tau$ generates $\mathscr{T}(S)$.

(2) We say that $\mathscr{T}$ is *compactly generated by* $\mathscr{P}$ *and* τ if $\mathscr{T}$ is generated by $\mathscr{P}$ and τ and for any $\mathscr{P}$-morphism $X \to S$ and $i \in \tau$, $M_S(X)\{i\}$ is compact.

(3) We say that $\mathscr{T}$ is *well generated* if $\mathscr{T}(S)$ is well generated in the sense of [Nee01, 8.1.7] for any object S of $\mathscr{S}$.

(4) We say that $\mathscr{T}$ is *well generated by* $\mathscr{P}$ *and* τ if $\mathscr{T}$ is well generated and generated by $\mathscr{P}$ and τ.

Note that $\mathscr{T}$ is compactly generated by $\mathscr{P}$ and τ if and only if $\mathscr{T}$ is generated by $\mathscr{P}$ and τ and compactly τ-generated in the sense of [CD12, 1.3.16].

1.4.4. Let $\mathscr{T}$ be a well generated $\mathscr{P}$-premotivic triangulated category over $\mathscr{S}$. Recall from [CD12, 1.3.17] that a family of objects $\mathcal{G}$ of $\mathscr{T}$ generates $\mathscr{T}$ if and only if $\mathscr{T}$ is the localizing subcategory of $\mathscr{T}$ generated by $\mathcal{G}$.

1.4.5. Assume that $\mathscr{S}$ be a full subcategory of the category fs log schemes satisfying the conditions of (1.2.7). Assume also that Λ is a $\mathbb{Q}$-algebra. Let P be any combination of unions of the additive, plain lower, Zariski, strict Nisnevich, quasi-piercing, additive+Galois, and plain lower+winding ecd-structures. Then BC_P in (1.3.4) is a *bounded generating family for* t_P*-hypercovering* in $\mathrm{Sh}_{t_P}(ft,\Lambda)$ in the sense of [CD12, 5.1.28]. We will use this in (1.7.5).

1.5 Localizing subcategories

1.5.1. Through this section, we fix a category $\mathscr{S}$ and classes of morphisms $\mathscr{P}' \subset \mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks. We fix also a τ-twisted $\mathscr{P}$-premotivic triangulated category $\mathscr{T}$ well generated by $\mathscr{P}$ and τ.

For an object S of $\mathscr{S}$, we denote by $\mathcal{F}_{\mathscr{P}'/S}$ the family of motives of the form

$$M_S(X)\{i\}$$

for $\mathscr{P}'$-morphism $X \to S$ and twist $i \in \tau$. Then we denote by $\mathscr{T}(\mathscr{P}'/S)$ the localizing subcategory of $\mathscr{T}(S)$ generated by $\mathcal{F}_{\mathscr{P}'/S}$, and we denote by $\mathscr{T}(\mathscr{P}')$ the collection of $\mathscr{T}(\mathscr{P}'/S)$ for object S of $\mathscr{S}$. The purpose of this section is to show that $\mathscr{T}(\mathscr{P}')$ has a structure of $\mathscr{P}'$-premotivic triangulated category.

1.5.2. For a $\mathscr{P}$-morphism $X \to S$, we denote by $M_{\mathscr{P}'/S}(X)$ the image of $M_S(X)$ in $\mathscr{T}_{\mathscr{P}'}$, and we denote by $\rho_\sharp$ the inclusion functor

$$\mathscr{T}(\mathscr{P}') \to \mathscr{T}.$$

Then the set of twists τ for $\mathscr{T}$ gives a set of twists for $\mathscr{T}(\mathscr{P}')$. It is denoted by τ again. Since $\mathscr{T}$ is well generated by $\mathscr{P}$ and τ by assumption, $\mathscr{T}(\mathscr{P}')$ is well generated by $\mathscr{P}'$ and τ. By [Nee01, 8.4.4], $\rho_\sharp$ has a right adjoint

$$\rho^* : \mathscr{T} \to \mathscr{T}(\mathscr{P}')$$

since $\rho_\sharp$ respects small sums. For any object S of $\mathscr{S}$, we denote by

$$\rho_{\sharp,S} : \mathscr{T}(\mathscr{P}'/S) \rightleftarrows \mathscr{T}(S) : \rho_S^*$$

the specification of $\rho_\sharp$ and ρ^* to S.

1.5.3. Let X and S be objects of $\mathscr{S}$. Consider a diagram

$$\begin{array}{ccc}
\mathscr{T}(\mathscr{P}'/X) & & \mathscr{T}(\mathscr{P}'/S) \\
{\scriptstyle \rho_{\sharp,X}}\downarrow\uparrow{\scriptstyle \rho_X^*} & & {\scriptstyle \rho_{\sharp,S}}\downarrow\uparrow{\scriptstyle \rho_S^*} \\
\mathscr{T}(X) & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \mathscr{T}(S)
\end{array}$$

such that α is left adjoint to β. Suppose that α maps $\mathcal{F}_{\mathscr{P}'/S}$ into $\mathscr{T}(\mathscr{P}'/S)$ and that α commutes with twists. Then we define

$$\alpha_{\mathscr{P}'} : \mathscr{T}(\mathscr{P}'/X) \to \mathscr{T}(\mathscr{P}'/S),$$

$$\beta_{\mathscr{P}'} : \mathscr{T}(\mathscr{P}'/S) \to \mathscr{T}(\mathscr{P}'/X)$$

as $\alpha_{\mathscr{P}'} = \rho_S^* \alpha \rho_{\sharp,X}$ and $\beta_{\mathscr{P}'} = \rho_X^* \beta \rho_{\sharp,S}$. We often omit $\mathscr{P}'$ in $\alpha_{\mathscr{P}'}$ and $\beta_{\mathscr{P}'}$ for brevity.

Proposition 1.5.4. *Under the notations and hypotheses of (1.5.3),*

(1) α *commutes with* $\rho_\sharp$, *i.e.,* $\rho_{\sharp,S}\alpha_{\mathscr{P}'} \cong \alpha\rho_{\sharp X}$,

(2) $\alpha_{\mathscr{P}'}$ *is left adjoint to* $\beta_{\mathscr{P}'}$,

Proof. (1) The counit

$$\rho_{\sharp,S}\rho_S^*\alpha\rho_{\sharp,X} \longrightarrow \alpha\rho_{\sharp,X}$$

is an isomorphism since $\rho_{\sharp,S}$ is fully faithful and the essential image of $\alpha\rho_{\sharp,X}$ is in the essential image of $\rho_{\sharp,S}$. This proves the statement.

(2) We will show this by constructing the unit and counit. The unit

$$\mathrm{id} \xrightarrow{ad} \beta_{\mathscr{P}'}\alpha_{\mathscr{P}'}$$

is constructed by

$$\mathrm{id} \xrightarrow{ad} \rho_X^*\rho_{\sharp,X} \xrightarrow{ad} \rho_X^*\beta\alpha\rho_{\sharp,X} \xrightarrow{ad'^{-1}} \rho_X^*\beta\rho_{\sharp,S}\rho_S^*\alpha\rho_{\sharp,X}.$$

Here, the third arrow is defined and an isomorphism by (1). The counit

$$\alpha_{\mathscr{P}'}\beta_{\mathscr{P}'} \xrightarrow{ad'} \mathrm{id}$$

is constructed by

$$\rho_S^*\alpha\rho_{\sharp,X}\rho_X^*\beta\rho_{\sharp,S} \xrightarrow{ad'} \rho_S^*\alpha\beta\rho_{\sharp,S} \xrightarrow{ad'} \rho_S^*\rho_{\sharp,S} \xrightarrow{ad^{-1}} \mathrm{id}.$$

Here, the third arrow is defined and an isomorphism since $\rho_{\sharp,S}$ is fully faithful. These two satisfy the counit-unit equations, so $\alpha_{\mathscr{P}'}$ is left adjoint to $\beta_{\mathscr{P}'}$. □

Proposition 1.5.5. *Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$*-schemes. Assume that* g^* *and* g'^* *commutes with* ρ^* *and that the exchange transformation*

$$g^*f_* \xrightarrow{Ex} f'_*g'^*$$

is an isomorphism. Then the exchange transformation

$$g^*_{\mathscr{P}'}f_{*,\mathscr{P}'} \xrightarrow{Ex} f'_{*,\mathscr{P}'}g'^*_{\mathscr{P}'}$$

is also an isomorphism.

Proof. Since ρ^* is essentially surjective, it suffices to show that the natural transformation

$$g^*_{\mathscr{P}'} f_{*,\mathscr{P}'} \rho^*_{S'} \xrightarrow{Ex} f'_{*,\mathscr{P}'} g'^*_{\mathscr{P}'} \rho^*_{S'}$$

is an isomorphism. By the condition that g^* and g'^* commutes with ρ^*, it is equivalent to the assertion that the natural transformation

$$\rho^*_X g^* f_* \xrightarrow{Ex} \rho^*_X f'_* g'^*$$

is an isomorphism. This follows from the other condition. □

1.5.6. We will define operations ($f_{\sharp,\mathscr{P}'}$ for $f \in \mathscr{P}'$, $f^*_{\mathscr{P}'}$, $f_{*,\mathscr{P}'}$, $\otimes$, and Hom) and prove the axioms of $\mathscr{P}'$-premotivic categories for $\mathscr{T}$ as follows.

(1) For any object S of $\mathscr{S}$, we put $1_{\mathscr{P}',S} = \rho_{\sharp,S} 1_S$. We often omit $\mathscr{P}'$ in the notation for brevity.

(2) *Construction of $f_{\sharp,\mathscr{P}'}$ for $f \in \mathscr{P}'$, $f^*_{\mathscr{P}'}$, and $f_{*,\mathscr{P}'}$.* By (1.5.4), we have adjunctions

$$f_{\sharp,\mathscr{P}} : \mathscr{T}(\mathscr{P}/X) \rightleftarrows \mathscr{T}(\mathscr{P}/S) : f^*_{\mathscr{P}}$$

$$f^*_{\mathscr{P}} : \mathscr{T}(\mathscr{P}/X) \rightleftarrows \mathscr{T}(\mathscr{P}/S) : f_{*,\mathscr{P}}$$

where in the first one, we assume that $f \in \mathscr{P}'$. Then by (loc. cit), $f_{\sharp,\mathscr{P}'}$ for $f \in \mathscr{P}'$ and $f^*_{\mathscr{P}'}$ commute with $\rho_\sharp$.

(3) *Functoriality of $f^*_{\mathscr{P}'}$.* Let $f : X \to S$ and $g : Y \to X$ be morphisms in $\mathscr{S}$. Then the natural isomorphism

$$g^*_{\mathscr{P}'} f^*_{\mathscr{P}'} \longrightarrow (fg)^*_{\mathscr{P}'}$$

is constructed by

$$\rho^*_S g^* \rho_{\sharp,S} \rho^*_S f^* \rho_{\sharp,S} \xrightarrow{ad'} \rho^*_S g^* f^* \rho_{\sharp,S} \xrightarrow{\sim} \rho^*_S (fg)^* \rho_{\sharp,S}.$$

The usual cocycle condition for $f^*_{\mathscr{P}'}$ follows from the usual cocycle condition for f^*. Thus $\mathscr{T}$ is a fibered category over $\mathscr{S}$.

(4) *Construction of $\otimes$.* For an object S of $\mathscr{S}$ and objects K and L of $\mathscr{T}(\mathscr{P}'/S)$, we denote by $K \otimes_{\mathscr{P}'/S} L$ the object

$$\rho^*_S(\rho_{\sharp,S} K \otimes_S \rho_{\sharp,S} L)$$

in $\mathscr{T}(\mathscr{P}'/S)$.

(5) *Monoidality of $\rho_{\sharp,S}$.* The morphism

$$\rho_{\sharp,S}(K \otimes_{\mathscr{P}'/S} L) \longrightarrow (\rho_{\sharp,S} K) \otimes_S (\rho_{\sharp,S} L)$$

is constructed by

$$\rho_{\sharp,S}\rho_S^*(\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}L) \xrightarrow{ad'} (\rho_{\sharp,S}K) \otimes_S (\rho_{\sharp,S}L). \tag{1.5.6.1}$$

We will show that it is an isomorphism. To show this, since $\mathscr{T}(\mathscr{P}'/S)$ is well generated by $\mathscr{P}$ and τ, it suffices to show that the morphism

$$\rho_{\sharp,S}(M_{\mathscr{P}'/S}(V) \otimes_{\mathscr{P}'/S} M_{\mathscr{P}'/S}(W)) \longrightarrow (\rho_{\sharp,S}M_{\mathscr{P}'/S}(V)) \otimes_S (\rho_{\sharp,S}M_{\mathscr{P}'/S}(V))$$

is an isomorphism for $\mathscr{P}'$-morphisms $V \to S$ and $W \to S$. It follows from the commutative diagram

$$\begin{array}{ccc} \rho_{\sharp,S}(M_{\mathscr{P}'/S}(V) \otimes_{\mathscr{P}'/S} M_{\mathscr{P}'/S}(W)) & \longrightarrow & (\rho_{\sharp,S}M_{\mathscr{P}'/S}(V)) \otimes_S (\rho_{\sharp,S}M_{\mathscr{P}'/S}(V)) \\ & \searrow^{\sim} & \downarrow \\ & & M_S(V \times_S W) \end{array}$$

in $\mathscr{T}(S)$.

We can similarly construct the isomorphism

$$\rho_{\sharp,S}(1_{\mathscr{P}'/S}) \xrightarrow{\sim} 1_S.$$

We will show below that $- \otimes_{\mathscr{P}'/S} -$ gives a closed symmetric monoidal structure on $\mathscr{T}(\mathscr{P}'/S)$. With this structure, one can check that the coherence conditions given in [Ayo07, 2.1.79, 2.1.81] are satisfied, i.e., the functor $\rho_{\sharp,S}$ is monoidal.

(6) *Functoriality of* $\otimes$. The natural transformation

$$(- \otimes_{\mathscr{P}'/S} -) \otimes_{\mathscr{P}'/S} - \longrightarrow - \otimes_{\mathscr{P}'/S} (- \otimes_{\mathscr{P}'/S} -)$$

is constructed by the composition

$$\begin{aligned} \rho_S^*(\rho_{\sharp,S}\rho_S^*(\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}L) \otimes_S \rho_{\sharp,S}N) &\xrightarrow{ad'} \rho_S^*((\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}L) \otimes_S \rho_{\sharp,S}N) \\ &\xrightarrow{\sim} \rho_S^*(\rho_{\sharp,S}K \otimes_S (\rho_{\sharp,S}L \otimes_S \rho_{\sharp,S}N)) \\ &\xrightarrow{ad'^{-1}} \rho_S^*(\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}\rho_S^*(\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}N)) \end{aligned}$$

for objects K, L, and N of $\mathscr{T}(\mathscr{P}'/S)$. Here, the first and third arrows are defined and isomorphisms since (1.5.6.1) is an isomorphism, so the composition is an isomorphism. We can construct similarly isomorphisms

$$K \otimes_{\mathscr{P}'/S} 1_{\mathscr{P}'/S} \longrightarrow K, \quad 1_{\mathscr{P}'/S} \otimes_{\mathscr{P}'/S} K \longrightarrow K$$

in $\mathscr{T}(\mathscr{P}'/S)$. The coherence conditions given in [Ayo07, 2.1.79, 2.1.81] for $- \otimes_{\mathscr{P}'/S} -$ follows from the coherence conditions for $- \otimes_S -$. Thus $- \otimes_{\mathscr{P}'/S} -$ gives a symmetric monoidal structure on $\mathscr{T}(\mathscr{P}'/S)$.

(7) *Construction of Hom* For an object S of $\mathscr{S}$, and objects K and L of $\mathscr{T}(S)$, we construct the internal Hom

$$Hom_{\mathscr{P}'/S}(K, L)$$

by

$$\rho_S^* Hom_S(\rho_{\sharp,S}K, \rho_{\sharp,S}L).$$

Then $-\otimes_{\mathscr{P}'/S} K$ is left adjoint to $Hom_{\mathscr{P}'/S}(K,-)$, so $-\otimes_{\mathscr{P}'/S}-$ is a closed symmetric monoidal structure on $\mathscr{T}(\mathscr{P}'/S)$.

(8) *Monoidality of $f^*_{\mathscr{P}'}$.* Let $f : X \to S$ be a morphism in $\mathscr{S}$. For objects K and L of $\mathscr{T}(S)$, the isomorphism

$$f^*_{\mathscr{P}'}K \otimes_{\mathscr{P}'/S} f^*_{\mathscr{P}'}L \xrightarrow{\sim} f^*_{\mathscr{P}'}(K \otimes_{\mathscr{P}'/S} L)$$

is constructed by the composition

$$\begin{aligned} \rho_S^*(\rho_{\sharp,S}\rho_S^* f^* \rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}\rho_S^* f^* \rho_{\sharp,S}L) &\xrightarrow{ad^{-1}} \rho_S^*(f^*\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}\rho_S^* f^*\rho_{\sharp,S}L) \\ &\xrightarrow{ad^{-1}} \rho_S^*(f^*\rho_{\sharp,S}K \otimes_S f^*\rho_{\sharp,S}L) \\ &\xrightarrow{\sim} \rho_S^* f^*(\rho_{\sharp,S}K \otimes_s \rho_{\sharp,S}L) \\ &\xrightarrow{ad} \rho_S^* f^* \rho_{\sharp,S}\rho_S^*(\rho_{\sharp,S}K \otimes_S \rho_{\sharp,S}L) \end{aligned} .$$

in $\mathscr{T}(\mathscr{P}'/S)$. The first and second arrows are defined and isomorphisms since f^* commutes with $\rho_\sharp$, and the fourth arrow is an isomorphism since (1.5.6.1) is an isomorphism.

We can similarly construct the isomorphism

$$f^*_{\mathscr{P}'}(1_{\mathscr{P}'/S}) \xrightarrow{\sim} 1_{\mathscr{P}'/S'}.$$

The coherence conditions given in [Ayo07, 2.1.85, 2.1.86] for these follows from the coherence conditions for f^*. Thus the functor

$$f^*_{\mathscr{P}'} : \mathscr{T}(\mathscr{P}/S) \to \mathscr{T}(\mathscr{P}/S')$$

is monoidal.

(9) *Proof of ($\mathscr{P}'$-BC).* The $\mathscr{P}'$-base change property for $\mathscr{T}(\mathscr{P}')$ follows from (1.5.5).

(10) *Proof of ($\mathscr{P}'$-PF).* Let $f : X \to S$ be a $\mathscr{P}$-morphism. For objects K of $\mathscr{T}(X)$ and L of $\mathscr{T}(S)$, we want to show that the morphism

$$f_{\sharp,\mathscr{P}'}(K \otimes_{\mathscr{P}'/X} f^*_{\mathscr{P}'}L) \xrightarrow{Ex} f_{\sharp,\mathscr{P}'}K \otimes_{\mathscr{P}'/S} L$$

is an isomorphism. Since $\rho_\sharp$ is fully faithful and monoidal, applying $\rho_{\sharp,S}$ to the above morphism, it suffices to show that the morphism

$$f_\sharp(\rho_{\sharp,X}K \otimes_X f^*\rho_{\sharp,S}L) \xrightarrow{Ex} f_\sharp\rho_{\sharp,X}K \otimes_S \rho_{\sharp,S}L$$

is an isomorphism. This follows from the $\mathscr{P}$-projection formula for $\mathscr{T}$.

(11) *Twists.* The set of twists τ on $\mathscr{T}$ induces a set of twists on $\mathscr{T}(\mathscr{P})$. It is also denoted by τ.

1.5.7. Thus we have proven that

(i) $\mathscr{T}(\mathscr{P}')$ is a τ-twisted $\mathscr{P}$-premotivic triangulated category,

(ii) $\mathscr{T}(\mathscr{P}')$ is well generated by $\mathscr{P}'$ and τ.

1.6 Bousfield localization

1.6.1. Through this section, we fix a category $\mathscr{S}$ and class of morphisms $\mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks. We fix also a τ-twisted $\mathscr{P}$-premotivic triangulated category $\mathscr{T}$ well generated by $\mathscr{P}$ and τ. For any object S of $\mathscr{S}$, we also fix an essentially small family of morphisms $\mathscr{W}_S$ in $\mathscr{T}(S)$ stable by twists in τ, $f_\sharp$ for $\mathscr{P}$-morphism f, and f^*. The collection of $\mathscr{W}_S$ is denoted by $\mathscr{W}$.

Definition 1.6.2. Let S be an object of $\mathscr{S}$.

(1) We denote by $\mathcal{T}_{\mathscr{W},S}$ the localizing subcategory of $\mathscr{T}(S)$ generated by the cones of the arrows of $\mathscr{W}$.

(2) We denote by $\mathscr{T}(S)[\mathscr{W}^{-1}]$ the Verdier Quotient $\mathscr{T}(S)/\mathcal{T}_{\mathscr{W},S}$. Then we denote by $\mathscr{T}[\mathscr{W}^{-1}]$ the collection of $\mathscr{T}(S)[\mathscr{W}^{-1}]$ for object S of $\mathscr{S}$.

(3) We say that an object L of $\mathscr{T}(S)$ is *$\mathscr{W}$-local* if

$$\mathrm{Hom}_{\mathscr{T}(S)}(K, L) = 0$$

for any object K of $\mathscr{T}(S)$ which is the arrow of a morphism in $\mathscr{W}$. Equivalently,

$$\mathrm{Hom}_{\mathscr{T}(S)}(K, L) = 0$$

for any object K of $\mathcal{T}_{\mathscr{W},S}$.

(4) We say that a morphism $K \to K'$ in $\mathscr{T}(S)$ is a *$\mathscr{W}$-weak equivalence* if the cone of the morphism is in $\mathcal{T}_{\mathscr{W},S}$. Equivalently, the induced homomorphism

$$\mathrm{Hom}_{\mathscr{T}(S)}(K', L) \to \mathrm{Hom}_{\mathscr{T}(S)}(K, L)$$

is an isomorphism for any $\mathscr{W}$-local object L of $\mathscr{T}(S)$. This equivalence follows from [Nee01, 9.1.14].

1.6.3. The purpose of this section is to show that $\mathscr{T}[\mathscr{W}^{-1}]$ has a structure of a τ-twisted $\mathscr{P}$-premotivic triangulated category. Let S be an object of $\mathscr{S}$. First note that $\mathscr{T}(S)[\mathscr{W}^{-1}]$ is well generated by [Nee01, Introduction 1.16] and that it is generated by $\mathscr{P}$ and τ. Then $\mathscr{T}(S)[\mathscr{W}^{-1}]$ is well generated by $\mathscr{P}$ and τ. By [Nee01, 9.1.19], we have the adjunction

$$\pi_S : \mathscr{T}(S) \rightleftarrows \mathscr{T}(S)[\mathscr{W}^{-1}] : \mathcal{O}_S$$

of triangulated categories where π_S denotes the Verdier quotient functor and $\mathcal{O}_S$ denotes the Bousfield localization functor. Note that by [Nee01, 9.1.16], the functor $\mathcal{O}_S$ is fully faithful, and its essential images are exactly $\mathscr{W}$-local objects of $\mathscr{T}(S)$.

For any $\mathscr{P}$-morphism $X \to S$, we put

$$M_{S,\mathscr{W}}(X) = \pi_S(M_S(X)).$$

Then we denote by

$$\pi : \mathscr{T} \rightleftarrows \mathscr{T}[\mathscr{W}^{-1}] : \mathcal{O}$$

the collection of the functors π_S and $\mathcal{O}_S$.

Because $\mathscr{W}$ is stable by $f_\sharp$ for $\mathscr{P}$-morphism f and f^*, if $f : X \to S$ is a $\mathscr{P}$-morphism, then the functor

$$f_\sharp f^* = M_S(X) \otimes_S -$$

preserves $\mathscr{W}$. Thus it preserves $\mathscr{T}_{\mathscr{W},S}$. This means that the functor

$$K \otimes_S -$$

preserves $\mathscr{W}$-weak equivalence for any object K of $\mathscr{T}(S)$.

Note also that a morphism $K \to K'$ in $\mathscr{T}(S)$ is a $\mathscr{W}$*-weak equivalence* if and only if the induced morphism $\pi_S K \to \pi_S K'$ in $\mathscr{T}(S)[\mathscr{W}^{-1}]$ is an isomorphism.

1.6.4. Let X and S be objects of $\mathscr{S}$, and consider a diagram

$$\begin{array}{ccc} \mathscr{T}(X) & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \mathscr{T}(S) \\ {\scriptstyle \pi_X}\downarrow\uparrow{\scriptstyle \mathcal{O}_X} & & {\scriptstyle \pi_S}\downarrow\uparrow{\scriptstyle \mathcal{O}_S} \\ \mathscr{T}(X)[\mathscr{W}^{-1}] & & \mathscr{T}(S)[\mathscr{W}^{-1}] \end{array}$$

such that α is left adjoint to β. Suppose that α maps the cones of $\mathscr{W}_X$ into $\mathscr{T}_{\mathscr{W},Y}$ and commutes with twists. Then β preserves $\mathscr{W}$-local objects, so α preserves $\mathscr{W}$-weak equivalences. Then we define

$$\alpha_{\mathscr{W}} : \mathscr{T}(X)[\mathscr{W}^{-1}] \to \mathscr{T}(S)[\mathscr{W}^{-1}],$$
$$\beta_{\mathscr{W}} : \mathscr{T}(S)[\mathscr{W}^{-1}] \to \mathscr{T}(X)[\mathscr{W}^{-1}]$$

as $\alpha_{\mathscr{W}} = \pi_S \alpha \mathcal{O}_X$ and $\beta_{\mathscr{W}} = \pi_X \beta \mathcal{O}_X$. We often omit $\mathscr{W}$ in $\alpha_{\mathscr{W}}$ and $\beta_{\mathscr{W}}$ for brevity.

Proposition 1.6.5. *Under the notations and hypotheses of (1.6.4),*

(1) α commutes with π, i.e., $\pi_S \alpha \cong \alpha_{\mathscr{W}} \pi_X$,

(2) $\alpha_{\mathscr{W}}$ is left adjoint to $\beta_{\mathscr{W}}$,

Proof. (1) For any object K of $\mathscr{T}(X)$, the morphism $K \to \mathcal{O}_X \pi_X K$ is a $\mathscr{W}$-weak equivalence, so $\alpha K \to \alpha \mathcal{O}_X \pi_X K$ is a $\mathscr{W}$-weak equivalence since α presrerves $\mathscr{W}$-weak equivalences. Then the morphism $\pi_S \alpha K \to \pi_S \alpha \mathcal{O}_X \pi_X K$ is an isomorphism, i.e., $\pi_S \alpha = \alpha_{\mathscr{W}} \pi_X$.

(2) We will show this by constructing the unit and counit. The unit

$$\mathrm{id} \xrightarrow{ad} \beta_{\mathscr{W}}\alpha_{\mathscr{W}}$$

is constructed by

$$\mathrm{id} \xrightarrow{ad'^{-1}} \pi_X\mathscr{O}_X \xrightarrow{ad} \pi_X\beta\alpha\mathcal{O}_X \xrightarrow{ad} \pi_X\beta\mathcal{O}_S\pi_S\alpha\mathcal{O}_X.$$

Here, the first arrow is defined and an isomorphism since $\mathcal{O}_S$ is fully faithful. The counit

$$\alpha_{\mathscr{W}}\beta_{\mathscr{W}} \xrightarrow{ad'} \mathrm{id}$$

is constructed by

$$\pi_S\alpha\mathcal{O}_X\pi_X\beta\mathcal{O}_S \xrightarrow{ad^{-1}} \pi_S\alpha\beta\mathcal{O}_S \xrightarrow{ad} \pi_S\mathcal{O}_S \xrightarrow{ad} \mathrm{id}.$$

Here, the first arrow is defined and an isomorphism by (1). These two satisfy the counit-unit equations, so $\alpha_{\mathscr{W}}$ is left adjoint to $\beta_{\mathscr{W}}$. □

Proposition 1.6.6. *Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. Assume that g^* and g'^* commutes with $\mathcal{O}$ and that the exchange transformation

$$g^*f_* \xrightarrow{Ex} f'_*g'^*$$

is an isomorphism. Then the exchange transformation

$$g^*_{\mathscr{W}}f_{*,\mathscr{W}} \xrightarrow{Ex} f'_{*,\mathscr{W}}g'^*_{\mathscr{W}}$$

is also an isomorphism.

Proof. Since $\mathcal{O}$ is fully faithful, it suffices to show that the natural transformation

$$\mathcal{O}_{S'}g^*_{\mathscr{W}}f_{*,\mathscr{W}} \xrightarrow{Ex} \mathcal{O}_{S'}f'_{*,\mathscr{P}'}g'^*_{\mathscr{P}'}$$

is an isomorphism. By the condition that g^* and g'^* commutes with $\mathcal{O}$, it is equivalent to the assertion that the natural transformation

$$g^*f_*\mathcal{O}_X \xrightarrow{Ex} f'_*g'^*\mathcal{O}_X$$

is an isomorphism. This follows from the other condition. □

1.6.7. Now we will show that $\mathscr{T}[\mathscr{W}^{-1}]$ is a $\mathscr{P}$-premotivic triangulated category by constructing $f_\sharp$ for $f \in \mathscr{P}$, f^*, f_*, $\otimes$, and Hom.

(1) For any object S of $\mathscr{S}$, we put $1_{S,\mathscr{W}} = \pi_S 1_S$. We often omit $\mathscr{W}$ in $1_{S,\mathscr{W}}$ for brevity.

(2) *Constructions of $f_{\sharp,\mathscr{W}}$ for $f \in \mathscr{P}$, $f^*_{\mathscr{W}}$, and $f_{*,\mathscr{W}}$.* The functors $f_\sharp$ for $f \in \mathscr{P}$ and f^* preserve $\mathscr{W}$, so by (1.6.5), we have the adjunctions

$$f_{\sharp,\mathscr{W}} : \mathscr{T}(X)[\mathscr{W}^{-1}] \rightleftarrows \mathscr{T}(S)[\mathscr{W}^{-1}] : f^*_{\mathscr{W}}$$

$$f^*_{\mathscr{W}} : \mathscr{T}(X)[\mathscr{W}^{-1}] \rightleftarrows \mathscr{T}(S)[\mathscr{W}^{-1}] : f_{*,\mathscr{W}}$$

where in the first one, we assume that f is a $\mathscr{P}$-morphism. Then by (loc. cit), $f_{\sharp,\mathscr{W}}$ for $f \in \mathscr{P}$ and $f^*_{\mathscr{W}}$ commute with π.

(3) *Functoriality of $f^*_{\mathscr{W}}$.* Let $f : X \to S$ and $g : Y \to X$ be morphisms in $\mathscr{S}$. Then the natural isomorphism

$$g^*_{\mathscr{W}} f^*_{\mathscr{W}} \longrightarrow (fg)^*_{\mathscr{W}}$$

is constructed by

$$\pi_{S''} g^* \mathcal{O}_{S'} \pi_{S'} f^* \mathcal{O}_S \xrightarrow{ad^{-1}} \pi_{S''} g^* f^* \mathcal{O}_S \xrightarrow{\sim} \pi_{S''} (fg)^* \mathcal{O}_S.$$

Here, the first arrow is an isomorphism since g^* preserves $\mathscr{W}$-weak equivalence and the unit

$$\mathrm{id} \xrightarrow{ad} \mathcal{O}_{S'} \pi_{S'}$$

is a $\mathscr{W}$-weak equivalence. The usual cocycle condition for $f^*_{\mathscr{W}}$ follows from the usual cocycle condition for f^*. Thus $\mathscr{T}[\mathscr{W}^{-1}]$ is a fibered category over $\mathscr{S}$.

(4) *Construction of $\otimes$.* For an object S of $\mathscr{S}$ and objects K and L of $\mathscr{T}(\mathscr{P}'/S)$, we denote by $K \otimes_{S,\mathscr{W}} L$ the object

$$\pi_S(\mathcal{O}_S K \otimes \mathcal{O}_S L)$$

in $\mathscr{T}(S)[\mathscr{W}^{-1}]$.

(5) *Functoriality of $\otimes$.* Then the natural transformation

$$(- \otimes_{S,\mathscr{W}} -) \otimes_{S,\mathscr{W}} - \to - \otimes_{S,\mathscr{W}} (- \otimes_{S,\mathscr{W}} -) \tag{1.6.7.1}$$

is constructed by the composition

$$\begin{aligned}\pi_S(\mathcal{O}_S \pi_S(\mathcal{O}_S K \otimes_S \mathcal{O}_S L) \otimes_S \mathcal{O}_S N) &\xrightarrow{ad^{-1}} \pi_S(\mathcal{O}_S K \otimes \mathcal{O}_S L \otimes \mathcal{O}_S N) \\ &\xrightarrow{ad} \pi_S(\mathcal{O}_S K \otimes_S \mathcal{O}_S \pi_S(\mathcal{O}_S L \otimes_S \mathcal{O}_S N))\end{aligned}$$

for objects K, L, and N of $\mathscr{T}(S)[\mathscr{W}^{-1}]$. Here, the first arrow is defined and an isomorphism because π_S and $\otimes_S$ preserve $\mathscr{W}$-weak equivalence and the morphism

$$\mathcal{O}_S K \otimes_S \mathcal{O}_S L \xrightarrow{ad} \mathcal{O}_S \pi_S(\mathcal{O}_S K \otimes_S \mathcal{O}_S L)$$

is a $\mathscr{W}$-weak equivalence. Similarly, the second arrow is an isomorphism, so (1.6.7.1) is an isomorphism.

We can similarly construct isomorphisms

$$K \otimes_{S,\mathscr{W}} 1_S \longrightarrow K, \quad 1_S \otimes_{S,\mathscr{W}} K \longrightarrow K$$

in $\mathscr{T}(S)[\mathscr{W}^{-1}]$. The coherence conditions given in [Ayo07, 2.1.79, 2.1.81] for $-\otimes_{S,\mathscr{W}}-$ follows from the coherence conditions for $-\otimes-$. Thus $-\otimes_{S,\mathscr{W}}-$ gives a monoidal structure on $\mathscr{T}(S)[\mathscr{W}^{-1}]$.

(6) *Construction of Hom.* For an object S of $\mathscr{S}$, and objects K and L of $\mathscr{T}(S)$, we construct the internal Hom

$$Hom_{S,\mathscr{W}}(K, L)$$

by

$$\pi_S Hom_{\mathscr{T}(S)}(\mathcal{O}_S K, \mathcal{O}_S L).$$

Then $-\otimes_{S,\mathscr{W}} K$ is left adjoint to $Hom_{S,\mathscr{W}}(K,-)$, so $-\otimes_{S,\mathscr{W}}-$ is a symmetric closed monoidal structure on $\mathscr{T}(S)[\mathscr{W}^{-1}]$.

(7) *Monoidality of f^*.* Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes. For any objects K and L of $\mathscr{T}(S)[\mathscr{W}^{-1}]$, we construct the morphism

$$f^*_{\mathscr{W}} K \otimes_{S,\mathscr{W}} f^*_{\mathscr{W}} L \to f^*_{\mathscr{W}}(K \otimes_{S,\mathscr{W}} L) \tag{1.6.7.2}$$

by the composition

$$\begin{aligned}\pi_X(\mathcal{O}_X \pi_X f^* \mathcal{O}_S K \otimes_X \mathcal{O}_X \pi_X f^* \mathcal{O}_{S'} L) &\xrightarrow{ad^{-1}} \pi_X(f^* \mathcal{O}_S K \otimes_X \mathcal{O}_X \pi_X f^* \mathcal{O}_X L)\\ &\xrightarrow{ad^{-1}} \pi_X(f^* \mathcal{O}_S K \otimes_X f^* \mathcal{O}_X L)\\ &\xrightarrow{\sim} \pi_X f^*(\mathcal{O}_S K \otimes_X f^* \mathcal{O}_X L)\\ &\xrightarrow{ad} \pi_X f^*(\mathcal{O}_S \pi_S(\mathcal{O}_S K \otimes_X f^* \mathcal{O}_{S'} L)).\end{aligned}$$

The first and second arrows are defined and isomorphisms since π_X and $\otimes_S$ preserve $\mathscr{W}$-weak equivalences and the unit

$$\mathrm{id} \xrightarrow{ad} \mathcal{O}_X \pi_X$$

is a $\mathscr{W}$-weak equivalence. The fourth arrow is an isomorphism by the same reason. Thus (1.6.7.2) is an isomorphism.

We can similarly construct the isomorphism

$$f^*_{\mathscr{W}}(1_{S,\mathscr{W}}) \xrightarrow{\sim} 1_{X,\mathscr{W}}.$$

The coherence conditions given in [Ayo07, 2.1.85, 2.1.86] for these follows from the coherence conditions for f^*. Thus the functor

$$f^*_{\mathscr{W}} : \mathscr{T}(S)[\mathscr{W}^{-1}] \to \mathscr{T}(X)[\mathscr{W}^{-1}]$$

is monoidal.

(8) *Monoidality of* π. For an object S of $\mathscr{S}$, and objects K and L of $\mathscr{T}(S)[\mathscr{S}^{-1}]$, we construct the morphism

$$\pi_S(K \otimes_S L) \to \pi_S K \otimes_{S,\mathscr{W}} \pi_S L$$

by the composition

$$\pi_S(K \otimes_S L) \xrightarrow{ad} \pi_S(\mathcal{O}_S \pi_S K \otimes_S L) \xrightarrow{ad} \pi_S(\mathcal{O}_S \pi_S K \otimes_S \mathcal{O}_S \pi_S L).$$

Here, the arrows are isomorphisms since π_S and $\otimes_S$ preserve $\mathscr{W}$-weak equivalences and the unit

$$\mathrm{id} \xrightarrow{ad} \mathcal{O}_S \pi_S$$

is a $\mathscr{W}$-weak equivalence.

We can similarly construct the isomorphism

$$\pi_S(1_S) \xrightarrow{\sim} 1_{S,\mathscr{W}}.$$

Note that the coherence conditions given in [Ayo07, 2.1.85, 2.1.86] are satisfied, i.e., the functor

$$\pi_S : \mathscr{T}(S) \to \mathscr{T}(S)[\mathscr{W}^{-1}]$$

is monoidal.

(9) *Proof of ($\mathscr{P}$-BC).* The $\mathscr{P}$-base change property for $\mathscr{T}[\mathscr{W}^{-1}]$ follows from (1.6.6).

(10) *Proof of ($\mathscr{P}$-PF).* The $\mathscr{P}$-projection formula for $\mathscr{T}[\mathscr{W}^{-1}]$ can be obtained by applying π to the $\mathscr{P}$-projection formula for $\mathscr{T}$ since π is monoidal and essentially surjective.

(11) *Twists.* The set of twists τ on $\mathscr{T}$ induces a set of twists on $\mathscr{T}[\mathscr{W}^{-1}]$. It is also denoted by τ.

1.6.8. Thus we have proven that

(i) $\mathscr{T}[\mathscr{W}^{-1}]$ is a $\mathscr{P}$-premotivic triangulated category,

(ii) $\mathscr{T}[\mathscr{W}^{-1}]$ is well generated by $\mathscr{P}$ and τ.

1.7 *log*-localization

1.7.1. Throughout this section, we fix a full subcategory $\mathscr{S}$ of the category of noetherian fs log schemes satisfying the conditions of (1.2.7).

Definition 1.7.2. For an $\mathscr{S}$-scheme S, we will consider the following situations for morphisms

$$Y' \xrightarrow{h} Y \xrightarrow{g} X \xrightarrow{f} S$$

of $\mathscr{S}$-schemes.

(a) The morphism f is of finite type, the morphism g is the identity, and the morphism h is the projection $\mathbb{A}^1_Y \to Y$.

(b) The morphism f is of finite type, the morphism g is the identity, and the morphism h is a dividing cover.

(c) The morphism f is *log smooth*, the morphism g is an exact log smooth morphism, and the morphism h is the verticalization $Y^{\mathrm{ver}} \to Y$ of X via fg.

(d) The morphism f is *log smooth*, the $\mathscr{S}$-scheme X has a neat fs chart P, and the morphism g is the projection

$$X \times_{\mathbb{A}_P} \mathbb{A}_Q \to X$$

where the homomorphism $\theta : P \to Q$ is a locally exact vertical homomorphism of fs monoids such that g is an exact log smooth morphism. The morphism h is the morphism

$$X \times_{\mathbb{A}_P} \mathbb{A}_{Q_G} \to X \times_{\mathbb{A}_P} \mathbb{A}_Q$$

induced by the localization $Q \to Q_G$ where G is a maximal θ-critical face of Q.

Let $\mathscr{T}$ be a τ-twisted $\mathscr{P}$-premotivic triangulated category over $\mathscr{S}$. Then let $\mathscr{W}_{\mathbb{A}^1,S}$ (resp. $\mathscr{W}_{log',S}$, resp. $\mathscr{W}_{log'',S}$, resp. $\mathscr{W}_{log,S}$) denote the family of morphisms

$$M_S(Y')\{i\} \to M_S(Y)\{i\}$$

in $\mathscr{T}(S)$ where $i \in \tau$ and the morphism $Y' \to Y$ is of the type (a) (resp. of the types (a)–(b), resp. of the type (b), resp. of the types (a)–(d)). Note that $\mathscr{W}_{\mathbb{A}^1}$ (resp. $\mathscr{W}_{log'}$, resp. $\mathscr{W}_{log}$) is stable by the operations $f_\sharp$ for $f \in ft$ (resp. $f \in ft$, resp. $f \in lSm$) and f^*. To ease the notations, we often remove $\mathscr{W}$ in the notations. For example, we write *log*-weak equivalences instead of $\mathscr{W}_{log}$-weak equivalences.

Definition 1.7.3. Let t be a topology on $\mathscr{S}$ such that any t-covering consists of morphisms of finite type. Consider the category

$$\mathrm{D}_{\mathbb{A}^1}(\mathrm{Sh}_t(ft/S, \Lambda))$$

(see [CD12, 5.3.22, 5.1.4] for the definitions). It is a ft-premotivic triangulated category. They are also denoted by $\mathrm{D}_{\mathbb{A}^1,t}(ft, \Lambda)$ and $\mathrm{D}_{\mathbb{A}^1,t}(ft, \Lambda)$.

Let $\mathscr{P}$ be a class of morphisms of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks and contained in the class ft, and let $\mathscr{W}$ be an essentially small family of morphisms in $\mathrm{D}_{\mathbb{A}^1,t}(ft, \Lambda)$ containing $\mathbb{A}^1$-weak equivalences and stable by $f_\sharp$ for $f \in \mathscr{P}$ and f^*. Then we denote by

$$\mathrm{D}_{\mathscr{W},t}(ft, \Lambda)$$

the category obtained by inverting $\mathscr{W}$-weak equivalences as in (1.6.2).

We also denote by

$$\mathrm{D}_{\mathscr{W},t}(\mathscr{P}, \Lambda)$$

the localizing subcategory of $\mathrm{D}_{\mathscr{W},pw}(ft,\Lambda)$ obtained by applying (1.5.1) to the inclusion $\mathscr{P}\subset ft$.

If t' is another topology on $\mathscr{S}$ finer than t, then we have an adjunction

$$a_{t'}:\mathrm{Sh}_t(ft,\Lambda)\rightleftarrows \mathrm{Sh}_{t'}(ft,\Lambda):\iota$$

where $a_{t'}$ denotes the sheafification functor and ι denotes the inclusion functor. From this, we obtain the adjunction

$$a_{t'}^*:\mathrm{D}_{\mathscr{W},t}(\mathscr{P},\Lambda)\rightleftarrows \mathrm{D}_{\mathscr{W},t'}(\mathscr{P},\Lambda):a_{t',*}$$

of $\mathscr{P}$-premotivic categories.

1.7.4. By (1.4.5) and [CD12, 5.1.32], the ft-premotivic triangulated category

$$\mathrm{D}_{qw}(ft,\Lambda)$$

is compactly generated by ft and τ, so it is well generated by ft and τ. Then (1.5.2) and (1.6.3), if $\mathscr{P}$ is a class of morphisms of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks and contained in the class ft, then the $\mathscr{P}$-premotivic triangulated category

$$\mathrm{D}_{\mathbb{A}^1,pw}(\mathscr{P},\Lambda),\quad \mathrm{D}_{log',pw}(\mathscr{P},\Lambda),\quad \mathrm{D}_{log,pw}(\mathscr{P},\Lambda)$$

are well generated by $\mathscr{P}$ and τ.

1.7.5. By the proofs of [CD12, 5.2.38, 5.3.39], the ft-premotivic triangulated categories

$$\mathrm{D}_{\mathbb{A}^1,qw}(ft,\Lambda),\quad \mathrm{D}_{log',qw}(ft,\Lambda),\quad \mathrm{D}_{log,qw}(ft,\Lambda)$$

are compactly generated by ft and τ. Thus if $\mathscr{P}$ is a class of morphisms of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks and contained in the class ft, then the $\mathscr{P}$-premotivic triangulated category

$$\mathrm{D}_{\mathbb{A}^1,qw}(\mathscr{P},\Lambda),\quad \mathrm{D}_{log',qw}(\mathscr{P},\Lambda),\quad \mathrm{D}_{log,qw}(\mathscr{P},\Lambda)$$

are compactly generated by $\mathscr{P}$ and τ.

The above method is not applicable for $\mathrm{D}_{log',pw}(lSm,\Lambda)$ since we cannot apply (1.4.5) for the pw-topology. To show that it is compactly generated by lSm and τ, we will circumvent this obstacle by showing that the morphism

$$a_{qw}^*:\mathrm{D}_{log',pw}(lSm,\Lambda)\to \mathrm{D}_{log',qw}(lSm,\Lambda)$$

of lSm-premotivic triangulated categories is an isomorphism under some assumption as follows.

1.7.6. Let S be an $\mathscr{S}$-scheme, and let K be an object of $\mathrm{D}_{log',pw}(ft/S,\Lambda)$. For any commutative diagram

$$C = \begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of S-schemes over S, we denote by $L_{C,K}$ the homotopy pullback in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ of the lower right corner of the commutative diagram

$$\begin{array}{ccc} p_*p^*K & \xrightarrow{ad} & p_*g_*g^*p^*K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ p_*f_*f^*p^*K & \xrightarrow{ad} & p_*h_*h^*p^*K \end{array} \tag{1.7.6.1}$$

where $p : T \to S$ is the structural morphism and $h = fg'$. Then we denote by

$$q_C : p_*p^*K \to L_{C,K}$$

the induced morphism in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$.

Proposition 1.7.7. *Under the notations and hypotheses of (1.7.6), assume that C is a piercing distinguished square. If K is plain lower flasque, then the morphism q_C is an isomorphism, i.e., the diagram (1.7.6.1) is homotopy Cartesian in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$.*

Proof. Note that K satisfies the pw-descent by [CD12, 5.3.30]. Let C' denote the commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X' \times_{T'} X' & \longrightarrow & X \times_T X \end{array}$$

where the vertical arrows are the diagonal morphisms and the horizontal arrows are induced by g and g'. We put $X_0 = X$, and we denote by $(X_i)_{i\in\mathbb{N}}$ the Čech cover associated to $X_0 \to T$. As in the proof of [Voe10a, 5.3], it suffices to show that $q_{C'\times_T X_i,K}$ is an isomorphism in $\mathrm{D}_{log',pw}(lSm/S,\Lambda)$ for all i.

The diagram C' is a pullback of (1.2.8.2), which has the decomposition

$$\begin{array}{ccc} \mathrm{pt}_{\mathbb{N}} & \longrightarrow & \mathbb{A}_{\mathbb{N}} \\ \Big\downarrow & & \downarrow{\scriptstyle i} \\ & & \mathbb{A}_{\mathbb{N}\oplus\mathbb{Z}} \\ \Big\downarrow & & \downarrow{\scriptstyle \mathbb{A}_u} \\ V' & \longrightarrow & \mathbb{A}_M \\ \downarrow & & \downarrow{\scriptstyle \mathbb{A}_v} \\ \mathrm{pt}_{\mathbb{N}^2} & \xrightarrow{t} & \mathbb{A}_{\mathbb{N}} \times_{\mathbb{A}^1} \mathbb{A}_{\mathbb{N}} \end{array}$$

where

(i) each square is Cartesian,

(ii) t denotes a pullback of the diagonal morphism $\mathbb{A}^1 \to \mathbb{A}^2$ via the morphism $\mathbb{A}_{\mathbb{N}^2} \to \mathbb{A}^2$ removing the log structure,

(iii) M denotes the fs monoscheme which is the gluing of

$$\mathrm{spec}(\mathbb{N}x \oplus \mathbb{N}(x^{-1}y)), \quad \mathrm{spec}(\mathbb{N}y \oplus \mathbb{N}(y^{-1}x))$$

along $\mathrm{spec}(\mathbb{N}x \oplus \mathbb{Z}(x^{-1}y))$,

(iv) $u : \mathrm{spec}(\mathbb{N}x \oplus \mathbb{Z}(x^{-1}y)) \to M$ denotes the obvious open immersion of fs monoschemes,

(v) $v : M \to \mathrm{spec}(\mathbb{N}x \oplus \mathbb{N}y)$ denotes the obvious proper birational morphism of fs monoschemes,

(vi) i denotes the 1-section.

Then C' has a decomposition

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\ X' \times_{T'} X' & \longrightarrow & X \times_T X \end{array} \tag{1.7.7.1}$$

such that the upper square is a plain lower distinguished square and a and a' are dividing covers. Since we inverted *log'*-weak equivalences, the adjunctions

$$\mathrm{id} \xrightarrow{ad} a_*a^*, \quad \mathrm{id} \xrightarrow{ad} a'_*a'^*$$

are isomorphisms. Thus if we denote by C'' the upper diagram of (1.7.7.1), then it suffices to show that $q_{C'' \times_T X_i, K}$ is an isomorphism. It follows from (1.3.8) since C'' is a plain lower distinguished square. □

1.7.8. In (7.6.2), we will show that the essential image of the functor

$$\rho_\sharp : \mathrm{D}_{log',pw}(lSm, \Lambda) \to \mathrm{D}_{log',pw}(ft, \Lambda)$$

satisfies the plain lower descent. Let K be an object in its essential image, and let C be a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of $\mathscr{S}$-schemes over S. When C is a plain lower distinguished square, by (1.3.8), the condition (i) of (1.3.7) for C is satisfied. When C is a piercing distinguished square, by (1.7.7), the condition (i) of (1.3.7) for C is satisfied, and when C is a pullback of (1.2.8.2), the condition (i) of (1.3.7) for C is satisfied by the proof of (1.7.7). Thus the condition (i) of (1.3.7) for C when C is a quasi-piercing distinguished square is satisfied, so by (1.7.7), C satisfies qw-descent. Then [CD12, 5.3.30] implies that the functor

$$a_{qw}^* : \mathrm{D}_{log',pw}(lSm, \Lambda) \to \mathrm{D}_{log',qw}(lSm, \Lambda)$$

is an equivalence of lSm-premotivic triangulated categories. This implies that the functor

$$a_{qw}^* : \mathrm{D}_{\mathscr{W},pw}(\mathscr{P}, \Lambda) \to \mathrm{D}_{log',qw}(lSm, \Lambda)$$

is an equivalence of $\mathscr{P}$-premotivic triangulated categories for $\mathscr{W} = \mathscr{W}_{log'}, \mathscr{W}_{log}$ and $\mathscr{P} = lSm, eSm$.

In particular, for such $\mathscr{W}$ and $\mathscr{P}$, by (1.7.5), $\mathrm{D}_{\mathscr{W},pw}(\mathscr{P}, \Lambda)$ is compactly generated by $\mathscr{P}$ and τ.

1.7.9. One of the purposes of this thesis is to study the eSm-premotivic triangulated category $\mathrm{D}_{log,pw}(eSm, \Lambda)$. For brevity, it is also denoted by $\mathrm{D}_{log,pw}(-, \Lambda)$.

Chapter 2

Properties of premotivic triangulated categories

2.0.1. Through this section, fix a base fs log scheme $\mathcal{S}$. Then fix a full subcategory $\mathscr{S}$ of the category of noetherian fs log schemes over $\mathcal{S}$ such that

(i) $\mathscr{S}$ is closed under finite sums and pullbacks via morphisms of finite type,

(ii) if S belongs to $\mathscr{S}$ and $X \to S$ is strict quasi-projective, then X belongs to $\mathscr{S}$,

(iii) if S belongs to $\mathscr{S}$, then $S \times \mathbb{A}_M$ belongs to $\mathscr{S}$ for every fs monoscheme M,

(iv) If S belongs to $\mathscr{S}$, then $\underline{S}$ is belongs to $\mathscr{S}$,

(v) for any separated morphism $f : X \to S$ of $\mathscr{S}$-schemes, the morphism $\underline{f} : \underline{X} \to \underline{S}$ of underlying schemes admits a compactification in the sense of [SGA4, 3.2.5], i.e., we have a factorization

$$\underline{X} \to Y \to \underline{S}$$

in $\mathscr{S}$ such that the first arrow is an open immersion and the second arrow is a strict proper morphism.

We also fix a class $\mathscr{P}$ of morphisms of $\mathscr{S}$ containing all strict smooth morphisms of $\mathscr{S}$-schemes and stable by compositions and pullbacks. Then we fix a $\mathscr{P}$-premotivic triangulated category $\mathscr{T}$.

For example, as in [CD12, 2.0], $\mathcal{S}$ can be the spectrum of a prime field or Dedekind domain, and then $\mathscr{S}$ can be the category of noetherian fs log schemes over $\mathcal{S}$.

2.0.2. In [Ayo07] and [CD12], the adjoint property, base change property, $\mathbb{A}^1$-homotopy property, localization property, projection formula, purity, t-separated property, stability, and support property are discussed. Many of them can be trivially generalized to properties for *strict* morphisms. We also introduce base change properties for non strict morphisms and other homotopy properties. In the last section, we introduce the notion of log motivic triangulated categories, which will be the central topic in later chapters.

2.1 Elementary properties

2.1.1. Recall from [CD12, §2.1, 2.2.13] the following definitions.

(1) We say that $\mathscr{T}$ is *additive* if for any $\mathscr{S}$-schemes S and S', the obvious functor

$$\mathscr{T}(S \amalg S') \to \mathscr{T}(S) \times \mathscr{T}(S')$$

is an equivalence of categories.

(2) Let $f : X \to P$ be a proper morphism of $\mathscr{S}$-schemes. We say that f satisfies the *adjoint property*, denoted by (Adj_f), if the functor

$$f_* : \mathscr{T}(X) \to \mathscr{T}(S)$$

has a right adjoint. When (Adj_f) is satisfied for any proper morphism f, we say that $\mathscr{T}$ satisfies the *adjoint property*, denoted by (Adj).

(3) Let t be a topology on $\mathscr{S}$ generated by a pretopology t_0 on $\mathscr{S}$. We say that T is *t-seperated*, denoted by (t-sep), if for any t_0-cover $\{u_i : X_i \to S\}_{i \in I}$ of S, the family of functors $(f_i^*)_{i \in I}$ is conservative.

2.1.2. Let t be a topology on $\mathscr{S}$ generated by a pretopology t_0 on $\mathscr{S}$ such that any t_0-cover is consisted with $\mathscr{P}$-morphisms. Assume that $\mathscr{T}$ satisfies (t-sep) and that $\mathscr{T}$ is generated by $\mathscr{P}$ and τ. Let S be an $\mathscr{S}$-scheme, and let $\mathscr{P}'/S$ be a class of $\mathscr{P}$-morphisms $X \to S$ such that for any $\mathscr{P}$-morphism $g : Y \to S$, there is a t-cover $\{u_i : Y_i \to Y\}_{i \in I}$ such that each composition $gu_i : Y_i \to S$ is in $\mathscr{P}'/S$. In this setting, we will show that the family of objects of the form

$$M_S(X)\{i\}$$

for morphism $X \to S$ in $\mathscr{P}'/S$ and $i \in \tau$ generates $\mathscr{T}(S)$.

Since $\mathscr{T}$ is generated by $\mathscr{P}$ and τ, the family of functors

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(X)\{i\}, -) = \mathrm{Hom}_{\mathscr{T}(X)}(1_X\{i\}, f^*(-))$$

for $\mathscr{P}$-morphism $f : X \to S$ and $i \in \tau$ is conservative. By assumption, there is a t_0-cover $\{u_j : X_j \to X\}_{j \in I}$ such that each composition $fu_j : X_j \to S$ is in $\mathscr{P}'/S$. Applying (t-sep), we see that the family of functors

$$\mathrm{Hom}_{\mathscr{T}(X_j)}(u_j^* 1_X\{i\}, u_j^* f^*(-)) = \mathrm{Hom}_{\mathscr{T}(S)}(M_S(X_j)\{i\}, -)$$

for $\mathscr{P}$-morphism $f : X \to S$, $j \in I$, and $i \in \tau$ is conservative. This implies the assertion.

Proposition 2.1.3. *Let t be a topology on $\mathscr{S}$ generated by a pretopology t_0 on $\mathscr{S}$ such that any t_0-cover is consisted with $\mathscr{P}$-morphisms. Assume that $\mathscr{T}$ satisfies (t-sep) and that $\mathscr{T}$ is* well generated *by $\mathscr{P}$ and τ. Let S be an $\mathscr{S}$-scheme, and let $\{u_i : S_i \to S\}_{i \in I}$ be a t_0-cover. Then $\mathscr{T}(S)$ is the localizing subcategory of $\mathscr{T}(S)$ generated by the essential images of $u_{i\sharp}$ for $i \in I$.*

Proof. We denote by $\mathscr{P}'/S$ the class of morphisms of the form

$$X \to S_i \xrightarrow{u_i} S$$

where $i \in I$ and the first arrow is in $\mathscr{P}$. Then for any $\mathscr{P}$-morphism $Y \to S$, we have the Cartesian diagram

$$\begin{array}{ccc} Y \times_S S_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S_i & \longrightarrow & S \end{array}$$

of $\mathscr{S}$-schemes. From this diagram, we see that the hypotheses of (2.1.2) is verified. Then by (loc. cit) and (1.4.4), $\mathscr{T}(S)$ is the localizing subcategory of $\mathscr{T}(S)$ generated by objects of the form

$$M_S(X)\{i\}$$

for morphism $X \to S$ in $\mathscr{P}'/S$ and $i \in \tau$. The conclusion follows from this. $\square$

2.2 Localization property

2.2.1. Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, and let $j : U \to S$ be its complement. Recall from (1.1.3) that $\mathscr{T}$ satisfies ($\mathscr{P}$-BC). According to [CD12, 2.3.1], we have the following consequences of ($\mathscr{P}$–BC):

(1) the unit $\mathrm{id} \xrightarrow{ad} j^*j_\sharp$ is an isomorphism,

(2) the counit $j^*j_* \xrightarrow{ad'} \mathrm{id}$ is an isomorphism,

(3) $i^*j_\sharp = 0$,

(4) $j^*i_* = 0$,

(5) the composition $j_\sharp j^* \xrightarrow{ad'} \mathrm{id} \xrightarrow{ad} i_*i^*$ is zero.

Definition 2.2.2. We say that $\mathscr{T}$ satisfies the *localization property*, denoted by (Loc), if

(1) $\mathscr{T}(\emptyset) = 0$,

(2) For any *strict* closed immersion i of $\mathscr{S}$-schemes and its complement j, the pair of functors (j^*, i^*) is conservative, and the counit $i^*i_* \xrightarrow{ad'} \mathrm{id}$ is an isomorphism.

2.2.3. Assume that $\mathscr{T}$ satisfies (Loc). Consequences formulated in [CD12, §2.3] and in the proof of [CD12, 3.3.4] are as follows.

(1) For any closed immersion i of $\mathscr{S}$-schemes, the functor i_* admits a right adjoint $i^!$.

(2) For any closed immersion i of $\mathscr{S}$-schemes and its complement j, there exists a unique natural transformation $\partial_i : i_*i^* \to j_\sharp j^*[1]$ such that the triangle

$$j_\sharp j^* \xrightarrow{ad'} \mathrm{id} \xrightarrow{ad} i_*i^* \xrightarrow{\partial_i} j_\sharp j^*[1]$$

is distinguished.

(3) For any closed immersion i of $\mathscr{S}$-schemes and its complement j, there exists a unique natural transformation $\partial_i : j_*j^* \to i_*i^![1]$ such that the triangle

$$i_*i^! \xrightarrow{ad'} \mathrm{id} \xrightarrow{ad} j_*j^* \xrightarrow{\partial_i} i_*i^![1]$$

is distinguished.

(4) Let S be an $\mathscr{S}$-scheme, and let S_{red} denote the reduced scheme associated with S. The closed immersion $\nu : S_{red} \to S$ induces an equivalence of categories

$$\nu^* : \mathscr{T}(S) \to \mathscr{T}(S_{red}).$$

(5) For any partition $(S_i \xrightarrow{\nu_i} S)_{i\in I}$ of S by locally closed subsets, the family of functors $(\nu_i^*)_{i\in I}$ is conservative.

(6) The category $\mathscr{T}$ is additive.

(7) The category $\mathscr{T}$ satisfies the strict Nisnevich separation property (in the case of usual schemes, note that (Loc) implies the cdh separation property).

(8) For any $\mathscr{S}$-scheme S and any strict Nisnevich distinguished square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of $\mathscr{P}/S$-schemes, the associated Mayer-Vietoris sequence

$$p_\sharp h_\sharp h^* p^* K \longrightarrow p_\sharp f_\sharp f^* p^* K \oplus p_\sharp g_\sharp g^* p^* K \longrightarrow p_\sharp p^* K \longrightarrow p_\sharp h_\sharp h^* p^* K[1]$$

is a distinguished triangle for any object K of $\mathscr{T}(S)$ where $h = fg'$ and p denotes the structural morphism $T \to S$.

2.3 Support property

Definition 2.3.1. Following [CD12, 2.2.5], we say that a proper morphism f of $\mathscr{S}$-schemes satisfies the support property, denoted by (Supp_f), if for any Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes such that g is an open immersion, the exchange transformation

$$Ex : g_\sharp f'_* \to f_* g'_\sharp$$

is an isomorphism. We say that $\mathscr{T}$ satisfies the support property (resp. the strict support property), denoted by (Supp) (resp. (sSupp)), if the support property is satisfied for any proper morphism (resp. for any strict proper morphism).

2.3.2. In this section, from now, we assume that for any morphism $f : X \to S$ of $\mathscr{S}$-schemes, the morphism $\underline{f} : \underline{X} \to \underline{S}$ admits a compactification in the sense of [SGA4, 3.2.5], i.e., we have a factorization

$$\underline{X} \to Y \to \underline{S}$$

in $\mathscr{S}$ such that the first arrow is an open immersion and the second arrow is a strict proper morphism.

2.3.3. Let $f : X \to S$ be a separated morphism of $\mathscr{S}$-schemes. Then choose a compactification

$$\underline{X} \to \underline{S'} \to \underline{S}$$

of $\underline{f}$. Then following [Chi99, 5.4], f can be factored as

$$X \xrightarrow{f_1} \underline{X} \times_{\underline{S}} S \xrightarrow{f_2} \underline{S'} \times_{\underline{S}} S \xrightarrow{f_3} S$$

where

(i) f_1 denotes the morphism induced by $X \to \underline{X}$ and $X \to S$,

(ii) f_2 denotes the morphism induced by $\underline{X} \to \underline{S'}$,

(iii) f_3 denotes the projection.

The morphisms f_1 and f_3 are proper, and the morphism f_2 is an open immersion. Hence we can use the argument of [CD12, §2.2]. A summary of [loc. cit] is as follows.

Assume that $\mathscr{T}$ satisfies (Supp). For any separated morphism of finite type $f : X \to S$ of $\mathscr{S}$-schemes, we can associate a functor

$$f_! : \mathscr{T}(X) \to \mathscr{T}(S)$$

with the following properties:

(1) For any separated morphism of finite types $f : X \to Y$ and $g : Y \to Z$ of $\mathscr{S}$-schemes, there is a natural isomorphism

$$(gf)_! \to g_! f_!$$

with the usual cocycle condition with respect to the composition.

(2) For any separated morphism of finite type $f : X \to S$ of $\mathscr{S}$-schemes, there is a natural transformation

$$f_! \to f_*,$$

which is an isomorphism when f is proper. Moreover, it is compatible with compositions.

(3) For any open immersion $j : U \to S$, there is a natural isomorphism

$$f_! \to f_\sharp.$$

It is compatible with compositions.

(4) For any Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes such that f is separated of finite type, there is an exchange transformation

$$Ex : g^* f_! \to f'_! g'^*$$

compatible with horizontal and vertical compositions of squares such that the diagrams

$$\begin{array}{ccc} g^* f_! & \xrightarrow{Ex} & f'_! g'^* \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ g^* f_* & \xrightarrow{Ex} & f'_* g'^* \end{array} \qquad \begin{array}{ccc} g^* f_! & \xrightarrow{Ex} & f'_! g'^* \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ g^* f_\sharp & \xrightarrow{Ex^{-1}} & f'_\sharp g^* \end{array}$$

of functors commutes when f is proper in the first diagram and is open immersion in the second diagram.

(5) For any Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes such that f is separated of finite type and g is a $\mathscr{P}$-morphism, there is an exchange transformation

$$Ex : g_\sharp f'_! \to f_! g'_\sharp$$

compatible with horizontal and vertical compositions of squares such that the diagrams

$$\begin{array}{ccc} g_\sharp f'_! & \xrightarrow{Ex} & f_! g'_\sharp \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ g_\sharp f'_* & \xrightarrow{Ex} & f_* g'_\sharp \end{array} \qquad \begin{array}{ccc} g_\sharp f'_! & \xrightarrow{Ex} & f_! g'_\sharp \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ g_\sharp f'_\sharp & \xrightarrow{\sim} & f_\sharp g'_\sharp \end{array}$$

of functors commutes when f is proper in the first diagram and is open immersion in the second diagram.

(6) For any separated morphism of finite type $f : X \to S$ of $\mathscr{S}$-schemes, and for any $K \in \mathscr{T}(X)$ and $L \in \mathscr{T}(S)$, there is an exchange transformation

$$Ex : f_! K \otimes_S L \to f_!(K \otimes_X f^* L) \tag{2.3.3.1}$$

compatible with compositions such that the diagrams

$$\begin{array}{ccc} f_!K\otimes_S L & \xrightarrow{Ex} & f_!(K\otimes_X f^*L) \\ \downarrow\wr & & \downarrow\wr \\ f_!K\otimes_S L & \xrightarrow{Ex} & f_*(K\otimes_X f^*L) \end{array} \qquad \begin{array}{ccc} f_!K\otimes_S L & \xrightarrow{Ex} & f_!(K\otimes_X f^*L) \\ \downarrow\wr & & \downarrow\wr \\ f_\sharp K\otimes_S L & \xrightarrow{Ex^{-1}} & f_\sharp(K\otimes_X f^*L) \end{array}$$

of functors commutes when f is proper in the first diagram and is open immersion in the second diagram.

(7) Assume that $\mathscr{T}$ satisfies (Adj). For any separated morphism of finite type $f : X \to S$ of $\mathscr{S}$-schemes, the functor $f_!$ admits a right adjoint

$$f^! : \mathscr{T}(S) \to \mathscr{T}(X).$$

2.4 Homotopy properties

Definition 2.4.1. Let S be an $\mathscr{S}$-scheme. Let us introduce the following homotopy properties.

(Htp–1) Let f denote the projection $\mathbb{A}^1_S \to S$. Then the counit

$$f_\sharp f^* \xrightarrow{ad'} \mathrm{id}$$

is an isomorphism.

(Htp–2) Let $f : X \to S$ be an exact log smooth morphism of $\mathscr{S}$-schemes, and let $j : X^{\mathrm{ver}} \to X$ denote the verticalization of X via f. Then the natural transformation

$$f_\sharp j_\sharp j^* \xrightarrow{ad'} f_\sharp$$

is an isomorphism.

(Htp–3) Let S be an $\mathscr{S}$-scheme with a fs chart P, let $\theta : P \to Q$ be a vertical homomorphism of exact log smooth over S type (see (3.1.2) for the definition), and let G be a θ-critical face of Q. Consider the induced morphisms

$$S \times_{\mathbb{A}_P} \mathbb{A}_{Q_G} \xrightarrow{j} S \times_{\mathbb{A}_P} \mathbb{A}_Q \xrightarrow{f} S.$$

Then the natural transformation

$$f_\sharp j_\sharp j^* f^* \xrightarrow{ad'} f_\sharp f^*$$

is an isomorphism.

(Htp–4) Let $f : X \to S$ be a dividing cover of $\mathscr{S}$-schemes. Then the unit

$$\mathrm{id} \xrightarrow{ad} f_* f^*$$

is an isomorphism.

(Htp–5) Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes with the same underlying schemes such that the induced homomorphism $\overline{\mathcal{M}}^{\rm gp}_{S,f(x)} \to \overline{\mathcal{M}}^{\rm gp}_{X,x}$ is an isomorphism for all $x \in X$. Then f^* is an equivalence of categories. is an isomorphism.

(Htp–6) Let S be an $\mathscr{S}$-scheme, let $f : S \times \mathbb{A}_{\mathbb{N}} \to S$ denote the projection, and let $i : S \times {\rm pt}_{\mathbb{N}} \to S \times \mathbb{A}_{\mathbb{N}}$ denote the 0-section. Then the natural transformation

$$f_* f^* \xrightarrow{ad} f_* i_* i^* f^*$$

is an isomorphism.

(Htp–7) Assume that $\mathscr{T}$ satisfies (Supp). Under the notations and hypotheses of (Htp–3), the natural transformation

$$f_! j_\sharp j^* f^* \xrightarrow{ad'} f_! f^*$$

is an isomorphism.

2.4.2. Note that the right adjoint versions of (Htp–1), (Htp–2), and (Htp–3) are as follows.

(1) Under the notations and hypotheses of (Htp–1), the unit

$$\mathrm{id} \xrightarrow{ad} f_* f^*$$

is an isomorphism.

(2) Under the notations and hypotheses of (Htp–2), the natural transformation

$$f^* \xrightarrow{ad} j_* f^* f^*$$

is an isomorphism.

(3) Under the notations and hypotheses of (Htp–3), the unit

$$f_* f^* \xrightarrow{ad} f_* j_* j^* f^*$$

is an isomorphism.

2.5 Purity

Definition 2.5.1. Let S be an $\mathscr{S}$-scheme, let p denote projection $\mathbb{A}^1_S \to S$, and let a denote the zero section $S \to \mathbb{A}^1_S$. Then we denote by $1_S(1)$ the element $p_\sharp a_* 1_S[2]$, and we say that $\mathscr{T}$ satisfies the *stability property*, denoted by (Stab), if $1_S(1)$ is $\otimes$-invertible.

Remark 2.5.2. Note that our definition is different from the definition in [CD12, 2.4.4], but if we assume (Loc) and (Zar-Sep), then they are equivalent by the following result.

Proposition 2.5.3. *Assume that $\mathscr{T}$ satisfies (Loc), (Zar-Sep), and (Stab). Let $f : X \to S$ be a strict smooth separated morphism of $\mathscr{S}$-schemes, and let $i : S \to X$ be its section. Then the functor*

$$f_\sharp i_*$$

is an equivalence of categories.

Proof. It follows from the implication (i)⇔(iv) of [CD12, 2.4.14]. □

Definition 2.5.4. Let $f : X \to S$ be a separated $\mathscr{P}$-morphism of $\mathscr{S}$-schemes. We denote by a the diagonal morphism $X \to X \times_S X$ and p_2 the second projection $X \times_S X \to X$. Then we put

$$\Sigma_f = p_{2\sharp} a_*.$$

If we assume (Adj), then we put

$$\Omega_f = a^! p_2^*.$$

Note that Σ_f is left adjoint to Ω_f.

Definition 2.5.5. Let $f : X \to S$ be a $\mathscr{P}$-morphism of $\mathscr{S}$-schemess. Assume that (f is proper) or (f is separated and $\mathscr{T}$ satisfies (Supp)). Consider the Cartesian diagram

$$\begin{array}{ccc} X \times_S X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & S \end{array}$$

of $\mathscr{S}$-schemes, and let $a : X \to X \times_S X$ denote the diagonal morphism. Following [CD12, 2.4.24], we define the natural transformation

$$\mathfrak{p}_f : f_\sharp \xrightarrow{\sim} f_\sharp p_{1!} a_* \xrightarrow{Ex} f_! p_{2\sharp} a_* = f_! \Sigma_f.$$

The right adjoint of $\mathfrak{p}_f$ is denoted by

$$\mathfrak{q}_f : \Omega_f f^! \longrightarrow f^*.$$

Definition 2.5.6. Let f be a $\mathscr{P}$-morphism of $\mathscr{S}$-schemes, and assume that (f is proper) or (f is separated and $\mathscr{T}$ satisfies (Supp)). We say that f is *pure*, denoted by (Pur_f), if the natural transformation $\mathfrak{p}_f$ is an isomorphism. Note that if we assume (Adj), then f is pure if and only if $\mathfrak{q}_f$ is an isomorphism. We say that $\mathscr{T}$ satisfies the purity, denoted by (Pur), if $\mathscr{T}$ satisfies (Pur_f) for any exact log smooth separated morphism f.

Remark 2.5.7. Note that our definition is different from the definition in [CD12, 2.4.25] in which the additional condition that Σ_f is an isomorphism is assumed. However, if we assume (Loc), (Zar-Sep), and (Stab), then the definitions are equivalent by (2.5.3).

Theorem 2.5.8. *Assume that $\mathscr{T}$ satisfies (Htp–1), (Loc), and (Stab). Then $\mathscr{T}$ satisfies (sSupp).*

Proof. The conditions of the theorem of Ayoub [CD12, 2.4.28] are satisfied, and the same proof of [loc. cit] can be applied even if S is not a usual scheme. The consequence is that the projection $\mathbb{P}^1_S \to S$ is pure for any $\mathscr{S}$-scheme S. Then the conclusion follow from the proof of [CD12, 2.4.26(2)]. □

Theorem 2.5.9. *Assume that $\mathscr{T}$ satisfies (Htp–1), (Loc), (Stab), and (Supp). Then any strict smooth separated morphism is pure.*

Proof. As in the proof of (2.5.8), $\mathbb{P}^1_S \to S$ is pure for any $\mathscr{S}$-scheme S. Then the conclusion follows from the proof of [CD12, 2.4.26(3)]. □

Theorem 2.5.10. *Assume that $\mathscr{T}$ satisfies (Htp–1), (Loc), (Stab) and (Supp). Consider a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes such that f is strict smooth separated and g is separated. Then the exchange transformation

$$f_\sharp g'_! \xrightarrow{Ex} f'_\sharp g_!$$

is an isomorphism.

Proof. It follows from (2.5.9) and the proof of [CD12, 2.4.26(3)]. □

2.6 Base change property

2.6.1. Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. When f is proper, let us introduce the following base change properties.

(BC$_{f,g}$) The exchange transformation $g^*f_* \to f'_*g'^*$ is an isomorphism.

(BC–1') For all f and g such that f is strict and proper, (BC$_{f,g}$) is satisfied.

(BC–2') For all f and g such that f is an exact log smooth morphism and proper, (BC$_{f,g}$) is satisfied.

(BC–3') For all f and g such that g is strict and f is proper, (BC$_{f,g}$) is satisfied.

(BC–4') For all f and g such that g is a $\mathscr{P}$-morphism and f is proper, (BC$_{f,g}$) is satisfied.

On the other hand, when f is just assumed separated but $\mathscr{T}$ satisfies (Supp), we have the following base change properties.

($BC_{f,g}$) The exchange transformation $g^* f_! \to f'_! g'^*$ is an isomorphism.

(BC–1) For all f and g such that f is strict, ($BC_{f,g}$) is satisfied.

(BC–2) For all f and g such that f is an exact log smooth morphism, ($BC_{f,g}$) is satisfied.

(BC–3) For all f and g such that g is strict, ($BC_{f,g}$) is satisfied.

(BC–4) For all f and g such that g is a $\mathscr{P}$-morphism, ($BC_{f,g}$) is satisfied.

Proposition 2.6.2. *If $\mathscr{T}$ satisfies (Loc), then ($BC_{f,g}$) is satisfied for all f and g such that f is a strict closed immersion.*

Proof. It follow from the proof of [CD12, 2.3.13(1)], but we repeat the proof for the convenience of reader. Let h' denote the complement of f'. Then by (Loc), the pair (f'^*, h'^*) of functors is conservative, so it suffices to show that the natural transformations

$$f'^* g^* f_* \xrightarrow{ad} f'^* f'_* g'^*,$$

$$h'^* g^* f_* \xrightarrow{ad} h'^* f'_* g'^*$$

are isomorphisms. The first one is an isomorphism since the counits $f^* f_* \xrightarrow{ad} \mathrm{id}$ and $f'^* f'_* \xrightarrow{ad'} \mathrm{id}$ are isomorphisms by (Loc), and the second one is an isomorphism by (2.2.1(4)). □

Proposition 2.6.3. *If $\mathscr{T}$ satisfies (Supp), then the property (BC–n) implies (BC–n') for $n = 1, 2, 3, 4$.*

Proof. It follows from (2.3.3(4)). □

Proposition 2.6.4. *If $\mathscr{T}$ satisfies (Supp), then the category $\mathscr{T}$ satisfies (BC–4').*

Proof. It follows from (2.6.3) and ($\mathscr{P}$-BC). □

Proposition 2.6.5. *Assume that*

(i) $\mathscr{T}$ satisfies (Loc) and (Supp),

(ii) for any $\mathscr{S}$-scheme S, the projection $\mathbb{P}^1_S \to S$ is pure.

Then the category $\mathscr{T}$ satisfies (BC–1).

Proof. The conditions of [CD12, 2.4.26(2)] are satisfied, and the same proof of [loc. cit] can be applied even if S is a fs log scheme. □

Proposition 2.6.6. *Assume that $\mathscr{T}$ satisfies (Loc), (Supp), and (Zar-Sep). Then the category $\mathscr{T}$ satisfies (BC–3).*

Proof. By (2.6.3), it suffices to show (BC–3'). Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes such that g is strict and that f is proper. By (Zar-Sep), the question is Zariski local on S', so we reduce to the case when S' is affine. Then the morphism g is quasi-projective, so we reduce to the cases when

(1) g is an open immersion,

(2) g is a strict closed immersion,

(3) g is the projection $\mathbb{P}^1_S \to S$.

In the cases (1) and (3), we are done by (BC–4), so the remaining is the case (2). Hence assume that g is a strict closed immersion.

Let $h : S'' \to S$ denote the complement of g, and consider the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & \xleftarrow{h'} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S & \xleftarrow{h} & S'' \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. Since the pair $(g'_*, h'_\sharp)$ generates $\mathscr{T}(X)$ by (Loc), it suffices to show that the natural transformations

$$g^* f_* g'_* \to f'_* g'^* g'_*, \quad g^* f_* h'_\sharp \to f'_* g'^* h_\sharp$$

are isomorphisms. The first arrow is an isomorphism by (Loc), and the second arrow is an isomorphism by (Supp) and ($\mathscr{P}$-BC). This completes the proof. □

Proposition 2.6.7. *Assume that $\mathscr{T}$ satisfies (Loc). Consider a plain lower distinguished square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes, i.e., f and g are closed immersions such that $f(X) \cup g(X) = S'$, and the diagram is Cartesian. We put $h = fg'$. Then for any object K of $\mathscr{T}(S)$, the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{ad} & f_* f^* K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ g_* g^* K & \xrightarrow{ad} & h_* h^* K \end{array}$$

in $\mathscr{T}(S)$ is homotopy Cartesian.

Proof. Let $u : S'' \to S$ denote the complement of g. Then u factors through X by assumption, and let $u' : S'' \to X$ denote the morphism. and consider the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & \xleftarrow{u'} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g''} \\ S' & \xrightarrow{g} & S & \xleftarrow{u} & S'' \end{array}$$

where each square is Since the pair (g^*, u^*) of functors is conservative, it suffices to prove that the diagrams

$$\begin{array}{ccc} g^*K & \xrightarrow{ad} & g^*f_*f^*K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ g^*g_*g^*K & \xrightarrow{ad} & g^*h_*h^*K \end{array} \qquad \begin{array}{ccc} u^*K & \xrightarrow{ad} & u^*f_*f^*K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ u^*g_*g^*K & \xrightarrow{ad} & u^*h_*h^*K \end{array}$$

are homotopy Cartesian. The first diagram is isomorphic to

$$\begin{array}{ccc} g^*K & \xrightarrow{ad} & f'_*h^*K \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ g^*K & \xrightarrow{ad} & f'_*h^*K \end{array}$$

by (Loc) and (2.6.2), and it is homotopy Cartesian. For the second diagram, since its lower horizontal arrow is an isomorphism by ($\mathscr{P}$-BC), it suffices to show that the natural transformation

$$u^* \xrightarrow{ad} u^*f_*f^*K$$

is an isomorphism. Since $u = fu'$, we have the natural transformations

$$u'^*f^* \xrightarrow{ad} u'^*f^*f_*f^*K \xrightarrow{ad'} u'^*f^*K,$$

whose composition is an isomorphism. The second arrow is an isomorphism by (Loc), so the first arrow is also an isomorphism. □

2.7 Projection formula

Definition 2.7.1. For a proper morphism $f : X \to S$ of $\mathscr{S}$-schemes, we say that f satisfies the *projection formula*, denoted by (PF_f), if the exchange transformation

$$f_*K \otimes_X L \xrightarrow{Ex} f_*(K \otimes_Y f^*L)$$

is an isomorphism for any objects K of $\mathscr{T}(X)$ and L of $\mathscr{T}(S)$. We say that $\mathscr{T}$ satisfies the *projection formula*, denoted by (PF), if (PF_f) is satisfied for any proper morphism f.

2.7.2. Assume that $\mathscr{T}$ satisfies (PF) and (Supp). Let $f : X \to Y$ be a separated morphism of $\mathscr{S}$-schemes. Then by the proof of [CD12, 2.2.14(5)], the exchange transformation

$$f_! K \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^* L)$$

is an isomorphism for any objects K of $\mathscr{T}(X)$ and L of $\mathscr{T}(S)$. If we assume further (Adj), then by taking adjunctions of the above exchange transformation, we obtain the natural transformations

$$Hom_S(f_! K, L) \xrightarrow{\sim} f_* Hom_X(K, f^! L),$$
$$f^! Hom_X(L, M) \xrightarrow{\sim} Hom_X(f^* L, f^! M).$$

for any objects K of $\mathscr{T}(X)$ and L and M of $\mathscr{T}(S)$.

2.8 Orientation

Definition 2.8.1. Let $p : E \to S$ be a vector bundle of rank n of $\mathscr{S}$-schemes, and let $i_0 : S \to E$ denote its 0-section. Then an isomorphism

$$\mathfrak{t}_E : p_\sharp i_{0*} \longrightarrow 1_S(n)[2n]$$

is said to be an *orientation* of E. When $\mathscr{T}$ satisfies (Adj) and (Stab), we denote by

$$\mathfrak{t}'_E : 1_S(-n)[-2n] \longrightarrow i_0^! p^*$$

its right adjoint.

Recall from [CD12, 2.4.38] that a collection $\mathfrak{t}$ of orientations for all vector bundles $E \to S$ of $\mathscr{S}$-schemes with the compatibility conditions (a)–(c) in [loc. cit] is said to be an *orientation* of $\mathscr{T}$.

2.8.2. Note that by (2.5.3), if $\mathscr{T}$ satisfies (Loc), (Zar-Sep), and (Stab), then any vector bundle has an orientation.

2.9 Log motivic categories

Definition 2.9.1. Let $\mathscr{T}$ be a *eSm*-premotivic triangulated category. Borrowing a terminology from [CD12, 2.4.45], we say that $\mathscr{T}$ is a *log motivic triangulated category* if

(i) $\mathscr{T}$ satisfies (Adj), (Htp–1), (Htp–2), (Htp–3), (Htp–4), (Loc), (két-Sep), and (Stab).

(ii) for any $\mathscr{S}$-scheme S with the trivial log structure, the morphism $S \times \mathbb{A}_{\mathbb{N}} \to S \times \mathbb{A}^1$ removing the log structure satisfies the support property.

2.9.2. In [CD12, 2.4.45], motivic triangulated category is defined, and in [CD12, 2.4.50], the six operations formalism is given for motivic triangulated cateogires. Following this spirit, we introduced our notion of log motivic triangulated category, which will satisfy the log version of the six operations formalism (1)–(5) in (0.5).

Now, we state our main theorems.

Theorem 2.9.3. *A log motivic triangulated category satisfies the properties (1)–(6) in (0.5), the homotopy properties (Htp–5), (Htp–6), and (Htp–7), and (Pur).*

Theorem 2.9.4. *The eSm-premotivic category* $\mathrm{D}_{log,pw}(-,\Lambda)$ *is a log motivic triangulated category.*

2.9.5. Here is the outline of the proofs of the above theorems. Let $\mathscr{T}$ be a log motivic triangulated category over $\mathscr{S}$.

(1) In (2.6.3), (2.6.4), (2.6.5), and (2.6.6), we have proven that $\mathscr{T}$ satisfies (BC–1), (BC–3), and (BC–4).

(2) In (5.3.4), we will show that $\mathscr{T}$ satisfies (PF).

(3) In (5.6.5), we will show that $\mathscr{T}$ satisfies (Supp).

(4) In (6.1.9), we will show that $\mathscr{T}$ satisfies (Htp–5).

(5) In (6.2.1), we will show that $\mathscr{T}$ satisfies (Htp–6).

(6) In (6.3.1), we will show that $\mathscr{T}$ satisfies (Htp–7).

(7) In (6.4.4), we will show that $\mathscr{T}$ satisfies (BC–2).

(8) In (10.5.5), we will show that $\mathscr{T}$ satisfies (Pur).

(9) $\mathrm{D}_{log,pw}(-,\Lambda)$ satisfies (sét-Sep) and (Stab) by construction.

(10) In (7.5.3), we will show that $\mathrm{D}_{log,pw}(-,\Lambda)$ satisfies (Loc).

(11) In (7.6.3), we will show that $\mathrm{D}_{log,pw}(-,\Lambda)$ is compactly generated by eSm and τ. This implies (Adj) by [CD12, 1.3.20].

(12) In (8.3.3), we will show that $\mathrm{D}_{log,pw}(-,\Lambda)$ satisfies (Htp–1), (Htp–2), (Htp–3), and (Htp–4).

(13) In (8.4.3), we will show that $\mathrm{D}_{log,pw}(-,\Lambda)$ satisfies the axiom (ii) of (2.9.1).

Chapter 3

Some results on log geometry and motives

3.1 Charts of log smooth morphisms

Definition 3.1.1. Let $f : X \to S$ be a morphism of fine log schemes with a fine chart $\theta : P \to Q$. Consider the following conditions:

(i) θ is injective, the order of the torsion part of the cokernel of θ^{gp} is invertible in $\mathcal{O}_X$, and the induced morphism $X \to S \times_{\mathbb{A}_P} \mathbb{A}_Q$ is strict étale,

(ii) θ is locally exact,

(iii) $\overline{\theta}$ is Kummer.

Then we say that

(1) θ is of log smooth type if (i) is satisfied,

(2) θ is of exact log smooth type if (i) and (ii) are satisfied,

(3) θ is of Kummer log smooth type if (i) and (iii) are satisfied,

Definition 3.1.2. Let S be a fine log schemes with a fine chart P. Let $\theta : P \to Q$ be a homomorphism of fine monoids. Consider the following conditions:

(i) θ is injective and the order of the torsion part of the cokernel of θ^{gp} is invertible in $\mathcal{O}_S$,

(ii) θ is locally exact,

(iii) $\overline{\theta}$ is Kummer.

Then we say that

(1) θ is of log smooth over S type if (i) is satisfied,

(2) θ is of exact log smooth over S type if (i) and (ii) are satisfied,

(3) θ is of Kummer log smooth over S type if (i) and (iii) are satisfied,

Proposition 3.1.3. *Let $f : X \to S$ be a morphism of fs log schemes, and let P be a fs chart of S. If f is log smooth, then strict étale locally on X, there is a fs chart $\theta : P \to Q$ of f of log smooth type.*

Proof. By [Ogu14, IV.3.3.1], there is a *fine* chart $\theta : P \to Q$ of f of log smooth type. We may further assume that Q is exact at some point x of X by [Ogu14, II.2.3.2]. Then Q is saturated since $\overline{Q} \cong \overline{\mathcal{M}}_{X,x}$ is saturated, so θ is a fs chart. □

Proposition 3.1.4. *Let $f : X \to S$ be a morphism of fs log schemes, let x be a point of X, and let P be a fs chart of S exact at $s := f(x)$. If f is exact log smooth (resp. Kummer log smooth), then strict étale locally on X, there is a fs chart $\theta : P \to Q$ of f of exact log smooth type (resp. Kummer log smooth type).*

Proof. Let us use the notations and hypotheses of the proof of (3.1.3). When f is exact, by the proof of [NO10, 3.5], the homomorphism θ is critically exact. Then by [Ogu14, I.4.6.5], θ is locally exact. Thus we are done for the case when f is exact log smooth.

When f is Kummer, the homomorphism $\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{S,s}$ is Kummer. Thus the homomorphism $\overline{\theta}$ is Kummer since P is exact at s and Q is exact at x. This proves the remaining case. □

Proposition 3.1.5. *Let $g : S' \to S$ be a strict closed immersion of fs log schemes, and let $f' : X' \to S'$ be a log smooth (resp. exact log smooth, resp. Kummer log smooth) morphism of fs log schemes. Then strict étale locally on X', there is a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of fs log schemes such that f is log smooth (resp. exact log smooth, resp. Kummer log smooth).

Proof. Let x' be a point of X', and we put $s' = f'(x')$ and $s = g(s')$. We can choose a fs chart P of S exact at s by [Ogu14, II.2.3.2]. Then P is also a fs chart of S' exact at s'. By (3.1.3) and (3.1.4), there is a fs chart $\theta : P \to Q$ of log smooth type (resp. exact log smooth type, resp. Kummer log smooth type) such that the induced morphism

$$h' : X' \to S' \times_{\mathbb{A}_P} \mathbb{A}_Q$$

is strict étale. Then by [EGA, IV.18.1.1], Zariski locally on X', there is a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ S' \times_{\mathbb{A}_P} \mathbb{A}_Q & \xrightarrow{g''} & S \times_{\mathbb{A}_P} \mathbb{A}_Q \end{array}$$

such that h is strict étale and g'' denotes the morphism induced by g. The remaining is to put $f = ph$ where $p : S \times_{\mathbb{A}_P} \mathbb{A}_Q \to S$ denotes the projection. □

3.2 Change of charts

3.2.1. Let S be a fine log scheme, let $\alpha : P \to \Gamma(S, \mathcal{M}_S)$ and $\alpha' : P' \to \Gamma(S, \mathcal{M}_S)$ be fine charts of S, and let s be a geometric point of S. Assume that one of the following conditions is satisfied:

(a) α is neat at s,

(b) α is exact at s, and P'^{gp} is torsion free.

In this setting, strict étale locally on S near s, we will explicitly construct a chart $\alpha'' : P'' \to \Gamma(S, \mathcal{M}_S)$ and homomorphisms $\beta : P \to P''$ and $\beta' : P' \to P''$ such that $\alpha''\beta = \alpha$ and $\alpha''\beta' = \alpha'$.

By [Ogu14, II.2.3.9], strict étale locally on S near s, there exist homomorphisms

$$\kappa : P' \to P, \quad \gamma : P' \to \mathcal{M}_S^*$$

such that $\alpha' = \alpha \circ \kappa + \gamma$. Consider the homomorphisms

$$\beta : P \to P \oplus P'^{\mathrm{gp}}, \quad a \mapsto (a, 0),$$

$$\beta' : P' \to P \oplus P'^{\mathrm{gp}}, \quad a \mapsto (\kappa(a), a),$$

$$\alpha'' : P \oplus P'^{\mathrm{gp}} \to \Gamma(S, \mathcal{M}_S), \quad (a, b) \mapsto \alpha(a) + \gamma^{\mathrm{gp}}(b)$$

of fs monoids. Then $\alpha''\beta = \alpha$ and $\alpha''\beta' = \alpha'$. The remaining is to show that α'' is a chart of S. This follows from the fact that the morphism

$$\mathbb{A}_\beta : \mathbb{A}_{P \oplus P'^{\mathrm{gp}}} \to \mathbb{A}_P$$

is strict. So far we have discussed the way to compare charts of S. In the following two propositions, we will discuss the way to compare charts of birational morphisms.

Proposition 3.2.2. *Let S be a fs log scheme with fs charts $\alpha : P \to \Gamma(S, \mathcal{M}_S)$ and $\alpha' : P' \to \Gamma(S, \mathcal{M}_S)$, and let $\theta' : P' \to Q'$ be a homomorphism of fs monoid with a homomorphism $\varphi : Q'^{\mathrm{gp}} \to P'^{\mathrm{gp}}$ such that $\varphi \circ \theta'^{\mathrm{gp}} = \mathrm{id}$. Assume that P is neat at some geometric point $s \in S$. We denote by κ the composition*

$$P' \to \overline{\mathcal{M}}_{S,s} \to P$$

where the first (resp. second) arrow is the morphism induced by α (resp. α'). Consider the coCartesian diagram

$$\begin{array}{ccc} P' & \xrightarrow{\kappa} & P \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ Q' & \xrightarrow{\kappa'} & Q \end{array}$$

of fs monoids. Then strict étale locally on S, there is an isomorphism

$$S \times_{\mathbb{A}_{P'}} \mathbb{A}_{Q'} \cong S \times_{\mathbb{A}_P} \mathbb{A}_Q.$$

Proof. Choose homomorphisms β, β', and α'' as in (3.2.1). Consider the commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{\beta'} & P \oplus P'^{\mathrm{gp}} \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta''} \\ Q' & \xrightarrow{\delta'} & Q \oplus P'^{\mathrm{gp}} \end{array} \tag{3.2.2.1}$$

of fs monoids where θ'' and δ' denote the homomorphisms

$$\theta'' : P \oplus P'^{\mathrm{gp}} \to Q \oplus P'^{\mathrm{gp}}, \quad (a, b) \mapsto (\theta(a), b),$$

$$\delta' : Q' \to Q \oplus P'^{\mathrm{gp}}, \quad a \mapsto (\kappa'(a), \varphi(a)).$$

We will show that the above diagram is coCartesian. The induced commutative diagram

$$\begin{array}{ccc} P'^{\mathrm{gp}} & \xrightarrow{\beta'^{\mathrm{gp}}} & P^{\mathrm{gp}} \oplus P'^{\mathrm{gp}} \\ \downarrow{\scriptstyle \theta'^{\mathrm{gp}}} & & \downarrow{\scriptstyle \theta''^{\mathrm{gp}}} \\ Q'^{\mathrm{gp}} & \xrightarrow{\delta'^{\mathrm{gp}}} & Q^{\mathrm{gp}} \oplus P'^{\mathrm{gp}} \end{array}$$

of finitely generated abelian groups is coCartesian. Hence from the description of pushout in the category of fs monoid, to show that (3.2.2.1) is coCartesian, it suffices to show that the images of η and δ generate $Q \oplus P'^{\mathrm{gp}}$. This follows from the fact that $\kappa' : Q' \to Q$ is surjective.

We also have the coCartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta} & P \oplus P'^{\mathrm{gp}} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta''} \\ Q & \xrightarrow{\delta} & Q \oplus P'^{\mathrm{gp}} \end{array}$$

of fs monoids where δ denotes the first inclusion. Then we have isomorphisms $S \times_{\mathbb{A}_{P'}} \mathbb{A}_{Q'} \cong S \times_{\mathbb{A}_{P \oplus P'^{\mathrm{gp}}}} \mathbb{A}_{Q \oplus Q'^{\mathrm{gp}}} \cong S \times_{\mathbb{A}_P} \mathbb{A}_Q$. □

Proposition 3.2.3. *Let S be a fs log scheme with fs charts $\alpha : P \to \Gamma(S, \mathcal{M}_S)$ and $\alpha' : P' \to \Gamma(S, \mathcal{M}_S)$, and let $u' : M' \to \operatorname{spec} P'$ be a birational homomorphism of fs monoschemes. Assume that P is neat at some point $s \in S$. Consider the Cartesian diagram*

$$\begin{array}{ccc} M & \xrightarrow{v} & M' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} \\ \operatorname{spec} P & \xrightarrow{\operatorname{spec} \kappa} & \operatorname{spec} P' \end{array}$$

of fs monoschemes where κ denotes the homomorphism defined in (3.2.2). Then there is an isomorphism

$$S \times_{\mathbb{A}_{P'}} \mathbb{A}_{M'} \cong S \times_{\mathbb{A}_P} \mathbb{A}_M.$$

Proof. Let $\operatorname{spec} Q' \to M'$ be an open immersion, and let $\operatorname{spec} Q \to M$ denote the pullback of it via $v : M \to M'$. Then by (3.2.2), there is an isomorphism

$$S \times_{\mathbb{A}_{P'}} \mathbb{A}_{Q'} \cong S \times_{\mathbb{A}_P} \mathbb{A}_Q.$$

Its construction is compatible with any further open immersion $\operatorname{spec} Q'_1 \to \operatorname{spec} Q' \to M'$, so by gluing the isomorphisms, we get an isomorphism $S \times_{\mathbb{A}_{P'}} \mathbb{A}_{M'} \cong S \times_{\mathbb{A}_P} \mathbb{A}_M$. $\square$

3.3 Sections of log smooth morphisms

Lemma 3.3.1. *Let $f : X \to S$ be a log étale morphism of fs log schemes, and let $i : S \to X$ be its section. Then i is an open immersion.*

Proof. From the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{i} & X & \xrightarrow{f} & S \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{a} & X \times_S X & \xrightarrow{p_2} & X \end{array}$$

of fs log schemes where

(i) a denotes the diagonal morphism,

(ii) p_2 denotes the second projection,

(iii) each square is Cartesian,

it suffices to show that $a : X \to X \times_S X$ is an open immersion. Since the diagonal morphism

$$\underline{X} \to \underline{X} \times_{\underline{S}} \underline{X}$$

is radiciel, it suffices to show that a is strict étale by [EGA, IV.17.9.1]. As in [EGA, IV.17.3.5], the morphism a is log étale. Thus it suffices to show that a is strict. We will show this in several steps.

(I) *Locality on S.* Let $g : S' \to S$ be a strict étale cover of fs log schemes, and we put $X' = X \times_S S'$. Then the commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X' \times_{S'} X' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_S X \end{array}$$

of fs log schemes is Cartesian, so the question is strict étale local on S.

(II) *Locality on X.* Let $h : X' \to X$ be a strict étale cover of fs log schemes. Then we have the commutative diagram

$$\begin{array}{ccc} & & X' \times_S X' \\ & \overset{a''}{\nearrow} & \downarrow h'' \\ X' & \xrightarrow{a'} & X \times_S X' \\ \downarrow h & & \downarrow h' \\ X & \xrightarrow{a} & X \times_S X \end{array}$$

of fs log schemes where

(i) the small square is Cartesian,

(ii) a'' denotes the diagonal morphism,

(iii) h' and h'' denote the morphism induced by $h : X' \to X$.

Assume that a'' is strict. Then a' is strict since h'' is strict, so a is strict since h is a strict étale cover. Conversely, assume that a is strict. Then a' is strict, so a'' is strict. Thus the question is strict étale local on X

(III) Final step of the proof. By [Ogu14, IV.3.3.1], strict étale locally on X and S, we have a fs chart $\theta : P \to Q$ of f such that

(i) θ is injective, and the cokernel of θ^{gp} is finite,

(ii) the induced morphism $X \to S \times_{\mathbb{A}_P} \mathbb{A}_Q$ is strict étale.

Hence by (I) and (II), we may assume that $(X, S) = (\mathbb{A}_Q, \mathbb{A}_P)$. Then it suffices to show that the diagonal homomorphism

$$\mathbb{A}_Q \to \mathbb{A}_Q \oplus_{\mathbb{A}_P} \mathbb{A}_Q$$

is strict. To show this, it suffices to show $a \oplus (-a) \in (Q \oplus_P Q)^*$ for any $a \in Q$. Choose $n \in \mathbb{N}^+$ such that $na \in P^{\mathrm{gp}}$. Because the summation homomorphism

$$P^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} P^{\mathrm{gp}} \to P^{\mathrm{gp}}$$

is an isomorphism, the two elements $(na) \oplus 0$ and $0 \oplus (na)$ of $Q \oplus_P Q$ are equal. Thus $n(a \oplus (-a)) = 0$. Since $Q \oplus_P Q$ is a fs monoid, we have $a \oplus (-a) \in Q \oplus_P Q$. This means $a \oplus (-a) \in (Q \oplus_P Q)^*$ since $n(a \oplus (-a)) = 0$. □

Lemma 3.3.2. *Let $f : X \to S$ and $h : Y \to X$ be morphisms of fine log schemes. Assume that h is surjective. If h and fh are strict, then f is strict.*

Proof. By [Ogu14, III.1.2.10], it suffices to show that the induced homomorphism

$$\overline{\mathcal{M}}_{S,\overline{f(x)}} \to \overline{\mathcal{M}}_{X,\overline{x}}$$

of fine monoids is an isomorphism for any point x of X. Since h is surjective, we can choose a ray $y \in Y$ whose image in X is x. Then we have the induced homomorphisms

$$\overline{\mathcal{M}}_{S,\overline{f(x)}} \to \overline{\mathcal{M}}_{X,\overline{x}} \to \overline{\mathcal{M}}_{Y,\overline{y}}$$

of fine monoids. The second arrow (resp. the composition of the two arrows) is an isomorphism since h (resp. fh) is strict. Thus the first arrow is also an isomorphism. $\square$

Lemma 3.3.3. *Let $f : X \to S$ be a morphism of fine log schemes. Then there is a maximal open subscheme U of X such that the composition $U \to X \xrightarrow{f} S$ is strict.*

Proof. Let U denote the set of points $x \in X$ such that the induced homomorphism

$$\overline{\mathcal{M}}_{S,\overline{f(x)}} \to \overline{\mathcal{M}}_{X,\overline{x}}$$

of fine monoids is an isomorphism. If U is open in X, then by [Ogu14, III.1.2.10], U is the maximal open subscheme U of X such that the composition $U \to X \xrightarrow{f} S$ is strict. Thus the remaining is to show that U is open.

By [Kat00, 1.5], there is a strict étale morphism $h : Y \to X$ of fine log schemes such that the image of h contains U and that fh is strict. Let V denote the image of h, which can be considered as an open subscheme of X. Then by (3.3.2), the composition $V \to X \xrightarrow{f} S$ is strict where the first arrow is the open immersion. Thus $V \subset U$ by the construction of U. Then $V = U$, and in particular U is open in X. $\square$

Lemma 3.3.4. *Let S be a fs log scheme such that the underlying scheme $\underline{S}$ is henselian, let $i : Z \to S$ a strict closed immersion of fs log schemes, and let $f : X \to S$ be a log étale morphism of fs log schemes. Then any partial section $s : Z \to X$ of $X \to S$ can be uniquely extended to a section $S \to X$.*

Proof. The graph morphism $t : Z \to Z \times_S X$ of S is a section of the projection $Z \times_S X \to Z$, so t is an open immersion by (3.3.1). We denote by U the set of points x of X such that the induced homomorphism

$$\overline{\mathcal{M}}_{S,\overline{f(x)}} \to \overline{\mathcal{M}}_{X,\overline{x}}$$

is an isomorphism. Then by the proof of (3.3.3), U is an open subset of X, and we consider it as an open subscheme of X. Since t is strict, t factors through $Z \times_S U$, so we have the commutative diagram

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ Z & \xrightarrow{i} & S \end{array}$$

of fs log schemes. Then since $\underline{S}$ is henselian, there exists a section of f' extending s', which makes a section of f extending s. Hence the remaining is the uniqueness of a section.

If $s' : S \to X$ is a section of f extending s, then by (3.3.1), it is an open immersion, so s' should factor through U. The morphism $U \to S$ is strict étale, so a section s' is unique since $\underline{S}$ is henselian. $\square$

Lemma 3.3.5. *Let $f : X \to S$ be a Kummer log smooth separated morphism of fs log schemes, and let $i : S \to X$ be its section. Then i is a strict regular embedding.*

Proof. Since i is a pullback of the diagonal morphism $d : X \to X \times_S X$, it suffices to show that d is a strict regular embedding. The new question is strict étale local on X and S, so we may assume that f has a fs chart $\theta : P \to Q$ of Kummer log smooth type by (3.1.4). Then $\overline{\theta} : \overline{P} \to \overline{Q}$ is Kummer, so by (1.2.18), the summation homomorphism $Q \to Q \oplus_P Q$ is strict. Thus the first inclusion $Q \oplus_P Q \to Q$ is also strict. In particular, the first projection $p_1 : X \times_S X \to X$ is strict smooth. Then d is the section of a strict smooth separated morphism, so d is a strict regular embedding. □

Lemma 3.3.6. *Let $\theta : P \to Q$ be a Kummer homomorphism of fs monoids, and let $\eta : Q \to P$ be a homomorphism of fs monoids such that $\eta\theta = \mathrm{id}$. If Q is sharp, then θ is an isomorphism.*

Proof. Let $q \in Q$ be an element not in $\theta(P)$. Since θ is $\mathbb{Q}$-surjective, we can choose $n \in \mathbb{N}^+$ such that $nq = \theta(p)$ for some $p \in P$. Then

$$n(q - \theta\eta(q)) = \theta(p) - \theta\eta\theta(p) = 0,$$

so $q - \theta\eta(q) \in Q^*$ since Q is saturated. Then $q - \theta\eta(q) = 0$ since Q is sharp, which proves the assertion. □

3.4 Log étale monomorphisms

Proposition 3.4.1. *Let $f : X \to S$ be a log étale monomorphism of fs log schemes, and let P be a fs chart of S. Then Zariski locally on X, there exists a chart $\theta : P \to Q$ of f with the following properties:*

(i) the induced morphism $X \to S \times_{\mathbb{A}_P} \mathbb{A}_Q$ is an open immersion,

(ii) $\theta^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is an isomorphism.

Proof. Let x be a point of X. By [Ogu14 IV.3.3.1], there is a strict étale neighborhood $g : X' \to X$ of x such that fg has a fs chart $\theta' : P \to Q'$ of log étale type. Let $x' \in X'$ be a point over x. By [Ogu14, II.2.3.1], we may further assume that the chart $Q' \to \mathcal{M}_{X'}$ is exact at x'. We may also assume that g is a strict étale cover because the question is Zariski local on X.

We put $Q = P^{\mathrm{gp}} \cap Q'$. Then Q is a fs monoid by Gordon's lemma [Ogu14, I.2.3.17], and the induced homomorphism $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is an isomorphism. The inclusion $Q \to Q'$ is Kummer since the inclusion $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is Kummer, so replacing $S \to \mathbb{A}_P$ by $S \times_{\mathbb{A}_P} \mathbb{A}_Q \to \mathbb{A}_Q$, we may assume that θ' is Kummer.

Since f is a monomorphism, the diagonal morphism $X \to X \times_S X$ is an isomorphism, so the morphism $X' \times_X X' \to X' \times_S X'$ induced by f is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} \underline{X' \times_X X'} & \xrightarrow{\sim} & \underline{X' \times_S X'} \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle v} \\ \underline{X'} \times_{\underline{X}} \underline{X'} & \xrightarrow{u} & \underline{X'} \times_{\underline{S}} \underline{X'} \\ \downarrow & & \downarrow \\ \underline{X} & \xrightarrow{\quad d \quad} & \underline{X} \times_{\underline{S}} \underline{X} \end{array}$$

of fs log schemes where d denotes the diagonal morphism. Since d is an immersion, u is immersion, so v is an immersion. Consider the factorizaton

$$\underline{X' \times_S X'} \xrightarrow{w} \underline{X' \times_S^{\mathrm{int}} X'} \xrightarrow{w'} \underline{X'} \times_{\underline{S}} \underline{X'}$$

of v where $X' \times_S^{\mathrm{int}} X'$ denotes the fiber product computed in the category of *fine* log schemes. Then w is an immersion since v is an immersion.

If θ' is not an isomorphism, then let $a \in Q' - \theta'(P)$ be an element. For some $n \in \mathbb{N}^+$, we have $na \in P$. Then the monoid

$$Q'' := Q' \oplus_P^{\mathrm{int}} Q'$$

is *not* saturated since $(a, -a) \notin Q''$ but $n(a, -a) = 0 \in Q''$. Thus for any morphism $t : \operatorname{Spec} k \to \underline{\mathbb{A}_{(Q'', Q''+)}}$ where k is a field, the pullback $T \to \operatorname{Spec} k$ of the induced morphism

$$\underline{\mathbb{A}_{(Q''^{\mathrm{sat}}, Q''+)}} \to \underline{\mathbb{A}_{(Q'', Q''+)}}$$

via t is not an isomorphism. This contradicts to the fact that w is an immersion. Thus θ' is an isomorphism. Then fg is strict, so f is strict by (3.3.2) since g is a strict étale cover. Thus f is a strict étale monomorphism, which is an open immersion by [EGA, IV.17.9.1]. This completes the proof. □

Corollary 3.4.2. *Let $(f_i : X_i \to S)_{i \in I}$ be a finite family of log étale monomorphisms such that each X_i is quasi-compact, and let P be a fs chart of S. Then there is a birational morphism $u : M \to \operatorname{spec} P$ of fs monoschemes such that for each i, the induced morphism*

$$X_i \times_{\mathbb{A}_P} \mathbb{A}_M \to S \times_{\mathbb{A}_P} \mathbb{A}_M$$

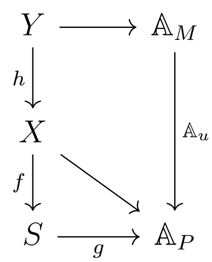
is open immersion. We may also assume that u is proper.

Proof. Note that the question is Zariski local on each X_i. By (3.4.1), for each i, Zariski locally on X_i, there is a homomorphism $\theta_i : P \to Q_i$ of fs monoids such that θ_i^{gp} is an isomorphism and that the induced morphism $X_i \to S \times_{\mathbb{A}_P} \mathbb{A}_{Q_i}$ is an open immersion.

Choose a fan Σ of the dual lattice $(P^{\mathrm{gp}})^\vee$ such that for each element σ of Σ, there is $i \in I$ such that $\sigma \subset Q_i^\vee$. Let M denote the monoscheme associated to the fan Σ. Then for each i, the induced morphism $X_i \times_{\mathbb{A}_P} \mathbb{A}_M \to S \times_{\mathbb{A}_P} \mathbb{A}_Q$ is open immersion.

If we choose a fan Σ such that its support is equal to $(P^{\mathrm{gp}})^\vee$, then $u : M \to \operatorname{spec} P$ is proper. □

Corollary 3.4.3. *Let $f : X \to S$ be a proper log étale monomorphism of fs log schemes such that X is quasi-compact, and let $g : S \to \mathbb{A}_P$ be a fs chart. Then there are a proper birational morphism $M \to \operatorname{spec} P$ of fs monoschemes and a commutative diagram*



such that the outside diagram and the upper square are Cartesian

Proof. By (3.4.2), there are a Zariski cover $(v_i : X_i \to X)_{i \in I}$ with finite I and a proper birational morphism $u : M \to \operatorname{spec} P$ of monoschemes such that for each $i \in I$, the induced morphism $X_i \times_{\mathbb{A}_P} \mathbb{A}_M \to S \times_{\mathbb{A}_P} \mathbb{A}_M$ is an open immersion. Then the induced morphism

$$g' : X \times_{\mathbb{A}_P} \mathbb{A}_M \to S \times_{\mathbb{A}_P} \mathbb{A}_M$$

is also an open immersion. Since f is proper, g' should be an isomorphism. If we put $Y = S \times_{\mathbb{A}_P} \mathbb{A}_M$, then we get the wanted diagram. □

3.5 Structure of Kummer homomorphisms

Lemma 3.5.1. *Let P be a fine monoid such that the group $\overline{P}^{\text{gp}}$ is torsion free (e.g. when P is a fs monoid). Then there is a section $\overline{P} \to P$ of the quotient homomorphism $\theta : P \to \overline{P}$.*

Proof. We follow the proof of [Ogu14, II.2.3.7]. Since $\overline{P}^{\text{gp}}$ is torsion free, we can choose a section

$$\eta : \overline{P}^{\text{gp}} \to P^{\text{gp}}$$

of θ^{gp}. Let p be an element of $\overline{P}$, and choose an element $p' \in P$ such that $\theta(p') = p$. Then

$$\theta^{\text{gp}} \eta(p) = p = \theta^{\text{gp}}(p'),$$

so $\eta(p) - p' \in \ker \theta^{\text{gp}} = P^*$. Thus $\eta(p) \in P$. This means $\eta(\overline{P}) \subset P$, so η induces a section of θ. □

Lemma 3.5.2. *Under the notations and hypotheses of (3.5.1), the homomorphism*

$$\lambda : \overline{P} \oplus P^* \to P$$

induced by a section $\eta : \overline{P} \to P$ of θ and the inclusion $P^* \to P$ is an isomorphism.

Proof. Let p be an element of P. Then $p = \eta\theta(p) + p'$ for some $p' \in P^*$, so $p = \lambda(\theta(p), p')$, i.e., λ is surjective. For the injectivity, let (p_1, p_2) and (p_1', p_2') be elements of $\overline{P} \oplus P^*$ such that $\lambda(p_1, p_2) = \lambda(p_1', p_2')$. Then we have

$$p_1 = \theta\lambda(p_1, p_2) = \theta\lambda(p_1', p_2') = p_1'.$$

The above two equations implies $p_2 = p_2'$, and this shows the injectivity of λ. $\square$

Lemma 3.5.3. *Let $\theta : P \to Q$ be a homomorphism of fs monoids such that $\overline{\theta}$ is Kummer, and for $n \in \mathbb{N}^+$, let $\mu_n : P \to P$ denote the multiplication homomorphism $a \mapsto na$. Then there is $n \in \mathbb{N}^+$ such that in the coCartesian diagram*

$$\begin{array}{ccc} P & \xrightarrow{\theta} & Q \\ \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \eta} \\ P & \xrightarrow{\theta'} & Q' \end{array}$$

of fs monoids, the homomorphism θ' is strict, i.e., $\overline{\theta'}$ is an isomorphism.

Proof. By (3.5.1), there is a section $\lambda : \overline{P} \to P$ of the quotient homomorphism $P \to \overline{P}$. Replacing $P \xrightarrow{\theta} Q$ by $\overline{P} \xrightarrow{\lambda} P \xrightarrow{\theta} Q$, we may assume that P is sharp.

If $\theta'(p) \in Q'^*$ for some $p \in P$, then $n\theta(p) \in Q^*$. Thus $\theta(p) \in Q^*$, contradicting to the assumption that $\overline{\theta}$ is Kummer. Thus the remaining is to show the surjectivity of $\overline{\theta'}$.

Choose $n \in \mathbb{N}^+$ such that $nQ \subset \theta(P) + Q^*$. If $q \in Q$ is an element, then $nq = \theta(p) + q'$ for some $p \in P$ and $q' \in Q^*$. We have

$$n\eta(q) - n\theta'(p) \in Q'^*,$$

so $\overline{\eta(q)} = \overline{\theta'(p)}$ in $\overline{Q'}$ because Q' is saturated. This shows the surjectivity of $\overline{\theta'}$. $\square$

3.5.4. We will study the structure of Kummer homomorphisms of fs monoids as follows. Let $\theta : P \to Q$ be a Kummer homomorphism of fs monoids, and let $\lambda : \overline{P} \to P$ be a section of the quotient homomorphism $P \to \overline{P}$. Such a section exists by (3.5.1).

By (3.5.3), there is $n \in \mathbb{N}^+$ such that in the diagram

$$\begin{array}{ccccc} \overline{P} & \xrightarrow{\lambda} & P & \xrightarrow{\theta} & Q \\ \downarrow{\scriptstyle \overline{\mu}_n} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ \overline{P} & \longrightarrow & P' & \xrightarrow{\theta'} & Q' \end{array}$$

of fs monoids where each square is coCartesian and $\mu_n : P \to P$ denotes the multiplication homomorphism $a \mapsto na$, the homomorphism θ' is strict. Then by (3.5.2), we obtain the commutative diagram

$$\begin{array}{ccc} P \cong \overline{P} \oplus P^* & \longrightarrow & Q \\ \downarrow{\scriptstyle \overline{\mu}_n \oplus \mu'^*} & & \downarrow \\ P' \cong \overline{P} \oplus P'^* & \xrightarrow{\mathrm{id} \oplus \theta'^*} & \overline{P} \oplus Q'^* \end{array}$$

of fs monoids.

3.6 Generating motives

3.6.1. Throughout this section, we fix a full subcategory $\mathscr{S}$ of the category of fs log schemes satisfying the conditions (2.0.1). We also fix a τ-twisted eSm-premotivic triangulated category $\mathscr{T}$ over $\mathscr{S}$ generated by eSm and τ satisfying (Loc) and (sét-Sep).

3.6.2. Let S be an $\mathscr{S}$-scheme with a fs chart $\alpha : P \to \mathcal{M}_S$. We denote by $\mathcal{F}_{S,\alpha}$ the family of motives in $\mathscr{T}(S)$ of the form

$$M_S(S' \times_{\mathbb{A}_{P'},\mathbb{A}_{\theta'}} \mathbb{A}_Q)\{r\}$$

where

(i) $S' \to S$ is a Kummer log smooth morphism with a fs chart $\eta : P \to P'$ of Kummer log smooth type,

(ii) $\theta' : P' \to Q$ is an injective homomorphism of fs monoids such that the cokernel of θ'^{gp} is torsion free,

(iii) θ' is logarithmic and locally exact,

(iv) r is a twist in τ.

Proposition 3.6.3. *Under the notations and hypotheses of (3.6.2), the family $\mathcal{F}_{S,\alpha}$ generates $\mathscr{T}(S)$.*

Proof. Let $f : X \to S$ be an exact log smooth morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ of exact log smooth type. It suffices to show that the motive $M_S(X)\{r\}$ is in $\langle \mathcal{F}_{S,\alpha} \rangle$ where r is a twist in τ. Here, $\langle \mathcal{F}_{S,\alpha} \rangle$ denotes the localizing subcategory of $\mathcal{T}(S)$ generated by $\mathcal{F}_{S,\alpha}$.

(I) *Reduction of S.* Note first that the question is strict étale local on S by (2.1.2). Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, let $j : U \to S$ denote its complement, and let $\beta : P \to \mathcal{M}_Z$ denote the fs chart induced by α. Assume that the question is true for Z and U. Then by (Loc), to show the question for S, it suffices to show that the motive

$$i_* M_Z(Z' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)\{r\}$$

with the similar conditions as in (i)–(iv) of (3.6.2) is in $\langle F_{Z,\beta} \rangle$.

The induced morphism $Z' \to Z \times_{\mathbb{A}_P} \mathbb{A}_{P'}$ is open since it is smooth, and let W denote its image. Choose an open immersion $Y \to S \times_{\mathbb{A}_P} \mathbb{A}_{P'}$ such that $W \cong Z \times_S Y$ and that η^{gp} is invertible in $\mathcal{O}_Y$. By [EGA, IV.18.1.1], we may assume $Z' \cong W \times_Y S'$ for some strict smooth morphism $S' \to Y$ since the question is Zariski local on Z'. Then we have $Z' \cong Z \times_S S'$. By (Loc), we have the distinguished triangle

$$M_S(U \times_S S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q) \longrightarrow M_S(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q) \longrightarrow i_* M_Z(Z' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q) \longrightarrow M_S(U \times_S S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)[1]$$

in $\mathscr{T}(S)$, and this proves the question since $M_S(U \times_S S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)$ and $M_S(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)$ are in $\langle \mathcal{F}_{S,\alpha} \rangle$.

By the proof of [Ols03, 3.5(ii)], there is a stratification $\{S_i \to S\}$ of S such that each S_i has a constant log structure. Hence applying the above argument, we reduce to the case when $\alpha : P \to \mathcal{M}_S$ induces a constant log structure.

(II) *Construction of P'.* We will use induction on

$$d := \max_{x \in X} \operatorname{rk} \overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}.$$

If $d = \dim P$, then f is Kummer log smooth, so we are done. Hence let us assume $d > \dim P$.

We denote by P' the submonoid of Q consisting of elements $p \in Q$ such that $np \in \theta(P) + Q^*$ for some $n \in \mathbb{N}^+$. Then P' is a fs monoid by Gordon's lemma [Ogu14, I.2.3.17]. Let $\theta' : P' \to Q$ denote the inclusion. Then the cokernel of θ'^{gp} is torsion free by construction. We will check the conditions (ii) and (iii) of (3.6.2). Since $P'^{\mathrm{gp}} = Q^{\mathrm{gp}}$, θ' is logarithmic. For the locally exactness, it suffices to show that $\theta'_{\mathbb{Q}} : \overline{P}'_{\mathbb{Q}} \to \overline{Q}_{\mathbb{Q}}$ is integral. This follows from [Ogu14, I.4.5.3(2), I.4.5.3(1)]. The remaining is to show that θ'^{gp} is torsion free.

Let G be a maximal θ'-critical face of Q. Then we have $(\overline{Q})_{\mathbb{Q}}^{\mathrm{gp}} = (\overline{P'})_{\mathbb{Q}}^{\mathrm{gp}} \oplus (\overline{G})_{\mathbb{Q}}^{\mathrm{gp}}$ by [Ogu14, I.4.6.6]. Thus, for any $q \in Q^{\mathrm{gp}}$ such that $nq \in P'^{\mathrm{gp}}$ for some $n \in \mathbb{N}^+$, the image of q in $\overline{P'})_{\mathbb{Q}}^{\mathrm{gp}} \oplus (\overline{G})_{\mathbb{Q}}^{\mathrm{gp}}$ should be in $\overline{P'})_{\mathbb{Q}}^{\mathrm{gp}}$. This means $nq + p' \in P' + Q^*$ for some $p' \in P'$, so $n(q+p) \in P' + Q^*$. Thus $q + p \in P'$ by the construction of P', so $q \in P'^{\mathrm{gp}}$. This proves that θ'^{gp} is torsion free.

(III) *Construction of S'.* The induced morphism $X \to S \times_{\mathbb{A}_P} \mathbb{A}_{P'}$ is open by [Nak09, 5.7], and let Y denote its image. Then Y has the chart P'. Note that the induced morphism $Y \to S$ is Kummer log smooth and that the order of the torsion part of the cokernel of η^{gp} is invertible in $\mathcal{O}_Y$.

The closed immersion $Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(P',P'^+)} \to Y$ is an isomorphism since S has a constant log structure. Thus the projection

$$\underline{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)}} \to \underline{Y}$$

of underlying schemes is an isomorphism since $\theta' : P' \to Q$ is logarithmic. Consider the pullback

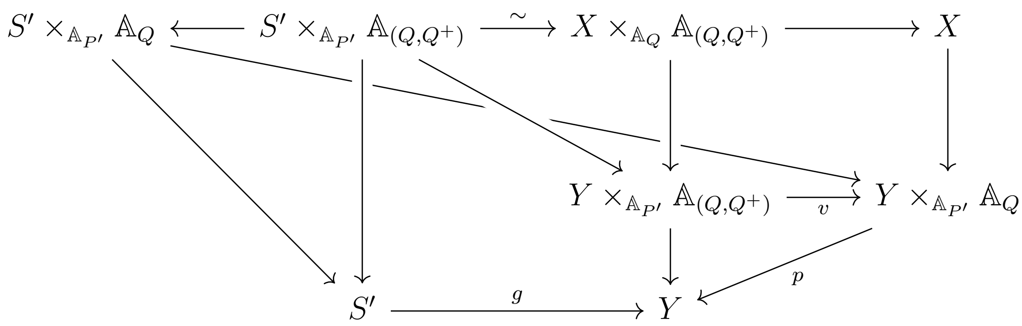
$$g' : X \times_{\mathbb{A}_Q} \mathbb{A}_{(Q,Q^+)} \to Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)}$$

of the induced morphism $h : X \to Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q$. Since θ is exact log smooth type, h is strict étale, so g' is also strict étale. Then there is a unique Cartesian diagram

$$\begin{array}{ccc} X \times_{\mathbb{A}_Q} \mathbb{A}_{(Q,Q^+)} & \xrightarrow{g'} & Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)} \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\quad g \quad} & Y \end{array}$$

of $\mathscr{S}$-schemes where the right vertical arrow is the projection. The morphism g is automatically strict étale. This verifies the condition (i) of (3.6.2), so we have checked the conditions (i)–(iii) of (loc. cit) for our constructions of P' and S'.

(IV) *Final step of the proof.* Then we have the commutative diagram



of $\mathscr{S}$-schemes. Note that the projection p is log smooth by the conditions (ii) and (iii) of (loc. cit). Let u denote the complement of the closed immersion $v : Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)} \to Y$. Then by (Loc), we have distinguished triangles

$$p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(X) \longrightarrow M_S(X) \longrightarrow p_\sharp v_* v^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(X) \longrightarrow p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(X)[1],$$

$$p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q) \longrightarrow M_S(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q) \longrightarrow p_\sharp v_* v^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q) \longrightarrow p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)[1].$$

Let r be a twist in τ. We have isomorphisms

$$\begin{aligned} v^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(X) &\cong M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)}}(X \times_{\mathbb{A}_Q} \mathbb{A}_{(Q,Q^+)}) \\ &\cong M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)}}(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_{(Q,Q^+)}) \\ &\cong v^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q), \end{aligned}$$

and $M_S(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)$ is in $\langle \mathcal{F}_{S,\alpha} \rangle$ by definition. Moreover, by induction on d,

$$p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(X), \quad p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_{P'}} \mathbb{A}_Q}(S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)\{r\}$$

are in $\langle \mathcal{F}_{S,\alpha} \rangle$. Thus from the above triangles, we conclude that $M_S(X)\{r\}$ is also in $\langle \mathcal{F}_{S,\alpha} \rangle$. □

Corollary 3.6.4. *Assume that $\mathscr{T}$ satisfies (Htp–3). Let S be an $\mathscr{S}$-scheme with a fs chart $\alpha : P \to \mathcal{M}_S$. Consider the family of motives in $\mathscr{T}(S)$ of the form*

$$M_S(S')\{r\}$$

where r is a twist and $S' \to S$ is a Kummer log smooth morphism with a fs chart $\theta : P \to P'$ of Kummer log smooth type. Then the family generates $\mathscr{T}(S)$.

Proof. Let $S' \to S$ be a Kummer log smooth morphism with a fs chart $\theta : P \to P'$ of Kummer log smooth type, let $\theta' : P' \to Q$ is a logarithmic, locally exact, and injective homomorphism of fs monoids such that the cokernel of θ'^{gp} is torsion free, and let G be a θ'-critical face of Q. Then the induced morphism

$$M_S(S' \times_{\mathbb{A}_{P'}, \mathbb{A}_{\theta'}} \mathbb{A}_{Q_G}) \to M_S(S' \times_{\mathbb{A}_{P'}, \mathbb{A}_{\theta'}} \mathbb{A}_Q)$$

in $\mathscr{T}(S)$ is an isomorphism by (Htp–3). Since the induced morphism

$$S' \times_{\mathbb{A}_{P'}, \mathbb{A}_{\theta'}} \mathbb{A}_{Q_G} \to S$$

is Kummer log smooth and has the fs chart $P \to Q_G$ of Kummer log smooth type, we are done. $\square$

Chapter 4

Purity

4.0.1. Throughout this chapter, we fix a full subcategory $\mathscr{S}$ of the category of fs log schemes satisfying the conditions of (2.0.1). We also fix a eSm-premotivic triangulated category satisfying (Adj), (Htp–1), (Loc), and (Stab).

4.1 Thom transformations

Definition 4.1.1. Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes, and let $i : S \to X$ be its section. Assume that i is a strict regular embedding. We have the following definitions.

(1) $B_S X$ denotes the blow-up of X with center S,

(2) $B_S(X \times \mathbb{A}^1)$ denotes the blow-up of $X \times \mathbb{A}^1$ with center $S \times \{0\}$,

(3) $D_S X = B_S(X \times \mathbb{A}^1) - B_S X$,

(4) $N_S X$ denotes the normal bundle of S in X.

The morphisms $S \xrightarrow{i} X \xrightarrow{f} S$ induces the morphisms $D_S S \to D_S X \to D_S S$, which is

$$S \times \mathbb{A}^1 \to D_S X \to S \times \mathbb{A}^1 \tag{4.1.1.1}$$

since $D_S S = S \times \mathbb{A}^1$.

Definition 4.1.2. Let $h : X \to Y$ and $g : Y \to S$ be morphisms of $\mathscr{S}$-schemes, and we put $f = gh$. Consider a commutative diagram

$$\begin{array}{ccccc} D_0 & \xrightarrow{b} & D & & \\ \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u} & \searrow^{q_2} & \\ X & \xrightarrow[a]{} & Y \times_S X & \xrightarrow[p_2]{} & X \end{array}$$

of $\mathscr{S}$-schemes where a denotes the graph morphism and p_2 denotes the second projection. Assume that b is proper. Then we define the following functors:

$$\Sigma_{g,f} := p_{2\sharp} a_*, \quad \Omega_{g,f} := a^! p_2^*, \quad \Omega_{g,f,D} := u_{0*} b^! q_2^*.$$

The third notation depends on the morphisms, so we will use it only when no confusion arises.

When b is a strict regular embedding, consider the diagrams

$$
\begin{array}{ccccc}
D_0 & \xrightarrow{b} & D & \xrightarrow{q_2} & X \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \alpha_1} \\
D_0 \times \mathbb{A}^1 & \xrightarrow{d} & D_{D_0}D & \xrightarrow{s_2} & X \times \mathbb{A}^1 \\
\downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \pi} \\
D_0 & & & & X
\end{array}
\qquad
\begin{array}{ccccc}
D_0 & \xrightarrow{e} & N_{D_0}D & \xrightarrow{t_2} & X \\
\downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \alpha_0} \\
D_0 \times \mathbb{A}^1 & \xrightarrow{d} & D_{D_0}D & \xrightarrow{s_2} & X \times \mathbb{A}^1 \\
\downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \pi} \\
D_0 & & & & X
\end{array}
\tag{4.1.2.1}
$$

of $\mathscr{S}$-schemes where

(a) each square is Cartesian,

(b) α_0 denotes the 0-section, and α_1 denotes the 1-section,

(c) d and s_2 are the morphisms constructed by (4.1.1.1),

(d) ϕ and π denotes the projections.

Then we define the following functors:

$$\Omega^d_{g,f,D} := u_{0*}\pi_* d^! s_2^* \pi^*, \quad \Omega^n_{g,f,D} := u_{0*} e^! t_2^*.$$

Now, assume $u_0 = \mathrm{id}$. Then we define the following functor:

$$\Omega^o_{g,f,D} := \mathfrak{t}'_{N_X D}.$$

Here, $\mathfrak{t}'_{N_X D}$ is the right adjoint of an orientation of $N_X D$, and it exists by (2.8.2). By (2.5.3), the functor $\Omega^n_{g,f,D}$ is an equivalence, and by a theorem of Morel and Voevodsky [CD12, 2.4.35], $\Omega^d_{g,f,D}$ is also an equivalence. We denote by

$$\Sigma^d_{g,f,D}, \quad \Sigma^n_{g,f,D}, \quad \Sigma^o_{g,f,D}$$

the left adjoints (or equivalently right adjoints) of $\Omega^d_{g,f,D}$, $\Omega^n_{g,f,D}$, and $\Omega^o_{g,f,D}$ respectively.

When h is the identity morphism, we simply put

$$\Sigma_f := \Sigma_{f,f}, \quad \Omega_f := \Omega_{f,f}, \quad \Omega_{f,D} := \Omega_{f,f,D},$$

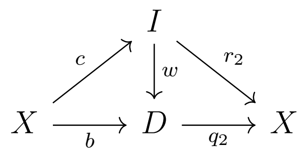
$$\Omega^d_{f,D} := \Omega^d_{f,f,D}, \quad \Omega^n_{f,D} := \Omega^n_{f,f,D}, \quad \Omega^o_{f,D} := \Omega^o_{f,f,D},$$
$$\Sigma^d_{f,D} := \Sigma^d_{f,f,D}, \quad \Sigma^n_{f,D} := \Sigma^n_{f,f,D}, \quad \Sigma^o_{f,D} := \Sigma^o_{f,f,D},$$

and when a is a strict regular embedding, we simply put

$$\Sigma^d_f := \Sigma^d_{f,X\times_S X}, \quad \Sigma^n_f := \Sigma^n_{f,X\times_S X}.$$

These functors are called *Thom transformations*.

4.1.3. Under the notations and hypotheses of (4.1.2), we will frequently assume that $u_0 = \mathrm{id}$ and there is a commutative diagram



with the following properties:

(i) w is an open immersion,

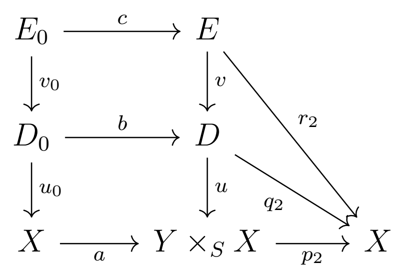
(ii) c is a strict closed immersion,

(iii) r_2 is a *strict* smooth morphism.

4.2 Transition transformations

4.2.1. In this section, we will develop various functorial properties of Thom transformations.

4.2.2. Under the notations and hypotheses of (4.1.2), consider a commutative diagram



of $\mathscr{S}$-schemes, and assume that b and c are proper. Then we have a natural transformation

$$T_{D,E} : \Omega_{g,f,E} \longrightarrow \Omega_{g,f,D}$$

in the below two cases. This is called a *transition transformation*. Here, when $D = Y \times_S X$, we put $T_{Y\times_S X,E} = T_E$ for simplicity.

(i) Assume that v_0 is the identity and that the exchange transformation

$$\mathrm{id}^* b^! \xrightarrow{Ex} c^! v^*$$

is defined and an isomorphism. Then the natural transformation $\Omega_{g,f,E} \xrightarrow{T_{D,E}} \Omega_{g,f,D}$ is given by

$$u_{0*} c^! r_2^* \xrightarrow{\sim} u_{0*} c^! v^* q_2^* \xrightarrow{Ex^{-1}} u_{0*} b^! r_2^*.$$

Note that when v is an open immersion and (Supp) is satisfied, then the condition is satisfied.

(ii) Assume that the unit $\mathrm{id} \xrightarrow{ad} v_* v^*$ is an isomorphism. Then the natural transformation $\Omega_{g,f,E} \xrightarrow{T_{D,E}} \Omega_{g,f,D}$ is given by

$$u_{0*} v_{0*} c^! r_2^* \xrightarrow{Ex} u_{0*} b^! v_* r_2^* \xrightarrow{\sim} u_{0*} b^! v_* v^* q_2^* \xrightarrow{ad^{-1}} u_{0*} b^! q_2^*.$$

(iii) Assume that v is strict étale, v_0 is the identity, and (Supp) is satisfied. The purity transformation

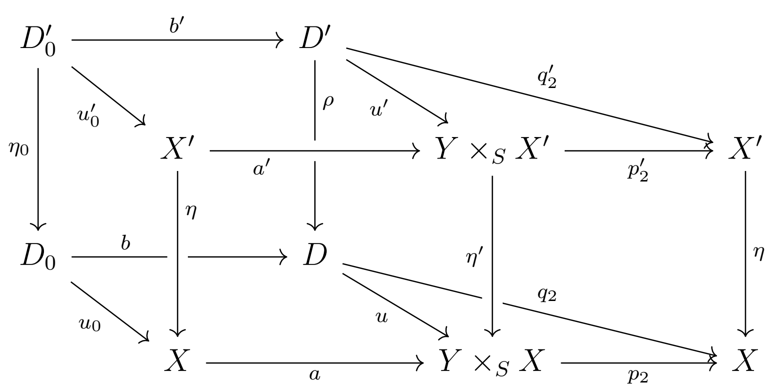
$$v^! \xrightarrow{\mathfrak{q}_v^n} v^*$$

whose description is given in (4.4.2) is an isomorphism by [CD12, 2.4.50(3)]. Then the natural transformation $\Omega_{g,f,E} \xrightarrow{T_{D,E}} \Omega_{g,f,D}$ is given by

$$u_{0*} c^! r_2^* \xrightarrow{\sim} u_{0*} c^! v^* q_2^* \xrightarrow{(\mathfrak{q}_v^n)^{-1}} u_{0*} c^! v^! q_2^* \xrightarrow{\sim} u_{0*} b^! q_2^*.$$

Note that $T_{D,E}$ is an isomorphism.

4.2.3. Under the notations and hypotheses of (4.1.2), let $\eta : X' \to X$ be a morphism of $\mathscr{S}$-schemes, and we put $f' = f\eta$. Consider the commutative diagram



of $\mathscr{S}$-schemes where the upper layer is a pullback of the lower layer. Then we have the natural transformations

$$\eta^* \Omega_{g,f,D} \xrightarrow{Ex} \Omega_{g,f',D'} \eta^*, \quad \Omega_{g,f,D} \eta_* \xrightarrow{Ex} \eta_* \Omega_{g,f',D'}, \quad \Omega_{g,f',D'} \eta^! \xrightarrow{Ex} \eta^! \Omega_{g,f,D}$$

given by

$$\eta^* u_{0*} b^! q_2^* \xrightarrow{Ex} u'_{0*} \eta_0^* b^! q_2^* \xrightarrow{Ex} u'_{0*} b'^! \rho^* q_2^* \xrightarrow{\sim} u'_{0*} b'^! q_2'^* \eta^*,$$

$$u_{0*} b^! q_2^* \eta_* \xrightarrow{Ex} u_{0*} b^! \rho_* q_2'^* \xrightarrow{Ex^{-1}} u_{0*} \eta_{0*} b'^! q_2'^* \xrightarrow{\sim} \eta_* u'_{0*} b'^! q_2'^*,$$

$$u'_{0*} b'^! q_2'^* \eta^! \xrightarrow{Ex} u'_{0*} b'^! \rho^! q_2^* \xrightarrow{\sim} u'_{0*} \eta_0^! b^! q_2^* \xrightarrow{Ex} \eta^! u_{0*} b^! q_2^*$$

respectively. These are called *exchange transformations.* Here, to define the first (resp. second, resp. third) natural transformation, we assume the condition (CE^*) (resp. (CE_*), resp. $(CE^!)$) whose definition is given below:

(CE^*) The exchange transformation $\eta_0^* b^! \xrightarrow{Ex} b'^! \rho^*$ is defined.

(CE_*) The exchange transformation $\eta_{0*} b'^! \xrightarrow{Ex} b^! \rho_*$ is an isomorphism.

($CE^!$) (η is proper) or (η is separated and (Supp) is satisfied). Moreover, the exchange transformation $q_2'^* \eta^! \xrightarrow{Ex} \rho^! q_2^*$ is defined.

For example, if b is a strict closed immersion, then by (2.6.2), (CE^*) and (CE_*) are satisfied. On the other hand, if η is a strict closed immersion, then by (2.6.2), ($CE^!$) is satisfied.

When b is a strict regular embedding, we similarly have the natural transformations

$$\eta^* \Omega^d_{g,f,D} \xrightarrow{Ex} \Omega^d_{g,f',D'} \eta^*, \quad \Omega^d_{g,f,D} \eta_* \xrightarrow{Ex} \eta_* \Omega^d_{g,f',D'},$$

$$\eta^* \Omega^n_{g,f,D} \xrightarrow{Ex} \Omega^n_{g,f',D'} \eta^*, \quad \Omega^n_{g,f,D} \eta_* \xrightarrow{Ex} \eta_* \Omega^n_{g,f',D'}$$

because the corresponding versions (CE^*) and (CE_*) are always satisfied since the morphisms $d : D_0 \times \mathbb{A}^1 \to D_{D_0} D$ and $e : D_0 \to N_{D_0} D$ are strict regular embeddings. Since t_2 is exact log smooth, if (η is proper) or (η is separated and (Supp) is satisfied), we have the natural transformation

$$\Omega^n_{g,f',D'} \eta^! \xrightarrow{Ex} \eta^! \Omega^n_{g,f,D}.$$

When s_2 is exact log smooth, if (η is proper) or (η is separated and (Supp) is satisfied), we also have the natural transformation

$$\Omega^d_{g,f',D'} \eta^! \xrightarrow{Ex} \eta^! \Omega^d_{g,f,D}$$

because the corresponding version ($CE^!$) is satisfied. However, s_2 may not be exact log smooth morphism. In this case, assume that (Supp) is satisfied and that the conditions of (4.1.3) are satisfied. We put $I' = I \times_D D'$, and consider the diagram

$$\begin{array}{ccc} \Omega_{g,f',I'} \eta^! & \xrightarrow{Ex} & \eta^! \Omega_{g,f,I} \\ \downarrow{\scriptstyle T_{D',I'}} & & \downarrow{\scriptstyle T_{D,I}} \\ \Omega_{g,f',D'} \eta^! & & \eta^! \Omega_{g,f,D} \end{array}$$

of functors. The horizontal arrow is defined since the induced morphism $D_X I \to X \times \mathbb{A}^1$ is strict smooth. The vertical arrows are isomorphism by (4.2.2). Now, the definition of

$$\Omega^d_{g,f',D'} \eta^! \xrightarrow{Ex} \eta^! \Omega^d_{g,f,D}$$

is given by the composition

$$\Omega_{g,f',D'} \eta^! \xrightarrow{(T_{D',I'})^{-1}} \Omega_{g,f',I'} \eta^! \xrightarrow{Ex} \eta^! \Omega_{g,f,I} \xrightarrow{T_{D,I}} \eta^! \Omega_{g,f,D}.$$

Lemma 4.2.4. *Under the notations and hypotheses of (4.2.3), the exchange transformation*

$$\Omega^n_{g,f,D} \eta_* \xrightarrow{Ex} \eta_* \Omega^n_{g,f',D'}$$

is an isomorphism.

Proof. It follows from (eSm-BC) because the morphism $N_{D_0}D \to X$ in (4.1.2.1) is exact log smooth. □

Lemma 4.2.5. *Under the notations and hypotheses of (4.2.3), assume that u_0 is the identity. If η is proper, then the exchange transformation*

$$\Omega^n_{g,f',D'}\eta^! \xrightarrow{Ex} \eta^!\Omega^n_{g,f,D}$$

is an isomorphism.

Proof. Note first that $\Omega^n_{g,f,D}$ and $\Omega^n_{g,f',D'}$ are equivalences of categories by (2.8.2). Consider the natural transformation

$$\eta^!\Sigma^n_{g,f',D'} \xrightarrow{Ex} \Sigma^n_{g,f,D}\eta^!$$

given by the left adjoint of the exchange transformation

$$\Omega^n_{g,f,D}\eta_* \xrightarrow{Ex} \eta_*\Omega^n_{g,f',D'}.$$

Then consider the commutative diagram

$$\begin{array}{ccc}
\Omega^n_{g,f',D'}\eta^!\Sigma^n_{g,f',D'} & \xrightarrow{Ex} & \eta^!\Omega^n_{g,f,D}\Sigma^n_{g,f',D'} \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle ad'} \\
\Omega^n_{g,f',D'}\Sigma^n_{g,f,D}\eta^! & \xrightarrow{ad'} & \eta^!
\end{array}$$

of functors. By (4.2.4), the left vertical arrow is an isomorphism. The right vertical and lower horizontal arrows are also isomorphisms since $\Omega^n_{g,f,D}$ and $\Omega^n_{g,f',D'}$ are equivalences of categories. Thus $\Omega^n_{g,f,D}$ and $\Omega^n_{g,f',D'}$ are equivalences of categories. Then the conclusion follows from the fact that $\Sigma^n_{g,f',D'}$ is an equivalence of categories. □

Lemma 4.2.6. *Under the notations and hypotheses of (4.2.3), assume that u_0 is the identity. if q_2 is strict smooth separated and η is separated, then the exchange transformation*

$$\Omega_{g,f',D'}\eta^! \xrightarrow{Ex} \eta^!\Omega_{g,f,D}$$

is defined and an isomorphism.

Proof. It is a direct consequence of (2.5.10). □

Lemma 4.2.7. *Under the notations and hypotheses of (4.2.3), assume that u_0 is the identity. if η is an open immersion and (Supp) is satisfied, then the exchange transformation*

$$\Omega^n_{g,f',D'}\eta^! \xrightarrow{Ex} \eta^!\Omega^n_{g,f,D}$$

is an isomorphism.

Proof. It is a direct consequence of (Supp). □

Lemma 4.2.8. *Under the notations and hypotheses of (4.2.3), assume that u_0 is the identity. if η is separated and (Supp) is satisfied, then then the exchange transformation*

$$\Omega^n_{g,f',D'}\eta^! \xrightarrow{Ex} \eta^!\Omega^n_{g,f,D}$$

is an isomorphism.

Proof. It follows from (4.2.5) and (4.2.7). □

4.2.9. Under the notations and hypotheses of (4.1.2), we have the natural transformations

$$\Omega^n_{g,f,D} \xleftarrow{(T^n)^{-1}} \Omega^d_{g,f,D} \xrightarrow{T^d} \Omega_{g,f,D}$$

whose descriptions are given below. These are called *transition transformations* again.

(1) The natural transformation

$$T^d : \Omega^d_{g,f,D} \longrightarrow \Omega_{g,f,D}$$

is given by

$$\begin{aligned}\pi_*\Omega_{g\times\mathbb{A}^1,f\times\mathbb{A}^1,D_XD}\pi^* &\xrightarrow{ad} \pi_*\alpha_{1*}\alpha_1^*\Omega_{g\times\mathbb{A}^1,f\times\mathbb{A}^1,D_XD}\pi^* \xrightarrow{\sim} u_{0*}\alpha_1^*\Omega_{g\times\mathbb{A}^1,f\times\mathbb{A}^1,D_XD}\pi^* \\ &\xrightarrow{Ex} \Omega_{g,f,D}\alpha_1^*\pi^* \xrightarrow{\sim} \Omega_{g,f,D}.\end{aligned}$$

(2) The natural transformation

$$(T^n)^{-1} : \Omega^d_{g,f,D} \longrightarrow \Omega^n_{g,f,D}$$

is given by

$$\begin{aligned}\pi_*\Omega_{g\times\mathbb{A}^1,f\times\mathbb{A}^1,D_XD}\pi^* &\xrightarrow{ad} \pi_*\alpha_{0*}\alpha_0^*\Omega_{g\times\mathbb{A}^1,f\times\mathbb{A}^1,D_XD}\pi^* \xrightarrow{\sim} \alpha_0^*\Omega_{g\times\mathbb{A}^1,f\times\mathbb{A}^1,D_XD}\pi^* \\ &\xrightarrow{Ex} \Omega^n_{g,f,D}\alpha_0^*\pi^* \xrightarrow{\sim} \Omega^n_{g,f,D}.\end{aligned}$$

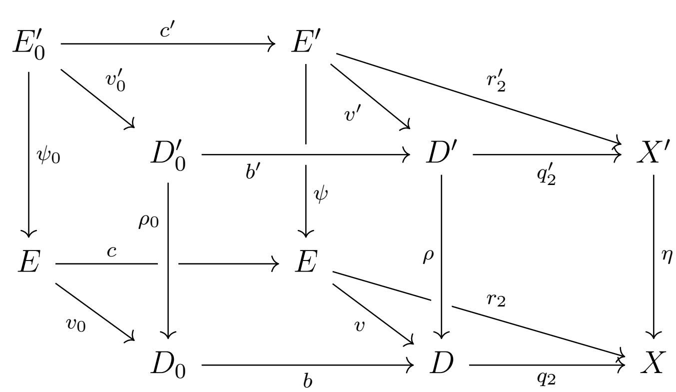
When $(T^n)^{-1}$ is an isomorphism, its inverse is denoted by T^n.

When u_0 is the identity, we also have the natural transformation

$$T^o : \Omega^o_{g,f,D} \longrightarrow \Omega^n_{g,f,D}$$

given by the right adjoint of an orientation of N_XD. It depends on the orientation.

4.2.10. Under the notations and hypotheses of (4.2.3), consider the commutative diagram



of $\mathscr{S}$-schemes where each small square is Cartesian. Assume that one of the conditions (i) and (ii) of (4.2.2) is simultaneously satisfied for both (D, E) and (D', E'). Consider the diagrams

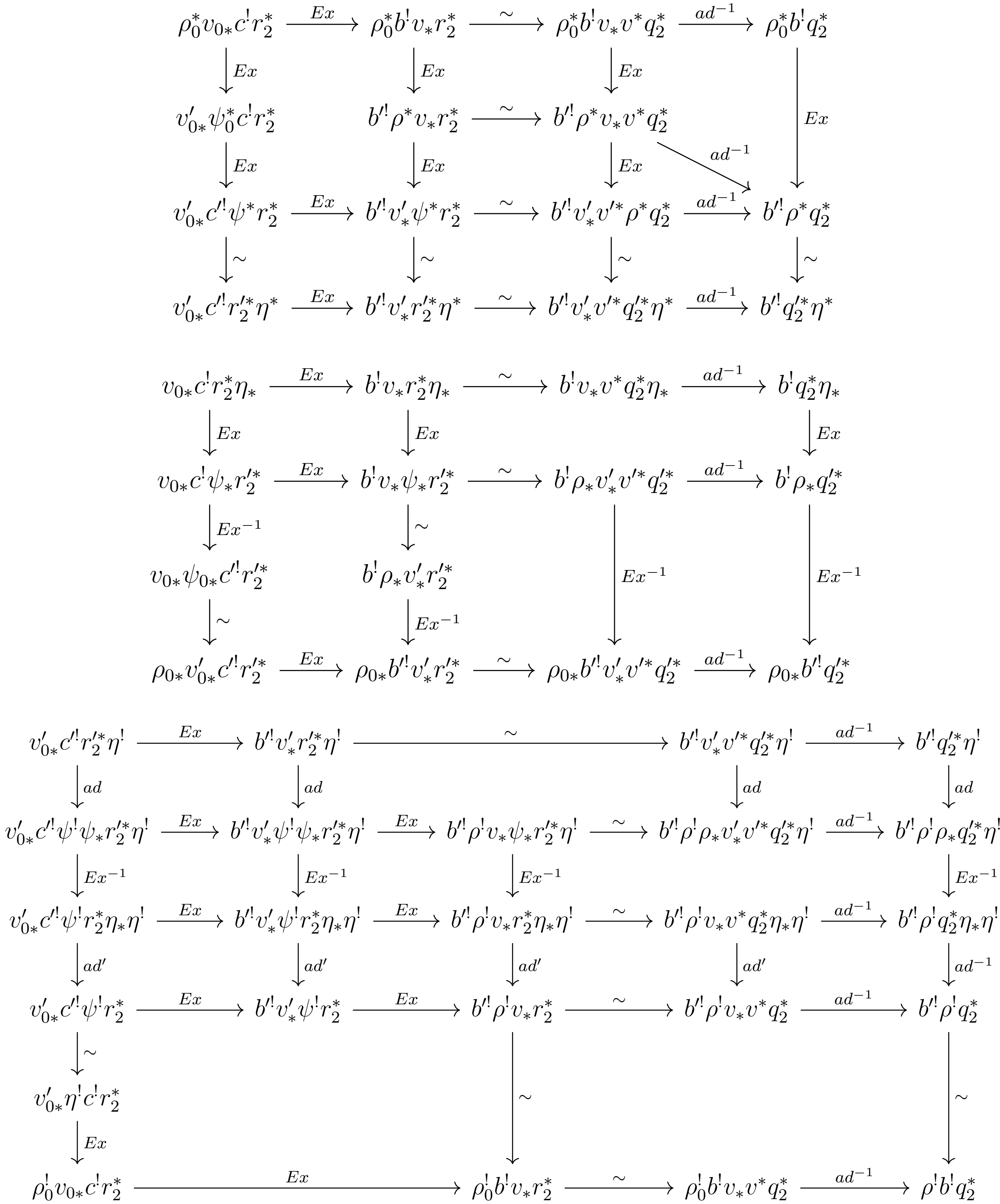
$$\begin{CD} \eta^*\Omega_{g,f,E} @>{T_{D,E}}>> \eta^*\Omega_{g,f,D} \\ @V{Ex}VV @V{Ex}VV \\ \Omega_{g,f',E'}\eta^* @>{T_{D',E'}}>> \Omega_{g,f',D'}\eta^* \end{CD} \qquad \begin{CD} \Omega_{g,f,E}\eta_* @>{T_{D,E}}>> \Omega_{g,f,D}\eta_* \\ @V{Ex}VV @V{Ex}VV \\ \eta_*\Omega_{g,f',E'} @>{T_{D',E'}}>> \eta_*\Omega_{g,f',D'} \end{CD} \qquad \begin{CD} \Omega_{g,f',E'}\eta^! @>{T_{D',E'}}>> \Omega_{g,f',D'}\eta^! \\ @V{Ex}VV @V{Ex}VV \\ \eta^!\Omega_{g,f,E} @>{T_{D,E}}>> \Omega_{g,f,D} \end{CD}$$

of functors. Here, in the first (resp. second, resp. third) case, we assume the condition (CE^*) (resp. (CE_*), resp. $(CE^!)$) in (4.2.3) for η and η'. We will show that the above diagrams commute under suitable conditions. If the condition (i) of (4.2.2) is satisfied, then note that v_0 and v_0' are the identity, and the assertion can be checked by considering the diagrams

$$\begin{CD} \rho_0^*c^!r_2^* @>{\sim}>> \rho_0^*c^!v^*q_2^* @>{Ex^{-1}}>> \rho_0^*b^!q_2^* \\ @V{Ex}VV @V{Ex}VV @V{Ex}VV \\ c'^!\psi^*r_2^* @>{\sim}>> c'^!\psi^*v^*q_2^* @. @| \\ @V{\sim}VV @V{\sim}VV @VVV \\ @| c'^!v'^*\rho^*q_2^* @>{Ex^{-1}}>> b'^!\rho^*q_2^* \\ @VVV @V{\sim}VV @V{\sim}VV \\ c'^!r_2'^*\eta^* @>{\sim}>> c'^!v'^*q_2'^*\eta^* @>{Ex^{-1}}>> b'^!q_2'^*\eta^* \end{CD} \qquad \begin{CD} c^!r_2^*\eta_* @>{\sim}>> c^!v^*q_2^*\eta_* @>{Ex^{-1}}>> b^!q_2^*\eta_* \\ @V{Ex}VV @V{Ex}VV @V{Ex}VV \\ @| c^!v^*\rho_*q_2'^* @>{Ex^{-1}}>> b^!\rho_*q_2'^* \\ @VVV @V{Ex}VV @V{Ex^{-1}}VV \\ c^!\psi_*r_2'^* @>{\sim}>> c^!\psi_*v'^*q_2'^* @. @| \\ @V{Ex^{-1}}VV @V{Ex^{-1}}VV @VVV \\ \rho_{0*}c'^!r_2'^* @>{\sim}>> \rho_{0*}c'^!v'^*q_2'^* @>{Ex^{-1}}>> \rho_{0*}b'^!q_2'^* \end{CD}$$

$$\begin{CD} c'^!r_2'^*\eta^! @>{\sim}>> c'^!v'^*q_2'^*\eta^! @>{Ex^{-1}}>> b'^!q_2'^*\eta^! \\ @V{Ex}VV @V{Ex}VV @V{Ex}VV \\ @| c'^!v'^*\rho^!q_2^* @>{Ex^{-1}}>> b'^!\rho^!q_2^* \\ @VVV @V{Ex}VV @V{\sim}VV \\ c'^!\psi^!r_2^* @>{\sim}>> c'^!\psi^!v^*q_2^* @. @| \\ @V{\sim}VV @V{\sim}VV @VVV \\ \rho_0^!c^!r_2^* @>{\sim}>> \rho_0^!c^!v^*q_2^* @>{Ex^{-1}}>> \rho_0^!b^!q_2^* \end{CD}$$

of functors. If the condition (ii) of (loc. cit) is satisfied, then the assertion can be checked by considering the diagrams



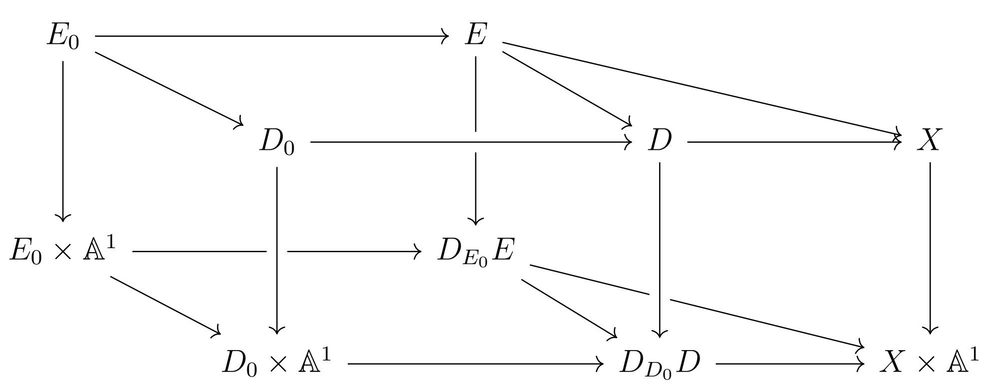
of functors. When b is a strict regular embedding, we similarly have the commutative diagrams

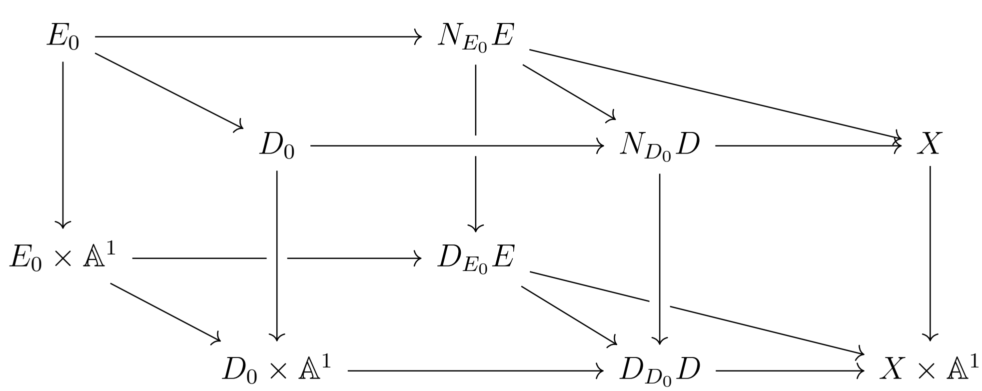
$$
\begin{array}{ccc}
\eta^*\Omega^d_{g,f,D} & \xrightarrow{T^d} & \eta^*\Omega_{g,f,D} \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\Omega^d_{g,f',D'}\eta^* & \xrightarrow{T^d} & \Omega_{g,f',D'}\eta^*
\end{array}
\qquad
\begin{array}{ccc}
\Omega^d_{g,f,D}\eta_* & \xrightarrow{T^d} & \Omega_{g,f,D}\eta_* \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\eta_*\Omega^d_{g,f',D'} & \xrightarrow{T^d} & \eta_*\Omega_{g,f',D'}
\end{array}
\qquad
\begin{array}{ccc}
\Omega^d_{g,f',D'}\eta^! & \xrightarrow{T^d} & \Omega_{g,f',D'}\eta^! \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\eta^!\Omega^d_{g,f,D} & \xrightarrow{T^d} & \Omega_{g,f,D}
\end{array}
$$

$$
\begin{array}{ccc}
\eta^*\Omega^n_{g,f,D} & \xleftarrow{(T^n)^{-1}} & \eta^*\Omega^d_{g,f,D} \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\Omega^n_{g,f',D'}\eta^* & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f',D'}\eta^*
\end{array}
\qquad
\begin{array}{ccc}
\Omega^n_{g,f,D}\eta_* & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f,D}\eta_* \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\eta_*\Omega^n_{g,f',D'} & \xleftarrow{(T^n)^{-1}} & \eta_*\Omega^d_{g,f',D'}
\end{array}
\qquad
\begin{array}{ccc}
\Omega^n_{g,f',D'}\eta^! & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f',D'}\eta^! \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\eta^!\Omega^n_{g,f,D} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f,D}
\end{array}
$$

of functors. Here, in the third and sixth diagram, we assume that (η and η' are proper) or (η and η' are separated and (Supp) is satisfied).

4.2.11. Under the notations and hypotheses of (4.2.2), if b and c are strict regular embeddings, we have the commutative diagrams





of $\mathscr{S}$-schemes as in (4.1.2). Thus we similarly obtain the natural transformations

$$\Omega^d_{g,f,E} \longrightarrow \Omega^d_{g,f,E}, \quad \Omega^n_{g,f,E} \longrightarrow \Omega^n_{g,f,E}$$

as in (4.2.2) when one of the conditions (i)–(iii) of (loc. cit) is satisfied. These are again denoted by $T_{D,E}$ and called *transition transformations.* We also have the commutative diagram

$$\begin{array}{ccccc}
\Omega^n_{g,f,E} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f,E} & \xrightarrow{T^d} & \Omega_{g,f,E} \\
\downarrow{\scriptstyle T_{D,E}} & & \downarrow{\scriptstyle T_{D,E}} & & \downarrow{\scriptstyle T_{D,E}} \\
\Omega^n_{g,f,D} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f,D} & \xrightarrow{T^d} & \Omega_{g,f,D}
\end{array}$$

of functors. Note that in the case (iii), the horizontal arrows are isomorphisms as in (loc. cit). In the case (i), if (Supp) is satisfied, then the horizontal arrows are isomorphisms.

4.2.12. Under the notations and hypotheses of assume that the conditions of (4.1.3) are satisfied. Consider the commutative diagram

$$\begin{array}{ccccc}
\Omega^n_{g,f,I} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f,I} & \xrightarrow{T^d} & \Omega_{g,f,I} \\
\downarrow{\scriptstyle T_{D,I}} & & \downarrow{\scriptstyle T_{D,I}} & & \downarrow{\scriptstyle T_{D,I}} \\
\Omega^n_{g,f,D} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{g,f,D} & \xrightarrow{T^d} & \Omega_{g,f,D}
\end{array}$$

of functors. By the proof of [CD12, 2.4.35], the upper horizontal arrows are isomorphisms. The vertical arrows are isomorphisms by (4.2.2(i)), so the lower horizontal arrows are also isomorphisms. In particular, the natural transformation

$$\Omega^n_{g,f,D} \xleftarrow{(T^n)^{-1}} \Omega^d_{g,f,D}$$

has the inverse T^n.

4.2.13. Under the notations and hypotheses of (4.2.2), assume that we have a commutative diagram

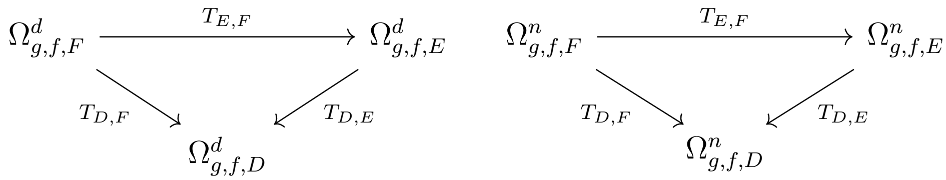
$$\begin{array}{ccccc}
F_0 & \xrightarrow{c'} & F & & \\
\downarrow{\scriptstyle w_0} & & \downarrow{\scriptstyle w} & \searrow{\scriptstyle r'_2} & \\
E_0 & \xrightarrow{c} & E & & \\
\downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v} & \searrow{\scriptstyle r_2} & \\
D_0 & \xrightarrow[b]{} & D & \xrightarrow[q_2]{} & X
\end{array}$$

of $\mathscr{S}$-schemes and that $w : F \to E$ and $v : E \to D$ simultaneously satisfy one of the conditions (i)–(iii) of (loc. cit). Then the composition $vw : F \to D$ also satisfies it, and the diagram

$$\begin{array}{ccc}
\Omega_{g,f,F} & \xrightarrow{T_{E,F}} & \Omega_{g,f,E} \\
& {\scriptstyle T_{D,F}}\searrow \quad \swarrow{\scriptstyle T_{D,E}} & \\
& \Omega_{g,f,D} &
\end{array}$$

of functors commutes.

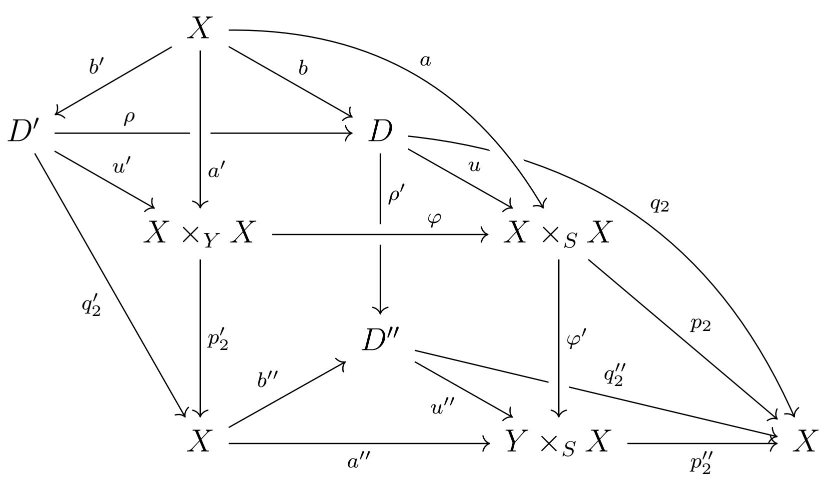
Assume further that b, c, and d are strict regular embeddings. Then we similarly have the commutative diagrams



of functors.

4.3 Composition transformations

4.3.1. Let $h : X \to Y$ and $g : Y \to S$ be morphisms of $\mathscr{S}$-schemes, and we put $f = gh$. Consider a commutative diagram



of $\mathscr{S}$-schemes. Assume that the exchange transformation

$$q_2'^* b''^! \xrightarrow{Ex} \rho^! \rho'^* \tag{4.3.1.1}$$

is defined. For example, when the diagram

$$\begin{array}{ccc} D' & \xrightarrow{\rho} & D \\ \downarrow{\scriptstyle q_2'} & & \downarrow{\scriptstyle \rho'} \\ X & \xrightarrow{b''} & D'' \end{array}$$

is Cartesian and the exchange transformation

$$b''^* \rho'_* \xrightarrow{Ex} q'_{2*} \rho^*$$

is an isomorphism, (4.3.1.1) is defined. Then the *composition transformation*

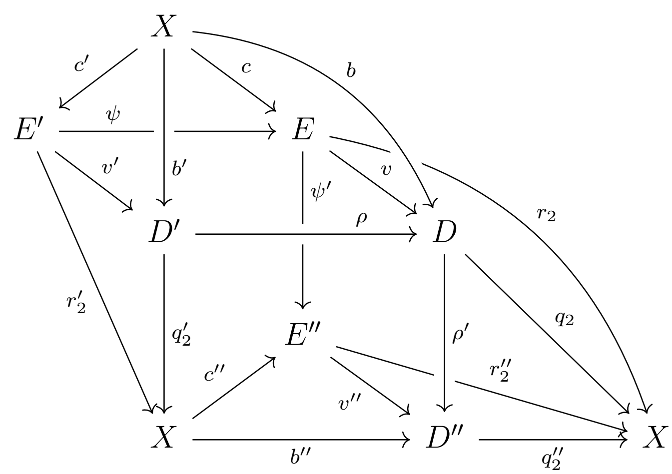
$$\Omega_{h,D'}\Omega_{g,f,D''} \xrightarrow{C} \Omega_{f,D}$$

is given by

$$b'^! q_2'^* b''^! q_2''^* \xrightarrow{Ex} b'^! \rho^! \rho'^* q_2''^* \xrightarrow{\sim} b^! q_2^!.$$

Note that it is an isomorphism when the first arrow is an isomorphism. For example, if (Supp) is satisfied and ρ' is strict smooth separated, then the first arrow is an isomorphism by (2.5.10).

4.3.2. Under the notations and hypotheses of (4.3.1), consider a commutative diagram



of $\mathscr{S}$-schemes. Assume that the exchange transformation

$$r_2'^* c''^! \xrightarrow{Ex} \psi^! \psi'^*$$

is also defined. Then in the below two cases, we will show that the diagram

$$\begin{array}{ccc} \Omega_{h,E'}\Omega_{g,f,E''} & \xrightarrow{C} & \Omega_{f,E} \\ \downarrow{\scriptstyle T_{D',E'}T_{D'',E''}} & & \downarrow{\scriptstyle T_{D,E}} \\ \Omega_{h,D'}\Omega_{g,f,D''} & \xrightarrow{C} & \Omega_{f,D} \end{array} \tag{4.3.2.1}$$

of functors commutes where the horizontal arrows are define in (loc. cit).

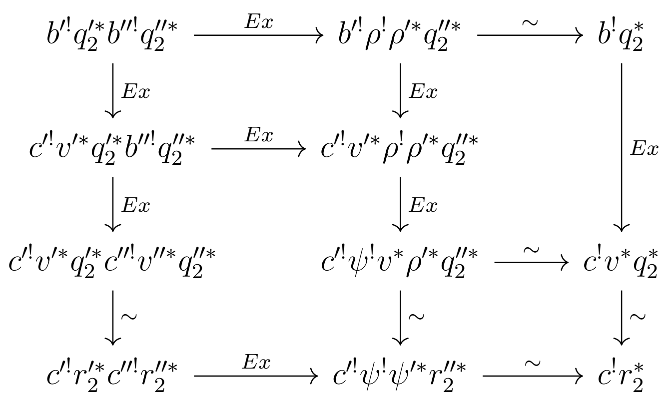
(i) Assume that the exchange transformations

$$\mathrm{id}^* b^! \xrightarrow{Ex} c^! v^*, \quad \mathrm{id}^* b'^! \xrightarrow{Ex} c'^! v'^*, \quad \mathrm{id}^* b''^! \xrightarrow{Ex} c''^! v''^*$$

are defined and isomorphisms. Assume *further* that the exchange transformation

$$v'^* \rho^! \xrightarrow{Ex} \psi^! v^*$$

is defined. Then the commutativity of (4.3.2.1) is equivalent to the commutativity of the big outside diagram of the diagram

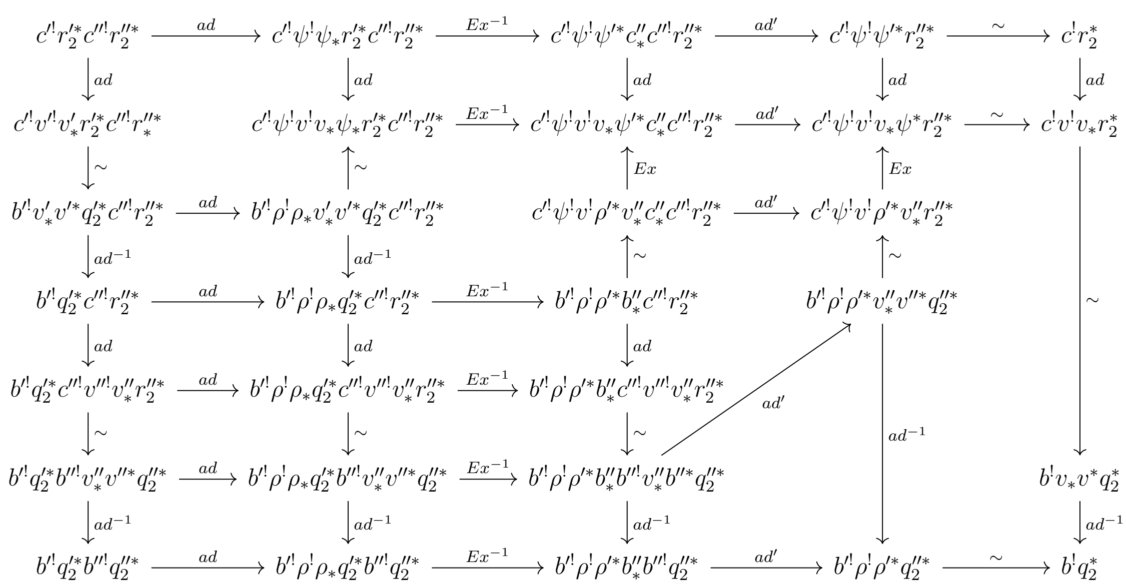


of functors. It is true since each small diagram commutes.

(ii) Assume that the units

$$\mathrm{id} \xrightarrow{ad} v_* v^*, \quad \mathrm{id} \xrightarrow{ad} v'_* v'^*, \quad \mathrm{id} \xrightarrow{ad} v''_* v''^*$$

are isomorphisms. Then the commutativity of (4.3.2.1) is equivalent to the commutativity of the big outside diagram of the diagram



of functors. It is true since each small diagram commutes.

4.4 Purity transformations

4.4.1. In this section, we will introduce purity transformations and their functorial properties.

Definition 4.4.2. Let $f : X \to S$ be an exact log smooth morphism of $\mathscr{S}$-schemes. Assume that (f is proper) or (f is separated and $\mathscr{T}$ satisfies (Supp)). We also assume that we have a commutative diagram

$$\begin{array}{ccccc} & & D & & \\ & {}^{b}\nearrow & \downarrow u & \searrow^{q_2} & \\ X & \xrightarrow{a} & X \times_S X & \xrightarrow{p_2} & X \end{array}$$

of $\mathscr{S}$-schemes where

1. b is a strict regular embedding,
2. u satisfies one of the conditions (i)–(iii) of (4.2.2).

Then we denote by

$$\mathfrak{q}^n_{f,D} : \Omega^n_{f,D} f^! \longrightarrow f^*, \quad \mathfrak{q}^o_{f,D} : \Omega^o_f f^! \longrightarrow f^*$$

the compositions

$$f^* \xrightarrow{\mathfrak{q}_f} \Omega_f f^! \xrightarrow{T_D} \Omega_{f,D} f^! \xrightarrow{T^d} \Omega^d_{f,D} f^! \xrightarrow{T^n} \Omega^n_{f,D} f^!,$$

$$f^* \xrightarrow{\mathfrak{q}_f} \Omega_f f^! \xrightarrow{T_D} \Omega_{f,D} f^! \xrightarrow{T^d} \Omega^d_{f,D} f^! \xrightarrow{T^n} \Omega^n_{f,D} f^! \xrightarrow{T^o} \Omega^o_{f,D} f^!$$

respectively. Their left adjoints are denoted by

$$\mathfrak{p}^n_{f,D} : f_\sharp \longrightarrow f_! \Sigma^n_{f,D}, \quad \mathfrak{p}^o_{f,D} : f_\sharp \longrightarrow f_! \Sigma^o_{f,D}$$

respectively.

4.4.3. Let $h : X \to Y$ and $g : Y \to S$ be separated $\mathscr{P}$-morphisms of $\mathscr{S}$-schemes, and we put $f = gh$. Then we have the commutative diagram

$$\begin{array}{ccccc} X & & & & \\ \downarrow a' & \searrow^{a} & & & \\ X \times_Y X & \xrightarrow{\varphi} & X \times_S X & & \\ \downarrow p_2 & & \downarrow \varphi' & \searrow^{p_2} & \\ X & \xrightarrow{a''} & Y \times_S X & \xrightarrow{p_2''} & X \end{array}$$

of $\mathscr{S}$-schemes, and the exchange transformation

$$p_2'^* a''^! \xrightarrow{Ex} \varphi^! \varphi'^*$$

is defined by (eSm-BC). Thus by (4.3.1), we have the composition transformation

$$C : \Omega_h \Omega_{g,f} \to \Omega_f.$$

Then the diagram

$$\begin{array}{ccc}
\Omega_h h^!\Omega_g g^! & \xrightarrow{\mathfrak{q}_h\mathfrak{q}_g} & h^*g^* \\
\uparrow{\scriptstyle Ex} & & \Big| \\
\Omega_h\Omega_{g,f}h^!g^! & & \Big| \\
\downarrow{\scriptstyle \sim} & & \Big|{\scriptstyle \sim} \\
\Omega_h\Omega_{g,f}f^! & & \Big| \\
\downarrow{\scriptstyle C} & & \downarrow \\
\Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & f^*
\end{array}$$

of functors commutes by the proof of [Ayo07, 1.7.3].

4.4.4. Let $f : X \to S$ be a separated and vertical $\mathscr{P}$-morphism of $\mathscr{S}$-schemes, and let $i : S \to X$ be its section. Then we have a commutative diagram

$$\begin{array}{ccccc}
S & \xrightarrow{i} & X & \xrightarrow{f} & S \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\
X & \xrightarrow{a} & X\times_S X & \xrightarrow{p_2} & X \\
 & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\
 & & X & \xrightarrow{f} & S
\end{array}$$

of $\mathscr{S}$-schemes where

(i) a denotes the diagonal morphism, and p_2 denotes the second projection,

(ii) each square is Cartesian.

Consider the diagram

$$\begin{array}{ccc}
\Omega_{f,\mathrm{id}}i^!f^! & \xrightarrow{\sim} & \Omega_{f,\mathrm{id}} \\
\downarrow{\scriptstyle Ex} & & \| \\
i^!\Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & i^!f^*
\end{array}$$

of functors. It commutes since the big outside diagram of the diagram

$$\begin{array}{ccccc}
i^!f^*i^!f^! & \xrightarrow{\sim} & i^!f^*(fi)^! & \xrightarrow{\sim} & i^!f^* \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & \nearrow{\scriptstyle \sim} & \| \\
i^!i'^!p_2^*f^! & & i^!(p_1a)^!f^* & & \| \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \| \\
i^!a^!p_2^*f^! & \xrightarrow{Ex} & i^!a^!p_1^!f^* & \xrightarrow{\sim} & i^!f^*
\end{array}$$

of $\mathscr{S}$-schemes commutes.

4.4.5. Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes where f is an exact log smooth morphism. Assume that (f is proper) or (f is separated and (Supp) is satisfied). Then we have the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{a'} & X' \times_{S'} X' & \xrightarrow{p_2'} & X' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{a} & X \times_S X & \xrightarrow{p_2} & X \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian and p_2 denotes the second projection. We also denote by p_1 (resp. p_1') the first projection $X \times_S X \to X$ (resp. $X' \times_{S'} X' \to X'$). In this setting, we will show that the diagram

$$\begin{array}{ccc} f'_\sharp g'^* & \xrightarrow{\mathfrak{p}_{f'}} & f'_! \Sigma_{f'} g'^! \\ & & \downarrow{\scriptstyle Ex} \\ \downarrow{\scriptstyle Ex} & & f'_! g'^* \Sigma_f \\ & & \uparrow{\scriptstyle Ex} \\ g^* f_\sharp & \xrightarrow{\mathfrak{p}_f} & g^* f_! \Sigma_f \end{array}$$

of functors commutes. It is the big outside diagram of the diagram

$$\begin{array}{ccccccc} f'_\sharp g'^* & \xrightarrow{\sim} & f'_\sharp p'_{1!} a'_* g'^* & & \xrightarrow{Ex} & & f'_! p'_{2\sharp} a'_* g'^* \\ & \searrow{\scriptstyle \sim} & & \nwarrow{\scriptstyle Ex} & & & \uparrow{\scriptstyle Ex} \\ & & f'_\sharp g'^* p_{1!} a_* & \xrightarrow{Ex} & f'_\sharp p'_{1!} g''^* a_* & \xrightarrow{Ex} & f'_! p'_{2\sharp} g''^* a_* \\ \downarrow{\scriptstyle Ex} & & & & & & \downarrow{\scriptstyle Ex} \\ & & \downarrow{\scriptstyle Ex} & & & & f'_! g'^* p_{2\sharp} a_* \\ & & & & & & \uparrow{\scriptstyle Ex} \\ g^* f_\sharp & \xrightarrow{\sim} & g^* f_\sharp p_{1!} a_* & & \xrightarrow{Ex} & & g^* f_! p_{2\sharp} a_* \end{array}$$

of functors. Thus the assertion follows from the fact that each small diagram commutes.

Chapter 5

Support property

5.0.1. Throughout this chapter, we fix a full subcategory $\mathscr{S}$ of the category of fs log schemes satisfying the conditions of (2.0.1). We also fix a eSm-premotivic triangulated category satisfying (Adj), (Htp–1), (Htp–2), (Htp–3), (Loc), (sét-Sep), and (Stab). In §5.6, we assume also the axiom (ii) of (2.9.1) and (Htp–4).

5.1 Elementary properties of the support property

5.1.1. We will define the universal and semi-universal support property for not necessarily proper morphisms, and we will show that our definition coincides with the usual definition for proper morphisms in (5.1.4). Then we will study elementary properties of the universal support property. Recall from (2.5.8) that any proper strict morphism of $\mathscr{S}$-schemes satisfies the support property.

Proposition 5.1.2. *Let $g : Y \to X$ and $f : X \to S$ be proper morphisms of $\mathscr{S}$-schemes. If f and g satisfy the support property, then fg also satisfies the support property.*

Proof. Consider a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{g'} & V & \xrightarrow{f'} & U \\ \downarrow{\scriptstyle j''} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{g} & X & \xrightarrow{f} & S \end{array}$$

of $\mathscr{S}$-schemes where j is an open immersion and each square is Cartesian. Then the conclusion follows from the commutativity of the diagram

$$\begin{array}{ccccc} j_\sharp f'_* g'_* & \xrightarrow{Ex} & f_* j'_\sharp g_* & \xrightarrow{Ex} & f_* g_* j''_\sharp \\ \downarrow{\scriptstyle \sim} & & & & \downarrow \\ j_\sharp (f'g')_* & & \xrightarrow{Ex} & & (fg)_* j''_\sharp \end{array}$$

of functors. □

Definition 5.1.3. Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes. We say that f satisfies the *universal* (resp. semi-universal) support property if any pullback of the *proper* morphism $X \to \underline{X} \times_{\underline{S}} S$ (resp. any pullback of the *proper* morphism $X \to \underline{X} \times_{\underline{S}} S$ via strict morphism) satisfies the support property.

Proposition 5.1.4. *Let $f : X \to S$ be a proper morphism of $\mathscr{S}$-schemes. Then f satisfies the universal (resp. semi-universal) support property if and only if any pullback of f (resp. any pullback of f via strict morphism) satisfies the support property.*

Proof. If f satisfies the universal (resp. semi-universal) support property, let $f' : X' \to S'$ be a pullback of f via a morphism (resp. strict morphism) $S' \to S$. Consider the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{u'} & \underline{X} \times_{\underline{S}} S' & \xrightarrow{v'} & S' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u} & \underline{X} \times \underline{S} S & \xrightarrow{v} & S \end{array}$$

of $\mathscr{S}$-schemes. By assumption, u' satisfies the support property. Since v' is strict proper, it satisfies the support property by (5.1.1). Thus $f = v'u'$ satisfies the support property by (5.1.2).

Conversely, if the support property is satisfied for any pullback of f (resp. for any pullback of f via strict morphism), we put $T = \underline{X} \times_{\underline{S}} S$, and let $p' : X \times_T T' \to T'$ be a pullback of $X \to T$ via a morphism (resp. strict morphism) $T' \to T$. The morphism p' has the factorization

$$X \times_T T' \xrightarrow{r} X \times_S T' \xrightarrow{q} T'$$

where r denotes the morphism induced by $T \to S$, and q denotes the projection. Then the morphism r is a closed immersion since it is a pullback of the diagonal morphism $T \to T \times_S T$, so r satisfies the support property, and the morphism q satisfies the support property since it is a pullback of f via the morphism (resp. strict morphism) $T' \to T$. Thus by (5.1.2), the morphism $p' = qr$ satisfies the support property. □

Proposition 5.1.5. *Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes. Then the question that f satisfies the universal (resp. semi-universal) support property is strict étale local on X.*

Proof. Replacing f by $X \to \underline{X} \times_{\underline{S}} S$, we may assume that $\underline{f}$ is an isomorphism. Then the question is strict étale local on S by (sét-Sep), which implies that the question is strict étale local on X. □

Proposition 5.1.6. *Let $g : Y \to X$ and $f : X \to S$ be morphisms of $\mathscr{S}$-schemes.*

(1) If f is strict, then f satisfies the universal support property.

(2) If f and g satisfy the universal (resp. semi-universal) support property, then fg also satisfies the universal (resp. semi-universal) support property.

(3) Assume that g is proper and that for any pullback h of g via strict morphism, the unit

$$\mathrm{id} \xrightarrow{ad} h_*h^*$$

is an isomorphism. If fg satisfy the semi-universal support property, then f satisfies the semi-universal support property.

Proof. (1) It is true since the morphism $X \to \underline{X} \times_{\underline{S}} S$ is an isomorphism when f is strict.

(2) The induced morphism $p : Y \to \underline{Y} \times_{\underline{S}} S$ has the factorization

$$Y \xrightarrow{r} \underline{Y} \times_{\underline{X}} X \xrightarrow{q} \underline{Y} \times_{\underline{S}} S$$

where r denotes the morphism induced by $Y \to \underline{Y}$ and $Y \to X$, and q denotes the morphism induced by $X \to S$. Any pullback of r (resp. any pullback of r via strict morphism) satisfies the support property by assumption, and any pullback of q (resp. any pullback of q via strict morphism) satisfies the support property by assumption since q is a pullback of the morphism $X \to \underline{X} \times_{\underline{S}} S$. Thus by (5.1.2), any pullback of p (resp. any pullback of p via strict morphism) satisfies the support property, i.e., fg satisfies the universal support property.

(3) Replacing f by $X \to \underline{X} \times_{\underline{S}} S$, we may assume that $\underline{f}$ is an isomorphism. The question is also preserved by any base change via strict morphism to S, so we only need to prove that f satisfies the support property. Consider the commutative diagram

$$\begin{array}{ccccc}
W & \xrightarrow{g'} & V & \xrightarrow{f'} & U \\
\downarrow{\scriptstyle j''} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\
Y & \xrightarrow{g} & X & \xrightarrow{f} & S
\end{array}$$

of $\mathscr{S}$-schemes where j is an open immersion and each square is Cartesian. We have the commutative diagram

$$\begin{array}{ccccccccc}
j_\sharp f'_* & & & \xrightarrow{\qquad\qquad\qquad Ex \qquad\qquad\qquad} & & & & & f_*j'_\sharp \\
\downarrow{\scriptstyle ad} & & & & & & & & \downarrow{\scriptstyle ad} \\
j_\sharp f'_* g'_* g'^* & \xrightarrow{\sim} & j_\sharp (f'g')_* g'^* & \xrightarrow{Ex} & (fg)_* j''_\sharp g'^* & \xrightarrow{Ex} & (fg)_* g^* j_\sharp & \xrightarrow{\sim} & f_* g_* g^* j'_\sharp
\end{array}$$

of functors, and we want to show that the upper horizontal arrow is an isomorphism. The right vertical arrow is isomorphisms by assumption, and the third bottom horizontal arrow is an isomorphism by (eSm-BC). Moreover, the morphism $f'g' : W \to U$ satisfies the support property by assumption, so the second bottom horizontal arrow is an isomorphism. Thus to show that the upper horizontal arrow is an isomorphism, it suffices to show the left vertical arrow is an isomorphism. To show this, we will show that the unit

$$\mathrm{id} \xrightarrow{ad} g'_*g'^*$$

is an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{id} & & \xrightarrow{ad} & & & & g'_*g'^* \\
\downarrow{\scriptstyle ad} & & & & & & \downarrow{\scriptstyle ad} \\
j'^*j'_\sharp & \xrightarrow{ad} & j'^*g_*g^*j'_\sharp & \xrightarrow{Ex} & g'_*j''^*g^*j'_\sharp & \xrightarrow{Ex} & g_*j''^*j''_\sharp g'^*
\end{array}$$

of functors. The vertical arrows are isomorphisms by (2.2.1), and the lower left horizontal arrow is an isomorphism by assumption. Moreover, the lower middle and right horizontal arrows are isomorphisms by (eSm-BC). Thus the upper horizontal arrow is an isomorphism. □

5.1.7. Let $g : S' \to S$ be a morphism of $\mathscr{S}$-schemes. We will sometimes assume that

(i) for any pullback g' of g, g'^* is conservative,

(ii) for any commutative diagram

$$\begin{array}{ccc}
Y' & \xrightarrow{g''} & Y \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\
X' & \xrightarrow{g'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{g} & S
\end{array}$$

of $\mathscr{S}$-schemes such that each square is Cartesian, the exchange transformation

$$g'^*h_* \xrightarrow{Ex} h'_*g''^*$$

is an isomorphism.

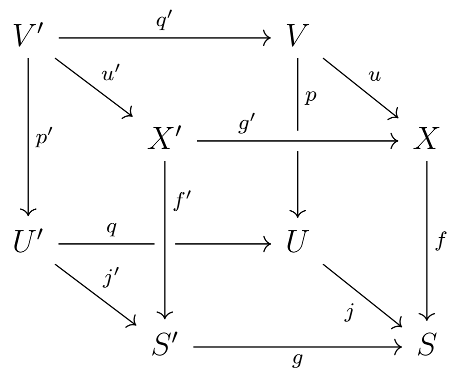
Proposition 5.1.8. *Consider a Cartesian diagram*

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{g} & S
\end{array}$$

of $\mathscr{S}$-schemes. Assume that g satisfies the conditions of (5.1.7). If f' satisfies the universal support property, then f satisfies the universal support property.

Proof. Replacing f by $X \to \underline{X} \times_{\underline{S}} S$, we may assume that $\underline{f}$ is an isomorphism. Then f' is proper, so it satisfies the support property by (5.1.4). Since the question is preserved by any base change of S, it suffices to show that f satisfies the support property.

Consider a commutative diagram



where j is an open immersion and each small square is Cartesian. We want to show that the natural transformation

$$j_\sharp p_* \xrightarrow{Ex} f_* u_\sharp$$

is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccc} j'_\sharp q^* p_* & \xrightarrow{Ex} & j'_\sharp p'_* q'^* & \xrightarrow{Ex} & f'_* u'_\sharp q'^* \\ \downarrow{\scriptstyle Ex} & & & & \downarrow{\scriptstyle Ex} \\ g^* j_\sharp p_* & \xrightarrow{Ex} & g^* f_* u_\sharp & \xrightarrow{Ex} & f'_* g'^* u_\sharp \end{array}$$

of functors. The vertical arrows are isomorphisms by (eSm-BC), and the upper left and lower right horizontal arrows are isomorphisms by the assumption that g satisfies the conditions of (5.1.7). Moreover, the upper right horizontal arrow is an isomorphism since f' satisfies the support property. Thus the lower left horizontal arrow is an isomorphism. Then the conservativity of g^* implies the support property for f. □

5.2 Conservativity

Lemma 5.2.1. *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ be functors of categories. Assume that for any objects X and Y of $\mathcal{C}$, the function*

$$\tau_{XY} : \mathrm{Hom}_{\mathcal{C}'}(FX, FY) \to \mathrm{Hom}_{\mathcal{C}''}(GFX, GFY)$$

defined by

$$f \mapsto Gf$$

is bijective. If F is conservative, then GF is also conservative.

Proof. Let X and Y be objects of $\mathcal{C}$, and let $\alpha : X \to Y$ be a morphism in $\mathcal{C}$ such that $GF\alpha$ is an isomorphism. We put $\beta = GF\alpha$. Choose the inverse of $\phi : GFY \to GFX$ of α. Then

$$\mathrm{id} = \tau_{XX}^{-1}(\mathrm{id}) = \tau_{XX}^{-1}(\phi \circ \beta) = \tau_{YX}^{-1}(\phi) \circ \tau_{XY}^{-1}(\beta),$$

so $F\alpha = \tau_{XY}^{-1}(\beta)$ has a left inverse. Similarly, $F\alpha$ has a right inverse. Thus $F\alpha$ is an isomorphism. Then the conservativity of F implies that α is an isomorphism. □

5.2.2. Let $f : X \to S$ be a Kummer log smooth morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ of Kummer log smooth type. We will construct a homomorphism $\eta : P \to P'$ of Kummer log smooth over S type with the following properties.

(i) Let $g : S' \to S$ denotes the projection $S \times_{\mathbb{A}_P} \mathbb{A}_{P'} \to S$. For any pullback u of g, u^* is conservative.

(ii) In the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes, f' is strict smooth.

This will be used in the proof of (5.3.1).

Consider the homomorphisms

$$\lambda : P \to P \oplus Q, \quad \eta : P \to P^{\mathrm{gp}} \oplus Q$$

defined by $p \mapsto (p, \theta(p))$. Using these homomorphism, we construct the fiber products

$$S'' = S \times_{\mathbb{A}_P} \mathbb{A}_{P \oplus Q}, \quad S' = S \times_{\mathbb{A}_P} \mathbb{A}_{P^{\mathrm{gp}} \oplus Q}.$$

Consider the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{s} & S'' & \xleftarrow{j} & S' \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle h} & \swarrow {\scriptstyle g} & \\ & & S & & \end{array}$$

of $\mathscr{S}$-schemes where s denotes the morphism constructed by the homomorphism $P \oplus Q \to P$ defined by $(p, q) \mapsto p$, h denotes the projection, and j denotes the open immersion induced by the inclusion $P \oplus Q \to P^{\mathrm{gp}} \oplus Q$.

From $s^* h^* \cong \mathrm{id}$, we see that h^* is conservative. We will show that g^* is also conservative. By (Htp–3), the composition

$$h_\sharp h^* \xrightarrow{\sim} g_\sharp j_\sharp j^* g^* \xrightarrow{ad'} g_\sharp g^*$$

is an isomorphism, so for $F, G \in \mathscr{T}(S)$, the homomorphism

$$\mathrm{Hom}_{\mathscr{T}(S'')}(h^* F, h^* G) \to \mathrm{Hom}_{\mathscr{T}(S')}(g^* F, g^* G) \qquad (*)$$

is an isomorphism. Then (5.2.1) implies that g^* is conservative. The same proof can be applied to show that for any pullback u of g, u^* is conservative.

The remaining is to show that f' is strict. The homomorphism $\theta : P \to Q$ factors through $P' = P^{\mathrm{gp}} \oplus Q$ via $p \mapsto (p, \theta(p))$ and $(p, q) \mapsto q$, so the morphism $f : X \to S$ factors through S'. Consider the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & S' \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. The morphism $X \to S'$ is strict, so to show that f' is strict, it suffices to show that p_1 is strict. This follows from (1.2.18).

5.2.3. Let $f : X \to S$ be a Kummer log smooth morphism of $\mathscr{S}$-schemes. By (3.1.4), we can choose a strict étale cover $\{u_i : S_i \to S\}_{i \in I}$ such that for each i, there is a commutative diagram

$$\begin{array}{ccc} X_i & \xrightarrow{v_i} & X \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f} \\ S_i & \xrightarrow{u_i} & S \end{array}$$

of $\mathscr{S}$-schemes such that f_i has a fs chart of log smooth type and $\{v_i\}_{i \in I}$ is a strict étale cover. Then by (5.2.2), there is a Kummer log smooth morphism satisfying the conditions (i) and (ii) of (loc. cit). Let $g : S' \to S$ denote the union of $g_i u_i : S_i' \to S$. Then g satisfies the condition (i) of (loc. cit) by (két-Sep), and g satisfies the condition (ii) of (loc. cit) by construction.

5.3 Support property for Kummer log smooth morphisms

Proposition 5.3.1. *Let f be a Kummer log smooth morphisms of $\mathscr{S}$-schemes. Then f satisfies the universal support property.*

Proof. By (5.1.5) and (3.1.4), we may assume that f has a fs chart $\theta : P \to Q$ of Kummer log smooth type. As in (5.2.2), choose a Kummer log smooth morphism $g : S' \to S$ satisfying the condition (i) of (5.1.7) such that the pullback of f via $g : S' \to S$ is strict. Since g is an exact log smooth morphism, the condition (ii) of (loc. cit) is satisfied by (eSm-BC). Now we can apply (5.1.8), so replacing f by the projection $X \times_S S' \to S'$, we may assume that f is strict. Then the conclusion follows from (5.1.6(1)). □

Proposition 5.3.2. *Let $f : X \to S$ and $g : Y \to X$ be morphisms of $\mathscr{S}$-schemes such that g is proper. If f and fg satisfy the semi-universal support property, then g satisfies the semi-support property.*

Proof. Consider a commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow{\quad\beta\quad} & T \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\quad g\quad} & X \\ & {\scriptstyle fg}\searrow \qquad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

of $\mathscr{S}$-schemes where the small square is Cartesian and α is strict. By (5.1.4), it suffices to show that β satisfies the support property for any α. The morphisms α and α' satisfy the semi-universal support property by (5.1.6(1)) since they are strict, so the morphisms $f\alpha$ and $fg\alpha'$satisfies the semi-universal support property by (5.1.6(2)). Hence replacing (Y, X, S) by (T', T, S), we reduce to showing that g satisfies the support property.

Consider a Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{w} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{v} & X \end{array}$$

of $\mathscr{S}$-schemes where v is an open immersion. By (3.6.4), it suffices to show that for any Kummer log smooth morphism $p : X' \to X$ of $\mathscr{S}$-schemes and any object K of $\mathscr{T}(W)$, the homomorphism

$$\mathrm{Hom}_{\mathscr{T}(X)}(M_X(X'), v_\sharp g'_* K) \to \mathrm{Hom}_{\mathscr{T}(X)}(M_X(X'), g_* w_\sharp K)$$

is an isomorphism. It is equivalent to the assertion that

$$\mathrm{Hom}_{\mathscr{T}(X')}(1_{X'}, p^* v_\sharp g'_* K) \to \mathrm{Hom}_{\mathscr{T}(X')}(1_{X'}, p^* g_* w_\sharp K)$$

is an isomorphism since $M_X(X') = p_\sharp 1_{X'}$. By (5.3.1), p satisfies the universal support property, so fp and gp satisfy the semi-universal support property by (5.1.6(2)). Since p^* commutes with $v_\sharp$, g'_*, g_*, and $w_\sharp$, replacing $Y \to X \to S$ by $Y \times_X X' \to X' \to S$, we reduce to showing that

$$\mathrm{Hom}_{\mathscr{T}(X)}(1_X, v_\sharp g'_* K) \to \mathrm{Hom}_{\mathscr{T}(X)}(1_X, g_* w_\sharp K)$$

is an isomorphism. It is equivalent to showing that

$$\mathrm{Hom}_{\mathscr{T}(S)}(1_S, f_* v_\sharp g'_* K) \to \mathrm{Hom}_{\mathscr{T}(S)}(1_S, f_* g_* w_\sharp K)$$

is an isomorphism. Hence it suffices to show that the natural transformation

$$v_\sharp g'_* \xrightarrow{Ex} g_* w_\sharp$$

is an isomorphism.

The induced morphism $Y \to \underline{X} \times_{\underline{S}} S$ has the factorization

$$Y \to \underline{Y} \times_{\underline{S}} S \to \underline{X} \times_{\underline{S}} S.$$

The first arrow satisfies the semi-universal support property by assumption, and the second arrow satisfies the semi-universal support property by (5.1.6(1)) since it is strict. Thus the composition also satisfies the semi-universal support property by (5.1.6(2)). Since the induced morphism $X \to \underline{X} \times_{\underline{S}} S$ also satisfies the semi-universal support property by assumption, replacing (Y, X, S) by $(Y, X, \underline{X} \times_{\underline{S}} S)$, we may assume that $\underline{f}$ is an isomorphism.

Then there is a unique commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{w} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{v} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{u} & S \end{array}$$

of $\mathscr{S}$-schemes such that the lower square is also Cartesian. Since v is an open immersion, u is automatically an open immersion. In the commutative diagram

$$\begin{array}{ccccc} u_\sharp f'_* g'_* & \xrightarrow{Ex} & f_* v_\sharp g'_* & \xrightarrow{Ex} & f_* g_* w_\sharp \\ \downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim} \\ u_\sharp (f'g')_* & & \xrightarrow{\qquad Ex \qquad} & & (fg)_* w_\sharp \end{array}$$

of functors, the upper left arrow and lower arrows are isomorphisms by assumption. Thus the upper right arrow is an isomorphism. This completes the proof. □

Proposition 5.3.3. *Let $f : X \to S$ be a proper morphism of $\mathscr{S}$-schemes satisfying the semi-universal support. Let $g : S' \to S$ be a Kummer log smooth morphism of $\mathscr{S}$-schemes, and consider the Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. Then the exchange transformation

$$g_\sharp f'_* \xrightarrow{Ex} f_* g'_\sharp$$

is an isomorphism.

Proof. Note first that by (5.3.1), g and g' satisfies the universal support property, so fg' satisfies the semi-universal support property by (5.1.6(2)). Then f' satisfies the support property by (5.3.2). By (5.2.3), there is a Kummer log smooth morphism $h : T \to S$ of $\mathscr{S}$-schemes such that h^* is conservative and that the pullback of $g : S' \to S$ via h is strict smooth. Note also that h satisfies the universal support property by (5.3.1). Thus replacing (X', X, S', S) by $(X' \times_S T, X \times_S T, S' \times_S T, T)$, we may assume that g is strict smooth.

The question is Zariski local on S' since f' satisfies the support property, so we may assume that g is a strict smooth morphism of relative dimension d. Choose a compactification

$$S' \xrightarrow{j} S'' \xrightarrow{p} S$$

where j is an open immersion and p is a strict proper morphism of $\mathscr{S}$-schemes. Consider the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{j'} & X'' & \xrightarrow{p'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{j} & S'' & \xrightarrow{p} & S \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. Since g is strict smooth, as in [CD12, 2.4.50], we have the purity isomorphisms

$$g_\sharp \xrightarrow{\sim} p_* j_\sharp(d)[2d],$$
$$g'_\sharp \xrightarrow{\sim} p'_* j'_\sharp(d)[2d].$$

with the commutative diagram

$$\begin{array}{ccccc} g_\sharp p'_* & & \xrightarrow{\qquad\sim\qquad} & & p_* j_\sharp f'_*(d)[2d] \\ \downarrow{\scriptstyle Ex} & & & & \downarrow{\scriptstyle Ex} \\ f_* g'_\sharp & \xrightarrow{\sim} & f_* p'_* j'_\sharp(d)[2d] & \xrightarrow{\sim} & p_* f''_* j'_\sharp(d)[2d] \end{array}$$

of functors. The morphisms p and p' satisfies the semi-support property by (5.1.6(1)) since they are strict, so the morphism fp' satisfies the semi-support property by (5.1.6(2)). Then f'' satisfies the semi-support property by (5.3.2), so the right vertical arrow is an isomorphism. Thus the left vertical arrow is an isomorphism. □

Proposition 5.3.4. *Let $f : X \to S$ be a proper morphism of $\mathscr{S}$-schemes satisfying the semi-universal support property. Then f satisfies the projection formula.*

Proof. We want to show that for any objects K of $\mathscr{T}_X$ and L of $\mathscr{T}_S$, the morphism

$$f_* K \otimes_S L \xrightarrow{Ex} f_*(K \otimes_X f^* L)$$

is an isomorphism. By (3.6.4), it suffices to show that for any Kummer log smooth morphism $g : S' \to S$ of $\mathscr{S}$-schemes, the morphism

$$f_* K \otimes_S g_\sharp 1_{S'} \xrightarrow{Ex} f_*(K \otimes f^* g_\sharp 1_{S'})$$

is an isomorphism. Consider the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. In the commutative diagram

$$\begin{array}{ccccc} f_*K \otimes_S g_\sharp 1_{S'} & & \xrightarrow{Ex} & & f_*(K \otimes f^* g_\sharp 1_{S'}) \\ & & & & \downarrow{\scriptstyle Ex} \\ \downarrow{\scriptstyle Ex} & & & & f_*(K \otimes g'_\sharp f'^* 1_{S'}) \\ & & & & \downarrow{\scriptstyle Ex} \\ g_\sharp(g^* f_* K \otimes 1_{S'}) & & & & f_* g'_\sharp (g'^* K \otimes f'^* 1_{S'}) \\ \downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim} \\ g_\sharp g^* f_* K & \xrightarrow{Ex} & g_\sharp f'_* g'^* K & \xrightarrow{Ex} & f_* g'_\sharp g'^* K \end{array}$$

of $\mathscr{S}$-schemes, the upper left vertical and the middle right vertical arrows are isomorphisms by (eSm-PF), and the lower left horizontal and the upper right vertical arrows are isomorphisms by (eSm-BC). Moreover, the lower right horizontal arrow is an isomorphism by (5.3.3), so the upper horizontal arrow is an isomorphism. □

Proposition 5.3.5. *Let $f : X \to S$ be a proper morphism of $\mathscr{S}$-schemes satisfying the semi-universal support property. Then the property* ($\mathrm{BC}_{f,g}$) *holds for any strict morphism $g : S' \to S$ of $\mathscr{S}$-schemes.*

Proof. By (Zar-Sep), we may assume that g is quasi-projective. Then g has a factorization

$$S' \xrightarrow{i} T \xrightarrow{p} S$$

where i is a strict closed immersion and p is strict smooth. By (eSm-BC), we only need to deal with the case when g is a strict closed immersion.

Let $h : S'' \to S$ denote the complement of g. Then we have the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & \xleftarrow{h'} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ S' & \xrightarrow{g} & S & \xleftarrow{h} & S'' \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. By (Loc), we have the commutative diagram

$$\begin{array}{ccccccc} g^* f_* h'_\sharp h'^* & \xrightarrow{ad'} & g^* f_* & \xrightarrow{ad} & g^* f_* g'_* g'^* & \xrightarrow{\partial_{g'}} & g^* f_* h'_\sharp h'^*[1] \\ \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\ f'_* g'^* h'_\sharp h'^* & \xrightarrow{ad'} & f'_* g'^* & \xrightarrow{ad} & f'_* g'^* g'_* g'^* & \xrightarrow{\partial_{g'}} & f'_* g'^* h'_\sharp h'^*[1] \end{array}$$

of functors where the two rows are distinguished triangles. To show that the second vertical arrow is an isomorphism, it suffices to show that the first and third vertical arrows are isomorphisms.

We have an isomorphism
$$f_*h'_\sharp \cong h_\sharp f'_*$$
since f satisfies the support property, so we have
$$g^*f_*h'_\sharp h'^* \cong g^*h_\sharp f'_*h'^* = 0, \quad f_*g'^*h'_\sharp h'^* = 0$$
since $g^*h_\sharp = g'^*h'_\sharp = 0$ by (eSm-BC). Thus the first vertical arrow is an isomorphism. The assertion that the third arrow is an isomorphism follows from (Loc), which completes the proof. □

Proposition 5.3.6. *Let $g : Y \to X$ and $f : X \to S$ be morphisms of $\mathscr{S}$-schemes. Assume that g is proper and that the unit*
$$\mathrm{id} \xrightarrow{ad} g_*g^*$$
is an isomorphism. If g and fg satisfy the semi-universal support propoerty, then f satisfies the semi-universal support property.

Proof. By (5.1.6(3)), it suffices to show that for any Cartesian diagram
$$\begin{array}{ccc} Y' & \xrightarrow{g'} & X' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & X \end{array}$$
of $\mathscr{S}$-schemes such that h is a strict, the unit
$$\mathrm{id} \xrightarrow{ad} g'_*g'^*$$
is an isomorphism.

By (5.3.4), for any object K of $\mathscr{T}(X')$, the composition
$$g'_*g'^*1_{X'} \otimes_X K \xrightarrow{Ex} g'_*(g'^*1_S \otimes_X g'^*K) \xrightarrow{\sim} g'_*g'^*K$$
is an isomorphism, so we only need to show that the morphism
$$1_{X'} \xrightarrow{ad} g'_*g'^*1_{X'}$$
in $\mathscr{T}(X')$ is an isomorphism. It has the factorization
$$1_{X'} \xrightarrow{\sim} h^*1_X \xrightarrow{ad} h^*g_*g^*1_X \xrightarrow{Ex} g'_*h'^*g^*1_X \xrightarrow{\sim} g'_*g'^*1_{X'}$$
in $\mathscr{T}(X')$. The second arrow is an isomorphism by assumption, and the third arrow is an isomorphism by (5.3.5). Thus the morphism
$$1_{X'} \xrightarrow{ad} g'_*g'^*1_{X'}$$
in $\mathscr{T}(X')$ is an isomorphism. □

5.4 Poincaré duality for a compactification of $\mathbb{A}_{\mathbb{N}^2} \to \mathbb{A}_{\mathbb{N}}$

5.4.1. We fix an $\mathscr{S}$-scheme S over $\mathbb{A}_{\mathbb{N}}$, and we put

$$U = \mathbb{A}_{\mathbb{N}\oplus\mathbb{N}} \times_{\mathbb{A}_\theta, \mathbb{A}_{\mathbb{N}}} S$$

where $\theta : \mathbb{N} \to \mathbb{N}\oplus\mathbb{N}$ denotes the diagonal morphism. We want to compactify the projection

$$h : U \to S.$$

Then we will prove the Poincaé duality for the compactification.

5.4.2. Under the notations and hypotheses of (5.4.1), consider the lattice $L = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2$, and consider the dual coordinates

$$e_1 = x_1^\vee, \quad e_2 = x_2^\vee.$$

We denote by $\underline{T}$ the toric variety associated to the fan generated by

(a) $e_1, e_2 \geq 0$,

(b) $e_1 + e_2 \geq 0$, $e_1 \leq 0$,

(c) $e_1 + e_2 \geq 0$, $e_2 \leq 0$.

We give the log structure on (a), (b), and (c) by

$$\mathbb{N}x_1 \oplus \mathbb{N}x_2 \to \mathbb{Z}[x_1, x_2], \quad \mathbb{N}(x_1x_2) \to \mathbb{Z}[x_1x_2, x_1^{-1}], \quad \mathbb{N}(x_1x_2) \to \mathbb{Z}[x_1x_2, x_2^{-1}]$$

respectively. Then we denote by T the resulting $\mathscr{S}$-scheme. Because the support of this fan is $\{(e_1, e_2) : e_1 + e_2 \geq 0\}$, the morphism $T \to \mathbb{A}_{\mathbb{N}}$ induced by the diagonal homomorphism $\mathbb{N} \to \mathbb{N}x_1 \oplus \mathbb{N}x_2$ is proper, so we have the compactification

$$\begin{array}{ccc} \mathbb{A}_{\mathbb{N}^2} & \longrightarrow & T \\ & \searrow & \downarrow \\ & & \mathbb{A}_{\mathbb{N}} \end{array}$$

of the morphism $\mathbb{A}_{\mathbb{N}^2} \to \mathbb{A}_{\mathbb{N}}$. Thus if we put $X = S \times_{\mathbb{A}_{\mathbb{N}}} T$, then we have the compactification

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle f} \\ & & S \end{array}$$

of h. Here, the meaning of compactification is that j is an open immersion and f is proper.

5.4.3. Under the notations and hypotheses of (5.4.2), consider the lattice

$$(\mathbb{Z}x_1 \oplus \mathbb{Z}x_2) \oplus_{\mathbb{Z}} (\mathbb{Z}y_1 \oplus \mathbb{Z}y_2)$$

with $x_1 + x_2 = y_1 + y_2$, and consider the dual coordinates

$$f_1 = y_1^{\vee}, \quad f_2 = y_2^{\vee}.$$

We denote by $\underline{T'}$ the toric variety associated to the fan generated by

(a) $e_1, e_2, f_1, f_2 \geq 0$, $e_1 + e_2 = f_1 + f_2$,
(b) $e_1 + e_2 = f_1 + f_2 \geq 0$, $e_1, f_1 \leq 0$,
(c) $e_1 + e_2 = f_1 + f_2 \geq 0$, $e_2, f_2 \leq 0$.

Then we have an open immersion $\underline{T'} \to \underline{T} \times_{\mathbb{A}^1} \underline{T}$. Thus if we denote by T' the $\mathscr{S}$-scheme whose underlying scheme is $\underline{T'}$ and with the log structure induced by the open immersion, then we have the open immersion

$$T' \to T \times_S T.$$

We put $D = (X \times_S X) \times_{T \times_{\mathbb{A}_{\mathbb{N}}} T} T'$.

5.4.4. Under the notations and hypotheses of (5.4.3), we denote by $\underline{T''}$ the toric variety associated to the fan generated by

(a) $e_1, e_2, f_1, f_2 \geq 0$, $e_1 + e_2 = f_1 + f_2$, $e_1 \geq f_1$,
(a') $e_1, e_2, f_1, f_2 \geq 0$, $e_1 + e_2 = f_1 + f_2$, $e_1 \leq f_1$,
(b) $e_1 + e_2 = f_1 + f_2 \geq 0$, $e_1, f_1 \leq 0$,
(c) $e_1 + e_2 = f_1 + f_2 \geq 0$, $e_2, f_2 \leq 0$.

We give the log structure on (a), (a'), (b), and (c) by

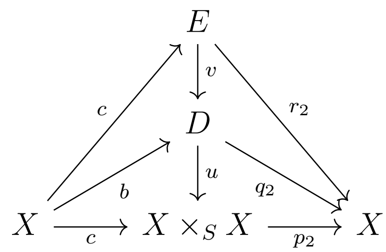
$$\begin{aligned}
&\mathbb{N}y_1 \oplus \mathbb{N}x_2 \oplus \mathbb{N}(x_1 y_1^{-1}) \to \mathbb{Z}[y_1, x_2, x_1 y_1^{-1}], \\
&\mathbb{N}x_1 \oplus \mathbb{N}y_2 \oplus \mathbb{N}(y_1 x_1^{-1}) \to \mathbb{Z}[x_1, y_2, y_1 x_1^{-1}], \\
&\mathbb{N}(x_1 x_2) \to \mathbb{Z}[x_1 x_2, x_1^{-1}, y_1^{-1}], \\
&\mathbb{N}(x_1 x_2) \to \mathbb{Z}[x_1 x_2, x_2^{-1}, y_2^{-1}]
\end{aligned}$$

respectively. Then we denote by T'' the resulting $\mathscr{S}$-scheme. The supports of this fan and the fan in (5.4.3) are equal, so the morphism $T'' \to T'$ induced by the fans is proper. Thus if we put $E = Y \times_{T'} T''$, we have the proper morphism

$$v : E \to D$$

of $\mathscr{S}$-schemes.

5.4.5. From (5.4.1) to (5.4.4), we obtain the commutative diagram



of $\mathscr{S}$-schemes where

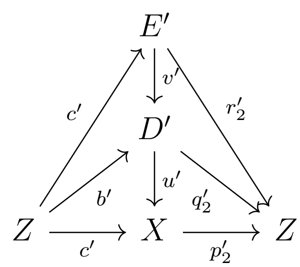
(i) p_2 denotes the second projection,

(ii) a denotes the diagonal morphism, and b and c denotes the morphisms induced by a.

Note that u is an open immersion and that v and p_2 are proper. The morphism $E \to D$ satisfies the condition of (4.2.2(2)), and the morphism $D \to X \times_S X$ satisfies the condition of (4.2.2(1)). Consider the natural transformation

$$\mathfrak{q}_f^o : \Omega_{f,E}^o f^! \longrightarrow f^*$$

in (4.4.2). Note that we have $\Omega_{f,E}^o = (-1)[-2]$. We also consider the pullback of the above commutative diagram via $i : Z \to X$ where i denotes the complement of $j : U \to X$:



Note that u' and v' are isomorphisms.

Proposition 5.4.6. *Under the notations and hypotheses of (5.4.5), the natural transformation*

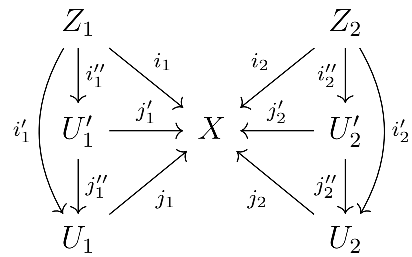
$$f_* f^!(-1)[-2] \xrightarrow{\mathfrak{q}_f^o} f_* f^*$$

is an isomorphism.

Proof. We put $\tau = (1)[2]$, and let

$$\mathfrak{p}_f^o : f_\sharp \longrightarrow f_* \tau$$

denote the left adjoint of $\mathfrak{q}_f^o$. We have the commutative diagram



of $\mathscr{S}$-schemes where

(i) j_1' and j_2' denote the open immersions induced by the convex sets (b) and (c) of (5.4.2) respectively,

(ii) $Z_1 = U_1' \times_X (X - U)$ and $Z_2 = U_2' \times_X (X - U)$,

(iii) U_1 and U_2 denotes the complements of Z_1 and Z_2 respectively,

(iv) i_1, i_1', i_1'', i_2, i_2', and i_2'' are the closed immersions,

(v) j_1, j_1'', j_2, j_2'' are the open immersions.

The *key property* of our compactification is that U_1 and U_1' (resp. U_2 and U_2') are *log-homotopic equivalent* over S. The meaning is that the morphisms

$$M_S(U_1') \to M_S(U_1), \quad M_S(U_2') \to M_S(U_2)$$

in $\mathscr{T}(S)$ are isomorphisms. More generally, we will show that the natural transformations

$$f_\sharp j_{1\sharp} j''_{1\sharp} j''^*_1 j_1^* f^* \xrightarrow{ad} f_\sharp j_{1\sharp} j_1^* f^*, \quad f_\sharp j_{2\sharp} j''_{2\sharp} j''^*_2 j_2^* f^* \xrightarrow{ad} f_\sharp j_{2\sharp} j_2^* f^* \tag{5.4.6.1}$$

are isomorphisms. To show that the first one is an isomorphism, consider the Mayer-Vietoris triangle

$$f_\sharp j'''_{1\sharp} j'''^*_1 f^* \to f_\sharp j'_{1\sharp} j'^*_1 f^* \oplus f_\sharp j_\sharp j^* f^* \to f_\sharp j_{1\sharp} j_1^* f^* \to f_\sharp j'''_{1\sharp} j'''^*_1 f^*[1]$$

in $\mathscr{T}(S)$ where $j_1''' : U_1' \times_X U \to X$ denotes the open immersion. It is a distinguished triangle by (2.2.3), so it suffices to show that the morphism

$$f_\sharp j'''_{1\sharp} j'''^*_1 f^* \to f_\sharp j_\sharp j^* f^*$$

is an isomorphism since $j_1' = j_1 j_1''$. It is true by (Htp–3) because

$$U_1' \times_X U \cong U \times_{\mathbb{A}_{\mathbb{N}^2}} \mathbb{A}_{(\mathbb{N}^2)_F}$$

where F is the face of $\mathbb{N}^2$ generated by $(1,0)$. Thus the first natural transformation of (5.4.6.1) is an isomorphism. We can show that the other one is also an isomorphism similarly.

Then guided by a method of [Ayo07, 1.7.9], consider the commutative diagram

$$\begin{array}{ccccccc} f_\sharp i_{1*}i_1^! f^* & \xrightarrow{ad'} & f_\sharp f^* & \xrightarrow{ad} & f_\sharp i_{2*}i_2^* f^* & \xrightarrow{\partial} & f_\sharp i_{1*}i_1^! f^*[1] \\ \downarrow \mathfrak{p}_f^o & & \downarrow \mathfrak{p}_f^o & & \downarrow \mathfrak{p}_f^o & & \downarrow \mathfrak{p}_f^o \\ f_* i_{1*}i_1^! f^*\tau & \xrightarrow{ad'} & f_* f^*\tau & \xrightarrow{ad} & f_* i_{2*}i_2^* f^* & \xrightarrow{\partial} & f_* i_{1*}i_1^! f^*\tau[1] \end{array} \tag{5.4.6.2}$$

of functors. Assume that we have proven that

(1) the rows are distinguished triangles,

(2) the first and third vertical arrows are isomorphisms.

Then the second arrow is also an isomorphism, so we are done. Hence in the remaining, we will prove (1) and (2).

(1) To show that the second row is a distinguished triangle, by (Loc), it suffices to show that the composition

$$f_* i_{2*} i_2^* f^* \xrightarrow{\sim} f_* j_{2*} i'_{2*} i'^*_2 j_2^* f^* \xrightarrow{ad'} f_* j_{2*} j_2^* f^*$$

is an isomorphism. We have shown that the natural transformation

$$f_* j_{2*} j_2^* f^* \xrightarrow{ad'} f_* j_{2*} j''_{2*} j''^*_2 j_2^* f^*$$

is an isomorphism, and we have $fi_2 = \mathrm{id}$. Hence, it suffices to show that the unit

$$g_{2\sharp} g_2^* \xrightarrow{ad'} \mathrm{id}$$

is an isomorphism where $g_2 = fj'_2$. It is true by (Htp–1) since the morphism $U'_2 \to S$ is the projection $\mathbb{A}^1_S \to S$.

For the first row, first note that by (sSupp), we have an isomorphism

$$j_{1\sharp} i'_{1*} i'^!_1 j_1^* \xrightarrow{\sim} i_{1*} i_1^!.$$

Hence to show that the first row is distinguished, by (Loc), it suffices to show that the natural transformation

$$f_\sharp j_{1\sharp} i'_{1*} i'^!_1 j_1^* f^* \xrightarrow{ad'} f_\sharp j_{1\sharp} j_1^* f^*$$

is an isomorphism. By (sSupp), we have an isomorphism

$$f_\sharp j'_{1\sharp} i''_{1*} i''^!_1 j'^*_1 f^* \xrightarrow{\sim} f_\sharp j_{1\sharp} i'_{1*} i'^!_1 j_1^* f^*,$$

and we have shown that the natural transformation

$$f_\sharp j_{1\sharp} j''_{1\sharp} j''^*_1 j_1^* f^* \xrightarrow{ad'} f_\sharp j_{1\sharp} j_1^* f^*$$

is an isomorphism. Hence it suffices to show that the natural transformation

$$g_{1\sharp} i''_{1*} i''^{!}_{1} g_1^* \xrightarrow{ad'} g_{1\sharp} g_1^*$$

is an isomorphism where $g_1 = f j'_1$. By (Htp–1), the counit

$$g_{1\sharp} g_1^* \xrightarrow{ad'} \mathrm{id}$$

is an isomorphism since the morphism $U'_1 \to S$ is the projection $\mathbb{A}^1_S \to S$, and by (2.5.3), the composition

$$g_{1\sharp} i''_{1*} i''^{!}_{1} g_1^* \xrightarrow{ad'} g_{1\sharp} g_1^* \xrightarrow{ad'} \mathrm{id}$$

is an isomorphism. Thus the first row is distinguished.

(2) Consider the diagram

$$\begin{array}{ccccccccccccc}
\Omega^o_{f,\mathrm{id},E'} i^! f^! & \xrightarrow{\sim} & \Omega^n_{f,\mathrm{id},E'} i^! f^! & \xrightarrow{\sim} & \Omega^d_{f,\mathrm{id},E'} i^! f^! & \xrightarrow{\sim} & \Omega_{f,\mathrm{id},E'} i^! f^! & \xrightarrow{\sim} & \Omega_{f,\mathrm{id},D'} i^! f^! & \xrightarrow{\sim} & \Omega_{f,\mathrm{id}} i^! f^! & \xrightarrow{\sim} & \Omega_{f,\mathrm{id}} \\
 & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \| \\
i^! \Omega^o_{f,E} f^! & \xrightarrow{\sim} & i^! \Omega^n_{f,E} f^! & \longrightarrow & i^! \Omega^d_{f,E} f^! & \longrightarrow & i^! \Omega_{f,E} f^! & \longrightarrow & i^! \Omega_{f,D} f^! & \longrightarrow & i^! \Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & i^! f^*
\end{array}$$

of functors. It commutes by (4.2.10), and the natural transformation $\Omega^n_{f,\mathrm{id},E'} i^! f^! \xrightarrow{Ex} i^! \Omega^n_{f,E} f^!$ is an isomorphism by (4.2.5). Thus the composition of arrows in the second row

$$i^! \Omega^o_{f,E} f^! \xrightarrow{\mathfrak{q}^o_f} i^! f^*$$

is also an isomorphism. Then the first vertical arrow of (5.4.6.2) is also an isomorphism. The third vertical arrow of (5.4.6.2) is also an isomorphism similarly. □

Theorem 5.4.7. *Under the notations and hypotheses of (5.4.5), the natural transformation*

$$\mathfrak{q}^o_f : f^!(-1)[-2] \longrightarrow f^*$$

is an isomorphism.

Proof. We put $\tau = (1)[2]$, and let

$$\mathfrak{p}^o_f : f_\sharp \longrightarrow f_* \tau$$

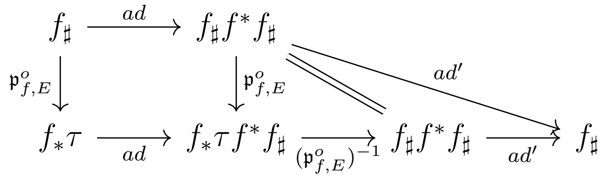
denote the left adjoint of $\mathfrak{q}^o_f$. Guided by a method of [CD12, 2.4.42], we will construct a right inverse ϕ_1 and a left inverse ϕ_2 to the morphism $\mathfrak{p}^o_f$. Note first that the natural transformation

$$f_\sharp f^* \xrightarrow{\mathfrak{p}^o_f} f_* f^* \tau$$

is an isomorphism by (5.4.6). The left inverse ϕ_2 is constructed by

$$\phi_2 : f_* \tau \xrightarrow{ad} f_* \tau f^* f_\sharp \xrightarrow{(\mathfrak{p}^o_{f,E})^{-1}} f_\sharp f^* f_\sharp \xrightarrow{ad'} f_\sharp.$$

To show $\phi_2 \circ \mathfrak{p}^o_f = \mathrm{id}$, it suffices to check that the outside diagram of the diagram



of functors commutes since the composition

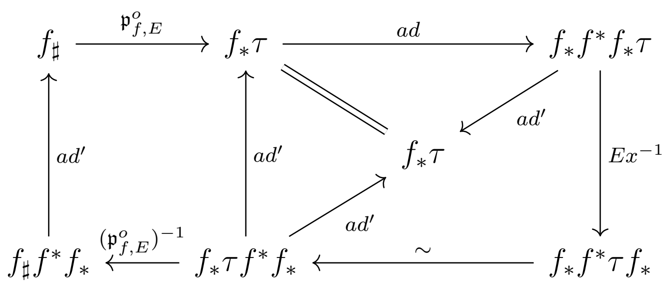
$$f_\sharp \xrightarrow{ad} f_\sharp f^* f_\sharp \xrightarrow{ad'} f_\sharp$$

is the identity. It is true since each small diagram commutes.

The right inverse ϕ_1 is constructed by

$$\phi_1 : f_*\tau \xrightarrow{ad} f_* f^* f_* \tau \xrightarrow{Ex^{-1}} f_* f^* \tau f_* \xrightarrow{\sim} f_* \tau f^* f_* \xrightarrow{(\mathfrak{p}^o_{f,E})^{-1}} f_\sharp f^* f_* \xrightarrow{ad'} f_\sharp.$$

To show $\mathfrak{p}^o_f \circ \phi_1 = \mathrm{id}$, it suffices to check that the composition of the outer cycle starting from upper $f_*\tau$ in the below diagram of functors is the identity:



It is true since each small diagram commutes.

Then from the existences of left and right inverses, we conclude that

$$\mathfrak{p}^o_f : f_\sharp \longrightarrow f_*\tau$$

is an isomorphism. □

Corollary 5.4.8. *Under the notations and hypotheses of (5.4.5), the universal support property holds for $f : X \to S$.*

Proof. We put $E'' = E \times_S V$. Consider the Cartesian diagram

$$\begin{array}{ccc} V' & \xrightarrow{f''} & V \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} \\ X & \xrightarrow{f} & S \end{array}$$

of $\mathscr{S}$-schemes where μ is an open immersion. Then by the above theorem, the support property for f follows from the commutativity of the diagram

$$\begin{array}{ccc}
\mu_\sharp f''_\sharp & \xrightarrow{\sim} & f_\sharp \mu'_\sharp \\
\Big\downarrow {\scriptstyle \mathfrak{p}^o_{f'',E''}} & & \Big\downarrow {\scriptstyle \mathfrak{p}^o_{f,E}} \\
 & & f_* \tau \mu'_\sharp \\
 & & \Big\downarrow {\scriptstyle Ex^{-1}} \\
\mu_\sharp f''_* \tau & \xrightarrow{Ex} & f_* \tau \mu'_\sharp
\end{array}$$

of functors.

Then because we can choose S arbitrary, f satisfies the universal support property. □

Corollary 5.4.9. *Under the notations and hypotheses of (5.4.5) the universal support property holds for $h : U \to S$.*

Proof. The conclusion follows from (5.4.8) and (5.1.6(1),(2)). □

5.5 Support property for the projection $\mathbb{A}_{\mathbb{N}} \times \mathrm{pt}_{\mathbb{N}} \to \mathrm{pt}_{\mathbb{N}}$

5.5.1. Let x and y denote the first and second coordinates of $\mathbb{N} \oplus \mathbb{N}$ respectively, and let S be an $\mathscr{S}$-scheme. Consider the morphisms

$$S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(x))} \xrightarrow{h} S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(xy))} \xrightarrow{g} S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(x))} \xrightarrow{f} S \times \mathrm{pt}_{\mathbb{N}}$$

of $\mathscr{S}$-schemes where

(i) h denotes the obvious closed immersion,

(ii) g denotes the morphism induced by

$$\mathbb{N} \oplus \mathbb{N} \mapsto \mathbb{N} \oplus \mathbb{N}, \quad (a, b) \mapsto (a, a + b),$$

(iii) f denotes the morphism induced by

$$\mathbb{N} \mapsto \mathbb{N} \oplus \mathbb{N}, \quad a \mapsto (a, 0).$$

To simplify the notations, we put

$$X = S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(x))}, \quad Y = S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(xy))}, \quad T = S \times \mathrm{pt}_{\mathbb{N}}.$$

Then we have the sequence

$$X \xrightarrow{h} Y \xrightarrow{g} X \xrightarrow{f} T$$

of morphisms of $\mathscr{S}$-schemes.

Proposition 5.5.2. *Under the notations and hypotheses of (5.5.1), the morphism gh satisfies the semi-universal support property.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{gh} & X \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & T & \end{array}$$

of $\mathscr{S}$-schemes where p and q denote the morphisms induced by the homomorphisms

$$\mathbb{N} \to \mathbb{N} \oplus \mathbb{N}, \quad a \mapsto (a, a),$$

$$\mathbb{N} \to \mathbb{N} \oplus \mathbb{N}, \quad a \mapsto (a, 2a),$$

respectively. By (5.3.2), it suffices to show that p and q satisfy the semi-universal support property.

The morphism fg satisfies the universal support property by (5.4.9), and the morphism h satisfies the universal support property by (5.1.6(1)) since it is strict. Thus the morphism $p = fgh$ satisfies the universal support property by (5.1.6(2)). Hence the remaining is to show that q satisfies the semi-universal support property.

The morphism q has the factorization

$$S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(x))} \xrightarrow{i} S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N}\oplus\mathbb{Z},(x))} \xrightarrow{u} S \times \mathbb{A}_{(\mathbb{N}\oplus\mathbb{N},(x))} \xrightarrow{p} S \times \mathrm{pt}_{\mathbb{N}}$$

where

(i) i denotes the morphism induced by the homomorphism

$$\mathbb{N} \oplus \mathbb{N} \oplus \mathbb{Z} \to \mathbb{N} \oplus \mathbb{N}, \quad (a, b, c) \mapsto (a, b),$$

(ii) u denotes the morphism induced by the homomorphism

$$\mathbb{N} \oplus \mathbb{N} \to \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{Z}, \quad (a, b) \mapsto (a, 2b, b).$$

We already showed that p satisfies the universal support property. The morphism i satisfies the universal support property by (5.1.6(1)) since it is strict, and the morphism u satisfies the support property by (5.3.1) since it is Kummer log smooth. Thus by (5.1.6(2)), the morphism $q = pui$ satisfies the universal support property. □

Corollary 5.5.3. *Under the notations and hypotheses of (5.5.1), the morphism gh satisfies the projection formula.*

Proof. It follows from (5.5.2) and (5.3.4). □

Proposition 5.5.4. *Under the notations and hypotheses of (5.5.1), the unit*

$$\mathrm{id} \xrightarrow{ad} (gh)_*(gh)^*$$

is an isomorphism.

Proof. To simplify the notation, we put $v = gh$. By (5.5.3), for any object K of $\mathscr{T}(X)$, the composition

$$v_*v^*1_X \otimes_X K \xrightarrow{Ex} v_*(v^*1_S \otimes_X v^*K) \xrightarrow{\sim} v_*v^*K$$

is an isomorphism, so we only need to show that the morphism

$$1_X \xrightarrow{ad} v_*v^*1_X$$

in $\mathscr{T}(X)$ is an isomorphism.

We denote by $j : U \to X$ the verticalization of X via f. Then we have the Cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{id}} & U \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{v} & X \end{array}$$

of $\mathscr{S}$-schemes, and by (Htp–2), the morphism

$$1_X \xrightarrow{ad} j_*j^*1_X \tag{5.5.4.1}$$

in $\mathscr{T}(X)$ is an isomorphism. In the commutative diagram

$$\begin{array}{ccc} 1_X & \xrightarrow{ad} & v_*v^*1_X \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ j_*j^*1_X & \xrightarrow{ad} & v_*j_*j^*v^*1_X \end{array}$$

of functors, the vertical arrows are isomorphisms since the morphism (5.5.4.1) is an isomorphism and $v^*1_X \cong 1_X$. The lower horizontal arrow is an isomorphism since $vj = j$. Thus the upper horizontal arrow is an isomorphism, which completes the proof. □

Proposition 5.5.5. *Under the notations and hypotheses of (5.5.1), the morphism f satisfies the semi-support property.*

Proof. By (5.4.9), the morphism fg satisfies the semi-universal support property. The morphism h satisfies the semi-universal support property by (5.1.6(1)) since it is strict, so by (5.1.6(2)), the morphism fgh satisfies the semi-universal support property. Then by (5.5.2) and (5.5.4), the morphism $gh : X \to X$ and $f : X \to S$ satisfy the condition of (5.3.6), so (loc. cit) implies that f satisfies the semi-universal support property. □

5.6 Proof of the support property

5.6.1. Throughout this section, we assume (Htp–4) and the axiom (ii) of (2.9.1) for $\mathscr{T}$.

Proposition 5.6.2. *Let S be an $\mathscr{S}$-scheme with a fs chart $\mathbb{N}$. Then the quotient morphism $f : S \to \underline{S}$ satisfies the support property.*

Proof. We have the factorization

$$S \xrightarrow{i} \underline{S} \times \mathbb{A}_M \xrightarrow{g} \underline{S} \times \mathbb{P}^1 \xrightarrow{p} \underline{S}$$

where

(i) i denotes the strict closed immersion induced by the chart $S \to \mathbb{A}_{\mathbb{N}}$,

(ii) p denotes the projection,

(iii) M denotes the fs monoscheme that is the gluing of $\operatorname{spec}\mathbb{N}$ and $\operatorname{spec}\mathbb{N}^{-1}$ along $\operatorname{spec}\mathbb{Z}$,

(iv) g denotes the morphism removing the log structure.

Then by (sSupp), i and p satisfies the support property. Hence by (5.1.2), the remaining is to show the support property for g. This question is Zariski local on $\underline{S} \times \mathbb{P}^1$, so we reduce to showing the support property for the morphism

$$\underline{S} \times \mathbb{A}_{\mathbb{N}} \to \underline{S} \times \mathbb{A}^1$$

removing the log structure. This is the axiom (ii) of (2.9.1). □

Proposition 5.6.3. *Let S be an $\mathscr{S}$-scheme with the trivial log structure. Then the semi-universal support property is satisfied for the projection $p : S \times \mathbb{A}_{\mathbb{N}} \to S$.*

Proof. Let q denote the morphism $S\times\mathbb{A}_{\mathbb{N}} \to S\times\mathbb{A}^1$ removing the log structure. By definition, we need to show that q satisfies the semi-universal support property. Any pullback of q via strict morphism is the quotient morphism $X \to \overline{X}$ for some $\mathscr{S}$-scheme X. Thus the conclusion follows from (5.6.2). □

Proposition 5.6.4. *Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes, and assume that S has the trivial log structure. Then f satisfies the semi-universal support property.*

Proof. By (5.1.6(3)) and (Htp–3), the question is dividing local on X. Hence by [CLS11, 11.1.9], we may assume that X has a fs chart $\mathbb{N}^r$. Then p has a factorization

$$X \xrightarrow{i} \underline{X} \times \mathbb{A}_{\mathbb{N}^r} \xrightarrow{\mathbb{A}_{\theta_r}} \underline{X} \times \mathbb{A}_{\mathbb{N}^{r-1}} \xrightarrow{\mathbb{A}_{\theta_{r-1}}} \cdots \to \underline{X} \times \mathbb{A}_{\mathbb{N}} \xrightarrow{q} \underline{X} \xrightarrow{f} S$$

where

(i) i denotes the morphism induced by the chart $S \to \mathbb{A}_{\mathbb{N}^r}$,

(ii) $\theta_s : \mathbb{N}^{s-1} \to \mathbb{N}^s$ denotes the homomorphism

$$(a_1, \ldots, a_{s-2}, a_{s-1}) \mapsto (a_1, \ldots, a_{s-2}, a_{s-1}, a_{s-1}),$$

(iii) q denotes the projection.

The morphism i and $\underline{f}$ satisfy the semi-universal support property by (5.1.6(1)) since they are strict, and the morphisms $\mathbb{A}_{\theta_s}$ for $s = 2, \ldots, r$ satisfy the universal support property by (5.4.9). Thus the conclusion follows from the (5.6.3) and (5.1.6(2)). □

Theorem 5.6.5. *The support property holds for $\mathscr{T}$.*

Proof. Let $f : X \to S$ be a proper morphism of $\mathscr{S}$-schemes. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \underline{X} & \xrightarrow{\underline{f}} & \underline{S} \end{array}$$

of $\mathscr{S}$-schemes where p and q denote the morphisms removing the log structures. Then p and q satisfy the semi-universal support property by (5.6.4), and $\underline{f}$ satisfies the semi-universal support property by (5.1.6(1)) since it is strict. Thus the composition $\underline{f}q$ satisfies the semi-support property by (5.1.6(2)), and then f satisfies the semi-universal support property by (5.3.2). □

Chapter 6

Homotopy and base change properties

6.0.1. Throughout this chapter, we fix a full subcategory $\mathscr{S}$ of the category of noetherian fs log schemes satisfying the conditions of (2.0.1), and we fix a log motivic triangulated category $\mathscr{T}$ over $\mathscr{S}$.

6.1 Homotopy property 5

6.1.1. Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes. In this section, we often consider the following conditions:

(i) the morphism $\underline{f} : \underline{X} \to \underline{S}$ of underlying schemes is an isomorphism,

(ii) the induced homomorphism $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\overline{x}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\overline{s}}$ is an isomorphism.

Proposition 6.1.2. *Consider the coCartesian diagram*

$$\begin{array}{ccc} P & \xrightarrow{\eta} & P' \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} \\ Q & \xrightarrow{\eta'} & Q' \end{array}$$

of sharp fs monoids such that

(i) θ'^{gp} is an isomorphism,

(ii) if F is a face of P' such that $F \cap \eta(P) = \langle 0 \rangle$, then $\theta'(F)$ is a face of Q',

(iii) if G is a face of Q' such that $G \cap \eta'(Q) = \langle 0 \rangle$, then $G = \theta'(F)$ for some face F of P'.

Then the induced morphism

$$f : \mathbb{A}_{(Q',(\eta'(Q^+)))} \to \mathbb{A}_{(P',(\lambda(P^+)))}$$

satisfies the conditions (i) and (ii) of (6.1.1).

Proof. By [Ogu14, I.3.2.3], $\mathbb{A}_{(P',(\lambda(P^+)))}$ has the stratification

$$\bigcup_F (\mathbb{A}_{F^*} \times \mathrm{pt}_{P'/F})$$

for face F of P' such that $F \cap \eta(P) = \langle 0 \rangle$. Similarly, $\mathbb{A}_{(Q',(\lambda'(Q^+)))}$ has the stratification

$$\bigcup_G (\mathbb{A}_{G^*} \times \mathrm{pt}_{Q'/G})$$

for face Q of Q' such that $G \cap \eta(Q) = \langle 0 \rangle$.

Thus by assumption, f is a union of the morphisms

$$f_F : \mathbb{A}_{\theta'(F)^*} \times \mathrm{pt}_{Q'/\theta'(F)} \to \mathbb{A}_{F^*} \times \mathrm{pt}_{P/F}.$$

This satisfies the condition (i) and (ii) of (6.1.1) because θ'^{gp} is an isomorphism. Then the conclusion follows from [EGA IV.18.12.6]. $\square$

Proposition 6.1.3. *Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes satisfying the conditions (i) and (ii) of (6.1.1). Then the unit*

$$\mathrm{id} \xrightarrow{ad} f_* f^*$$

is an isomorphism.

Proof. (I) *Locality of the question.* The question is strict étale local on S, so we may assume that S has a fs chart. Since $\underline{f}$ is an isomorphism, the question is also strict étale local on X, so we may assume that X has a fs chart.

Let $i : Z \to S$ be a closed immersion, and let $j : U \to S$ denote its complement. By (Loc) and (2.6.6), we reduce to the question for $X \times_S Z \to Z$ and $X \times_S U \to U$. Hence by the proof of [Ols03, 3.5(ii)], we reduce to the case when S has a constant log structure. Since $\underline{f}$ is an isomorphism, we can do the same method for X, and by [Ols03, 3.5(ii)], we reduce to the case when X has a constant log structure. Hence we reduce to the case when f is the morphism

$$\underline{S} \times \mathrm{pt}_Q \to \underline{S} \times \mathrm{pt}_P$$

induced by a homomorphism $\theta : P \to Q$ of sharp fs monoids. By assumption, θ^{gp} is an isomorphism.

(II) *Induction.* We will use an induction on $n = \dim P$. If $n = 1$, then we are done since $P = Q$, so we may assume $n > 1$.

(III) *Reduction to the case when $Q = (P + \langle a \rangle)^{\mathrm{sat}}$.* Choose generators $a_1, \ldots, a_m$ of Q. Then consider the homomorphisms

$$P \longrightarrow (P + \langle a_1 \rangle)^{\mathrm{sat}} \longrightarrow \cdots \longrightarrow (P + \langle a_1, \ldots, a_m \rangle)^{\mathrm{sat}}$$

of sharp fs monoids. If we show the question for each morphism

$$(P+\langle a_1,\dots,a_i\rangle)^{\mathrm{sat}} \longrightarrow (P+\langle a_1,\dots,a_{i+1}\rangle)^{\mathrm{sat}},$$

then we are done, so we reduce to the case when $Q=(P+\langle a\rangle)^{\mathrm{sat}}$ for some $a\in P^{\mathrm{gp}}$.

(IV) *Construction of fans.* We put

$$C=P_{\mathbb{Q}},\quad D=Q_{\mathbb{Q}},$$

and consider the dual cones $C^\vee$ and $D^\vee$. Choose a point v in the interior of $D^\vee$. We triangulate $D^\vee$, and then we triangulate $C^\vee$ such that the triangulations are compatible.

Let $\{\Delta_i\}_{i\in I}$ denote the set of $(n-1)$-simplexes of the triangulation $C^\vee$ contained in the boundary of $C^\vee$. We put

$$C_i^\vee=\Delta_i+\langle v\rangle,\quad D_i^\vee=C_i^\vee\cap D^\vee,$$

and we denote by C_i and D_i the dual cones of C_i and D_i respectively. Now we put

$$P_i=C_i\cup P^{\mathrm{gp}},\quad Q_i=D_i\cap P^{\mathrm{gp}},\quad H=(\langle v\rangle)^\perp,\quad r_i=\Delta_i^\perp.$$

Then r_i is a ray of C since Δ_i is an $(n-1)$-simplex, and H is an $(n-1)$-hyperplane such that $H\cap C=H\cap D=\langle 0\rangle$ since v is in the interior of $D^\vee$. For each $i\in I$, we have the following two cases: $C_i\neq D_i$ or $C_i=D_i$.

If $C_i\neq D_i$, then

$$C_i=\langle b_1,\dots,b_{n-1},r_i\rangle,\quad D_i=\langle b_1,\dots,b_{n-1},a\rangle$$

for some $b_1,\dots,b_{n-1}\in H$. Since $H\cap C=H\cap D=\langle 0\rangle$, if F (resp. G) is a face of C_i (resp. D_i), then

$$F\cap C=\langle 0\rangle \quad\Leftrightarrow\quad F\subset\langle b_1,\dots,b_{n-1}\rangle,$$
$$G\cap D=\langle 0\rangle \quad\Leftrightarrow\quad G\subset\langle b_1,\dots,b_{n-1}\rangle.$$

Thus the coCartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & P_i \\ \downarrow & & \downarrow \\ Q & \longrightarrow & Q_i \end{array}$$

satisfies the condition of (6.1.2), so by (loc. cit), the induced morphism

$$S\times\mathbb{A}_{Q_i}\times_{\mathbb{A}_Q}\mathrm{pt}_Q\to S\times\mathbb{A}_{P_i}\times_{\mathbb{A}_P}\mathrm{pt}_P \qquad (6.1.3.1)$$

satisfies the conditions (i) and (ii) of (6.1.1).

If $C_i=D_i$, then $C_i=D_i=\langle b_1,\dots,b_{n-1},r_i\rangle$ for some $b_1,\dots,b_{n-1}\in H$, and we can similarly show that (6.1.3.1) satisfies the conditions (i) and (ii) of (6.1.1).

Let M (resp. N) denote the fs monoscheme that is a gluing of $\operatorname{spec} P_i$ (resp. $\operatorname{spec} Q_i$) for $i \in I$. Then consider the induced commutative diagram

$$\begin{array}{ccc} \underline{S} \times \mathbb{A}_N \times_{\mathbb{A}_Q} \mathrm{pt}_Q & \xrightarrow{f'} & \underline{S} \times \mathbb{A}_M \times_{\mathbb{A}_P} \mathrm{pt}_P \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \underline{S} \times \mathrm{pt}_Q & \xrightarrow{f} & \underline{S} \times \mathrm{pt}_P \end{array}$$

of $\mathscr{S}$-schemes. We have shown that f' satisfies the conditions (i) and (ii) of (loc. cit).

Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{id} & & \xrightarrow{ad} & & f_* f^* \\ \downarrow{\scriptstyle ad} & & & & \downarrow{\scriptstyle ad} \\ g_* g^* & \xrightarrow{ad} & g_* f'_* f'^* g^* & \xrightarrow{\sim} & f_* g'_* g'^* f^* \end{array}$$

of functors. The vertical arrows are isomorphisms by (Htp–4) since g and g' are dividing covers. Thus the question for f reduces to the question for f'.

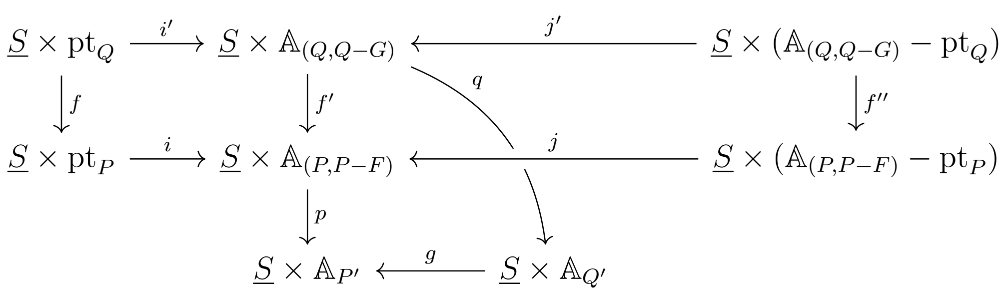
Then using Mayer-Vietoris triangle, by induction on $\dim P$, we reduce to the questions for $(P, Q) = (P_i, Q_i)$ for $i \in I$. In particular, we may assume

$$(P_i)_{\mathbb{Q}} = \langle b_1, \ldots, b_{r-1}, b_r \rangle, \quad (Q_i)_{\mathbb{Q}} = \langle b_1, \ldots, b_{r-1}, a \rangle.$$

(V) *Final step of the proof.* We put

$$F = \langle b_1 \rangle \cap P, \quad G = \langle b_1 \rangle \cap Q, \quad P' = \langle b_2, \ldots, b_r \rangle \cap P, \quad Q' = \langle b_2, \ldots, b_{r-1}, a \rangle \cap Q.$$

Consider the commutative diagram



where

(i) g denotes the morphism induced by the homomorphism $P' \to Q$ induced by θ

(ii) p denotes the morphism induced by the inclusion $P' \to P$,

(iii) j denotes the complement of i,

(iv) each square is Cartesian.

Then j (resp. j') is the verticalization of $\underline{S} \times \mathbb{A}_{(P,P-F)}$ (resp. $\underline{S} \times \mathbb{A}_{(Q,Q-G)}$) via p (resp. $f'p$). Thus by (Htp–2), the natural transformations

$$p^* \xrightarrow{ad} j_*j^*p^*, \quad q^* \xrightarrow{ad} j'_*j'^*q^*$$

are isomorphisms. From the commutative diagram

$$\begin{array}{ccccc}
p^* & & \xrightarrow{\quad ad \quad} & & f'_*f'^*p^* \\
{\scriptstyle\sim}\downarrow{\scriptstyle ad} & & & & \downarrow{\scriptstyle\sim} \\
j_*j^*p^* & \underset{\sim}{\xrightarrow{ad}} & j_*f''_*f''^*j^*p^* & \xrightarrow{\sim} & p'_*j'_*j'^*f'^*p^*
\end{array}$$

of functors, we see that the upper horizontal arrow is an isomorphism.

In the commutative diagram

$$\begin{array}{ccccc}
i^*p^* & \underset{\sim}{\xrightarrow{ad}} & i^*f'_*f'^*p^* & \xrightarrow{Ex} & f_*i'^*f'^*p^* \\
 & \searrow^{ad} & & & \downarrow{\scriptstyle\sim} \\
 & & & & f_*f^*i^*p^*
\end{array}$$

of functors, the upper right horizontal arrow is an isomorphism by (2.6.6). Thus the diagonal arrow is an isomorphism. In particular, the morpism

$$1_S \xrightarrow{ad} f_*f^*1_S$$

in $\mathscr{T}_S$ is an isomorphism.

For any object K of $\mathscr{T}_S$, we have the commutative diagram

$$\begin{array}{ccccc}
K & \xrightarrow{\sim} & 1_S \otimes K & \underset{\sim}{\xrightarrow{ad}} & f_*f^*1_S \otimes K \\
 & \searrow^{ad} & & & \downarrow{\scriptstyle Ex} \\
 & & & & f_*f^*K
\end{array}$$

in $\mathscr{T}_S$. By (PF), the right vertical arrow is an isomorphism. Thus the diagonal arrow is also an isomorphism, which completes the proof. □

6.1.4. So far, we have proven the half of (Htp–5). In the remaining, we will first prove a few lemmas. Then we will prove (Htp–5).

Lemma 6.1.5. *Let $\theta : P \to Q$ be a homomorphism of fs sharp monoids such that $\theta^{\rm gp} : P^{\rm gp} \to Q^{\rm gp}$ is an isomorphism, and let $\eta' : Q \to Q'$ be a homomorphism of fs monoids such that $\overline{\eta}' : Q \to \overline{Q'}$ is Kummer. Then there is a coCartesian diagram*

$$\begin{array}{ccc}
P & \xrightarrow{\theta} & Q \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta'} \\
P' & \longrightarrow & Q'
\end{array}$$

of fs monoids.

Proof. Let P' denote the submonoid of Q' consisting of elements $p' \in Q'$ such that $np' \in P + Q'^*$ for some $n \in \mathbb{N}^+$. Since $\overline{\eta}'$ is Kummer, by the construction of pushout in the category of fs monoids, it suffices to verify $P'^{\mathrm{gp}} = Q'^{\mathrm{gp}}$ to show that the above diagram is coCartesian.

Let $q' \in Q'$ be an element. Since $\overline{\eta}'$ is Kummer, we can choose $m \in \mathbb{N}^+$ such that $mq' = q + q''$ for some $q \in Q$ and $q'' \in Q'^*$. We put $r = \dim P$, and choose r linearly independent elements $p_1, \ldots, p_r \in P$ over $\mathbb{Q}$. Then let $(a_1, \ldots, a_r)$ denote the coordinate of $q \in Q$ according to the basis $\{p_1, \ldots, p_r\}$.

Choose $b_1, \ldots, b_r \in \mathbb{N}^+$ such that $a_1 + mb_1, \ldots, a_r + mb_r > 0$, and we put $p = (b_1, \ldots, b_r) \in P$. Then

$$q + mp = (a_1 + mb_1, \ldots, a_r + mb_r) \in P,$$

so $m(q' + p) = q + mp + q'' \in P + Q'^*$. Thus $q' + p \in P'$, so $q' \in P'^{\mathrm{gp}}$. This proves $P'^{\mathrm{gp}} = Q'^{\mathrm{gp}}$. □

Lemma 6.1.6. *Let P be a sharp fs monoid, and let $\eta : P \to P'$ be a homomorhpism of fs monoids such that $\overline{\eta} : P \to \overline{P'}$ is Kummer. We denote by I the ideal of P' generated by $\eta(P^+)$. Then the induced morphism*

$$\mathbb{A}_{(P', P'^+)} \to \mathbb{A}_{(P', I)}$$

is a bijective strict closed immersion.

Proof. For any element $p' \in P'^+$, for some $n \in \mathbb{N}^+$, we have $np' \in I$ since $\overline{\eta}$ is Kummer. Let $\mathfrak{m}$ (resp. $\mathfrak{n}$) denote the ideal of $\mathbb{Z}[P']$ induced by P'^+ (resp. I). Then by the above argument, for some $m \in \mathbb{N}^+$, we have $\mathfrak{n}^m \subset \mathfrak{m}$. This implies the assertion. □

6.1.7. Let $\theta : P \to Q$ be a homomorphism of sharp fs monoids such that θ^{gp} is an isomorphism, and consider a coCartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{\theta} & Q \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ P' & \xrightarrow{\theta'} & Q' \end{array}$$

of fs monoids where $\overline{\eta} : P \to \overline{P'}$ is Kummer. Note that η'^{gp} is also an isomorphism by the construction of the pushout in the category of fs monoids. Then we have the induced commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{(Q', Q'^+)} & \longrightarrow & \mathbb{A}_{(P', P'^+)} \\ \downarrow & & \downarrow \\ \mathbb{A}_{(Q', J)} & \longrightarrow & \mathbb{A}_{(P', I)} \end{array}$$

of schemes where I (resp. J) denote the ideal of P' (resp. Q') generated by $\eta(P)$ (resp. $\eta'(Q)$). By (6.1.6), the vertical arrows are bijective strict closed immersions, and the upper horizontal arrow is an isomorphism since they are isomorphic to $\mathbb{A}_{Q'^{\mathrm{gp}}} = \mathbb{A}_{P'^{\mathrm{gp}}}$. Thus the category of strict étale morphisms to $\mathbb{A}_{(Q,J)}$ is equivalent to that of $\mathbb{A}_{(P,I)}$ by [EGA, IV.18.1.2].

Proposition 6.1.8. *Let $\theta : P \to Q$ be a homomorphism of sharp fs monoids such that $\theta^{\rm gp}$ is an isomorphism, and let S be an $\mathscr{S}$-scheme with the trivial log structure. Consider the induced morphism $f : S \times {\rm pt}_Q \to S \times {\rm pt}_P$ of $\mathscr{S}$-schemes. Then the functor f^* is essentially surjective.*

Proof. By (3.6.4), it suffices to show that for any Kummer log smooth morphism $g' : Y' \to S \times {\rm pt}_Q$ with a fs chart $\eta' : Q \to Q'$ of Kummer log smooth type, there is a Cartesian diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ S \times {\rm pt}_Q & \xrightarrow{f} & S \times {\rm pt}_P \end{array}$$

of $\mathscr{S}$-schemes such that g is Kummer log smooth. Note that $\overline{\eta'} : Q \to \overline{Q'}$ is Kummer.

By definition (3.1.1), we can choose a factorization

$$Y' \to S \times {\rm pt}_Q \times_{\mathbb{A}_Q} \mathbb{A}_{Q'} \to S$$

of g' where the first arrow is strict étale and the second arrow is the projection. Then by (6.1.5), there is a coCartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{\theta} & Q \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ P' & \xrightarrow{\theta'} & Q' \end{array}$$

of fs monoids such that $\overline{\eta} : P \to \overline{P'}$ is Kummer. Now we have the commutative diagram

$$\begin{array}{ccc} Y' & & \\ \downarrow & & \\ S \times {\rm pt}_Q \times_{\mathbb{A}_Q} \mathbb{A}_{Q'} & \longrightarrow & S \times {\rm pt}_P \times_{\mathbb{A}_P} \mathbb{A}_{P'} \\ \downarrow & & \downarrow \\ S \times {\rm pt}_Q & \longrightarrow & S \times {\rm pt}_P \end{array}$$

of $\mathscr{S}$-schemes where the square is Cartesian. Since

$${\rm pt}_Q \times_{\mathbb{A}_Q} \mathbb{A}_{Q'} = \mathbb{A}_{(Q,J)}, \quad {\rm pt}_P \times_{\mathbb{A}_P} \mathbb{A}_{P'} = \mathbb{A}_{(P,I)}$$

where I (resp. J) denotes the ideal of P' (resp. Q') generated by $\eta(P)$ (resp. $\eta'(Q)$), by (6.1.7), there is a commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S \times {\rm pt}_Q \times_{\mathbb{A}_Q} \mathbb{A}_{Q'} & \longrightarrow & S \times {\rm pt}_P \times_{\mathbb{A}_P} \mathbb{A}_{P'} \\ \downarrow & & \downarrow \\ S \times {\rm pt}_Q & \longrightarrow & S \times {\rm pt}_P \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian and the arrow $Y \to S \times \mathrm{pt}_P \times_{\mathbb{A}_P} \mathbb{A}_{P'}$ is strict étale. Thus we have constructed the diagram we want. □

Theorem 6.1.9. *The log motivic category $\mathscr{T}$ satisfies (Htp–5).*

Proof. Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes satisfying the conditions (i) and (ii) of (6.1.1). By (6.1.3), it suffices to show that the counit

$$f^*f_* \xrightarrow{ad'} \mathrm{id}$$

is an isomorphism.

Let $\{g_i : S_i \to S\}_{i\in I}$ be a strict étale cover. Consider the Cartesian diagram

$$\begin{CD} X_i @>{g_i'}>> X \\ @V{f_i}VV @VV{f}V \\ S_i @>{g_i}>> S \end{CD}$$

of $\mathscr{S}$-schemes. Then we have the commutative diagram

$$\begin{array}{ccccc} g_i'^*f^*f_* & \xrightarrow{\sim} & f_i^*g_i^*f_* & \xrightarrow{Ex} & f_i^*f_{i*}g_i'^* \\ & \searrow^{ad'} & & & \downarrow{\scriptstyle ad'} \\ & & & & g_i'^* \end{array}$$

of functors, and the upper right horizontal arrow is an isomorphism by (eSm-BC). Since the family of functors $\{g_i'^*\}_{i\in I}$ is conservative by (két-Sep), we reduce to showing that for any $i \in I$, the counit $f_i^*f_{i*} \xrightarrow{ad'} \mathrm{id}$ is an isomorphism. Using this technique, we reduce to the case when f has a fs chart $\theta : P \to Q$.

Then let $i : S' \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, and let $j : S'' \to S$ denote its complement. Consider the commutative diagram

$$\begin{CD} X' @>{i'}>> X @<{j'}<< X'' \\ @V{f'}VV @V{f}VV @VV{f''}V \\ S' @>{i}>> S @<{j}<< S'' \end{CD}$$

of $\mathscr{S}$-schemes where each square is Cartesian. Then we have the commutative diagrams

$$\begin{array}{ccccc} i'^*f^*f_* & \xrightarrow{\sim} & f'^*i^*f_* & \xrightarrow{Ex} & f'^*f'_*i'^* \\ & \searrow^{ad'} & & & \downarrow{\scriptstyle ad'} \\ & & & & i'^* \end{array} \qquad \begin{array}{ccccc} j'^*f^*f_* & \xrightarrow{\sim} & f''^*j^*f_* & \xrightarrow{Ex} & f''^*f''_*j'^* \\ & \searrow^{ad'} & & & \downarrow{\scriptstyle ad'} \\ & & & & j^* \end{array}$$

of functors. As in the above paragraph, by (2.6.6) and (Loc), we reduce to showing that the counits

$$f'^*f'_* \xrightarrow{ad} \mathrm{id}, \quad f''^*f''_* \xrightarrow{ad} \mathrm{id}$$

are isomorphisms. Using this technique, by the proof of [Ols03, 3.5(ii)], we reduce to the case when $X \to S$ is the morphism $\underline{S} \times \mathrm{pt}_Q \to \underline{S} \times \mathrm{pt}_P$ induced by the homomorphism $P \to Q$. In this case, the conclusion follows from (6.1.8) since f^* is fully faithful by (6.1.3). □

6.2 Homotopy property 6

Theorem 6.2.1. *The log motivic category $\mathscr{T}$ satisfies (Htp–6).*

Proof. Let S be an $\mathscr{S}$-scheme, and we put $X = S \times \mathbb{A}_{\mathbb{N}}$ and $Y = S \times \mathrm{pt}_{\mathbb{N}}$. Consider the commutative diagram

$$\begin{array}{ccccc} S\times \mathrm{pt}_{\mathbb{N}} & \xrightarrow{i'} & S\times\mathbb{A}_{\mathbb{N}} & \xleftarrow{j'} & S\times\mathbb{G}_m \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \mathrm{id}} \\ S & \xrightarrow{i} & S\times \mathbb{A}^1 & \xleftarrow{j} & S\times\mathbb{G}_m \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle f} & & \\ & & S & & \end{array}$$

of $\mathscr{S}$-schemes where

(i) each small square is Cartesian,

(ii) f denotes the projection, and f' denotes the morphism removing the log structure.

(iii) i denotes the 0-section, and j denotes its complement.

We want to show that the natural transformation

$$f_*f'_*f'^*f^* \xrightarrow{ad} f_*f'_*i'_*i'^*f'^*f^* \xrightarrow{\sim} g'_*g'^*$$

is an isomorphism. By (Loc), it is equivalent to showing

$$f_*f'_*j'_\sharp j'^*f'^*f^* = 0.$$

Then by (Supp), it is equivalent to showing

$$f_*j_\sharp j^*f^* = 0.$$

Thus by (Loc), it is equivalent to showing that the natural transformation

$$f_*f^* \xrightarrow{ad} f_*i_*i^*f^*$$

is an isomorphism, which follows from (Htp–1). □

6.2.2. Here, we give an application of (Htp–6). Let S be an $\mathscr{S}$-scheme. Consider the commutative diagram

$$\begin{array}{ccc} S \times \mathrm{pt}_{\mathbb{N}} & \xrightarrow{i_0} & S \times \mathbb{A}_{\mathbb{N}} \\ & \searrow^{g} & \downarrow f \\ & & S \end{array}$$

of $\mathscr{S}$-schemes where f denotes the projection and i_0 denotes the 0-section. Let $i_1 : S \to S \times \mathbb{A}_{\mathbb{N}}$ denote the 1-section. By (Htp–6), the natural transformation

$$f_* f^* \xrightarrow{ad} f_* i_* i^* f^*$$

is an isomorphism, and f^* is conservative since $f i_1 = \mathrm{id}$. Thus by (5.2.1), $(fi)^* = g^*$ is conservative.

6.3 Homotopy property 7

Theorem 6.3.1. *Let S be an $\mathscr{S}$-scheme with a fs chart P, let $\theta : P \to Q$ be a vertical homomorphism of exact log smooth over S type, and let G be a θ-critical face of Q. We denote by*

$$f : S \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_G}) \to S$$

the projection. Then $f_! f^* = 0$. In other words, $\mathscr{T}$ satisfies (Htp–7).

Proof. (I) *Locality of the question.* Note first that the question is strict étale local on S by (eSm-BC).

Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, and let $j : U \to S$ denote its complement. Consider the commutative diagram

$$\begin{array}{ccccc} Z \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_G}) & \xrightarrow{i'} & S \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_G}) & \xleftarrow{j'} & U \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_G}) \\ \downarrow g & & \downarrow f & & \downarrow h \\ Z & \xrightarrow{i} & S & \xleftarrow{j} & U \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. Then by (BC–3), we reduce to showing $g_! g^* = 0$ and $h_! h^* = 0$. By the proof of [Ols03, 3.5(ii)], we reduce to the case when S has a constant log structure.

(II) *Reduction of G.* Let G_1 be a maximal θ-critical face of Q containing G. Consider the induced commutative diagram

$$\begin{array}{ccccc} S \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_G}) & \longrightarrow & S \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_{G_1}}) & \longleftarrow & S \times_{\mathbb{A}_P} (\mathbb{A}_{Q_G} - \mathbb{A}_{Q_{G_1}}) \\ & \searrow^{f} & \downarrow f' & \swarrow^{f''} & \\ & & S & & \end{array}$$

of $\mathscr{S}$-schemes. By (Loc), to show $f_! f^* = 0$, it suffices to show $f'_! f'^* = 0$ and $f''_! f''^* = 0$. Hence replacing f by f' or f'', we reduce to the case when G is a maximal θ-critical face of Q.

(III) *Reduction of P.* We denote by P' the submonoid of Q consisting of elements $q \in Q$ such that $nq \in P + Q^*$ for some $n \in \mathbb{N}^+$. The induced homomorphism $\theta' : P' \to Q$ is again a vertical homomorphism of exact log smooth over $S \times_{\mathbb{A}_P} \mathbb{A}_{P'}$ type. Replacing S by $S \times_{\mathbb{A}_P} \mathbb{A}_{P'}$, we reduce to the case when the cokernel of θ^{gp} is torsion free and θ is logarithmic.

Then by (3.2.2), since the question is strict étale local on S, we may assume that P is sharp and that the fs chart $S \to \mathbb{A}_P$ is neat at some point $s \in S$. Then P and Q are sharp, and with (I), we may further assume that the fs chart $S \to \mathbb{A}_P$ factors through pt_P.

(IV) *Homotopy limit.* Let $G_1 = G, \ldots, G_r$ denote the maximal θ-critical faces of Q. The condition that θ is vertical implies $r \geq 2$. For any nonempty subset $I = \{i_1, \ldots, i_s\} \subset \{2, \ldots, r\}$, we put

$$G_I = G_{i_1} \cap \cdots \cap G_{i_s},$$

and we denote by

$$f_I : S \times_{\mathbb{A}_P} \mathbb{A}_{(Q,Q-G_I)} \to S$$

the projection. For any face H of Q, $\mathbb{A}_H \subset \mathbb{A}_Q - \mathbb{A}_{Q_H}$ if and only if $H \neq Q, G$, which is equivalent to $H \subset G_2 \cup \cdots \cup G_r$. Thus the family

$$\{S \times_{\mathbb{A}_P} \mathbb{A}_{(Q,Q-G_2)}, \ldots, S \times_{\mathbb{A}_P} \mathbb{A}_{(Q,Q-G_r)}\}$$

forms a closed cover of $S \times_{\mathbb{A}_P} (\mathbb{A}_Q - \mathbb{A}_{Q_G})$, so for any object K of $\mathscr{T}(S)$, $f_* f^* K$ is the homotopy limit of the Čech-type sequence

$$\bigoplus_{|I|=1, |I| \subset \{2,\ldots,r\}} f_{I*} f_I^* K \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=r-1, |I| \subset \{2,\ldots,r\}} f_{I*} f_I^* K$$

in $\mathscr{T}(S)$. Hence we reduce to showing $f_{I*} f_I^* K = 0$ for any nonempty subset $I \subset \{2, \ldots, r\}$. This is proved in (6.3.1) below. □

Lemma 6.3.2. *Let S be an $\mathscr{S}$-scheme with a constant log structure $S \to \mathrm{pt}_P$ where P is a sharp fs monoid, let $\theta : P \to Q$ be a homomorphism of exact log smooth over S type, and let G be a θ-critical face of Q such that $G \neq Q^*$. We denote by*

$$f : S \times_{\mathbb{A}_P} \mathbb{A}_{(Q,Q-G)} \to S$$

the projection. Then $f_! f^* = 0$.

Proof. We will use induction on $\dim Q$. We have $\dim Q \geq 1$ since $G \neq Q^*$.

(I) *Locality of the question.* Note that by (eSm-BC), the question is strict étale local on S.

(II) *Reduction of G.* Let G' be a 1-dimensional face of G, and choose generators $b_1, \ldots, b_r$ of the ideal $Q - G'$ in Q. For any nonempty subset $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$, we denote by Q_I the localization $Q_{b_{i_1},\dots,b_{i_s}}$, and we denote by G_I the face of Q_I generated by G. The family

$$\{S\times_{\mathbb{A}_P}\mathbb{A}_{(Q_{b_1},Q_{b_1}-G_{b_1})},\dots,S\times_{\mathbb{A}_P}\mathbb{A}_{(Q_{b_s},Q_{b_s}-G_{b_s})}\}$$

forms an open cover of $S\times_{\mathbb{A}_P}(\mathbb{A}_{(Q,Q-G)}-\mathbb{A}_{(Q,Q-G')})$. Thus if we denote by

$$f_I : S\times_{\mathbb{A}_P}\mathbb{A}_{(Q_I,Q_I-G_I)}\to S,$$

$$f' : S\times_{\mathbb{A}_P}\mathbb{A}_{(Q,Q-G')}\to S,\quad f'' : S\times_{\mathbb{A}_P}(\mathbb{A}_{(Q,Q-G)}-\mathbb{A}_{(Q,Q-G')})\to S$$

the projections, then for any object K of $\mathscr{T}(S)$, $f''_! f''^* K$ is a homotopy colimit of

$$\bigoplus_{|I|=r} f_{I*}f_I^*K\longrightarrow\cdots\longrightarrow\bigoplus_{|I|=1} f_{I*}f_I^*K$$

in $\mathscr{T}(S)$. Since $\dim Q_I<\dim Q$, for any nonempty subset $I\subset\{1,\dots,r\}$, $f_{I!}f_I^*=0$ by induction on $\dim Q$. Thus $f''_! f''^! = 0$. Then by (Loc), $f_!f^*=0$ is equivalent to $f'_!f'^*=0$. Hence replacing G by G', we reduce to the case when $\dim G=1$.

(III) *Reduction of* θ. Let a_1 be a generator of G, let H be a maximal θ-critical face of Q containing G, and choose $a_2,\dots,a_d\in H$ where $d=\dim G$ such that $\overline{a_1},\dots,\overline{a_d}$ in $\overline{Q}$ are independent over $\mathbb{Q}$.

We denote by P' (resp. Q') the submonoid of Q consisting of elements $q\in Q$ such that $nq\in\langle a_2,\dots,a_n\rangle+P$ (resp. $nq\in\langle a_1,\dots,a_n\rangle+P$) for some $n\in\mathbb{N}^+$. Then we denote by G' the face of Q' generated by a_1, and we denote by $\theta' : P'\to Q'$ the induced homomorphism. Consider the induced morphisms

$$S\times_{\mathbb{A}_P}\mathbb{A}_{(Q,Q-G)}\xrightarrow{w}S\times_{\mathbb{A}_P}\mathbb{A}_{(Q',Q'-G')}\xrightarrow{v}S\times_{\mathbb{A}_P}\mathbb{A}_{(P',P')}\xrightarrow{u}S$$

of $\mathscr{S}$-schemes. The induced homomorphism $Q'^{\rm gp}\to Q^{\rm gp}$ is an isomorphism by [Ogu14, 4.6.6.4]. Thus by (6.1.3), the unit $\mathrm{id}\xrightarrow{ad}w_*w^*$ is an isomorphism. Hence to show $f_!f^*=0$, it suffices to show $v_!v^*=0$.

The cokernel of $\theta'^{\rm gp}$ is torsion free, and the diagram

$$\begin{array}{ccc} P' & \longrightarrow & \overline{P'} \\ \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\overline{\theta'}} \\ Q' & \longrightarrow & \overline{Q'} \end{array}$$

is coCartesian where the horizontal arrows are the quotient homomorphisms. Thus we can apply (3.2.2), so strict étale locally on S, we have a Cartesian diagram

$$\begin{array}{ccc} S\times_{\mathbb{A}_P}\mathbb{A}_{(Q',Q'-G')} & \longrightarrow & \mathbb{A}_{(\overline{Q'},\overline{Q'}-\overline{G'})} \\ \downarrow & & \downarrow \\ S\times_{\mathbb{A}_P}\mathbb{A}_{(P',P')} & \longrightarrow & \mathbb{A}_{\overline{P'}} \end{array}$$

of $\mathscr{S}$-schemes. The homomorphism $\overline{\theta'}:\overline{P'}\to\overline{Q'}$ is again a homomorphism of exact log smooth over $S\times_{\mathbb{A}_P}\mathbb{A}_{(P',P')}$ type. Replacing $(S\times_{\mathbb{A}_P}\mathbb{A}_{(Q,Q-G)}\to S,\ \theta:P\to Q)$ by

$$(S\times_{\mathbb{A}_P}\mathbb{A}_{(Q',Q'-G')}\to S\times_{\mathbb{A}_P}\mathbb{A}_{(P',P')},\ \overline{\theta'}:\overline{P'}\to\overline{Q'}),$$

we may assume that

(i) P and Q are sharp,

(ii) $Q_{\mathbb{Q}}=P_{\mathbb{Q}}\oplus G_{\mathbb{Q}}$,

(iii) $\dim G=1$.

(IV) *Further reduction of θ.* Since $P_{\mathbb{Q}}$ and $G_{\mathbb{Q}}$ generate $Q_{\mathbb{Q}}$, as in (5.2.2), we can choose a homomorphism $P\to P_1$ of Kummer log smooth over S type such that

1. P_1 and G generate $P_1\oplus_P Q$,
2. the functor g^* is conservative where $g:S\times_{\mathbb{A}_P}\mathbb{A}_{P_1}\to S$ denotes the projection.

We put $Q_1=P_1\oplus_P Q$, and we denote by G_1 the face of Q_1 generated by G. Consider the Cartesian diagram

$$\begin{array}{ccc} S\times_{\mathbb{A}_P}\mathbb{A}_{(Q_1,Q_1-G_1)} & \xrightarrow{f'} & S\times_{\mathbb{A}_P}\mathbb{A}_{P_1} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ S\times_{\mathbb{A}_P}\mathbb{A}_{(Q,Q-G)} & \xrightarrow{f} & S \end{array}$$

of $\mathscr{S}$-schemes. Since g^* is conservative, to show $f_!f^*=0$, it suffices to show $f'_!f'^*=0$ by (eSm-BC).

By (3.2.2), strict étale locally on $S\times_{\mathbb{A}_P}\mathbb{A}_{P'}$, there is a Cartesian diagram

$$\begin{array}{ccc} S\times_{\mathbb{A}_P}\mathbb{A}_{(Q_1,Q_1-G_1)} & \longrightarrow & \mathbb{A}_{(\overline{Q_1},\overline{Q_1}-\overline{G_1})} \\ \downarrow{\scriptstyle f'} & & \downarrow \\ S\times_{\mathbb{A}_P}\mathbb{A}_{P_1} & \longrightarrow & \mathbb{A}_{\overline{P_1}} \end{array}$$

of $\mathscr{S}$-schemes. Replacing $(S\times_{\mathbb{A}_P}\mathbb{A}_{(Q,Q-G)}\to S,\ \theta:P\to Q)$ by

$$(S\times_{\mathbb{A}_P}\mathbb{A}_{(Q_1,Q_1-G_1)}\to S\times_{\mathbb{A}_P}\mathbb{A}_{P_1},\ \overline{P_1}\to\overline{Q_1}),$$

we may assume that

(i) P and Q are sharp,

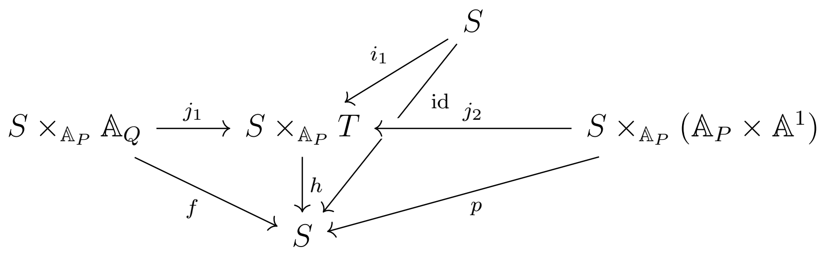
(ii) $Q=P\oplus G$,

(iii) $\dim G=1$.

(V) *Final step of the proof.* Then $S\times_{\mathbb{A}_P}\mathbb{A}_{(Q,Q-G)}=S\times_{\mathbb{A}_P}\mathbb{A}_Q$. Let a_1 denote the generator of G. We denote by T the gluing of

$$\mathrm{Spec}(Q\to\mathbb{Z}[Q]),\quad \mathrm{Spec}(P\to\mathbb{Z}[P,a_1^{-1}])$$

along $\operatorname{Spec}(P \to \mathbb{Z}[P, a_1^{\pm}])$. Consider the commutative diagram



of $\mathscr{S}$-schemes where

(i) j_1 and j_2 denote the induced open immersions,

(ii) h and p denote the projections,

(iii) i_1 is a complement of j_1.

Then h is exact log smooth, and j_2 is the verticalization of $S \times_{\mathbb{A}_P} T$ via h. Thus by (Htp–2), the natural transformation

$$h^* \xrightarrow{ad} j_{2*}j_2^*h^*$$

is an isomorphism. By (Htp–1), the composition

$$\mathrm{id} \longrightarrow p_*p^* \xrightarrow{\sim} h_*j_{2*}j_2^*h^*$$

is an isomorphism, so the unit $\mathrm{id} \xrightarrow{ad} h_*h^*$ is an isomorphism. Then by (Loc), for any object K of $\mathscr{T}(S)$, we have the distinguished triangle

$$h_*j_{1\sharp}j_1^*h^*K \xrightarrow{ad'} h_*h^*K \xrightarrow{ad} h_*i_{1*}i_1^*h^*K \longrightarrow h_*j_{1\sharp}j_1^*h^*K[1]$$

in $\mathscr{T}(S)$. Since $i_1h = \mathrm{id}$, the second arrow is the inverse of the unit $\mathrm{id} \xrightarrow{ad} h_*h^*$, which is an isomorphism. Thus $f_!f^* \cong h_*j_{1\sharp}j_1^*h^* = 0$. □

6.4 Base change property 2

Proposition 6.4.1. *Let T be an $\mathscr{S}$-scheme, and consider the commutative diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & \longrightarrow & T \times \mathbb{A}_{\mathbb{N}\oplus\mathbb{N}} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{g} & S & \longrightarrow & T \times \mathbb{A}_{\mathbb{N}} \end{array}$$

of $\mathscr{S}$-schemes where

(i) each square is Cartesian,

(ii) p denotes the morphism induced by the diagonal homomorphism $\mathbb{N} \to \mathbb{N} \oplus \mathbb{N}$ of fs monoids.

Then $(\mathrm{BC}_{f,g})$ *is satisfied.*

Proof. By (5.4.7), the purity transformations

$$f_\sharp \xrightarrow{\mathfrak{p}^o_f} f_!(1)[2], \quad f'_\sharp \xrightarrow{\mathfrak{p}^o_{f'}} f'_!(1)[2]$$

are isomorphisms. Thus to show that the exchange transformation $g^* f_! \xrightarrow{Ex} f'_! g'^*$ is an isomorphism, it suffices to show that the exchange transformation

$$f'_\sharp g'^* \xrightarrow{Ex} g^* f_\sharp$$

is an isomorphism. This follows from (eSm-BC). □

Proposition 6.4.2. *Let $f : X \to S$ be a separated morphism of $\mathscr{S}$-schemes, and let $g : S \times \mathrm{pt}_{\mathbb{N}} \to S$ denote the projection. Then* $(\mathrm{BC}_{f,g})$ *is satisfied.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
X'' & \xrightarrow{\quad i' \quad} & X' & \xrightarrow{\quad p' \quad} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f} \\
S \times \mathrm{pt}_{\mathbb{N}} & \xrightarrow{\quad i \quad} & S \times \mathbb{A}_{\mathbb{N}} & \xrightarrow{\quad p \quad} & S
\end{array}$$

of $\mathscr{S}$-schemes where

(i) each square is Cartesian,

(ii) i denotes the 0-section, and p denotes the projection.

Then $(\mathrm{BC}_{f,p})$ is satisfied by (eSm-BC), and $(\mathrm{BC}_{f'',i})$ is satisfied by (BC–3). These two imply $(\mathrm{BC}_{f,g})$. □

Proposition 6.4.3. *Let T be an $\mathscr{S}$-scheme, and let $\theta : P \to Q$ be a locally exact homomorphism of sharp fs monoids. We put $X = T \times \mathrm{pt}_Q$ and $S = T \times \mathrm{pt}_P$, and consider the morphism $f : X \to S$ induced by θ. Then* $(\mathrm{BC}_{f,g})$ *is satisfied for any morphism $g : S' \to S$ of $\mathscr{S}$-schemes.*

Proof. (I) *Reduction method 1.* Assume that we have a factorization

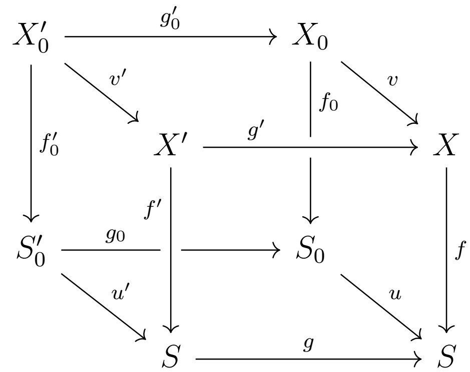
$$P \xrightarrow{\theta'} Q' \xrightarrow{\theta''} Q$$

of θ where θ' is locally exact, θ''^{gp} is an isomorphism, and θ' is a sharp fs monoid. Consider the morphism

$$f' : T \times \mathrm{pt}_{Q'} \to T \times \mathrm{pt}_P$$

induced by θ'. Then by (Htp–5), ($\mathrm{BC}_{f,g}$) is equivalent to ($\mathrm{BC}_{f',g}$).

(II) *Reduction method 2.* Let $u : S_0 \to S$ be a morphism of $\mathscr{S}$-schemes, and consider a commutative diagram



of $\mathscr{S}$-schemes where each small square is Cartesian. Then we have the commutative diagram

$$\begin{array}{ccccc}
u'^*g^*f_* & \xrightarrow{Ex} & u'^*f'_*g'^* & \xrightarrow{Ex} & f'_{0*}v'^*g'^* \\
\downarrow\wr & & & & \downarrow\wr \\
g_0^*u^*f_* & \xrightarrow{Ex} & g_0^*f_{0*}v^* & \xrightarrow{Ex} & f'_{0*}g_0'^*v^*
\end{array}$$

of functors. Assume that u'^* is conservative. If ($\mathrm{BC}_{f,u}$), ($\mathrm{BC}_{f',u'}$), and (BC_{f_0,g_0}) are satisfied, then the lower left horizontal, upper right horizontal, and lower right horizontal arrows are isomorphisms. Thus the upper left horizontal arrow is also an isomorphism. Then ($\mathrm{BC}_{f,g}$) is satisfied since u'^* is conservative.

We will apply this technique to the following two cases.

(a) When u is an exact log smooth morphism such that u'^* is conservative, then ($\mathrm{BC}_{f,u}$) and ($\mathrm{BC}_{f',u'}$) are satisfied by (eSm-BC). Thus (BC_{f_0,g_0}) implies ($\mathrm{BC}_{f,g}$).

(b) When u is the projection $S \times \mathrm{pt}_{\mathbb{N}} \to S$, u'^* is conservative by (6.2.2). We also have ($\mathrm{BC}_{f,u}$) and ($\mathrm{BC}_{f',u'}$) by (6.4.2). Thus (BC_{f_0,g_0}) implies ($\mathrm{BC}_{f,g}$).

(III) *Final step of the proof.* Let G be a maximal θ-critical face of Q, and we denote by Q' the submonoid of Q consisting of elements $q \in Q'$ susch that $nq \in P + G$ for some $n \in \mathbb{N}^+$. Then by [Ogu14, 4.6.6], $Q'^{\mathrm{gp}} = Q^{\mathrm{gp}}$. Thus by (I), we reduce to the case when $Q = Q'$. In this case, we have

$$Q_{\mathbb{Q}} = (P \oplus G)_{\mathbb{Q}}.$$

Then choose $n \in \mathbb{N}^+$ such that $nq \in P+G$ for any $q \in Q$, and consider the homomorphism

$$P \to P^{\mathrm{gp}} \oplus P, \quad a \mapsto (a, na).$$

We put $P' = P^{\mathrm{gp}} \oplus P$, and consider the projection $u : S \times_{\mathbb{A}_P} \mathbb{A}_{P'} \to S$. Then u^* is conservative as in (5.2.2). Thus by the case (a) in (II), we can replace $P \to Q$ by $P' \to P' \oplus_P Q$. Thus we reduce to the case when

$$Q = P \oplus G.$$

Since $X = S \times \mathrm{pt}_G$, we reduce to the case when $P = 0$.

By [CLS11, 11.1.9], there is a homomorphism $\lambda : \mathbb{N}^r \to Q$ of fs monoids such that λ^{gp} is an isomorphism. Thus by (I), we reduce to the case when $Q = \mathbb{N}^r$. Then we have the factorization

$$S \times \mathrm{pt}_{\mathbb{N}^r} \to \cdots \to S \times \mathrm{pt}_{\mathbb{N}} \to S$$

of f, so we reduce to the case when $Q = \mathbb{N}$. By the case (b) of (II), we reduce to the case when θ is the first inclusion $\mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}$. Composing with the homomorphism

$$\mathbb{N} \oplus \mathbb{N} \to \mathbb{N} \oplus \mathbb{N}, \quad (a, b) \mapsto (a, a + b)$$

of fs monoids, by (I), we reduce to the case when θ is the diagonal homomorphism $\mathbb{N} \to \mathbb{N} \oplus \mathbb{N}$.

Then f has the factorization

$$S \times \mathrm{pt}_{\mathbb{N}\oplus\mathbb{N}} \xrightarrow{i} S \times \mathbb{A}_{\mathbb{N}\oplus\mathbb{N}} \times_{\mathbb{A}_\theta, \mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}} \xrightarrow{p} S$$

where i denotes the induced strict closed immersion and p denotes the projection. By (6.4.1), $(\mathrm{BC}_{p,g})$ is satisfied. If g'' denotes the pullback of $g : S' \to S$ via p, $(\mathrm{BC}_{i,g''})$ is satisfied by (BC–3). These two implies $(\mathrm{BC}_{f,g})$. □

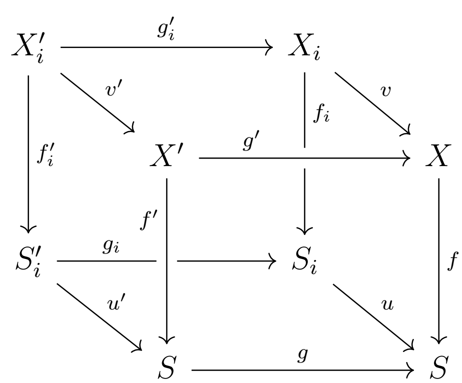
Theorem 6.4.4. *The log motivic triangulated category $\mathscr{T}$ satisfies* (BC–2)*.*

Proof. Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes where f is a separated exact log smooth morphism. We want to show $(\mathrm{BC}_{f,g})$.

(I) *Reduction method 1.* Let $\{u_i : S_i \to S\}_{i \in I}$ be a family of strict morphisms. For $i \in I$, consider the commutative diagram



of $\mathscr{S}$-schemes where each square is Cartesian. Assume that the family of functors $\{u_i^*\}_{i\in I}$ is conservative. Then we have the commutative diagram

$$\begin{array}{ccccc}
u_i'^* g^* f_! & \xrightarrow{Ex} & u_i'^* f_!' g'^* & \xrightarrow{Ex} & f_{i!}' v_i'^* g'^* \\
\downarrow\wr & & & & \downarrow\wr \\
g_i^* u_i^* f_! & \xrightarrow{Ex} & g_i^* f_{i!} v_i^* & \xrightarrow{Ex} & f_{i*}' g_i'^* v_i^*
\end{array}$$

of functors. By (BC–3), the lower left horizontal and upper right horizontal arrows are isomorphisms. If (BC_{f_i,g_i}) is satisfied for any i, then the lower right horizontal arrow is an isomorphism for any i, so the upper left horizontal arrow is an isomorphism for any i. This implies $(\mathrm{BC}_{f,g})$ since $\{u_i^*\}_{i\in I}$ is conservative.

We will apply this in the following two situations.

(a) When $\{u_i\}_{i\in I}$ is a strict étale cover, the family of functors $\{u_i^*\}_{i\in I}$ is conservative by (két-sep).

(b) When u_0 is a strict closed immersion and u_1 is its complement, the pair of functors (u_0^*, u_1^*) is conservative by (Loc).

(II) *Reduction method 2.* Let $\{v_i : X_i \to X\}_{i\in I}$ be a family of separated strict morphisms such that the family of functors $\{v_i^!\}_{i\in I}$ is conservative. Consider the commutative diagram

$$\begin{array}{ccc}
X_i' & \xrightarrow{g_i'} & X_i \\
\downarrow{\scriptstyle v_i'} & & \downarrow{\scriptstyle v_i} \\
X' & \xrightarrow{g'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{g} & S
\end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. Then we have the commutative diagram

$$\begin{array}{ccccc}
g^* f_! v_{i!} & \xrightarrow{Ex} & f_!' g'^* v_{i!} & \xrightarrow{Ex} & f_!' v_{i!}' g_i'^* \\
\downarrow\wr & & & & \downarrow\wr \\
g^*(f v_i)_! & & \xrightarrow{\qquad Ex \qquad} & & (f' v_i')_! g_i'^*
\end{array}$$

of functors. The upper right horizontal arrow is an isomorphism by (BC–1). If $(\mathrm{BC}_{f v_i, g})$ is satisfied, then the lower horizontal arrow is an isomorphism, so the upper left horizontal arrow is an isomorphism. This implies that the natural transformation

$$v_i^! g_*' f^! \xrightarrow{Ex} v_i^! f^! g_*$$

is an isomorphism. This implies $(\mathrm{BC}_{f,g})$ since $\{v_i^!\}_{i\in I}$ is conservative.

We will apply this technique in the following two situations.

(a) When $\{v_i\}_{i\in I}$ is a strict étale cover such that each v_i is separated, by (két-sep) and (2.5.9), the family of functors $\{v_i^!\}_{i\in I}$ is conservative.

(b) When $v_0 : S_0 \to S$ is a strict closed immersion and v_1 is its complement, by (Loc), the pair of functors $(v_0^!, v_1^!)$ is conservative.

(III) *Final step of the proof.* By the case (a) of (I) and the case (a) of (II), we reduce to the case when f has a fs chart $\theta : P \to Q$ of exact log smooth type. Then by the case (b) of (I) and the proof of [Ols03, 3.5(ii)], we may assume that S has a constant log structure.

Consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g''} & \mathrm{pt}_Q \\ \downarrow{\scriptstyle h} & & \downarrow \\ X & \xrightarrow{g'} & \mathbb{A}_Q \\ \downarrow{\scriptstyle f} & & \downarrow \\ S & \xrightarrow{g} & \mathbb{A}_P \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian. By the case (b) of (II) and the proof of [Ols03, 3.5(ii)], we reduce to showing $(\mathrm{BC}_{fh,g})$.

Now, consider the the commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g''} & Y \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ T' & \xrightarrow{g'''} & \underline{Y} \times_{\underline{S}} S \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes where

(i) each square is Cartesian,

(ii) q denotes the induced morphism and p denotes the projection.

Then since p is strict, $(\mathrm{BC}_{p,g})$ is satisfied by (BC–1). By (6.4.3), $(\mathrm{BC}_{q,g'''})$ is also satisfied. These two implies $(\mathrm{BC}_{f,g})$. □

Chapter 7

Localization property

7.1 Localization property for $\mathrm{D}_{\mathbb{A}^1,s\grave{e}t}(lSm,\Lambda)$

7.1.1. We will prove the theorem of Morel and Voevodsky [Ayo07, 4.5.36] in the logarithmic setting. We will closely follow the proof of [loc. cit] except Étape 2 of [Ayo07, 4.5.42] in which some additional log geometry is need.

7.1.2. Let S be an $\mathscr{S}$-scheme. Recall from (1.3.2) that for any presheaf F on ft/S, we denote by $\Lambda_S(F)$ the Λ-free presheaf

$$(X \in ft/S) \mapsto \Lambda^{F(S)}.$$

For any topology t on $\mathscr{S}$, we denote by $\Lambda_S^t(F)$ its associated t-sheaf.

7.1.3. Let t be a topology on $\mathscr{S}$, let S be an $\mathscr{S}$-scheme, and let $\mathscr{A}_S$ be $\mathrm{Sh}_t(ft/S,\Lambda)$ or $\mathrm{PSh}(ft/S,\lambda)$. Then we denote by $\mathscr{A}$ the collection of $\mathscr{A}_S$ for $\mathscr{S}$-schemes S. For any family $\mathscr{W}$ of morphisms in $C(\mathscr{A})$ stable by twists, $f_\sharp$ for $f \in ft$, and f^*, we refer readers [CD12, 5.2.2] the $\mathscr{W}$-local model structure on $C(\mathscr{A})$.

We denote by $\mathscr{W}_{\mathbb{A}^1,S}$ the family

$$M_S(X \times \mathbb{A}^1)[n] \to M_S(X)[n]$$

for $n \in \mathbb{Z}$ and morphisms $X \to S$ of finite type. For any topology t' on $\mathscr{S}$ finer than t, we denote by $\mathscr{W}_{t',s}$ the family

$$M_S(\mathscr{X})[n] \to M_S(X)[n]$$

for $n \in \mathbb{Z}$, morphisms $X \to S$ of finite type, and t'-hypercovers $\mathscr{X} \to X$. Then for brevity, the $\mathscr{W}_{t'} \cup \mathscr{W}_{\mathbb{A}^1}$-local model structure is called $(t',\mathbb{A}^1)$-local model structure.

7.1.4. Recall from [Ayo07, Following paragraph of 4.5.31] that the $t_\emptyset$ is the topology on $\mathscr{S}$ generated by the empty cover of $\emptyset$.

Theorem 7.1.5. *Let $i : Z \to S$ be a closed immersion, and let $j : U \to S$ denote its complement. For any log smooth morphism $f : X \to S$ of $\mathscr{S}$-schemes, the commutative diagram*

$$\begin{array}{ccc} \Lambda_S^{t_\emptyset}(U \times_S X) & \longrightarrow & \Lambda_S^{t_\emptyset}(X) \\ \downarrow & & \downarrow \\ * & \longrightarrow & i_*\Lambda_S^{t_\emptyset}(Z \times_S X) \end{array}$$

of $\mathscr{S}$-schemes is homotopy coCartesian in $\mathrm{C}(\mathrm{Sh}_{t_\emptyset}(ft/S, \Lambda))$ with the $(s\grave{e}t, \mathbb{A}^1)$-local model structure.

7.1.6. Before proving the theorem, we recall several results in [Ayo07, §4.5].

Proposition 7.1.7. *Let $i : Z \to S$ be a* strict *closed immersion of $\mathscr{S}$-schemes. Then the functor*

$$i_* : \mathrm{C}(\mathrm{Sh}_{t_\emptyset}(ft/S, \Lambda)) \to \mathrm{C}(\mathrm{Sh}_{t_\emptyset}(ft/S, \Lambda))$$

preserves $(s\grave{e}t, \mathbb{A}^1)$-weak equivalences.

Proof. In [Ayo07, 4.5.35], the statement is proved for the topos $\mathrm{Sh}_{t_\emptyset}(Sm/S, \Lambda))$ (with S usual scheme) instead of the topos $\mathrm{Sh}_{t_\emptyset}(ft/S, \Lambda)$. However, the proof of [loc. cit] can be applied to our situation trivially. □

Proposition 7.1.8. *Let $G \to F$ be a morphism of presheaves of sets over ft/S. To show that the morphism*

$$\Lambda_S(G) \to \Lambda_S(F)$$

in $\mathrm{C}(\mathrm{PSh}(ft/S, \Lambda))$ is $(s\grave{e}t, \mathbb{A}^1)$-weak equivalent, it suffices to show that for any morphism $p : P \to S$ of finite type of $\mathscr{S}$-schemes and a section $s \in F(P)$, the morphism

$$\Lambda_S(p^*G \times_{p^*F} P) \to \Lambda_S(P)$$

in $\mathrm{C}(\mathrm{PSh}(ft/S, \Lambda))$ is $(s\grave{e}t, \mathbb{A}^1)$-weak equivalent. Here, the morphism $X \to p^*F$ used in the fiber product is the right adjoint of

$$p_\sharp P \cong P \xrightarrow{s} F.$$

Proof. In [Ayo07, 4.5.40], the statement is proved for the topos $\mathrm{PSh}(Sm/S, \Lambda)$ (with S usual scheme) instead of the topos $\mathrm{PSh}(ft/S, \Lambda)$. However, the proof of [loc. cit] can be applied to our situation trivially. □

Lemma 7.1.9. *Under the notations and hypotheses of (7.1.5), to show (loc. cit), it suffices to show that the commutative diagram*

$$\begin{array}{ccc} \Lambda_S(U \times_S X) & \longrightarrow & \Lambda_S(X) \\ \downarrow & & \downarrow \\ \Lambda_S(U) & \longrightarrow & \Lambda_S(i_*(Z \times_S X)) \end{array}$$

is homotopy coCartesian in $\mathrm{C}(\mathrm{PSh}(ft/S,\Lambda))$ *with the* $(s\grave{e}t,\mathbb{A}^1)$*-local model structure. Here, the morphism* $\Lambda_S(U) \to \Lambda(i_*(Z\times_S X))$ *used in the above diagram is induced by a unique element of*

$$\mathrm{Hom}_S(U, i_*(Z\times_S X)) = \mathrm{Hom}_Z(\emptyset, Z\times_S X) = *.$$

Proof. It is due to [Ayo07, 4.5.41]. □

Lemma 7.1.10. *Let* X *be an* $\mathscr{S}$*-scheme over* S*, and let* $g : X' \to X$ *be a strict étale cover of* $\mathscr{S}$*-schemes. Then the functor*

$$g^* : \mathrm{C}(\mathrm{PSh}(ft/S,\Lambda)) \to \mathrm{C}(\mathrm{PSh}(ft/S,\Lambda))$$

preserves and detects $(s\grave{e}t,\mathbb{A}^1)$*-weak equivalences.*

Proof. In [Ayo07, 4.5.43], the statement is proved for the topos $\mathrm{PSh}(Sm/S,\Lambda)$ (with S usual scheme) instead of the topos $\mathrm{PSh}(ft/S,\Lambda)$. However, the proof of [loc. cit] can be applied to our situation trivially. □

7.1.11. Now we start the proof of (7.1.5). By (7.1.9), it suffices to prove that the morphism

$$\Lambda_S(X \coprod_{U\times_S X} U) \to \Lambda_S(i_*(Z\times_S X))$$

induced by the diagram of (loc. cit) is $(s\grave{e}t,\mathbb{A}^1)$-weak equivalent. Here, $X\coprod_{U\times_S X} U$ is the fibered coproduct of presheaves of sets. Note that for any morphism $Y \to S$ of $\mathscr{S}$-scheme, we have

$$\begin{aligned}(X \coprod_{U\times_S X} U)(Y) =& \mathrm{Hom}_S(Y,X) \coprod_{\mathrm{Hom}_S(Y, U\times_S X)} \mathrm{Hom}_S(Y,U)\\ =& \begin{cases} \mathrm{Hom}_S(Y,X) & \text{if} \quad Y\times_S Z \neq \emptyset \\ * & \text{if} \quad Y\times_S Z = \emptyset.\end{cases}\end{aligned}$$

By (7.1.8), it suffices to prove that for any morphism $p : P \to S$ of finite type of $\mathscr{S}$-schemes and a section $s : P \to i_*(Z\times_S X)$, the morphism

$$\Lambda_S(T_{X,P,s}) \to P$$

in $\mathrm{C}(\mathrm{PSh}(ft/S,\Lambda))$ is $(s\grave{e}t,\mathbb{A}^1)$-weak equivalent where

$$T_{X,P,s} = p^*(X \coprod_{U\times_S X} U) \times_{p^* i_*(Z\times_S X)} P.$$

Note that for any morphism $Y \to P$ of $\mathscr{S}$-schemes, we have

$$T_{X,P,s}(Y) = \begin{cases} \mathrm{Hom}_S(Y,X) \times_{\mathrm{Hom}_Z(Z\times_S Y, Z\times_S X)} * & \text{if} \quad Y\times_S Z \neq \emptyset \\ * & \text{if} \quad Y\times_S Z = \emptyset \end{cases}$$

where the function $* \to \mathrm{Hom}_Z(Z \times_S Y, Z \times_S X)$ used in the fiber product is obtained by the composition

$$Z \times_S Y \longrightarrow Z \times_S P \xrightarrow{s} Z \times_S X.$$

Because

$$T_{X,P,s} = T_{X \times_S P, P, s_P}$$

where s_P denotes the morphism $(s, \mathrm{id}) : Z \times_S P \to X \times_S P$, we may assume that $P = S$. Hence to prove (7.1.5), the remaining is to prove the following proposition.

Proposition 7.1.12. *Under the notations and hypotheses of (7.1.5), Let $s : Z \to X$ be a partial section of $f : X \to S$. We denote by $T_{X,s}$ the presheaf of sets defined by*

$$T_{X,s}(Y) = \begin{cases} \mathrm{Hom}_S(Y, X) \times_{\mathrm{Hom}_Z(Z \times_S Y, Z \times_S X)} * & \text{if} \quad Y \times_S Z \neq \emptyset \\ * & \text{if} \quad Y \times_S Z = \emptyset \end{cases}$$

for any morphism $Y \to S$ of $\mathscr{S}$-schemes. Then the morphism

$$\Lambda_S(T_{X,s}) \to \Lambda_S(S)$$

in $\mathrm{C}(\mathrm{PSh}(ft/S, \Lambda))$ *is* $(s\grave{e}t, \mathbb{A}^1)$*-weak equivalent.*

Proof. We denote by t the graph morphism $Z \to Z \times_S X$ of $s : Z \to X$. To help readers, we include the description of $T_{X,s}(Y)$ via diagrams as follows: when $Y \times_S Z \neq \emptyset$, the set $T_{X,s}(Y)$ is the set of morphisms $Y \to X$ of $\mathscr{S}$-schemes over S such that the diagram

$$\begin{array}{ccccc} & & Z \times_S Y & \xrightarrow{b} & Y \\ & \swarrow^{q} & \downarrow{h'} & & \downarrow{h} \\ Z & & & & \\ & \searrow_{t} & & & \\ & & Z \times_S X & \xrightarrow{a} & X \end{array} \qquad (7.1.12.1)$$

of $\mathscr{S}$-schemes commutes where a and q denotes the projections and the small square is Cartesian. We will prove the proposition in several steps.

(I) *Locality on S.* Let $\{u_i : S_i \to S\}_{i \in I}$ be a strict étale cover. Then the presheaf $u_i^*(T_{X,s})$ is isomorphic to T_{X_i, s_i} where s_i and X_i denote the pullbacks of s and X via u_i respectively. Then (7.1.10) implies that the question is strict étale local on S. Hence from now, we will assume that S has a fs chart.

(II) *Comparison of presheaves.* Consider a commutative diagram

$$\begin{array}{ccc} & & X' \\ & \nearrow^{s'} & \downarrow{g} \\ Z & \xrightarrow[s]{} & X \end{array}$$

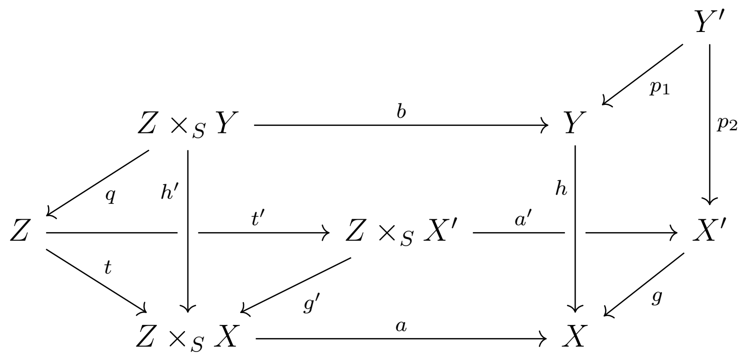
such that g is log étale. Then we will show that the evident morphism

$$T_{X',s'} \to T_{X,s}$$

becomes an isomorphism after sheafification. To show this, we will construct the inverse of

$$T_{X',s'}(Y) \to T_{X,s}(Y)$$

for any henselization Y of an $\mathscr{S}$-scheme W over S. Here, the henselization means the fiber product $W \times_{\underline{W}} (\underline{W})^h$. Consider the commutative diagram (7.1.12.1). We put $Y' = Y \times_X X'$. Then we have the commutative diagram



of $\mathscr{S}$-schemes where

(i) p_1 denotes the first projection, and p_2 denotes the second projection,

(ii) a' denotes the projection,

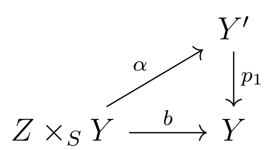
(iii) t and t' denotes the morphisms induced by s and s' respectively,

(iv) each small square is Cartesian.

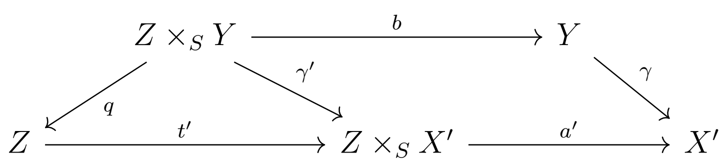
Then the two compositions

$$Z \times_S Y \to Y \to X,$$
$$Z \times_S Y \to Z \to Z \times_S X' \to X' \to X$$

are equal, so these two induce a morphism $\alpha : Z \times_S Y \to Y \times_S X' = Y'$ of $\mathscr{S}$-schemes. Thus we have the commutative diagram



of $\mathscr{S}$-schemes, and by (3.3.4), there is a unique section $\beta : Y \to Y'$ of p_1 extending α. Let γ denote the composition $p_2\beta$. Then the diagram



of $\mathscr{S}$-schemes commutes where the small square is Cartesian, so this gives an element of $T_{X',s'}(Y)$. Thus we have constructed the inverse of $T_{X',s'}(Y) \to T_{X,s}(Y)$.

(III) *Locality on X.* Note that we have assumed that S has a fs chart. Let $\{v_i : X_i \to X\}_{i \in I}$ be a strict étale cover. We denote by $w_i : Z_i \to Z$ the pullback of v_i via $s : Z \to X$. By [EGA, IV.18.1.1], for each $i \in I$, there is a Cartesian diagram

$$\begin{array}{ccc} Z_i & \longrightarrow & Z \\ \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle i} \\ S_i & \xrightarrow{u_i} & S \end{array}$$

such that u_i is strict étale. Then we have the commutative diagram

$$\begin{array}{ccc} & & X_i \times_S S_i \\ & \nearrow{\scriptstyle s'_i} & \downarrow \\ & & X \times_S S_i \\ & \nearrow{\scriptstyle s_i} & \downarrow \\ Z_i & \longrightarrow & S_i \end{array}$$

of $\mathscr{S}$-schemes where s'_i is the morphism induced by the morphisms $Z_i \to X_i$ and $Z_i \to S_i$. By (I), we reduce to the case when

$$(S, X, Z, s) = (S_i, X \times_S S_i, Z_i, s_i),$$

and by (II), we reduce to the case when

$$(S, X, Z, s) = (S_i, X_i \times_S S_i, Z_i, s'_i)$$

since the morphism $X_i \times_S S_i \to X \times_S S_i$ is strict étale.

We will apply this to the following two situations. Assume that $\{v_i : X_i \to X\}_{i \in I}$ be a strict étale cover such that each morphism $X_i \to S$ has a fs chart. Then each projection $X_i \times_S S_i \to S_i$ has also a fs chart, so we reduce to the case when the morphism $f : X \to S$ has a fs chart.

Another application of this process is that when $f : X \to S$ is strict smooth. By [EGA, IV.17.12.2], there is an open cover $\{v_i : X_i \to X\}_{i \in I}$ such that the composition $fv_i : X_i \to S$ has a factorization

$$X_i \xrightarrow{u'_i} \mathbb{A}^n_S \xrightarrow{u_i} S$$

where u'_i is strict étale and u_i denotes the projection such that the composition $u'_i s : Z \to \mathbb{A}^n_S$ is the 0-section. Then the projection $X_i \times_S S_i \to S_i$ has the factorization

$$X_i \times_S S_i \longrightarrow \mathbb{A}^n_{S_i} \longrightarrow S_i$$

induced by the above sequence. Hence when f is strict smooth, we reduce to the case when f has a factorization

$$X \xrightarrow{u'} \mathbb{A}_S^n \xrightarrow{u} S$$

where u' is strict étale and u denotes the projection such that the composition $u's : Z \to \mathbb{A}_S^n$ is the 0-section.

(IV) *Final step of the proof.* By (III), we may assume that f has a chart. Then by [Ogu14, IV.3.4.2], we have a commutative diagram

$$\begin{array}{ccc} & & X' \\ & \overset{s'}{\nearrow} & \downarrow g \\ Z & \xrightarrow{s} & X \end{array}$$

of $\mathscr{S}$-schemes where s' is strict closed immersion and g is a log étale morphism with a fs chart. By (3.3.3), we can choose a maximal open subscheme U of X such that the composition $U \to X' \overset{fg}{\to} S$ is strict. Then s' factors through U, so we have the commutative diagram

$$\begin{array}{ccc} & & U \\ & \overset{s''}{\nearrow} & \downarrow gj \\ Z & \xrightarrow{s} & X \end{array}$$

of $\mathscr{S}$-schemes where s'' denote the morphism induced by s', so by (II), we reduce to the case when $(X, s) = (U, s'')$. In particular, we may assume that f is strict. Then by (III), we may assume that $f : X \to S$ has a factorization

$$X \xrightarrow{u'} \mathbb{A}_S^n \xrightarrow{u} S$$

where u' is strict étale and u denotes the projection such that the composition $u's : Z \to \mathbb{A}_S^n$ is the 0-section. Then by (II) again, we may assume that $(X, s) = (\mathbb{A}_S^n, s_0)$ where $s_0 : Z \to \mathbb{A}_S^n$ denotes the 0-section. We have the morphism

$$T_{\mathbb{A}_S^n, 0} \times \mathbb{A}^1 \to T_{\mathbb{A}_S^n, 0}$$

of presheaves that maps $(f, t) \in T_{\mathbb{A}_S^n, 0}(Y) \times \mathbb{A}_S^1(Y)$ to the composite

$$Y \longrightarrow \mathbb{A}_S^n \times \mathbb{A}^1 \xrightarrow{(x_1, \ldots, x_n, t) \mapsto (tx_1, \ldots, tx_n)} \mathbb{A}_S^n.$$

This map forms a homotopy between the identity of $T_{\mathbb{A}^n, s_0}$ and the zero morphism, which completes the proof. □

Corollary 7.1.13. *Let $i : Z \to S$ be a strict closed immersion, and let $j : U \to S$ denote its complement. For any log smooth morphism $f : X \to S$, we have a distinguished triangle*

$$j_\sharp j^* M_S(X) \longrightarrow M_S(X) \longrightarrow i_* i^* M_S(X) \longrightarrow j_\sharp j^* M_S(X)[1]$$

in $\mathrm{D}_{\mathbb{A}^1, s\grave{e}t}(ft/S, \Lambda)$.

Proof. Because (7.1.5) and (7.1.7) are proved, we can argue as in the proofs of [Ayo07, 4.5.47]. □

Proposition 7.1.14. *Let $i : Z \to S$ be a strict closed immersion, and let $j : U \to S$ denote its complement. Then the functor*

$$i_* : \mathrm{D}_{\mathbb{A}^1,s\grave{e}t}(ft/Z,\Lambda) \to \mathrm{D}_{\mathbb{A}^1,s\grave{e}t}(ft/S,\Lambda)$$

admits a right adjoint.

Proof. It follows from the proof of [Ayo07, 4.5.46]. □

7.2 Localization property for $\mathrm{D}_{\mathbb{A}^1,pw}(lSm,\Lambda)$

7.2.1. Our final goal of this chapter is to show the localization property for $\mathrm{D}_{log,pw}(lSm,\Lambda)$. Our strategy is to use the following result.

Proposition 7.2.2. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, let $j : U \to S$ denote its complement, let $f : V \to Z$ be a log smooth morphism of $\mathscr{S}$-schemes, let t be a topology on $\mathscr{S}$, and let $\mathscr{W}$ be a family of morphisms stable by twists, $f_\sharp$ for $f \in lSm$, and f^*. Assume that*

(i) the functor

$$i_* : \mathrm{D}_{\mathbb{A}^1,t}(ft/Z,\Lambda) \to \mathrm{D}_{\mathbb{A}^1,t}(ft/S,\Lambda)$$

maps $\mathscr{W}$ to $\mathscr{W}$-weak equivalences, and it admits a right adjoint.

(ii) there is a distinguished triangle

$$j_\sharp j^* M_S(X) \longrightarrow M_S(X) \longrightarrow i_* i^* M_S(X) \longrightarrow j_\sharp j^* M_S(X)[1] \qquad (7.2.2.1)$$

in $\mathrm{D}_{\mathbb{A}^1,t}(ft/S,\Lambda)$.

Then

(1) the functor

$$i_* : \mathrm{D}_{\mathscr{W},t}(ft/Z,\Lambda) \to \mathrm{D}_{\mathscr{W},t}(ft/S,\Lambda)$$

admits a right adjoint,

(2) there is a distinguished triangle

$$j_\sharp j^* M_S(X) \longrightarrow M_S(X) \longrightarrow i_* i^* M_S(X) \longrightarrow j_\sharp j^* M_S(X)[1] \qquad (7.2.2.2)$$

in $\mathrm{D}_{\mathscr{W},t}(ft/S,\Lambda)$.

Proof. The assertion (1) follows from (1.6.4), and by (loc. cit), i_* commutes with the functor

$$\pi : \mathrm{D}_{\mathbb{A}^1,t}(ft,\Lambda) \to \mathrm{D}_{\mathscr{W},t}(ft,\Lambda).$$

Applying π to (7.2.2.1), we get a distinguished triangle

$$\pi j_\sharp j^* M_S(X) \longrightarrow \pi M_S(X) \longrightarrow \pi i_* i^* M_S(X) \longrightarrow \pi j_\sharp j^* M_S(X)[1]$$

in $\mathrm{D}_{\mathscr{W},t}(ft/S,\Lambda)$. It is exactly (7.2.2.2) because π commutes with $j_\sharp$, j^*, i^*, and i_*. □

7.2.3. For the localization property for $\mathrm{D}_{\mathbb{A}^1,pw}(lSm,\Lambda)$, out strategy is to apply Ayoub's following result.

Proposition 7.2.4. *Let t be a topology on $\mathscr{S}$ such that any t-cover consists of morphisms of finite type, and let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes. Assume that for any morphism $X \to S$ of finite type of $\mathscr{S}$-schemes such that $X \times_S Z \neq \emptyset$, the evident functor*

$$\mathrm{Cov}_t(X) \to \mathrm{Cov}_t(X \times_S Z)$$

is cofinal where $\mathrm{Cov}_t(X)$ denote the category of t-cover of X. Then the functor

$$i_* : \mathrm{D}_{\mathbb{A}^1,t_\emptyset}(ft/Z,\Lambda) \to \mathrm{D}_{\mathbb{A}^1,t_\emptyset}(ft/S,\Lambda)$$

preserves t-local equivalences.

Proof. It follows from the proof of [Ayo07, 4.5.34]. □

7.2.5. Hence the remaining is to study the cofinality for the pw-topology.

Proposition 7.2.6. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes. Then the evident functor*

$$\Phi : \mathrm{Cov}_{pw}(S) \to \mathrm{Cov}_{pw}(Z)$$

is cofinal.

Proof. Let $g : Z' \to Z$ be a morphism of $\mathscr{S}$-schemes. We divide the question into 3 cases.

(I) *Strict étale cover.* Assume that g is a strict étale cover. Then g has a refinement that is in the essential image of Φ by the proof of [Ayo07, 4.5.33].

(II) *Piercing cover.* Let $v : Z \to \mathbb{A}^1$ be a morphism of $\mathscr{S}$-schemes. We put

$$Z_1' = Z \times_{\mathbb{A}^1} \mathrm{Spec}\,\mathbb{Z}, \quad Z_2' = Z \times_{\mathbb{A}^1} \mathbb{A}_{\mathbb{N}}$$

where the morphisms $\mathrm{Spec}\,\mathbb{Z} \to \mathbb{A}^1$ and $\mathbb{A}_{\mathbb{N}} \to \mathbb{A}^1$ used above are the 0-sections and the morphism removing the log structure respectively. We want to show that the piercing cover

$$Z_1' \coprod Z_2' \to Z$$

has a refinement that is an essential image of Φ.

Zariski locally on S, the morphism $v : Z \to \mathbb{A}^1$ can be extended to a morphism $u : S \to \mathbb{A}^1$. We put similarly

$$S_1' = S \times_{\mathbb{A}^1} \mathrm{Spec}\,\mathbb{Z}, \quad S_2' = S \times_{\mathbb{A}^1} \mathbb{A}_{\mathbb{N}}.$$

Then the image of $S_1' \amalg S_2' \to S'$ via Φ is the cover $Z_1' \amalg Z_2 \to Z$.

(III) *Winding cover.* Assume that g is a pullback of a morphism

$$\mathbb{A}_{\theta'} : \mathbb{A}_{P_1'} \to \mathbb{A}_{P'}$$

where $\theta' : P' \to P'_1$ is a Kummer homomorphism of fs monoids. We want to show that g has a refinement that is an essential image of Φ.

Let x be a geometric point of Z. Strict étale locally on Z near x, we can choose a factorization

$$P' \to Q' \xrightarrow{\alpha'} \mathcal{M}_Z$$

of the homomorphism $P' \to \mathcal{M}_Z$ such that α' is a chart exact at x by [Ogu14, II.2.3.2]. Moreover, strict étale locally on S near x, we may assume that S has a fs chart

$$\alpha : Q \to \mathcal{M}_S$$

neat at x by [Ogu14, 2.3.7]. Choose homomorphisms

$$\beta : Q \to Q \oplus Q'^{\mathrm{gp}},$$

$$\beta' : Q' \to Q \oplus Q'^{\mathrm{gp}},$$

$$\beta'' : Q \oplus Q'^{\mathrm{gp}} \to \mathcal{M}_Z$$

as in (3.2.1). Then β' induces the homomorphism

$$P' \to Q \oplus Q'^{\mathrm{gp}}.$$

We put $Q_1 = P'_1 \oplus_{P'} (Q \oplus Q'^{\mathrm{gp}})$, and we denote by $\iota : Q \oplus Q'^{\mathrm{gp}} \to Q_1$ the second inclusion. For $n \in \mathbb{N}^+$, we denote by $\mu_n : Q \to Q$ the multiplication homomorphism $a \mapsto na$. By (3.5.4), there is a Kummer homomorphism $\zeta : Q'^{\mathrm{gp}} \to G$ of finitely generated abelian groups and a commutative diagram

$$\begin{array}{ccc} Q \oplus Q'^{\mathrm{gp}} & \xrightarrow{\iota} & Q_1 \\ & \searrow^{\mu_n \oplus \zeta} & \downarrow \\ & & Q \oplus G \end{array}$$

of fs monoids. Then g has a refinement

$$Z \times_{\mathbb{A}_{Q \oplus Q'^{\mathrm{gp}}}, \mathbb{A}_{\mu_n \oplus \zeta}} \mathbb{A}_{Q \oplus G} \to Z.$$

Choose a surjective homomorphism $\lambda : \mathbb{Z}^r \to Q'^{\mathrm{gp}}$ for some r. By (7.2.7) below, there is a Kummer homomorphism $\zeta' : \mathbb{Z}^r \to G'$ of finitely generated abelian groups and a coCartesian diagram

$$\begin{array}{ccc} Q \oplus \mathbb{Z}^r & \xrightarrow{\mathrm{id} \oplus \lambda} & Q \oplus Q'^{\mathrm{gp}} \\ \downarrow{\scriptstyle \mu_n \oplus \zeta'} & & \downarrow{\scriptstyle \mu_n \oplus \zeta} \\ Q \oplus G' & \longrightarrow & Q \oplus G \end{array}$$

of fs monoids. Thus g has a refinement

$$Z \times_{\mathbb{A}_{Q \oplus \mathbb{Z}^r}, \mathbb{A}_{\mu_n \oplus \zeta'}} \mathbb{A}_{Q \oplus G'} \to Z.$$

Strict étale locally on S near x, the composition

$$Q \oplus \mathbb{Z}^r \stackrel{\mathrm{id} \oplus \lambda}{\to} Q \oplus Q'^{\mathrm{gp}} \to \Gamma(Z, \mathcal{M}_Z)$$

factors through $\Gamma(S, \mathcal{M}_S)$ where the third arrow is the homomorphism induced by β'' because $\mathbb{Z}^r$ is free and β'' is induced by α. This gives a chart

$$Q \oplus \mathbb{Z}^r \to \mathcal{M}_S.$$

Then the image of the winding cover

$$S \times_{\mathbb{A}_{Q \oplus \mathbb{Z}^r}, \mathbb{A}_{\mu_n \oplus \zeta'}} \mathbb{A}_{Q \oplus G'} \to S$$

via Φ is a refinement of g. □

Lemma 7.2.7. *Let $\theta : G \to H$ be a surjective homomorphism of finitely generated abelian groups, and let $\eta' : H \to H'$ be a Kummer homomorphism of finitely generated abelian groups. Then there is a coCartesian diagram*

$$\begin{array}{ccc} G & \xrightarrow{\theta} & H \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ G' & \xrightarrow{\theta'} & H' \end{array}$$

of finitely generated abelian groups such that η' is Kummer.

Proof. Let a be an element of H'. By induction on $[H' : H]$, we may assume that H' is generated by a and H. We denote by $\lambda' : \mathbb{Z} \to H'$ the homomorphism maps 1 to a, and consider the Cartesian diagram

$$\begin{array}{ccc} K & \xrightarrow{\lambda} & H \\ \downarrow{\scriptstyle \eta''} & & \downarrow{\scriptstyle \eta'} \\ \mathbb{Z} & \xrightarrow{\lambda'} & H' \end{array}$$

of finitely generated abelian groups. Since $\eta'(H)$ and $\lambda'(\mathbb{Z})$ generate H', the above diagram is also coCartesian. The assumption that η' is Kummer implies that K is nontrivial, so K is isomorphic to $\mathbb{Z}$. Then λ has a factorization

$$K \stackrel{\mu}{\to} G \stackrel{\nu}{\to} H$$

since K is free. Consider the commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{\mu} & G & \xrightarrow{\nu} & H \\ \downarrow{\scriptstyle \eta''} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ \mathbb{Z} & \xrightarrow{\mu'} & G' & \xrightarrow{\nu} & H' \end{array}$$

of finitely generated abelian groups where the left square is coCartesian. Then the right square is also coCartesian, and η is Kummer. □

Corollary 7.2.8. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, let $j : U \to S$ be its complement, and let $f : V \to Z$ be a log smooth morphism of $\mathscr{S}$-schemes. Then*

(1) the functor

$$i_* : \mathrm{D}_{\mathbb{A}^1,pw}(ft/Z, \Lambda) \to \mathrm{D}_{\mathbb{A}^1,pw}(ft/S, \Lambda)$$

admits a right adjoint,

(2) there is a distinguished triangle

$$j_\sharp j^* M_S(X) \longrightarrow M_S(X) \longrightarrow i_* i^* M_S(X) \longrightarrow j_\sharp j^* M_S(X)[1]$$

in $\mathrm{D}_{\mathbb{A}^1,pw}(ft/S, \Lambda)$.

Proof. By (7.2.4) and (7.2.6), the functor

$$i_* : \mathrm{D}_{\mathbb{A}^1,s\grave{e}t}(ft/Z, \Lambda) \to \mathrm{D}_{\mathbb{A}^1,s\grave{e}t}(ft/S, \Lambda)$$

preserves pw-local equivalences. Then the conclusion follows from (7.1.13), (7.1.14), and (7.2.2). □

7.3 Dimensional density structure

Definition 7.3.1. Let S be a fs log scheme. Then we put

$$S^{(n)} = \{s \in S : \operatorname{rk} \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \leq n\}.$$

We consider it as an open subscheme of S.

Definition 7.3.2. Let S be a fs log scheme. We denote by $D_d^{\dim}(S)$ the family of open immersions $U \to S$ such that $\dim(S - U) \leq \dim S - d$. It is called *dimensional density structure.* Note that $D_*^{\dim}(-)$ satisfies the conditions of [Voe10a, 2.20], so it is a density structure whose definition is in [loc. cit]. Note also that any element of $D_d^{\dim}(S)$ is an isomorphism if $d > \dim X$. We also denote by $D_{d,(n)}^{\dim}(S)$ the family of open immersions $U \to S$ such that $U \cup S^{(n-1)} \in D_d^{\dim}(S)$.

Proposition 7.3.3. *The Zariski cd-structure is bounded by $D_*^{\dim}(-)$.*

Proof. Let S be an $\mathscr{S}$-scheme, and we put $n = \dim S$. Consider a Zariski distinguished square

$$\begin{array}{ccc} W & \xrightarrow{u'} & V \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ U & \xrightarrow{u} & S \end{array}$$

of $\mathscr{S}$-schemes with $W_0 \in D_{d-1}^{\dim}(W)$, $U_0 \in D_d^{\dim}(U)$, and $V_0 \in D_d^{\dim}(V)$. Then

$$\dim(U - U_0),\ \dim(V - V_0) \leq n - d,$$

so $\dim(\overline{U - U_0} \cup \overline{V - V_0}) \leq n - d$ where the closures are computed in S. Replacing S by $S - \overline{U - U_0} - \overline{V - V_0}$, we may assume that $U = U_0$ and $V = V_0$.

We put

$$Z = W - W_0, \quad C = S - U, \quad D = S - V.$$

Then $\dim Z \leq n - d + 1$. If Z' is an irreducible component of Z, then

$$Z' \cap (C \cap D) \subset C \cap D = \emptyset,$$

so $Z' \not\subset C$ or $Z' \not\subset D$. Thus we have a decomposition

$$Z = Z_1 \cup Z_2$$

such that

(i) Z_i is a union of irreducible components of Z for each $i = 1, 2$,

(ii) if Z' is an irreducible component of Z_1 (resp. Z_2), then $Z' \not\subset Z_2$ (resp. $Z' \not\subset Z_1$), and $Z' \not\subset D$ (resp. $Z' \not\subset C$).

The Cartesian diagram

$$\begin{array}{ccc} W_0 & \longrightarrow & V - \overline{Z_2} \\ \downarrow & & \downarrow \\ U - \overline{Z_1} & \longrightarrow & (U - \overline{Z_1}) \cup (V - \overline{Z_2}) \end{array}$$

of $\mathscr{S}$-schemes where the closures are computed in S is a Zariski distinguished squares, and we have

$$\begin{aligned} (U - \overline{Z_1}) \cup (V - \overline{Z_2}) &= S - (S - (U - \overline{Z_1})) \cap (S - (V - \overline{Z_2})) \\ &= S - (C \cup \overline{Z_1}) \cap (D \cup \overline{Z_2}) \\ &= S - ((C \cup \overline{Z_2}) \cup (\overline{Z_1} \cap D) \cup (\overline{Z_1} \cap \overline{Z_2})). \end{aligned}$$

By construction,

$$\dim(C \cup \overline{Z_2}), \dim(\overline{Z_1} \cap D), \dim(\overline{Z_1} \cap \overline{Z_2}) \leq n - d,$$

so $(U - \overline{Z_1}) \cup (V - \overline{Z_2}) \in D_d^{\dim}(S)$. We are done by putting $S_1 = (U - \overline{Z_1}) \cup (V - \overline{Z_2})$, $U_1 = U - \overline{Z_1}$, and $V_1 = V - \overline{Z_2}$. □

Definition 7.3.4. Let S be a fs log scheme with a Zariski log structure. We denote by $D_{d,(n)}^{\mathrm{dv}}(S)$ the family of log étale monomorphisms $U \to S$ such that for some dividing cover $T \to S$, the projection $U \times_S T \to T$ is an open immersion such that $U \times_S T \in D_{d,(n)}^{\dim}(T)$.

Lemma 7.3.5. *Let $f : X \to S$ be a log étale monomorphism such that $S = S^{(n)}$. Then the induced morphism*

$$X - X^{(n-1)} \to S - S^{(n-1)}$$

has fibers of dimension 0.

Proof. The question is Zariski local on S and X, so by (3.4.1), we may assume that there is a chart $\theta : P \to Q$ with the conditions (i) and (ii) of (loc. cit). By [Ogu14, II.2.3.2], we may assume that P is exact at some point of S. Then $\dim P \leq n$, so the conclusion follows from the fact that the induced morphism

$$\mathbb{A}_Q - \mathbb{A}_Q^{(n-1)} \to \mathbb{A}_P - \mathbb{A}_P^{(n-1)}$$

has fibers of dimension 0. □

Lemma 7.3.6. *Let $f : X \to S$ be a morphism of $\mathscr{S}$-schemes such that $S = S^{(n)}$. Assume that f is a dividing cover. If $S_0 \in D^{\dim}_{d,(n)}(S)$, then $S_0 \times_S X \in D^{\dim}_{d,(n)}(X)$.*

Proof. We have $\dim S \leq \dim X$, and by (7.3.5), we have

$$\dim(S - S_0 - S^{(n-1)}) \geq \dim(X - S_0 \times_S X - X^{(n-1)}).$$

Thus $S_0 \times_S X \in D^{\dim}_{d,(n)}(X)$. □

Definition 7.3.7. We denote by $\mathscr{S}_{Zar}$ the full subcategory of $\mathscr{S}$ consisting of $\mathscr{S}$-schemes having Zariski log structures.

7.3.8. Recall from [Voe10a, 2.1] that a B.G.-functor on $\mathscr{S}_{Zar}$ with respect to the union of the dividing and Zariski cd-structures is a family of contravariant functors T_q, $q \geq 0$ from $\mathscr{S}_{Zar}$ to the category of pointed sets together with pointed maps with pointed maps $\partial_C : T_{q+1}(X') \to T_q(S)$ for any Zariski or dividing distinguished square

$$C = \begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes such that

(i) the morphisms ∂_C are natural with respect to morphisms of distinguished squares,

(ii) for any $q \geq 0$ the sequence of pointed sets

$$T_{q+1}(X') \to T_q(S) \to T_q(X) \times T_q(S')$$

is exact.

When C is a dividing distinguished square, the condition (ii) means that the morphism $T_q(S) \to T_q(X)$ is an isomorphism.

Definition 7.3.9. We denote by dZar the union of dividing and Zariski cd-structures on $\mathscr{S}$. Then we have dZar-topology on $\mathscr{S}$

Proposition 7.3.10. *For any B.G.-functor (T_q, ∂_q) on $\mathscr{S}_{Zar}$ such that $T_q(\emptyset)$ is trivial and that the dZar-sheaves associated with T_q are trivial, T_q is trivial for all q. Here, we say that a pointed set is trivial if it is the one element set.*

Proof. Assume that $T_q(S)$ is trivial if S is an $\mathscr{S}_{Zar}$-scheme with $S = S^{(n-1)}$. We will show that $T_q(S)$ is trivial if S is an $\mathscr{S}_{Zar}$-scheme with $S = S^{(n)}$. Assume that we have shown this. Then this completes the proof by induction on n since the basic case (when $S = S^{(-1)}$) is true by the assumption that $T_q(\emptyset)$ is trivial.

(I) *Reduction of S.* Assume $T_q(S) = 0$ for any $\mathscr{S}_{Zar}$-scheme S with a fs chart. For general S, it has a finite Zariski cover $\{U_i \to X\}_{i\in I}$ such that each U_i has a fs chart. Then any intersection of U_i has also a fs chart, so we can apply the condition (ii) of (7.3.8) for the Zariski cover $\{U_i \to X\}_{i\in I}$. Thus $T_q(S) = 0$, so we reduce to the case when S has a fs chart. Loosening this, we may assume that there is a log étale monomorphism $S \to V$ such that V has a fs chart. We denote this condition as (*).

(II) *Voevodsky's argument.* Let $a \in T_q(S)$ be an element. Consider the following assertion: for any Zariski or dividing distinguished square

$$C = \begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array} \tag{7.3.10.1}$$

of $\mathscr{S}_{Zar}$-schemes such that S satisfies (*) and that for some

$$S_0' \in D^{dv}_{d+1,(n)}(S'), \quad X_0 \in D^{dv}_{d+1,(n)}(X),$$

the restrictions of a to them are trivial, there is $S_1 \in D^{dv}_{d+1,(n)}(S)$ such that the restriction of a to it is trivial.

If it is proven, then the proof of [Voe10a, 3.2] shows that for any $\mathscr{S}_{Zar}$-scheme S with the condition (*), there is $S_1 \in D^{dv}_{\infty,(n)}(S)$ such that the pullback of a to S_1 is trivial. Then the morphism $S_1 \to S$ has a factorization

$$S_1 \xrightarrow{j} V \xrightarrow{p} S$$

where

(a) j is an open immersion such that $V = S_1 \cup V^{(n-1)}$,

(b) p is a dividing cover.

By induction on n, $T_q(S^{(n-1)})$ and $T_{q+1}(S^{(n-1)} \cap S_1)$ are trivial. Thus the condition (ii) of (7.3.8) implies that the pullback of a to V is trivial. Since p is a dividing cover, $a = 0$ by the condition (ii) of (loc. cit). Therefore $T_q(S) = 0$. Hence the remaining is to show the above assertion.

If C is a dividing distinguished square, then $X_0 \in D^{dv}_{d+1,(n)}(S)$, so we are done. Hence the remaining case is when C is a Zariski distinguished square.

(III) *Reduction to the case when $S_0' = S'$ and $X_0 = X$.* Consider the square C in (II). Assume that for some $S_0' \in D^{dv}_{d+1,(n)}(S')$ and $X_0 \in D^{dv}_{d+1,(n)}(X)$, the pullbacks of a to them are trivial. We want to show that there is $S_1 \in D^{dv}_{d+1,(n)}(S)$ such that the pullback of a to it is trivial.

By definition, there are dividing covers $T' \to S$ and $T'' \to S$ such that the projections

$$X_0 \times_S T' \to T', \quad S'_0 \times_S T'' \to T''$$

are open immersions, $X_0 \times_S T' \in D^{\dim}_{d+1,(n)}(X \times_S T')$, and $S'_0 \times_S T'' \in \mathrm{D}^{\dim}_{d+1,(n)}(S' \times_S T'')$. We put $T = T' \times_S T''$. Then the projections

$$X_0 \times_S T \to T, \quad S'_0 \times_S T \to T$$

are open immersions, and by (7.3.6), $X_0 \times_S T \in D^{\dim}_{d+1,(n)}(X \times_S T)$ and $S'_0 \times_S T \in \mathrm{D}^{\dim}_{d+1,(n)}(S' \times_S T)$. Since T also satisfies the condition (*), replacing S by T, we may assume that the morphisms $X_0 \to S$ and $S_0 \to S$ are open immersions, $X_0 \in D^{\dim}_{d+1,(n)}(X)$, and $S_0 \in D^{\dim}_{d+1,(n)}(S)$. Then since $X' \in D^{\dim}_{d,(n)}(X')$ and C is reducing with respect to $D^{\dim}_*$ by (7.3.3), there is $S_1 \in D^{\dim}_{d+1,(n)}(S)$ such that the projections

$$S' \times_S S_1 \to S', \quad X \times_S S_1 \to X$$

factor through S'_0 and X_0 respectively. Replacing S by S_1, we may assume $S'_0 = S'$ and $X_0 = X$, i.e., the pullbacks of a to S' and X are trivial.

(IV) *Final step of the proof.* Then for some $b \in T_{q+1}(X')$, we have $\partial_C(b) = a$ by the condition (ii) of (7.3.8). Now by induction on d, there is $X_0 \in D^{dv}_{d,(n)}(X')$ such that the pullback of b to X'_0 is trivial. By definition, there is a dividing cover $T' \to S$ such that the projection

$$X'_0 \times_S T' \to T'$$

is an open immersion and $X'_0 \times_S T' \in D^{\dim}_{d,(n)}(X' \times_S T')$. Then T' also satisfies the condition (*), so replacing S by T', we may assume $X'_0 \in D^{\dim}_{d,(n)}(X')$.

By induction on n, $T_q(X'^{(n-1)})$ and $T_{q+1}(X'_0 \cap X'^{(n-1)})$ are trivial. Then by the condition (ii) of (7.3.8), the pullback of b to $X'_0 \cup X'^{(n-1)}$ is trivial. Thus we can replace X'_0 by $X'_0 \cup X'^{(n-1)}$, so we may assume

$$X'_0 \in D^{\dim}_d(X').$$

Since C is reducing with respect to $D^{\dim}_*(X')$ by (7.3.3), there is $S_2 \in D^{\dim}_{d+1}(S)$ such that the projection

$$X' \times_S S_2 \to X'$$

factors through X'_0. In particular, the pullback of b to $X' \times_S S_2$ is trivial. Then the restriction of a to S_2 is trivial by the condition (ii) of (7.3.8). This completes the proof of the assertion given in (II). □

Definition 7.3.11. For any $\mathscr{S}_{Zar}$-scheme S, we denote by ft_{Zar}/S the family of morphisms $X \to S$ of finite type of $\mathscr{S}_{Zar}$-schemes. Then note that $\mathrm{D}(\mathrm{PSh}(ft_{Zar}, \Lambda))$ is a ft_{Zar}-premotivic category over $\mathscr{S}_{Zar}$.

Corollary 7.3.12. *Let S be an $\mathscr{S}_{Zar}$-scheme, and let K be an object of* $\mathrm{D}(\mathrm{PSh}(ft_{Zar}/S,\Lambda))$*. Then the following conditions are equivalent.*

(i) For any morphism $p : T \to S$ of $\mathscr{S}_{Zar}$-schemes, and for any dividing or Zariski distinguished square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of $\mathscr{S}_{Zar}$-schemes, the commutative diagram

$$\begin{array}{ccc} p_*p^*K & \xrightarrow{ad} & p_*g_*g^*p^*K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ p_*f_*f^*p^*K & \xrightarrow{ad} & p_*h_*h^*p^*K \end{array}$$

is homotopy Cartesian where $h = fg'$.

(ii) K satisfies t_{dZar}-descent.

Proof. We have that (1.3.7) needs (1.3.6), (1.3.6) needs [Voe10a, 3.8], [Voe10a, 3.8] needs [Voe10a, 3.5], and [Voe10a, 3.5] needs [Voe10a, 3.2]. However, (7.3.10) can be used to [Voe10a, 3.5] instead of [Voe10a, 3.2], so (1.3.7) for $P = dZar$ with the restriction to $\mathscr{S}_{Zar}$ is also true. □

Corollary 7.3.13. *Let S be an $\mathscr{S}_{Zar}$-scheme, and let K be an object of* $\mathrm{D}_{Zar}(ft_{Zar}/S,\Lambda)$*. Then the following conditions are equivalent.*

(i) For any morphism $p : T \to S$ of $\mathscr{S}_{Zar}$-schemes, and for any dividing cover $f : X \to T$ of $\mathscr{S}_{Zar}$-schemes over S, the morphism

$$p_*p^*K \xrightarrow{ad} p_*f_*f^*p^*K$$

in $\mathrm{D}_{Zar}(ft_{Zar}/S,\Lambda)$ is an isomorphism.

(ii) K satisfies t_{dZar}-descent.

Proof. The conclusion follows from (1.3.7) for $P = Zar$ and (7.3.12). □

Remark 7.3.14. Note that the condition (i) of (7.3.13) is equivalent to the condition that K is log''-local. The restatement is that K is log''-local if any only if k is $dZar$-local.

7.4 Localization property for $\mathrm{D}_{log',pw}(lSm,\Lambda)$

Proposition 7.4.1. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}_{Zar}$-schemes. Then the evident functor*

$$\Phi : \mathrm{Cov}_{dZar}(S) \to \mathrm{Cov}_{dZar}(Z)$$

is cofinal.

Proof. Let $g : Z' \to Z$ be a morphism of $\mathscr{S}_{Zar}$-schemes. We divide the question into 2 cases.
(I) *Zariski cover.* Assume that g is a strict étale cover. Then g has a refinement that is in the essential image of Φ by the proof of [Ayo07, 4.5.33].

(II) *Dividing cover.* Assume that g is a pullback of a proper birational morphism $M' \to \operatorname{spec} P'$ of fs monoschemes. We want to show that g has a refinement that is in the essential image of Φ. Let x be a geometric point of Z. Strict étale locally near x, we can choose a factorization

$$P' \to Q' \stackrel{\alpha'}{\to} \mathcal{M}_Z$$

of the homomorphism $P' \to \mathcal{M}_Z$ such that α' is a chart exact at x by [Ogu14, II.2.3.2]. Then strict étale locally on S near x, we may assume that S has a fs chart

$$\alpha : Q \to \mathcal{M}_S$$

neat at x by [Ogu14, II.2.3.7]. We put $N' = M' \times_{\operatorname{spec} P'} \operatorname{spec} Q'$. By (3.2.3), there is a proper birational morphism

$$N \to \operatorname{spec} Q$$

of fs monoschemes such that

$$Z' = Z \times_{\mathbb{A}_{Q'}} \mathbb{A}_{N'} \cong Z \times_{\mathbb{A}_Q} \mathbb{A}_N.$$

Then the image of the projection $S \times_{\mathbb{A}_Q} \mathbb{A}_N \to S$ via Φ is the cover $g : Z' \to Z$. □

Corollary 7.4.2. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}_{Zar}$-schemes. Then the functor*

$$i_* : \mathrm{D}_{t_\emptyset}(ft_{Zar}/Z, \Lambda) \to \mathrm{D}_{t_\emptyset}(ft_{Zar}/S, \Lambda)$$

preserves $dZar$-local equivalences.

Proof. It follows from (7.4.1) and the proof of [Ayo07, 4.5.32]. □

Corollary 7.4.3. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}_{Zar}$-schemes. Then the functor*

$$i_* : \mathrm{D}_{Zar}(ft_{Zar}/Z, \Lambda) \to \mathrm{D}_{Zar}(ft_{Zar}/S, \Lambda)$$

preserves log''-weak equivalences.

Proof. By (7.3.14), log''-weak equivalences and t_{dZar}-local equivalences are equivalent. Then the conclusion follows from (7.4.2). □

Corollary 7.4.4. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes. Then the functor*

$$i_* : \mathrm{D}_{\mathbb{A}^1,pw}(ft/Z, \Lambda) \to \mathrm{D}_{\mathbb{A}^1,pw}(ft/S, \Lambda)$$

preserves log'-weak equivalences.

Proof. By (1.6.4), it suffices to show that for any dividing cover $W \to V$ in ft/Z, the induced morphism
$$i_* M_Z(W) \to i_* M_Z(V)$$
in $\mathrm{D}_{pw}(ft/S, \Lambda)$ is an isomorphism. The question is strict étale local on S and V, so we may assume that S and V have Zariski log structures. Then the conclusion follows from (7.4.3). □

Corollary 7.4.5. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, let $j : U \to S$ be its complement, and let $f : V \to Z$ be a log smooth morphism of $\mathscr{S}$-schemes. Then*

(1) the functor
$$i_* : \mathrm{D}_{log',pw}(ft/Z, \Lambda) \to \mathrm{D}_{log',pw}(ft/S, \Lambda)$$
admits a right adjoint,

(2) there is a distinguished triangle
$$j_\sharp j^* M_S(X) \longrightarrow M_S(X) \longrightarrow i_* i^* M_S(X) \longrightarrow j_\sharp j^* M_S(X)[1]$$
in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$.

Proof. Then the conclusion follows from (7.2.2), (7.2.8), and (7.4.4). □

Theorem 7.4.6. *The localization property is satisfied for*
$$\mathrm{D}_{log',pw}(lSm, \Lambda).$$

Proof. By (7.4.5) and the proof of [CD12, 2.3.15(iv)], the remaining is to show that for any strict closed immersion $i : Z \to S$ of $\mathscr{S}$-schemes, the functor
$$i_* : \mathrm{D}_{log',pw}(lSm/Z, \Lambda) \to \mathrm{D}_{log',pw}(lSm/S, \Lambda)$$
is conservative (here, the well-generatedness of the assumption of [loc. cit] can be ignored because the conclusion of [CD12, 1.3.18] holds for $\mathrm{D}_{log',pw}(lSm, \Lambda)$ by construction). The conservativity follows from (3.1.5) and the proof of [CD12, 2.3.16]. □

7.5 Localization property for $\mathrm{D}_{log,pw}(lSm, \Lambda)$

Proposition 7.5.1. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes. Then the functor*
$$i_* : \mathrm{D}_{\mathbb{A}^1,pw}(ft/Z, \Lambda) \to \mathrm{D}_{\mathbb{A}^1,pw}(ft/S, \Lambda)$$
preserves log-weak equivalences.

Proof. Consider the following situations for morphisms
$$W' \xrightarrow{h'} W \xrightarrow{g'} V \xrightarrow{f'} Z$$
of $\mathscr{S}$-schemes.

(c) The morphism f' is *log smooth*, the morphism g' is an exact log smooth morphism, and the morphism h' is the verticalization $W^{\mathrm{ver}} \to Y$ of W via $f'g'$.

(d) The morphism f' is *log smooth*, the $\mathscr{S}$-scheme V has a neat fs chart P, and the morphism g' is the projection

$$V \times_{\mathbb{A}_P} \mathbb{A}_Q \to V$$

where the homomorphism $\theta : P \to Q$ is a locally exact vertical homomorphism of fs monoids such that g is an exact log smooth morphism. The morphism h is the morphism

$$V \times_{\mathbb{A}_P} \mathbb{A}_{Q_G} \to V \times_{\mathbb{A}_P} \mathbb{A}_Q$$

induced by the localization $Q \to Q_G$ where G is a maximal θ-critical face of Q.

By (1.6.4) and (7.4.4), the remaining is to show that for each type (c) and (d), the morphism

$$i_*M_Z(W') \to i_*M_Z(W)$$

in $\mathrm{D}_{\mathbb{A}^1,pw}(ft/S, \Lambda)$ is a *log*-weak equivalence.

Strict étale locally on V, we will construct the following diagram

$$\begin{array}{ccccccc} W' & \xrightarrow{h'} & W & \xrightarrow{g'} & V & \xrightarrow{f'} & Z \\ \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle i} \\ Y' & \xrightarrow{h} & Y & \xrightarrow{g} & X & \xrightarrow{f} & S \end{array} \tag{7.5.1.1}$$

of $\mathscr{S}$-schemes such that

(i) the sequence $Y' \to Y \to X \to S$ is one of the types (c) and (d) in (1.7.2),

(ii) each square is Cartesian.

By (3.1.5), strict étale locally on V, there is a Cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{f'} & Z \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & S \end{array}$$

of $\mathscr{S}$-schemes such that f is log smooth. In the case (c), by (loc. cit), there is a Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{g'} & V \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow{g} & X \end{array}$$

of $\mathscr{S}$-schemes such that g is exact log smooth. Then to show the claim, we only need to put $Y' = Y^{\mathrm{ver/X}}$.

In the case (d), let x be a point in $v(V)$. We may assume that P is exact at x by [Ogu14, II.2.3.2]. we denote by P' the submonoid of Q consisting of elements $q \in Q$ such that $nq \in P + Q^*$ for some $n \in \mathbb{N}$. Then P' is a fs monoid by Gordon's lemma [Ogu14, I.2.3.17]. The morphism $P' \to Q$ is locally exact by [Ogu14, I.4.6.5], so the induced morphism

$$c : W = V \times_{\mathbb{A}_P} \mathbb{A}_Q \to V \times_{\mathbb{A}_P} \mathbb{A}_{P'}$$

is an open morphism by [Nak09, 5.7]. We denote by V' the image of c. Then the induced morphism $W \to V'$ is an exact log smooth morphism. Moreover, the order of the torsion part of the cokernel of $P^{\rm gp} \to P'^{\rm gp}$ is invertible in $\mathcal{O}_{V'}$, so the induced morphism $V' \to V$ is a Kummer log smooth morphism. Hence replacing $(W \to V \to Z, P \to Q)$ by $(W \to V' \to Z, P' \to Q)$, we may assume that the cokernel of $\theta^{\rm gp}$ is torsion free. In particular, there is a homomorphism $\varphi : Q^{\rm gp} \to P^{\rm gp}$ such that $\varphi \circ \theta^{\rm gp} = {\rm id}$. By [Ogu14, II.2.3.7], we may assume that X has a fs chart P'' neat at $\overline{x}$. Then by (3.2.2), there is a coCartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & P'' \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta''} \\ Q & \longrightarrow & Q'' \end{array}$$

of fs monoids where the upper arrow is the quotient homomorphism $P \to \overline{P} = P''$ such that we have an isomorphism

$$Z \times_{\mathbb{A}_P} \mathbb{A}_Q \cong Z \times_{\mathbb{A}_{P''}} \mathbb{A}_{Q''}.$$

If G'' denote the face of Q'' induced by G, then G'' is also a maximal θ''-critical face of Q''. Thus to show the claim, we only need to put

$$Y = X \times_{\mathbb{A}_{P''}} \mathbb{A}_{Q''}, \quad Y' = X \times_{\mathbb{A}_{P''}} \mathbb{A}_{Q''_{G''}}.$$

We have constructed (7.5.1.1). Then in the commutative diagram

$$\begin{array}{ccccccc} j_\sharp j^* M_S(Y') & \longrightarrow & M_S(Y') & \longrightarrow & i_* M_Z(V') & \longrightarrow & j_\sharp j^* M_S(Y')[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j_\sharp j^* M_S(Y) & \longrightarrow & M_S(Y) & \longrightarrow & i_* M_Z(V) & \longrightarrow & j_\sharp j^* M_S(Y)[1] \end{array}$$

of $\mathscr{S}$-schemes, the rows are distinguished triangles by (7.2.8). Moreover, the first and second vertical arrows are *log*-weak equivalences by construction. Thus the third vertical arrow is a *log*-weak equivalence. □

Corollary 7.5.2. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes, let $j : U \to S$ be its complement, and let $f : V \to Z$ be a log smooth morphism of $\mathscr{S}$-schemes. Then*

(1) the functor

$$i_* : {\rm D}_{log,pw}(ft/Z, \Lambda) \to {\rm D}_{log,pw}(ft/S, \Lambda)$$

admits a right adjoint,

(2) there is a distinguished triangle

$$j_\sharp j^* M_S(X) \longrightarrow M_S(X) \longrightarrow i_* i^* M_S(X) \longrightarrow j_\sharp j^* M_S(X)[1]$$

in $\mathrm{D}_{log,pw}(ft/S, \Lambda)$.

Proof. Then the conclusion follows from (7.2.2), (7.2.8), and (7.5.1). □

Theorem 7.5.3. *The localization property is satisfied for*

$$\mathrm{D}_{log,pw}(lSm, \Lambda), \quad \mathrm{D}_{log,pw}(-, \Lambda).$$

Proof. By (7.5.2) and the proof of [CD12, 2.3.15(iv)], the remaining is to show that for any strict closed immersion $i : Z \to S$ of $\mathscr{S}$-schemes, the functors

$$i_* : \mathrm{D}_{log,pw}(lSm/Z, \Lambda) \to \mathrm{D}_{log,pw}(lSm/S, \Lambda),$$

$$i_* : \mathrm{D}_{log,pw}(-/Z, \Lambda) \to \mathrm{D}_{log,pw}(-/S, \Lambda),$$

are conservative (here, the well-generatedness of the assumption of [loc. cit] can be ignored because the conclusion of [CD12, 1.3.18] holds for $\mathrm{D}_{log,pw}(lSm, \Lambda)$ and $\mathrm{D}_{log,pw}(-, \Lambda)$ by construction). The conservativity follows from (3.1.5) and the proof of [CD12, 2.3.16]. □

7.5.4. We have proven the localization property for $\mathrm{D}_{log,pw}(-, \Lambda)$. For future usage, we will construct *log‴*-weak equivalences and discuss the localization property for $\mathrm{D}_{log''',pw}(eSm, \Lambda)$.

Definition 7.5.5. For an $\mathscr{S}$-scheme S, we will consider the following situations for morphisms

$$Y' \xrightarrow{h} Y \xrightarrow{g} X \xrightarrow{f} S$$

of $\mathscr{S}$-schemes.

(a) The morphism f is of finite type, the morphism g is the identity, and the morphism h is the projection $Y \times \mathbb{A}^1 \to Y$.

(b) The morphism f is of finite type, the morphism g is the identity, and the morphism h is a dividing cover.

(c)' The morphism f is *exact log smooth*, the morphism g is an exact log smooth morphism, and the morphism h is the verticalization $Y^{\mathrm{ver}} \to Y$ of X via fg.

(d)' The morphism f is *exact log smooth*, the $\mathscr{S}$-scheme X has a neat fs chart P, and the morphism g is the projection

$$X \times_{\mathbb{A}_P} \mathbb{A}_Q \to X$$

where the homomorphism $\theta : P \to Q$ is a locally exact vertical homomorphism of fs monoids such that g is an exact log smooth morphism. The morphism h is the morphism

$$X \times_{\mathbb{A}_P} \mathbb{A}_{Q_F} \to X \times_{\mathbb{A}_P} \mathbb{A}_Q$$

induced by the localization $Q \to Q_F$ where F is a maximal θ-critical face of Q.

Let $\mathscr{T}$ be a τ-twisted $\mathscr{P}$-premotivic triangulated category over $\mathscr{S}$. Then let $\mathscr{W}_{log''',S}$ denote the family of morphisms

$$M_S(Y')\{i\} \to M_S(Y)\{i\}$$

in $\mathscr{T}(S)$ where $i \in \tau$ and the morphism $Y' \to Y$ is of the type (a), (b), (c)', and (d'). Note that $\mathscr{W}_{log'''}$ is stable by the operations $f_\sharp$ for $f \in eSm$. To ease the notations, we often remove $\mathscr{W}$ in the notation.

Proposition 7.5.6. *Let $i : Z \to S$ be a strict closed immersion of $\mathscr{S}$-schemes. Then the functor*

$$i_* : \mathrm{D}_{\mathbb{A}^1,pw}(ft/Z,\Lambda) \to \mathrm{D}_{\mathbb{A}^1,pw}(ft/S,\Lambda)$$

preserves log'''-weak equivalences.

Proof. The proof is parallel to the proof of (7.5.1). □

Theorem 7.5.7. *The localization property is satisfied for*

$$\mathrm{D}_{log''',pw}(eSm,\Lambda).$$

Proof. The proof is parallel to the proof of (7.5.3). □

7.6 Plain lower descent

7.6.1. As promised in (1.7.8), we will show the following result.

Proposition 7.6.2. *Let S be an $\mathscr{S}$-scheme. Consider the adjunction*

$$\rho_\sharp : \mathrm{D}_{log',pw}(lSm,\Lambda) \leftrightarrows \mathrm{D}_{log',pw}(ft,\Lambda) : \rho^*$$

of lSm-premotivic triangulated categories. For any object K of $\mathrm{D}_{log',pw}(lSm/S,\Lambda)$, the image $\rho_\sharp K$ satisfies the plain lower descent.

Proof. Let $p : T \to S$ be a morphism of finite type of $\mathscr{S}$-schemes, and consider a plain lower distinguished square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of $\mathscr{S}$-schemes. By (1.3.8), it suffices to show that the commutative diagram

$$\begin{array}{ccc} p^*\rho_\sharp K & \xrightarrow{ad} & f_*f^*p^*\rho_\sharp K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ g_*g^*p^*\rho_\sharp K & \xrightarrow{ad} & h_*h^*p^*\rho_\sharp K \end{array}$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ is homotopy Cartesian where $h = fg'$. By (7.4.6), we have a distinguished triangle

$$u_\sharp u^* p^* \rho_\sharp K \xrightarrow{ad'} p^* \rho_\sharp K \xrightarrow{ad} f_* f^* p^* \rho_\sharp K \longrightarrow u_\sharp u^* p^* \rho_\sharp K[1]$$

where u denotes the complement of f, so $f_* f^* p^* \rho_\sharp K$ is in the essential image of $\rho_\sharp$. In particular, since $\rho_\sharp$ is fully faithful, the natural transformation

$$\rho_\sharp \rho^* f_* f^* p^* \rho_\sharp K \xrightarrow{ad'} f_* f^* p^* \rho_\sharp K$$

is an isomorphism. The same is true for $p^* \rho_\sharp K$, $g_* g^* p^* \rho_\sharp K$, and $h_* h^* p^* \rho_\sharp K$, so it suffices to show that the commutative diagram

$$\begin{array}{ccc} \rho^* p^* \rho_\sharp K & \xrightarrow{ad} & \rho^* f_* f^* p^* \rho_\sharp K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ \rho^* g_* g^* p^* \rho_\sharp K & \xrightarrow{ad} & \rho^* h_* h^* p^* \rho_\sharp K \end{array}$$

in $\mathrm{D}_{log',pw}(lSm/S,\Lambda)$ is homotopy Cartesian. Since ρ^* commutes with f_*, g_*, h_* , and $\rho_\sharp$ commutes with p^*, f^*, g^*, h^*, it suffices to show that the commutative diagram

$$\begin{array}{ccc} \rho^* \rho_\sharp p^* K & \xrightarrow{ad} & f_* \rho^* \rho_\sharp f^* p^* K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ g_* \rho^* \rho_\sharp g^* p^* K & \xrightarrow{ad} & h_* \rho^* \rho_\sharp h^* p^* K \end{array}$$

in $\mathrm{D}_{log',pw}(lSm/S,\Lambda)$ is homotopy Cartesian. Then since $\rho_\sharp$ is fully faithful, it suffices to show that the commutative diagram

$$\begin{array}{ccc} p^* K & \xrightarrow{ad} & f_* f^* p^* K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ g_* g^* p^* K & \xrightarrow{ad} & h_* h^* p^* K \end{array}$$

in $\mathrm{D}_{log',pw}(lSm/S,\Lambda)$ is homotopy Cartesian. It follows from (2.6.7). □

7.6.3. Note that we have discussed in (1.7.8) that for $\mathscr{W} = \mathscr{W}_{log'}, \mathscr{W}_{log}$ and $\mathscr{P} = lSm, eSm$, (7.6.2) implies that we have an equivalence

$$\mathrm{D}_{\mathscr{W},pw}(\mathscr{P},\Lambda) \cong \mathrm{D}_{\mathscr{W},qw}(\mathscr{P},\Lambda)$$

of $\mathscr{P}$-premotivic triangulated categories. In particular, $\mathrm{D}_{\mathscr{W},pw}(\mathscr{P},\Lambda)$ is compactly generated by $\mathscr{P}$ and τ.

Chapter 8

Verification of the remaining axioms

8.0.1. In this chapter, we complete the proof that $\mathrm{D}_{log,pw}(-,\Lambda)$ is a log motivic triangulated category.

8.1 Isomorphisms in $\mathrm{D}_{log',pw}(ft,\Lambda)$

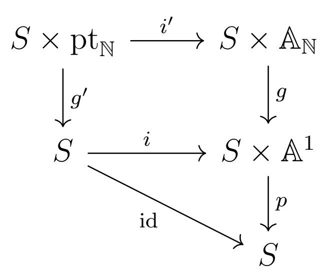
8.1.1. We will study various isomorphisms in $\mathrm{D}_{log',pw}(ft,\Lambda)$. Using these, in (8.1.15), we will prove that the functor

$$g^* : \mathrm{D}_{log',pw}(eSm/S,\Lambda) \to \mathrm{D}_{log',pw}(eSm/Y,\Lambda)$$

admits a left adjoint where S is an $\mathscr{S}$-scheme with a fs chart $\mathbb{N}$ and $g : Y \to S$ denotes the projection $S \times \mathrm{pt}_{\mathbb{N}} \to S$.

Proposition 8.1.2. *The ft-premotivic triangulated category $\mathrm{D}_{log',pw}(ft,\Lambda)$ satisfies (Htp–6).*

Proof. Let S be an $\mathscr{S}$-scheme, and consider the commutative diagram



of $\mathscr{S}$-schemes where

(i) g denotes the morphism removing the log structure,

(ii) the inside square is Cartesian,

(iii) p denotes the projection,

(iv) i denotes the 0-section.

By (7.6.3) and (1.3.8), for any object K of $\mathrm{D}_{log',pw}(ft/S,\Lambda)$, the diagram

$$\begin{array}{ccc} p_*p^*K & \xrightarrow{ad} & p_*i_*i^*p^*K \\ \downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\ p_*g_*g^*p^*K & \xrightarrow{ad} & p_*g_*i'_*i'^*g^*p^*K \end{array}$$

is homotopy Cartesian. The upper horizontal arrow is an isomorphism by (Htp–1), so the lower horizontal arrow is an isomorphism, which is (Htp–6). □

Proposition 8.1.3. *Let S be an $\mathscr{S}$-scheme, and let $\theta : \mathbb{N} \to P$ be a homomorphism of fs monoid such that $\theta(1)$ is not invertible, i.e., $\mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}$ is nonempty. Then the morphism*

$$M_{ft/S}(S \times \underline{\mathbb{A}_{(P,P^+)}}) \to M_{ft/S}(S \times \underline{\mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}})$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ induced by the closed immersion $i' : \mathbb{A}_{(P,P^+)} \to \mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}$ is an isomorphism.

Proof. Let I denote the ideal $(\theta(1))$ of P. Then $\mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N} \cong \mathbb{A}_{(P,I)}$. We argue as in [Ogu14, I.3.2.1.3]. By [Ogu14, I.2.2.1], we can choose a homomorphism $h : P \to \mathbb{N}$ such that $h^{-1}(0) = P^*$. Then we have a morphism

$$m' : \underline{\mathbb{A}_{(P,I)}} \times \mathbb{A}^1 \to \underline{\mathbb{A}_{(P,I)}}$$

induced by the homomorphism

$$\mathbb{Z}[P]/(I) \to \mathbb{Z}[P,t]/(I), \quad (p \in P) \mapsto pt^{h(p)}.$$

When we compose m' with the 0-sections and 1-sections, we get morphisms

$$\underline{\mathbb{A}_{(P,I)}} \to \underline{\mathbb{A}_{(P,P^+)}} \to \underline{\mathbb{A}_{(P,I)}}, \quad \underline{\mathbb{A}_{(P,I)}} \xrightarrow{\mathrm{id}} \underline{\mathbb{A}_{(P,I)}}.$$

Thus the closed immersions i' is an $\mathbb{A}^1$-homotopy equivalence, and this proves the statement. □

Proposition 8.1.4. *Let S be an $\mathscr{S}$-scheme, and let $\theta : \mathbb{N} \to P$ be a homomorphism of fs monoid. Then the morphism*

$$M_{ft/S}(S \times \underline{\mathbb{A}_{(P,P^+)}}) \to M_{ft/S}(S \times \underline{\mathbb{A}_P})$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ induced by the closed immersions $i : \mathbb{A}_{(P,P^+)} \to \mathbb{A}_P$ is an isomorphism.

Proof. As in the proof of (8.1.3), we can show that i is an $\mathbb{A}^1$-homotopy equivalence. □

Proposition 8.1.5. *Let S be an $\mathscr{S}$-scheme, and let $\theta : \mathbb{N} \to P$ be a homomorphism of fs monoid such that $\theta(1)$ is not invertible, i.e., $\mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}$ is nonempty. Then the morphism*

$$M_{ft/S}(S \times \mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}) \to M_{ft/S}(S \times \mathbb{A}_P)$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ induced by the projection $S \times \mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N} \to S \times \mathbb{A}_P$ is an isomorphism.

Proof. We put $p = \theta(1)$. We will use induction on $r'(P)$ where $r'(P)$ denotes the number of rays of P not containing p. Let $p_1, \ldots, p_r$ denote the rays of P not containing p.

(I) *Reduction method.* Let $\mu : P \to \overline{P}$ denote the quotient homomorphism. For $s = 0, \ldots, r$, we put

$$\begin{gathered}
R_s = (\mathbb{N}\overline{p} + \mathbb{N}(-\overline{p_1}) + \cdots + \mathbb{N}(-\overline{p_s}) + \mathbb{N}\overline{p_{s+1}} + \cdots + \mathbb{N}\overline{p_r})_\mathbb{Q} \cap \overline{P}^{\mathrm{gp}}, \\
R'_s = (\mathbb{N}\overline{p} + \mathbb{N}(-\overline{p_1}) + \cdots + \mathbb{N}(-\overline{p_{s-1}}) + \mathbb{N}\overline{p_{s+1}} + \cdots + \mathbb{N}\overline{p_r})_\mathbb{Q} \cap \overline{P}^{\mathrm{gp}}, \\
R''_s = (\mathbb{N}\overline{p} + \mathbb{N}(-\overline{p_1}) + \cdots + \mathbb{N}(-\overline{p_{s-1}}) + \mathbb{Z}\overline{p_s} + \mathbb{N}\overline{p_{s+1}} + \cdots + \mathbb{N}\overline{p_r})_\mathbb{Q} \cap \overline{P}^{\mathrm{gp}}. \\
P_s = \mu^{-1}(R_s), \quad P'_s = \mu^{-1}(R'_s), \quad P''_s = \mu^{-1}(R''_s).
\end{gathered}$$

Then the gluing of $\mathbb{A}_{P_s}$ and $\mathbb{A}_{P_{s-1}}$ along $\mathbb{A}_{P''_s}$ is a dividing cover of $\mathbb{A}_{P'_s}$, so we have the commutative diagram

$$\begin{array}{ccccccc}
M_{ft/S}(S \times \mathbb{A}_{P''_s} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}) & \longrightarrow & M_{ft/S}(S \times \mathbb{A}_{P_s} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}) \oplus M_{ft/S}(S \times \mathbb{A}_{P_{s-1}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}) & \longrightarrow & M_{ft/S}(S \times \mathbb{A}_{P'_s} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}) & \longrightarrow & M_{ft/S}(S \times \mathbb{A}_{P''_s} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{ft/S}(S \times \mathbb{A}_{P''_s}) & \longrightarrow & M_{ft/S}(S \times \mathbb{A}_{P_s}) \oplus M_{ft/S}(S \times \mathbb{A}_{P_{s-1}}) & \longrightarrow & M_{ft/S}(S \times \mathbb{A}_{P'_s}) & \longrightarrow & M_{ft/S}(S \times \mathbb{A}_{P''_s})[1]
\end{array}$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ where the rows are distinguished triangles by (2.2.3(8)). Assume

$$p \notin P^*_{s-1}, P^*_s, P'^*_s, P''^*_s.$$

Since $r'(P'_s), r'(P''_s) < r'(P)$, by induction on $r'(P)$, the question is true for P'_s and P''_s. Then the above diagram shows that the question for P_{s-1} is equivalent to the question for P_s.

(II) *Reduction of P.* Assume $r > \dim P - 1$. Then $\{\overline{p}, \overline{p_1}, \ldots, \overline{p_r}\}$ is linearly dependent over $\mathbb{Q}$, so we may assume

$$a_1 p_1 + \cdots + a_t p_t = ap + a_{t+1} p_{t+1} + \cdots + a_r p_r + p'$$

for some $0 \leq t \leq r$, $p' \in P^*$, and $a, a_1, \ldots, a_r \in \mathbb{N}$ with $a_t \neq 0$. In P_{t-1}, p_t is not a ray since

$$a_t p_t = a_1(-p_1) + \cdots + a_{t-1}(-p_{t-1}) + a_{t+1} p_{t+1} + \cdots + a_r p_r + p.$$

Thus we can choose least u such that $r'(P_u) < r'(P)$.

Assume $p \in P^*_s$ for some $1 \leq s \leq u$. Then

$$-bp = b_1(-p_1) + \cdots + b_s(-p_s) + b_{s+1} p_{s+1} + \cdots + b_r p_r$$

for some $b, b_1, \ldots, b_r \in \mathbb{N}$ with $b \neq 0$. In P_{s-1}, p_s is not a ray since

$$b_s p_s = bp + b_1(-p_1) + \cdots + b_{s-1}(-p_{s-1}) + b_{s+1} p_{s+1} + \cdots + b_r p_r,$$

contradicting to the fact that $r'(P_{s-1}) = r'(P)$. Thus $p \notin P_1^*, \dots, P_u^*$.

For $1 \leq s \leq u$, $P'_s \subset P_s$, so $p \notin P'^*_s$. If $p \in P''^*_s$, then for some

$$c_s \in \mathbb{Z}, \quad c, c_1, \dots, c_{s-1}, c_{s+1}, \dots, c_r \in \mathbb{N}, \quad c \neq 0,$$

we have

$$-cp = c_1(-p_1) + \cdots + c_{s-1}(-p_{s-1}) + c_s p_s + \cdots + c_r p_r.$$

This means $p \in P_s^*$ or $p \in P_{s-1}^*$, which is a contradiction. Thus $p \notin P''^*_s$.

Because $P_0 = P$ and $p \notin P_s^*, P'^*_s, P''^*_s$ for $1 \leq s \leq u$, the question is true for P if and only if the question is true for P_u by (I). Since $r'(P_u) < r'(P)$, by induction on $r'(P)$, the question is true for P_u.

Hence we reduce to the case when $r = \dim P - 1$. Then $P_{\mathbb{Q}}$ is simplicial, and p is a ray of P. The question is winding local on $\mathbb{A}_P$, so we may further assume that P is isomorphic to $\mathbb{N}^{r+1}$ by [CLS11, 11.1.9]. Choose minimal q such that $p \in \langle q \rangle$. Then $\overline{P} \cong \langle q \rangle \oplus \mathbb{N}^r$. By (3.5.2), we have an isomorphism

$$P \cong \langle q \rangle \oplus \mathbb{N}^r \oplus P^*.$$

(III) *Final step of the proof.* We put $P' = P/\langle q \rangle$. Then we have isomorphisms

$$S \times \mathbb{A}_P \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}} \cong S \times \mathbb{A}_{\langle q \rangle} \times_{\mathbb{A}_\eta, \mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}} \times \mathbb{A}_{P'}, \quad S \times \mathbb{A}_P \cong S \times \mathbb{A}_{\mathbb{N}} \times \mathbb{A}_{P'}$$

where $\eta : \mathbb{N} \to \langle q \rangle$ denotes the homomorphism $1 \mapsto p$. Hence replacing S by $S \times \mathbb{A}_{P'}$, we may assume $P = \mathbb{N}$. Then $\mathbb{A}_P \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}$ has $\mathrm{pt}_{\mathbb{N}}$ as a strict closed subscheme. Thus by (2.2.3(4)), it suffices to show that the morphism

$$M_{ft/S}(S \times \mathrm{pt}_{\mathbb{N}}) \to M_{ft/S}(S \times \mathbb{A}_{\mathbb{N}})$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ induced by the 0-section $\mathrm{pt}_{\mathbb{N}} \to \mathbb{A}_{\mathbb{N}}$ is an isomorphism. This follows from (8.1.2). □

Proposition 8.1.6. *Let S be an $\mathscr{S}$-scheme, and let $u : M \to \operatorname{spec} \mathbb{N}$ be a vertical morphism of fs monoschemes. Then we have the distinguished triangle*

$$\begin{aligned} &M_{ft/S}(S \times \mathbb{A}_M \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}) \to M_{ft/S}(S \times \mathbb{A}_M) \oplus M_{ft/S}(S \times \underline{\mathbb{A}_M \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}}) \\ \to &M_{ft/S}(S \times \underline{\mathbb{A}_M}) \to M_{ft/S}(S \times \mathbb{A}_M \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}})[1] \end{aligned}$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$.

Proof. The question is Zariski local on M, so we may assume that $M = \operatorname{Spec} P$ where P is a fs monoid. Let $\theta : \mathbb{N} \to P$ the morphism induced by u. When $\mathbb{A}_P \times_{\mathbb{N}} \mathrm{pt}_{\mathbb{N}} = \emptyset$, P is a group since θ is vertical. Thus $S \times \mathbb{A}_P = S \times \underline{\mathbb{A}_P}$. Hence the remaining case is when $\mathbb{A}_P \times_{\mathbb{N}} \mathrm{pt}_{\mathbb{N}} \neq \emptyset$.

In this case, the morphisms

$$M_{ft/S}(S \times \mathbb{A}_P \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}) \to M_{ft/S}(S \times \mathbb{A}_P),$$

$$M_{ft/S}(S \times \underline{\mathbb{A}_M \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}}) \to M_{ft/S}(S \times \underline{\mathbb{A}_M})$$

are isomorphisms by (8.1.5), (8.1.3), and (8.1.4). This implies the statement. □

Corollary 8.1.7. *Under the notations and hypotheses of (8.1.5), if $\theta : \mathbb{N} \to P$ is vertical, then the morphism*

$$M_{ft/S}(S \times \underline{\mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}}) \to M_{ft/S}(S \times \underline{\mathbb{A}_P})$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ *induced by the projection* $S \times \mathbb{A}_P \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N} \to S \times \mathbb{A}_P$ *is an isomorphism.*

Proof. It follows from (8.1.5) and (8.1.6). □

Proposition 8.1.8. *Under the notations and hypotheses of (8.1.5), the morphism*

$$M_{ft/S}(S \times \mathbb{A}_P^{\mathrm{ver}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}) \to M_{ft/S}(S \times \mathbb{A}_P^{\mathrm{ver}})$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ *induced by the closed immersion* $\mathbb{A}_P^{\mathrm{ver}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N} \to \mathbb{A}_P^{\mathrm{ver}}$ *is an isomorphism. Here,* $\mathbb{A}_P^{\mathrm{ver}}$ *denotes the verticalization of* $\mathbb{A}_P$ *via the morphism* $\mathbb{A}_\theta : \mathbb{A}_P \to \mathbb{A}_\mathbb{N}$.

Proof. (I) *Usage of (8.1.6).* By (8.1.6), we reduce to showing that the morphism

$$M_{ft/S}(S \times \underline{\mathbb{A}_P^{\mathrm{ver}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}}) \to M_{ft/S}(S \times \underline{\mathbb{A}_P^{\mathrm{ver}}})$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ induced by the projection $S \times \mathbb{A}_P \to S$ is an isomorphism. When P is already vertical over $\mathbb{N}$, we are done by (8.1.7). Hence we will assume that P is not vertical.

(II) *Dual cones.* We denote by $(\mathrm{Spec}\, P)^{\mathrm{ver}}$ the set of faces F such that P_F is vertical over $\mathbb{N}$. Then $(\mathrm{Spec}\, P)^{\mathrm{ver}}$ consists of the faces F of P such that $\langle F + \theta(1) \rangle = P$.

We also have the one-to-one correspondence

$$\Phi : \mathrm{Spec}\, P \to \mathrm{Spec}\, \overline{P}^\vee, \quad F \mapsto (\overline{P}_{\overline{F}})^\vee.$$

Then $\Phi((\mathrm{Spec}\, P)^{\mathrm{ver}})$ consists of the faces G of $\overline{P}^\vee$ such that $G \cap \Phi(\langle \theta(1) \rangle) = \langle 0 \rangle$.

(III) *Zariski descent.* Let $F_1, \ldots, F_r$ denote the elements of $(\mathrm{Spec}\, P)^{\mathrm{ver}}$, and for any nonempty subset $I = \{i_1, \ldots, i_l\}$ of $\{1, \ldots, r\}$, let F_I denote the face $\langle F_{i_1} + \cdots + F_{i_l} \rangle$ of P. Then we denote by $\mathscr{V}$ the set of nonempty subsets $I \subset \{1, \ldots, r\}$ such that $\mathbb{A}_{P_{F_I}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N} \neq \emptyset$. We have $\mathbb{A}_P^{\mathrm{ver}} = \mathbb{A}_{P_{F_1}} \cup \cdots \cup \mathbb{A}_{P_{F_r}}$, so the motives

$$M_{ft/S}(S \times \underline{\mathbb{A}_P^{\mathrm{ver}}}), \quad M_{ft/S}(S \times \underline{\mathbb{A}_P^{\mathrm{ver}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}})$$

are the homotopy colimits of the Čech-type sequences

$$\bigoplus_{|I|=r} M_{ft/S}(S \times \underline{\mathbb{A}_{P_{F_I}}}) \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=1} M_{ft/S}(S \times \underline{\mathbb{A}_{P_{F_I}}}), \tag{8.1.8.1}$$

$$\bigoplus_{|I|=r} M_{ft/S}(S \times \underline{\mathbb{A}_{P_{F_I}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}}) \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=1} M_{ft/S}(S \times \underline{\mathbb{A}_{P_{F_I}} \times_{\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N}}) \tag{8.1.8.2}$$

respectively. Then by (8.1.7), the sequence (8.1.8.2) is isomorphic to

$$\bigoplus_{|I|=r, I \in \mathscr{V}} M_{ft/S}(S \times \underline{\mathbb{A}_{P_{F_I}}}) \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=1, I \in \mathscr{V}} M_{ft/S}(S \times \underline{\mathbb{A}_{P_{F_I}}}). \tag{8.1.8.3}$$

Comparing (8.1.8.1) and (8.1.8.3), we get the following result: to show the question, it suffices to show that the homotopy colimit of the sequence

$$\bigoplus_{|I|=r, I\notin \mathscr{V}} M_{ft/S}(S\times \underline{\mathbb{A}_{P_{F_I}}}) \longrightarrow \cdots \longrightarrow \bigoplus_{|I|=1,\notin \mathscr{V}} M_{ft/S}(S\times \underline{\mathbb{A}_{P_{F_I}}}) \tag{8.1.8.4}$$

extracted from (8.1.8.1) is 0.

(IV) *Reduction to a topological problem.* If $I\notin \mathscr{V}$, then $\theta(1)\subset F_I$, so $F_I=P$ since $\langle F+\theta(1)\rangle = P$. Thus $\mathbb{A}_{P_{F_I}}=\mathbb{A}_{P^*}$, which has the trivial log structure. With this identification, morphisms

$$M_{ft/S}(S\times \underline{\mathbb{A}_{P_{F_I}}}) \to M_{ft/S}(S\times \underline{\mathbb{A}_{P_{F_{I'}}}})$$

in (8.1.8.4) are either $\mathrm{id} : M_{ft/S}(S\times \mathbb{A}_{P^*})\to M_{ft/S}(S\times \mathbb{A}_{P^*})$ or $-\mathrm{id}$. Thus to show that the homotopy colimit of the sequence (loc. cit) is 0, it suffices to show that the sequence

$$0 \xrightarrow{\alpha_r} \bigoplus_{|I|=r, I\notin \mathscr{V}} \mathbb{Z}_I \xrightarrow{\alpha_{r-1}} \cdots \xrightarrow{\alpha_1} \bigoplus_{|I|=1,\notin \mathscr{V}} \mathbb{Z}_I \xrightarrow{\alpha_0} 0 \tag{8.1.8.5}$$

is exact. Here, each $\mathbb{Z}_I$ is $\mathbb{Z}$, and morphisms $\mathbb{Z}_I\to\mathbb{Z}_{I'}$ in (8.1.8.5) are either id or $-$id. It is equivalent to the assertion that the morphism

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \bigoplus_{|I|=r, I\in \mathscr{V}} \mathbb{Z}_I & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{|I|=1, I\in \mathscr{V}} \mathbb{Z}_I & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \bigoplus_{|I|=r} \mathbb{Z}_I & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{|I|=1} \mathbb{Z}_I & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \tag{8.1.8.6}$$

of complexes of abelian groups is a quasi-isomorphism. Here, each $\mathbb{Z}_I\to\mathbb{Z}_{I'}$ in (8.1.8.6) is either id or $-$id, whose sign is the same as the corresponding sign in (8.1.8.1). By the universal coefficient theorem, it is equivalent to the assertion that the morphism

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \bigoplus_{|I|=r, I\in \mathscr{V}} \mathbb{Z}_I) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \bigoplus_{|I|=1, I\in \mathscr{V}} \mathbb{Z}_I) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \bigoplus_{|I|=r} \mathbb{Z}_I) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \bigoplus_{|I|=1} \mathbb{Z}_I) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \tag{8.1.8.7}$$

of complexes of abelian groups is a quasi-isomorphism.

We put

$$G_I = (\Phi(F_I))_{\mathbb{R}}, \quad G_I' = G_I - \{0\}, \quad K = G_1\cup\cdots\cup G_r, \quad K' = K-\{0\}.$$

Then $\{G_1,\ldots,G_r\}$ (resp. $\{G_1',\ldots,G_r'\}$) is a closed cover of K (resp. K'). Moreover, the topological space G_I is always contractible, and G_I' is contractible (resp. empty) if $I\in\mathscr{V}$ (resp. $I\notin\mathscr{V}$). Thus the cohomology of the first row (resp. second row) of (8.1.8.7) is exactly the Čech cohomology of K (resp. K') associated to the closed cover $\{G_1,\ldots,G_r\}$

(resp. $\{G'_1, \dots, G'_r\}$), and the cohomology is isomorphic to the singular cohomology $H^i(K, \mathbb{Z})$ (resp. $H^i(K', \mathbb{Z})$). Because K is contractible to $\{0\}$, to show that the morphism of complexes in (8.1.8.7) is a quasi-isomorphism, it suffices to show that the reduced singular cohomology

$$\widetilde{H}^i(K', \mathbb{Z})$$

vanishes for all i. It is equivalent to the assertion that the reduced singular homology

$$\widetilde{H}_i(K', \mathbb{Z})$$

vanishes for all i by the universal coefficient theorem.

(V) *Final step of the proof.* Choose a hyperplane H of $(\overline{P}^\vee)^{\mathrm{gp}}_{\mathbb{R}}$ such that

$$V := H \cap (\overline{P})_{\mathbb{R}}$$

is a polytope and that the cone generated by v is equal to $(\overline{P}^\vee)_{\mathbb{R}}$. Then $K' \cap H$ is homotopy equivalent to K', so it suffices to prove that the reduced singular homology

$$\widetilde{H}_i(K' \cap H, \mathbb{Z})$$

vanishes for all i. The topological space V is homeomorphic to D^d where $d = \dim P$, and the boundary ∂V is homeomorphic to S^d. We also have

$$V = V^{\mathrm{int}} \amalg \partial V = V^{\mathrm{int}} \amalg (K' \cap H) \amalg (\partial V - K' \cap H),$$

so by the Alexander duality, we have an isomorphism

$$\widetilde{H}_i(K' \cap H, \mathbb{Z}) \cong \widetilde{H}^{d-i-1}(\partial V - K' \cap H).$$

Hence it suffices to show that $\partial V - K' \cap H$ is contractible.

A face G of $\overline{P}^\vee$ is in the image of Φ if and only if $G \cap \Phi(\langle \theta(1) \rangle) = \langle 0 \rangle$, so $\partial V - K' \cap H$ is the union of $(G \cap H)^{\mathrm{int}}$ for faces G of $\overline{P}^\vee$ such that $G \cap \Phi(\langle \theta(1) \rangle) \neq \langle 0 \rangle$ and $G \neq \overline{P}^\vee$. Then the conclusion follows from this description and (8.1.9) below. □

Lemma 8.1.9. *Let P be a real polytope, let G be a face of P, and let $\mathcal{F}$ be a family of faces of P such that*

(i) G is in $\mathcal{F}$,

(ii) if F is in $\mathcal{F}$, then $F \cap G \neq \emptyset$,

(iii) if F is in $\mathcal{F}$, and if F' is a face of F, then $F' \cap G = \emptyset$ or $F' \in \mathcal{F}$.

Then the union

$$U := \bigcup_{F \in \mathcal{F}} F^{\mathrm{int}}$$

is contractible.

Proof. We will use an induction on $r = \dim G$ and $s = |\mathcal{F}|$. If s is equal to the number of faces of G, then we are done since $U = G$ is contractible. Hence assume that s is bigger than that number.

Choose an element $H \neq G$ of $\mathcal{F}$ maximal among $\mathcal{F}$. Then we put

$$U_1 = \bigcup_{F \in \mathcal{F} - \{H\}} F^{\text{int}}, \quad U_2 = \bigcup_{F \in \mathcal{F}, F \subset H} F^{\text{int}}, \quad U_{12} = \bigcup_{F \in \mathcal{F} - \{H\}, F \subset H} F^{\text{int}}.$$

We will show that these are contractible. If x is any point in H^{int}, then U_2 is contractible to x. If G is not a face of H, then $\dim(H \cap G) < \dim G$, so U_{12} is contractible by induction on r. If G is a face of H, then U_{12} is contractible by induction on s. Finally, U_1 is contractible by induction on s.

The topological space U is the gluing of U_1 and U_2 along U_{12}, so it is also contractible. $\square$

8.1.10. Let S be an $\mathscr{S}$-scheme with a fs chart $\alpha : \mathbb{N} \to \mathcal{M}_S$. We put

$$X = S \times \mathbb{A}_{\mathbb{N}}, \quad Y = S \times \text{pt}_{\mathbb{N}}, \quad P = \mathbb{N} \oplus \mathbb{N}, \quad F_1 = \mathbb{N} \oplus 0.$$

Then X has the fs chart $v : X \to \mathbb{A}_{\mathbb{N}} \times \mathbb{A}_{\mathbb{N}} \cong \mathbb{A}_P$ induced by α, and we have the commutative diagram Consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & \searrow^{g} & \downarrow f \\ & & S \end{array}$$

of $\mathscr{S}$-schemes where f and g denote the projections and i denote the 0-section.

Proposition 8.1.11. *Under the notations and hypotheses of (8.1.10), let $\theta : P \to Q$ be an injective homomorphism of monoids such that there is a face G_1 of Q such that $\theta^{-1}(G_1) = F_1$. Then the motive*

$$M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_Q)$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ is in the essential image of the functor

$$\rho_{\sharp,S} : \mathrm{D}_{log',pw}(eSm/S, \Lambda) \to \mathrm{D}_{log',pw}(ft/S, \Lambda)$$

Proof. We put $p = \theta(1, 0)$ and $p' = \theta(0, 1)$. We will use induction on $r'(Q)$ where $r'(Q)$ denotes the number of rays of Q not containing p and p'. By (8.1.12) below, there is a face G containing p and not containing p' such that $\dim G = \dim Q - 1$. Let $q_1, \ldots, q_{r'}$ denote the rays of Q not containing p. Among them, we may assume $q_1, \ldots, q_r$ are the rays contained in G.

(I) *Reduction of G.* We will first reduce to the case when $G_{\mathbb{Q}}$ is simplicial and p is a ray of G. Let $\mu : Q \to \overline{Q}$ denote the quotient homomorphism. For $s = 0, \ldots, r$, we put

$$R_s = (\mathbb{N}\overline{p} + \mathbb{N}\overline{p'} + \mathbb{N}(-\overline{q_1}) + \cdots + \mathbb{N}(-\overline{q_s}) + \mathbb{N}\overline{q_{s+1}} + \cdots + \mathbb{N}\overline{q_{r'}})_{\mathbb{Q}} \cap \overline{Q}^{\text{gp}},$$

$$R'_s = (\mathbb{N}\overline{p} + \mathbb{N}\overline{p'} + \mathbb{N}(-\overline{q_1}) + \cdots + \mathbb{N}(-\overline{q_{s-1}}) + \mathbb{N}\overline{q_{s+1}} + \cdots + \mathbb{N}\overline{q_{r'}})_{\mathbb{Q}} \cap \overline{Q}^{\mathrm{gp}},$$
$$R''_s = (\mathbb{N}\overline{p} + \mathbb{N}\overline{p'} + \mathbb{N}(-\overline{q_1}) + \cdots + \mathbb{N}(-\overline{q_{s-1}}) + \mathbb{Z}\overline{q_s} + \mathbb{N}\overline{q_{s+1}} + \cdots + \mathbb{N}\overline{q_{r'}})_{\mathbb{Q}} \cap \overline{Q}^{\mathrm{gp}},$$
$$Q_s = \mu^{-1}(R_s), \quad Q'_s = \mu^{-1}(R'_s), \quad Q''_s = \mu^{-1}(R''_s).$$

Then the gluing of $\mathbb{A}_{Q_s}$ and $\mathbb{A}_{Q_{s-1}}$ along $\mathbb{A}_{Q''_s}$ is a dividing cover of $\mathbb{A}_{Q'_s}$, so by (2.2.3(8)), we have a distinguished triangle

$$\begin{aligned} M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_{Q''_s}) &\longrightarrow M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_{Q_s}) \oplus M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_{Q_{s-1}}) \\ &\longrightarrow M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_{Q'_s}) \longrightarrow M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_{Q''_s})[1] \end{aligned}$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$. We have $r'(Q'_s), r'(Q''_s) < r'(Q)$, and the homomorphisms

$$P \to Q_s, \quad P \to Q_{s'}, \quad P \to Q_{s''}$$

again satisfy the condition of the statement. Thus by induction on $r'(Q)$, the question is true for Q'_s and Q''_s. Then the above diagram shows that the question for Q_{s-1} is equivalent to the question for Q_s. Because $Q_0 = Q$, to show the question for Q, it suffices to show the question for Q_s.

Assume $r > \dim G - 1$. Then $\{\overline{p}, \overline{q_1}, \ldots, \overline{q_r}\}$ is linearly dependent, so we may assume

$$a_1 q_1 + \cdots + a_s q_s = ap + a_{s+1} + \cdots + a_r q_r + q$$

with $a_s \neq 0$, $a_1, \ldots, a_r \in \mathbb{N}$, and $q \in Q^*$. In Q_{s-1}, q_s is not a ray since

$$a_s q_s = a_1(-q_1) + \cdots + a_{s-1}(-q_{s-1}) + a_{s+1} q_{s+1} + \cdots + a_r q_r + q.$$

Thus $r'(Q_{s-1}) < r'(Q)$, so by induction on $r'(Q)$, the question is true for Q_{s-1}, which implies the question for Q. Hence we may assume that $r = \dim G - 1$. In this case, p is a ray of G, and $G_{\mathbb{Q}}$ is simplicial.

(II) *Reduction of* Q. Assume that we have two different rays q_{r+1} and q_{r+2} not containing p' and not in G. Then $\{\overline{p}, \overline{q_1}, \ldots, \overline{q_{r+2}}\}$ is linearly dependent, and $\{\overline{p}, \overline{q_1}, \ldots, \overline{q_{r+1}}\}$ is linearly independent. Hence we may assume that

$$a_1 q_1 + \cdots + a_s q_s + a_{r+2} q_{r+2} = ap + a_{s+1} q_{s+1} + \cdots + a_{r+1} q_{r+1} + q$$

for some $0 \leq s \leq r$, $q \in Q^*$, and $a, a_1, \ldots, a_{r+2} \in \mathbb{N}$ with $a_{r+2} \neq 0$. In Q_s, q_{r+2} is not a ray since

$$a_{r+2} q_{r+2} = ap + a_1(-q_1) + \cdots + a_s(-q_s) + a_{s+1} q_{s+1} + \cdots + a_{r+1} q_{r+1} + q.$$

Then by induction, the question is true for Q_s, which implies the question for Q. Hence we may assume that there are no two different rays not containing p' and not in G.

Assume that $Q_{\mathbb{Q}}$ is not simplicial. In this case, there is a unique ray q_{r+1} not containing p' and not in G. Then $\{\overline{p}, \overline{p'}, \overline{q_1}, \ldots, \overline{q_{r+1}}\}$ is linearly dependent, and $\{\overline{p}, \overline{q_1}, \ldots, \overline{q_{r+1}}\}$ and $\{\overline{p}, \overline{p'}, \overline{q_1}, \ldots, \overline{q_r}\}$ are linearly independent. Moreover, there is a homomorphism $h : Q \to \mathbb{N}$

with $h^{-1}(0) = G$ by [Ogu14, I.2.2.1], and then we have $h(p'), h(q_{r+1}) > 0$. Hence we may assume that one of the two equations

$$a_1q_1 + \cdots + a_sq_s + a_{r+1}q_{r+1} = ap + a'p' + a_{s+1}q_{s+1} + \cdots + a_rq_r + q,$$

$$a'p' + a_1q_1 + \cdots + a_sq_s = ap + a_{s+1}q_{s+1} + \cdots + a_{r+1}q_{r+1} + q$$

holds for some $q \in Q^*$ and $a, a', a_1, \ldots, a_{r+1} \in \mathbb{N}$ with $a', a_{r+1} \neq 0$.

If the first equation holds, then q_{r+1} is not a ray in Q_s, so $r'(Q_s) < r'(Q)$. Thus by induction, the question is true for Q_s, which implies the question for Q. If the second equation holds, then p' is not a ray in Q_s, so $(Q_s)_{\mathbb{Q}}$ is simplicial. Since it suffices to show the question for Q_s, we reduce to the case when $Q_{\mathbb{Q}}$ is simplicial.

The question is winding local on $\mathbb{A}_Q$, so we may further assume that Q is isomorphic to $\mathbb{N}^{r+1}$ by [CLS11, 11.1.9]. Then $\overline{Q} \cong \langle q \rangle \oplus \mathbb{N}^r$ where q is a minimal element in Q such that $p \in \langle q \rangle$. By (3.5.2), we have an isomorphism

$$Q \cong \langle q \rangle \oplus \mathbb{N}^r \oplus Q^*.$$

(III) *Final step of the proof.* Let

$$\eta_1 : Q \to \langle q \rangle \cong \mathbb{N}, \quad \eta_2 : Q \to \mathbb{N}^r, \quad \eta_3 : Q \to Q^*$$

denote the projections. Consider the ideals

$$I := (\eta_1(p'), \eta_2(p'), 0), \quad I_1 := (\eta_1(p'), 0, 0), \quad I_2 := (0, \eta_2(p'), 0), \quad I_{12} := I_1 \cap I_2$$

of Q, and we put

$$W = X \times_{\mathbb{A}_P} \mathbb{A}_{(Q,I)}, \quad W_1 = X \times_{\mathbb{A}_P} \mathbb{A}_{(Q,I_1)}, \quad W_2 = X \times_{\mathbb{A}_P} \mathbb{A}_{(Q,I_2)}, \quad W_{12} = X \times_{\mathbb{A}_P} \mathbb{A}_{(Q,I_{12})}.$$

Then we put $Q' = Q/\langle p \rangle$, and let $Q' : \mathbb{N} \to Q'$ denote the composition

$$\mathbb{N} \to P \xrightarrow{\theta} Q \to Q'$$

where the first arrow is the second inclusion and the third arrow is the quotient homomorphism. We have

$$W \cong Y \times_{\mathbb{A}_P} \mathbb{A}_Q, \quad W_1 \cong (S \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}) \times \mathbb{A}_{Q'},$$

$$W_2 \cong S \times (\mathbb{A}_{Q'} \times_{\mathbb{A}_{\theta}, \mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}), \quad W_{12} \cong (S \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}) \times (\mathbb{A}_{Q'} \times_{\mathbb{A}_{\theta'} \mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}).$$

By (8.1.5), the morphism

$$M_{ft/S}(W_{12}) \to M_{ft/S}(W_1)$$

in $\mathrm{D}_{log',pw}(ft/S, \Lambda)$ induced by the closed immersion $W_{12} \to W_1$ is an isomorphism. Since $W_1 \amalg W_2 \to W$ is a plain lower cover, we have a distinguished triangle

$$M_{ft/S}(W_{12}) \to M_{ft/S}(W_1) \oplus M_{ft/S}(W_2) \to M_{ft/S}(W) \to M_{ft/S}(W_{12})[1]$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$. Thus the morphism

$$M_{ft/S}(W_2) \to M_{ft/S}(W)$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ is also an isomorphism. Since $W_2 \cong S \times (\mathbb{A}_{Q'} \times_{\mathbb{A}_\theta,\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N})$, by (8.1.5), the morphism

$$M_{ft/S}(W_2) \to M_{ft/S}(S \times \mathbb{A}_{Q'})$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ induced by the closed immersion $W_2 \to S \times \mathbb{A}_{Q'}$ is an isomorphism. This completes the proof since the projection $S \times \mathbb{A}_{Q'} \to S$ is exact log smooth. □

Lemma 8.1.12. *Let $\theta : \mathbb{N} \oplus \mathbb{N} \to Q$ be an injective homomorphism of fs monoids such that there is a face G_1 of Q with $\theta^{-1}(G_1) = \mathbb{N} \oplus 0$. We put $p = \theta(1,0)$ and $p' = \theta(0,1)$. Then there is a face G containing p and not containing p' such that* $\dim G = \dim Q - 1$.

Proof. We put $F_1 = \mathbb{N} \oplus 0$. Consider the homomorphism

$$\theta' : \mathbb{N} \cong (\mathbb{N} \oplus \mathbb{N})_{F_1} \to Q_{G_1}$$

of fs monoids induced by θ. Since $\theta^{-1}(G_1) = F_1$, θ' is injective. Choose a maximal proper face G'' of Q_{G_1} not containing $\theta'(1)$, and we denote by G the inverse image of G'' under the localization homomorphism $Q \to Q_{G_1}$. Then G satisfies the condition. □

Proposition 8.1.13. *Under the notations and hypotheses of (8.1.11), we assume further that the fs chart $\mathbb{N} \to \mathcal{M}_S$ induces a constant log structure. Then the morphism*

$$M_{ft/S}(Y \times_{\mathbb{A}_P} \mathbb{A}_Q) \to M_{ft/S}(X \times_{\mathbb{A}_P} \mathbb{A}_Q)$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ induced by $i : Y \to X$ is an isomorphism.

Proof. We can follow the proof of (8.1.11) until the end of the step (II). Hence we may assume that Q is isomorphic to $\mathbb{N}^{r+1}$. Let us use the notations in the step (III) of the proof of (loc. cit). We want to show that the homomorphism

$$M_{ft/S}(W) = M_{ft/S}(Y \times_{\mathbb{A}_P}) = M_{ft/S}(X \times_{\mathbb{A}_P} \mathbb{A}_{(Q,I)}) \to M_{ft/S}(X \times_{\mathbb{A}_P} \mathbb{A}_Q)$$

in $\mathrm{D}_{log',pw}(ft/S)$ is an isomorphism.

If $\eta_1(p') \neq 0$, then $W \cong W_1 \cong X \times_{\mathbb{A}_P} \mathbb{A}_Q$, so we are done. If $\eta_1(p') = 0$, then $W_2 \cong W$. As in the proof of (loc. cit), the morphism

$$M_{ft/S}(S \times (\mathbb{A}_{Q'} \times_{\mathbb{A}_\theta,\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N})) \to M_{ft/S}(S \times \mathbb{A}_{Q'})$$

in $\mathrm{D}_{log',pw}(ft/S,\Lambda)$ is an isomorphism by (8.1.5), so we are done because $W \cong W_2 \cong S \times (\mathbb{A}_{Q'} \times_{\mathbb{A}_\theta,\mathbb{A}_\mathbb{N}} \mathrm{pt}_\mathbb{N})$ and $X \times_{\mathbb{A}_P} \mathbb{A}_Q \cong S \times \mathbb{A}_{Q'}$. □

Proposition 8.1.14. *Under the notations and hypotheses of (8.1.10), the essential image of*

$$g_{\sharp,ft}\rho_{\sharp,Y} : \mathrm{D}_{log',pw}(eSm/Y,\Lambda) \to \mathrm{D}_{log',pw}(ft/S,\Lambda)$$

is in the essential image of

$$\rho_{\sharp,S} : \mathrm{D}_{log',pw}(eSm/S,\Lambda) \to \mathrm{D}_{log',pw}(ft/S,\Lambda).$$

Proof. We put $P = \mathbb{N} \oplus \mathbb{N}$. Then Y has the fs chart $P \to \mathcal{M}_S$. By (3.6.3), it suffices to prove that

$$M_{ft/S}(Y' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q)$$

is in the essential image of $\rho_{\sharp,S}$ where

(i) $h : Y' \to Y$ is a Kummer log smooth morphism with a fs chart $\eta : P \to P'$ of Kummer log smooth type,

(ii) $\theta' : P' \to Q$ is an injective homomorphism of fs monoids such that the cokernel of $\theta'^{\rm gp}$ is torsion free,

(iii) θ' is logarithmic and locally exact.

We put $T = \mathrm{pt}_{\mathbb{N}}$, and consider the diagram

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \downarrow g & & \\ S & & \end{array}$$

of $\mathscr{S}$-schemes where the horizontal arrow is the projection. Then the new question is winding local on S and T, so by (1.2.18), we may assume that $\overline{\eta}$ is an isomorphism. In this case, h is strict smooth, so there is a unique Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y \\ \downarrow & & \downarrow g \\ S' & \longrightarrow & S \end{array}$$

of $\mathscr{S}$-schemes since the morphism $\underline{g} : \underline{Y} \to \underline{S}$ of underlying schemes is an isomorphism. Then the morphism $S' \to S$ is automatically strict smooth. Replacing $Y' \to Y \to S$ by $Y' \to Y' \to S'$, we may assume that $Y = Y'$ and $P = P'$. Then we are done by (8.1.11). □

8.1.15. Under the notations and hypotheses of (8.1.10), by (8.1.14) and (1.5.4), we have the adjunction

$$g_\sharp : D_{log',pw}(eSm/Y, \Lambda) \rightleftarrows \mathrm{D}_{log',pw}(eSm/S, \Lambda) : g^*.$$

Moreover, $g_\sharp$ commutes with $\rho_\sharp$, and g_* commutes with ρ^*.

8.1.16. Let S be an $\mathscr{S}$-scheme with the trivial log structure. Consider the Cartesian diagram

$$\begin{array}{ccc} (S \times \mathrm{pt}_{\mathbb{N}}) \times \mathbb{A}_{\mathbb{N}} & \xrightarrow{g'} & S \times \mathbb{A}_{\mathbb{N}} \\ \downarrow f' & & \downarrow f \\ (S \times \mathrm{pt}_{\mathbb{N}}) \times \mathbb{A}^1 & \xrightarrow{g} & S \times \mathbb{A}^1 \end{array}$$

of $\mathscr{S}$-schemes where f denotes the morphism removing the log structure and g denotes the projection. Then by (8.1.15) and (1.5.5), the exchange transformation

$$g^* f_* \xrightarrow{Ex} f'_* g'^*$$

in $D_{log',pw}(eSm, \Lambda)$ is an isomorphism.

Proposition 8.1.17. *Under the notations and hypotheses of (8.1.10), the natural transformation*

$$f_*f^* \xrightarrow{ad} f_*i_*i^*f^*$$

in $\mathrm{D}_{log',pw}(eSm,\Lambda)$ *is an isomorphism.*

Proof. Let us add subscripts eSm and ft to functors for distinction. By (8.1.2), the natural transformation

$$f_{*,ft}f_{ft}^* \xrightarrow{ad} f_{*,ft}i_{*,ft}i_{ft}^*f_{ft}^*$$

is an isomorphism. Thus the natural transformation

$$\rho^*f_{*,ft}f_{ft}^*\rho_\sharp \xrightarrow{ad} \rho^*f_{*,ft}i_{*,ft}i_{ft}^*f_{ft}^*\rho_\sharp$$

is an isomorphism. Since ρ^* commutes with $f_{*,ft}$ and $i_{*,ft}$, and $\rho_\sharp$ commutes with f_{ft}^* and i_{ft}^*, the natural transformation

$$f_{*,eSm}\rho^*\rho_\sharp f_{eSm}^* \xrightarrow{ad} f_{*,eSm}i_{*,eSm}\rho^*\rho_\sharp i_{eSm}^*f_{eSm}^*$$

is an isomorphism. Then the conclusion follows from the fact that $\rho_\sharp$ is fully faithful. □

8.2 log'''-weak equivalences in $\mathrm{D}_{log',pw}(eSm,\Lambda)$

8.2.1. We will study various log'''-weak equivalences $\mathrm{D}_{log',pw}(eSm,\Lambda)$. Using these, in (8.2.9), we will prove that the functor

$$g^* : \mathrm{D}_{log''',pw}(eSm/S,\Lambda) \to \mathrm{D}_{log''',pw}(eSm/Y,\Lambda)$$

admits a left adjoint where S is an $\mathscr{S}$-scheme with a fs chart $\mathbb{N}$ and $g : Y \to S$ denotes the projection $S \times \mathrm{pt}_{\mathbb{N}} \to S$.

Proposition 8.2.2. *Let* S *be an* $\mathscr{S}$*-scheme with the trivial log structure, and we put* $Y = S \times \mathrm{pt}_{\mathbb{N}}$*. Let*

(i) $h : W \to Y$ *be a log smooth morphism of* $\mathscr{S}$*-schemes,*

(ii) Q *be a fs chart of* W*,*

(iii) $\eta : Q \to Q_0$ *be a vertical homomorphism of fs monoids of exact log smooth over* W *type,*

(iv) F *be a* η*-critical face of* Q*.*

We put $Q_1 = (Q_0)_F$*. Then the morphism*

$$g_\sharp M_Y(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1}) \to g_\sharp M_Y(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0}) \qquad (8.2.2.1)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ *induced by the open immersion* $\mathbb{A}_{Q_1} \to \mathbb{A}_{Q_0}$ *is an isomorphism.*

Proof. We will use an induction on

$$d := \max_{x \in W} \overline{\mathcal{M}}^{\mathrm{gp}}_{W,\overline{x}}.$$

If $d = 1$, then h is Kummer log smooth, so $F = 0$. Thus (8.2.2.1) is an isomorphism. Hence assume $d > 1$.

(I) *Reduction of Q.* We denote by R the submonoid of Q_0 consisting of elements $q \in Q_0$ such that $nq = \eta(q') + q''$ for some $nin\mathbb{N}^+$, $q' \in Q$, and $q'' \in Q_0^*$. Then the induced homomorphism $\nu : R \to Q_0$ is again a vertical homomorphism of fs monoids of exact log smooth over $W' := W \times_{\mathbb{A}_Q} \mathbb{A}_R$ type, and F is a ν-critical face of Q_0. Moreover, the projection $W' \to W$ is Kummer log smooth. Hence replacing

$$W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1} \to W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0} \to W$$

by

$$W' \times_{\mathbb{A}_R} \mathbb{A}_{Q_1} \to W' \times_{\mathbb{A}_R} \mathbb{A}_{Q_0} \to W',$$

we may assume that the cokernel of η^{gp} is torsion free.

(II) *Reduction of Y.* The question is strict étale local on W, so by [Ogu14, IV.3.3.1], we may assume that $h : W \to Y$ has a fs chart $\theta' : \mathbb{N} \to Q'$ of log smooth type. Let y be a point of Y. Then we may further assume that the chart $Q' \to \mathcal{M}_W$ is exact at y by [Ogu14, II.2.3.2]. We denote by P' the submonoid of Q' consisting of elements $q \in Q'$ such that $nq = \theta(p) + q'$ for some $n \in \mathbb{N}^+$, $p \in \mathbb{N}$, and $q \in Q'$. Then P' is a fs monoid by Gordon's lemma [Ogu14, I.2.3.17], and $\overline{P'} \cong \mathbb{N}$. Moreover, the induced homomorphism $\mu : P' \to Q'$ is logarithmic, so the commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{\mu} & Q' \\ \downarrow & & \downarrow \\ \overline{P'} & \xrightarrow{\overline{\mu}} & \overline{Q'} \end{array}$$

of fs monoids where the vertical arrows are the quotient homomorphisms is coCartesian.

Replacing $(W \to Y \to S, \theta : \mathbb{N} \to Q')$ by $(W \to Y \times_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_{P'} \to S \times \mathbb{A}_{P'^*}, \overline{\mu} : \overline{P'} \to \overline{Q'})$, we may assume further assume that

(i) Q' is sharp,

(ii) the cokernel of θ^{gp} is torsion free.

Then the chart $Q' \to \mathcal{M}_W$ is neat at y because Q' is sharp.

(III) *Further reduction of Q.* We denote by κ the composition $Q \to \overline{\mathcal{M}}_{Y,y} \xrightarrow{\sim} Q'$ where the first arrow is the chart homomorphism and the second arrow is the inverse of the chart homomorphism. We put

$$Q_0' = Q_0 \oplus_Q Q', \quad Q_1' = Q_1 \oplus_Q Q'.$$

Then since the cokernel of η^{gp} is torsion free by (I), we have isomorphisms

$$W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0} \cong W \times_{\mathbb{A}_{Q'}} \mathbb{A}_{Q_0'}, \quad W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1} \cong W \times_{\mathbb{A}_{Q'}} \mathbb{A}_{Q_1'}$$

by (3.2.2). Hence replacing $Q \to Q_0 \to Q_1$ by $Q' \to Q'_0 \to Q'_1$, we may assume that

(i) Q is neat at y,

(ii) $g : Y \to X$ has a fs chart $\theta : \mathbb{N} \to Q$ of log smooth type,

(iii) the cokernel of of $\theta^{\rm gp}$ is torsion free.

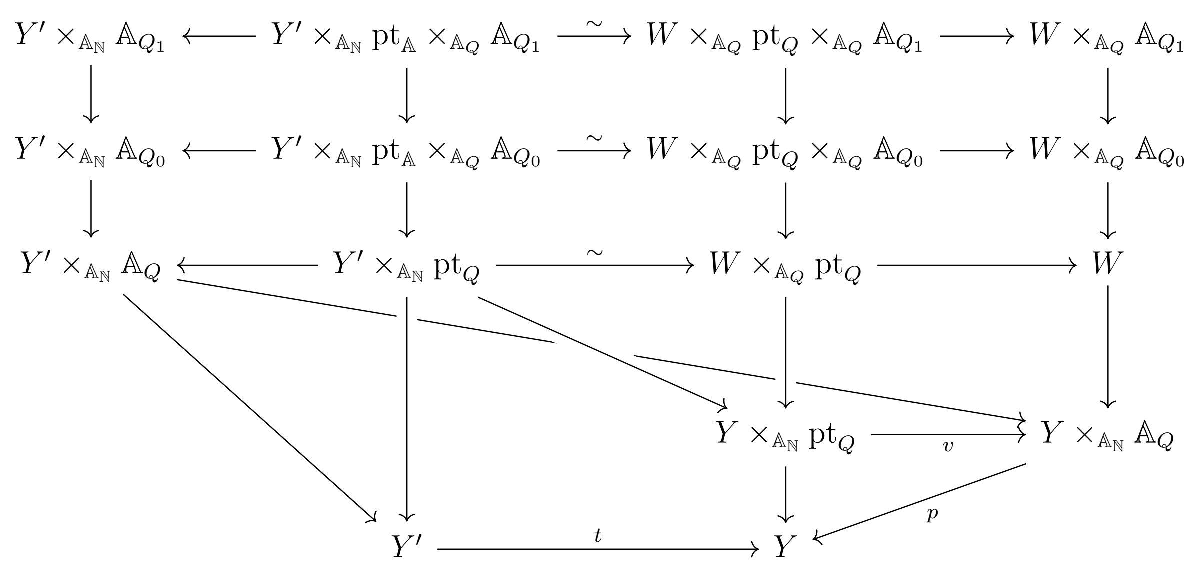
(IV) *Induction.* The induced morphism

$$\underline{Y \times_{\mathbb{A}_\mathbb{N}} {\rm pt}_Q} \to \underline{Y}$$

of schemes is an isomorphism since Q is sharp, so there is a unique Cartesian diagram

$$\begin{array}{ccc} W \times_{\mathbb{A}_Q} {\rm pt}_Q & \longrightarrow & Y \times_{\mathbb{A}_\mathbb{N}} {\rm pt}_Q \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{t} & Y \end{array}$$

of $\mathscr{S}$-schemes where the right vertical arrow is the projection and the upper horizontal arrow is a pullback of the strict étale morphism $W \to Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q$. Then the morphism $Y' \to Y$ is automatically strict étale. Now we have the commutative diagram



of $\mathscr{S}$-schemes. Let u denote the complement of the closed immersion $v : Y \times_{\mathbb{A}_\mathbb{N}} {\rm pt}_Q \to Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q$. Then by (Loc), we have the commutative diagrams

$$\begin{array}{ccccccc} p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1}) & \longrightarrow & M_Y(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1}) & \longrightarrow & p_\sharp v_* v^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1}) & \longrightarrow & p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_1})[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0}) & \longrightarrow & M_Y(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0}) & \longrightarrow & p_\sharp v_* v^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0}) & \longrightarrow & p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(W \times_{\mathbb{A}_Q} \mathbb{A}_{Q_0})[1] \end{array}$$

$$\begin{array}{ccccccc} p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_1}) & \longrightarrow & M_Y(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_1}) & \longrightarrow & p_\sharp v_* v^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_1}) & \longrightarrow & p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_1})[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_0}) & \longrightarrow & M_Y(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_0}) & \longrightarrow & p_\sharp v_* v^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_0}) & \longrightarrow & p_\sharp u_\sharp u^* M_{Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q}(Y' \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{Q_0})[1] \end{array}$$

of $\mathscr{S}$-schemes whose rows are distinguished triangles. We have isomorphisms

$$v^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(W\times_{\mathbb{A}_Q}\mathbb{A}_{Q_0})\cong M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathrm{pt}_Q}(W\times_{\mathbb{A}_Q}\mathrm{pt}_Q\times_{\mathbb{A}_Q}\mathbb{A}_{Q_0})\cong v^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_0}),$$

$$v^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(W\times_{\mathbb{A}_Q}\mathbb{A}_{Q_1})\cong M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathrm{pt}_Q}(W\times_{\mathbb{A}_Q}\mathrm{pt}_Q\times_{\mathbb{A}_Q}\mathbb{A}_{Q_0})\cong v^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_1}),$$

and by induction on d, the morphisms

$$g_\sharp p_\sharp u_\sharp u^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(W\times_{\mathbb{A}_Q}\mathbb{A}_{Q_1})\to g_\sharp p_\sharp u_\sharp u^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(W\times_{\mathbb{A}_Q}\mathbb{A}_{Q_0}),$$

$$g_\sharp p_\sharp u_\sharp u^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_1})\to g_\sharp p_\sharp u_\sharp u^*M_{Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q}(Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_0})$$

are log'''-weak equivalent in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$. Thus from the above commutative diagrams, we see that the question for $W\times_{\mathbb{A}_Q}\mathbb{A}_{Q_1}\to W\times_{\mathbb{A}_Q}\mathbb{A}_{Q_0}$ is equivalent to the question for $Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_1}\to Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_1}$. In other words, we may assume $W=Y'\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q$. Since $t:Y'\to Y$ is strict étale, there is a unique Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{t} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ S' & \longrightarrow & S \end{array}$$

of $\mathscr{S}$-schemes. Then the lower horizontal arrow is automatically strict étale. Replacing $W\to Y\to S$ by $W\to Y'\to S'$, we may further assume that $Y=Y'$, i.e., $W=Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_Q$.

(V) *Final step of the proof.* We put $X=S\times\mathbb{A}_\mathbb{N}$. Then X has the chart $\mathbb{N}$. We also put

$$W_0=Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_0},\quad W_1=Y\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_1},\quad V_0=X\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_0},\quad V_1=X\times_{\mathbb{A}_\mathbb{N}}\mathbb{A}_{Q_1}.$$

We want to show that the morphism

$$g_\sharp M_Y(W_1)\to g_\sharp M_Y(W_0)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ is a log'''-weak equivalence. If $\theta(1)$ is invertible in Q, then $W_1=W_0=\emptyset$, so we are done. If not, then the image of $\theta(1)$ in Q_1 is not invertible since $(\overline{Q_1})_\mathbb{Q}\cong Q_\mathbb{Q}$ by [Ogu14, 4.6.6]. Thus by (8.1.5), we have isomorphisms

$$g_\sharp M_Y(W_0)\to g_\sharp M_Y(V_0),\quad g_\sharp M_Y(W_1)\to g_\sharp M_Y(V_1)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$. This completes the proof since the morphism

$$g_\sharp M_Y(V_1)\to g_\sharp M_Y(V_0)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ is a log'''-weak equivalence by definition. □

Proposition 8.2.3. *Let S be an $\mathscr{S}$-scheme with the trivial structure, and we put $Y=S\times\mathrm{pt}_\mathbb{N}$. Let $h:W\to Y$ be a log smooth morphism. Consider the verticalization $W^{\mathrm{ver}}\to W$ of W via h. Then the morphism*

$$g_\sharp M_S(W^{\mathrm{ver}})\to g_\sharp M_S(W)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ is a log'''-weak equivalence.

Proof. The question is strict étale local on W, so we may assume that h has a fs chart $\theta : \mathbb{N} \to Q$ of log smooth type. We can follow the proof of (8.2.2) from Step (I) to Step (IV), so we reduce to the case when $W = Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q$. We put $X = S \times \mathbb{A}_\mathbb{N}$. Then X has the chart $\mathbb{N}$. We also put

$$W_0 = W = Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q, \quad W_1 = W^{\mathrm{ver}} = Y \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q^{\mathrm{ver}}, \quad V_0 = X \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q, \quad V_1 = X \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_Q^{\mathrm{ver}}.$$

We want to show that the morphism

$$g_\sharp M_Y(W_1) \to g_\sharp M_Y(W_0)$$

in $\mathrm{D}_{log',pw}(eSm/S, \Lambda)$ is a log'''-weak equivalence. If $\theta(1)$ is invertible in Q, then $W_1 = W_0 = \emptyset$, so we are done. If not, then by (8.1.8) and (8.1.5), we have isomorphisms

$$g_\sharp M_Y(W_0) \to g_\sharp M_Y(V_0), \quad g_\sharp M_Y(W_1) \to g_\sharp M_Y(V_1)$$

in $\mathrm{D}_{log',pw}(eSm/S, \Lambda)$. This completes the proof since the morphism

$$g_\sharp M_Y(V_1) \to g_\sharp M_Y(V_0)$$

in $\mathrm{D}_{log',pw}(eSm/S, \Lambda)$ is a log'''-weak equivalence by definition. □

Corollary 8.2.4. *Let S be an $\mathscr{S}$-scheme with the trivial log structure, and let $g : S \times \mathrm{pt}_\mathbb{N} \to S$ denote the projection. Then the functor*

$$g_\sharp : \mathrm{D}_{log',pw}(eSm/Y, \Lambda) \to \mathrm{D}_{log',pw}(eSm/S, \Lambda)$$

preserves log'''-weak equivalences.

Proof. It follows from (8.2.2) and (8.2.3). □

8.2.5. Under the notations and hypotheses of (8.1.10), assume further that the chart $\alpha : \mathbb{N} \to \mathcal{M}_S$ induces a constant log structure. We put

$$P' = \mathbb{N} \oplus \mathbb{N}, \quad X' = X \times_{\mathbb{A}_P, \mathbb{A}_\eta} \mathbb{A}_{P'}, \quad Y' = Y \times_{\mathbb{A}_P, \mathbb{A}_\eta} \mathbb{A}_{P'}$$

where $\eta : P \to P'$ denotes the homomorphism

$$(a, b) \mapsto (a + b, b).$$

Consider the commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{i} & X \\ \downarrow{\scriptstyle g} & \swarrow{\scriptstyle f} & \\ S & & \end{array}$$

where f' denotes the projection and the square is Cartesian. Then $gg' : Y' \to S$ is the projection $S \times \mathbb{A}_\mathbb{N} \to S$, and in particular it is exact log smooth.

Proposition 8.2.6. *Under the notations and hypotheses of (8.2.5), consider the natural transformation*

$$(gg')_\sharp g'^* \to g_\sharp$$

that is the left adjoint of

$$g^* \xrightarrow{ad} g'_* g'^* g^*.$$

Then for any object K of $\mathrm{D}_{\mathbb{A}^1,pw}(eSm/Y,\Lambda)$, the morphism

$$(gg')_\sharp g'^* K \to g_\sharp K$$

is a log'''-weak equivalence.

Proof. Note first that the functor $g_\sharp$ is defined by (8.1.15). As in the proof of (8.1.14), we reduce to the case when

$$K = M_Y(Y \times_{\mathbb{A}_P,\mathbb{A}_\theta} \mathbb{A}_Q)$$

where

(i) $\theta : P \to Q$ is an injective homomorphism of fs monoids such that the cokernel of θ'^{gp} is torsion free,

(ii) θ is logarithmic and locally exact.

By (8.1.13), the morphism

$$g_\sharp M_Y(Y \times_{\mathbb{A}_P} \mathbb{A}_Q) \to M_S(X \times_{\mathbb{A}_P} \mathbb{A}_Q)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ is an isomorphism. Hence to show the question, it suffices to show that the morphism

$$M_S(Y' \times_{\mathbb{A}_P} \mathbb{A}_Q) \to M_S(X \times_{\mathbb{A}_P} \mathbb{A}_Q)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ is a log'''-weak equivalence.

Consider the coCartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{\eta} & P' \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} \\ Q & \xrightarrow{\eta'} & Q' \end{array}$$

of fs monoids. Then θ' is again local and locally exact. Let G' be a maximal θ'-critical face of Q'. Then $\eta'^{-1}(G')$ is a maximal θ-critical face of Q. The morphisms

$$M_S(Y' \times_{\mathbb{A}_P} \mathbb{A}_{Q_G}) \to M_S(Y' \times_{\mathbb{A}_P} \mathbb{A}_Q), \quad M_S(X \times_{\mathbb{A}_P} \mathbb{A}_{Q_G}) \to M_S(X \times_{\mathbb{A}_P} \mathbb{A}_Q)$$

in $\mathrm{D}_{log',pw}(eSm/S,\Lambda)$ are log'''-weak equivalences by definition, so replacing Q by Q_G, we may assume $\overline{\theta}$ is Kummer by [Ogu14, I.4.6.6].

We put $T = \mathrm{pt}_{\mathbb{N}}$, and consider the diagram

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \downarrow{\scriptstyle g} & & \\ S & & \end{array}$$

of $\mathscr{S}$-schemes where the horizontal arrow is the projection. Then the question is winding local on S and T, so by (1.2.18), we may assume that $\overline{\theta}$ is an isomorphism. In this case, the projection $p : Y \times_{\mathbb{A}_P} \mathbb{A}_Q \to Y$ is strict smooth, so there is a unique Cartesian diagram

$$\begin{array}{ccc} Y \times_{\mathbb{A}_P} \mathbb{A}_Q & \xrightarrow{p} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ S' & \longrightarrow & S \end{array}$$

of $\mathscr{S}$-schemes since the morphism $\underline{g} : \underline{Y} \to \underline{S}$ of underlying schemes is an isomorphism. Then the morphism $S' \to S$ is automatically strict smooth. Replacing $Y \times_{\mathbb{A}_P} \mathbb{A}_Q \to Y \to S$ by $Y \times_{\mathbb{A}_P} \mathbb{A}_Q \to Y \times_{\mathbb{A}_P} \mathbb{A}_Q \to S'$, we may assume that $P = Q$.

Then the remaining is to show that the morphism

$$M_S(Y') \to M_S(X)$$

in $\mathrm{D}_{log',pw}(eSm/S, \Lambda)$ is log'''-weak equivalent. The morphisms

$$M_S(Y') \to M_S(X'), \quad M_S(X'^{\mathrm{ver}/S}) \to M_S(X'), \quad M_S(X^{\mathrm{ver}/S}) \to M_S(X)$$

in $\mathrm{D}_{log',pw}(eSm/S, \Lambda)$ is are log'''-weak equivalent by definition, so this proves the question since $X^{\mathrm{ver}/S} = X'^{\mathrm{ver}/S}$. □

Proposition 8.2.7. *Under the notations and hypotheses of (8.2.5), the functor*

$$g_\sharp : \mathrm{D}_{log',pw}(eSm/Y, \Lambda) \to \mathrm{D}_{log',pw}(eSm/S, \Lambda)$$

preserves log'''-weak equivalences.

Proof. By (8.2.6), it suffices to prove that the functor

$$(gg')_\sharp : \mathrm{D}_{log',pw}(eSm/Y', \Lambda) \to \mathrm{D}_{log',pw}(eSm/S, \Lambda)$$

preserves log'''-weak equivalences. This is true since gg' is exact log smooth. □

Proposition 8.2.8. *Under the notations and hypotheses of (8.1.10), the functor*

$$g_\sharp : \mathrm{D}_{log',pw}(eSm/Y, \Lambda) \to \mathrm{D}_{log',pw}(eSm/S, \Lambda)$$

preserves log'''-weak equivalences.

Proof. We put $S' = S \times_{\mathbb{A}_{\mathbb{N}}} \mathrm{pt}_{\mathbb{N}}$. Let $i : S' \to S$ denote the closed immersion, and let $j : S'' \to S$ denote its complement. Consider the commutative diagram

$$\begin{array}{ccccc} Y' & \xrightarrow{i'} & Y & \xleftarrow{j'} & Y'' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} \\ S' & \xrightarrow{i} & S & \xleftarrow{j} & S'' \end{array}$$

of $\mathscr{S}$-schemes where each square is Cartesian.

If $K \to L$ is a log'''-weak equivalence in $\mathrm{D}_{log',pw}(eSm/Y, \Lambda)$, then by (Loc), we have a commutative diagram

$$\begin{array}{ccccccc} j_\sharp j^* g_\sharp K & \xrightarrow{ad'} & g_\sharp K & \xrightarrow{ad} & i_* i^* g_\sharp K & \xrightarrow{\partial_i} & j_\sharp j^* g_\sharp K[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j_\sharp j^* g_\sharp L & \xrightarrow{ad'} & g_\sharp L & \xrightarrow{ad} & i_* i^* g_\sharp L & \xrightarrow{\partial_i} & j_\sharp j^* g_\sharp L[1] \end{array}$$

in $\mathrm{D}_{log',pw}(eSm/Y, \Lambda)$ whose rows are distinguished triangles. By (8.1.15) and (1.5.5), the exchange transformations

$$g'_\sharp i'^* \xrightarrow{Ex} i^* g_\sharp, \quad g''_\sharp j'^* \xrightarrow{Ex} j^* g_\sharp$$

are isomorphisms. Applying these to the above diagram, we get the commutative diagram

$$\begin{array}{ccccccc} j_\sharp g''_\sharp j'^* K & \xrightarrow{ad'} & g_\sharp K & \xrightarrow{ad} & i_* g'_\sharp i'^* K & \xrightarrow{\partial_i} & j_\sharp g''_\sharp j'^* K[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j_\sharp g''_\sharp j'^* L & \xrightarrow{ad'} & g_\sharp L & \xrightarrow{ad} & i_* g'_\sharp i'^* L & \xrightarrow{\partial_i} & j_\sharp j^* g_\sharp j_\sharp g''_\sharp j'^* L[1] \end{array}$$

in $\mathrm{D}_{log',pw}(eSm/Y, \Lambda)$ whose rows are distinguished triangles. Since i_* preserves log'''-weak equivalences by (7.5.6), to show that $g_\sharp$ preserves log'''-weak equivalences, it suffices to show that $g'_\sharp$ and $g''_\sharp$ preserves log'''-weak equivalences. It follows from (8.2.4) and (8.2.7). $\square$

8.2.9. Under the notations and hypotheses of (8.1.10), by (8.2.8) and (1.6.5), we have the adjunction

$$g_\sharp : D_{log''',pw}(eSm/Y, \Lambda) \rightleftarrows \mathrm{D}_{log''',pw}(eSm/S, \Lambda) : g^*.$$

Moreover, $g_\sharp$ commutes with $\pi : D_{log',pw}(eSm, \Lambda) \to D_{log''',pw}(eSm, \Lambda)$, and g^* commutes with $\mathcal{O} : D_{log''',pw}(eSm, \Lambda) \to D_{log',pw}(eSm, \Lambda)$.

8.2.10. Under the notations and hypotheses of (8.1.16), by (loc. cit), (8.2.8), and (1.6.6), the exchange transformation

$$g^* f_* \xrightarrow{Ex} f'_* g'^*$$

in $D_{log''',pw}(eSm, \Lambda)$ is an isomorphism.

Proposition 8.2.11. *Under the notations and hypotheses of (8.1.10), the natural transformation*

$$f_* f^* \xrightarrow{ad} f_* i_* i^* f^*$$

in $\mathrm{D}_{log''',pw}(eSm, \Lambda)$ *is an isomorphism.*

Proof. Let us add subscripts log''' and log' to functors for distinction. By (8.1.17), the natural transformation

$$f_{*,log'} f^*_{log'} \xrightarrow{ad} f_{*,log'} i_{*,log'} i^*_{log'} f^*_{log'}$$

is an isomorphism. Thus its adjunction

$$g_{\sharp,log'} g^*_{log'} \longrightarrow f_{\sharp,log'} f^*_{log'}$$

is an isomorphism. Then the natural transformation

$$\pi g_{\sharp,log'} g^*_{log'} \mathcal{O} \longrightarrow \pi f_{\sharp,log'} f^*_{log'} \mathcal{O}$$

is an isomorphism. Since π commutes with $f_{\sharp,log'}$ and $g_{\sharp,log'}$, and $\mathcal{O}$ commutes with $f^*_{log'}$ and $g^*_{log'}$, the natural transformation

$$g_{\sharp,log'''} \pi \mathcal{O} g^*_{log'''} \longrightarrow f_{\sharp,log'''} \pi \mathcal{O} f^*_{log'''}$$

is an isomorphism. Then the conclusion follows from the fact that $\mathcal{O}$ is fully faithful. □

Corollary 8.2.12. *Under the notations and hypotheses of (8.1.10), the functor*

$$g^* : \mathrm{D}_{log''',pw}(eSm/S, \Lambda) \to \mathrm{D}_{log''',pw}(eSm/Y, \Lambda)$$

is conservative.

Proof. The functor f^* is conservative since f has a section. Thus the conclusion follows from (8.2.11) and (5.2.1). □

8.3 Homotopy properties 1, 2, 3, and 4

Proposition 8.3.1. *The* lSm*-premotivic triangulated category*

$$\mathrm{D}_{log,pw}(lSm, \Lambda)$$

satisfies (Htp–1), (Htp–2), (Htp–3), and (Htp–4).

Proof. For any log smooth morphism $f : X \to S$ of $\mathscr{S}$-schemes and any object K of $\mathrm{D}_{log,pw}(lSm/S, \Lambda)$, we have

$$f_\sharp f^* K \cong M_S(X) \otimes K$$

by (lSm-PF). Thus to show (Htp–1), (Htp–3), and (Htp–4), it suffices to show that the morphism

$$M_S(Y') \to M_S(Y)$$

in $\mathrm{D}_{log,pw}(lSm/S,\Lambda)$ for each type (a), (c), and (d) in (1.7.2) is an isomorphism. It follows from the fact that the morphism is a *log*-weak equivalence.

For (Htp–2), let $f : X \to S$ be an exact log smooth morphism of $\mathscr{S}$-schemes, and let $j : X^{\mathrm{ver}/S} \to X$ denote its verticalization of X via f. Since $\mathrm{D}_{log,pw}(lSm,\Lambda)$ is generated by lSm and τ, it suffices to show that the morphism

$$f_\sharp j_\sharp j^* M_X(V) \to f_\sharp M_X(V)$$

in $\mathrm{D}_{log,pw}(lSm/S,\Lambda)$ is an isomorphism for any log smooth morphism $V \to S$. Consider the commutative diagram

$$\begin{array}{ccc} M_S((V \times_X X^{\mathrm{ver}/S})^{\mathrm{ver}/S}) & = & M_S(V^{\mathrm{ver}/S}) \\ \downarrow & & \downarrow \\ M_S(V \times_X X^{\mathrm{ver}/S}) & \longrightarrow & M_S(V) \end{array}$$

in $\mathrm{D}_{log,pw}(lSm/S,\Lambda)$. We want to show that the lower horizontal arrow is an isomorphism. This follows from the fact that the vertical arrows are *log*-weak equivalences, i.e., they are isomorphisms in $\mathrm{D}_{log,pw}(lSm/S,\Lambda)$. □

Remark 8.3.2. The method of (8.3.1) can be applied to $\mathrm{D}_{log''',pw}(eSm,\Lambda)$ to conclude that it satisfies (Htp–1), (Htp–2), and (Htp–3) (but not (Htp–4)).

Proposition 8.3.3. *The eSm-premotivic triangulated category*

$$\mathrm{D}_{log,pw}(-,\Lambda)$$

satisfies (Htp–1), (Htp–2), (Htp–3), and (Htp–4).

Proof. Consider the adjunction

$$\rho_\sharp : D_{log,pw}(-,\Lambda) \leftrightarrows D_{log,pw}(lSm,\Lambda) : \rho^*.$$

Let us prove (Htp–2). It suffices to show that the natural transformation

$$f_{*,eSm} \xrightarrow{ad} f_{*,eSm} j_{*,eSm} j^*_{eSm}$$

is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc}
f_{*,eSm} & \xrightarrow{ad} & f_{*,eSm}j_{*,eSm}j_{eSm}^{*} \\
 & & \downarrow{\scriptstyle ad}\,\wr \\
\downarrow{\scriptstyle ad}\,\wr & & f_{*,eSm}j_{*,eSm}j_{eSm}^{*}\rho^{*}\rho_{\sharp} \\
 & & \downarrow\wr \\
f_{*,eSm}\rho^{*}\rho_{\sharp} & & f_{*,eSm}j_{*,eSm}\rho^{*}j_{lSm}^{*}\rho_{\sharp} \\
 & & \downarrow\wr \\
\downarrow\wr & & f_{*,eSm}\rho^{*}j_{*,lSm}j_{lSm}^{*}\rho_{\sharp} \\
 & & \downarrow\wr \\
\rho^{*}f_{*,lSm}\rho_{\sharp} & \xrightarrow{ad} & \rho^{*}f_{*,lSm}j_{*,lSm}j_{lSm}^{*}\rho_{\sharp}
\end{array}$$

of functors. The lower horizontal arrow is an isomorphism by (Htp–2) for $\mathrm{D}_{log,pw}(lSm,\Lambda)$ proved in (8.3.1), so the upper horizontal arrow is also an isomorphism. This proves (Htp–2).

The other properties can be similarly proved. □

8.3.4. By (8.3.3), (7.6.3), (7.5.3) and (2.9.5), we have proved that $\mathrm{D}_{log,pw}(-,\Lambda)$ satisfies the axiom (i) of (2.9.1) for . We also have proved (Adj), (Htp–1), (Htp–2), (Htp–3), (sét-Sep), (Loc), and (Stab) by (2.9.5), (7.5.7), (7.6.3), and (8.3.2).

8.4 Axiom (ii) of (2.9.1)

Theorem 8.4.1. *Let S be an $\mathscr{S}$-scheme with the trivial log structure. Then the support property holds in $\mathrm{D}_{log''',pw}(eSm,\Lambda)$ for the morphism*

$$f : S \times \mathbb{A}_{\mathbb{N}} \to S \times \mathbb{A}^1$$

of $\mathscr{S}$-schemes removing the log structure.

Proof. Consider the Cartesian diagram

$$\begin{array}{ccc}
S \times \mathrm{pt}_{\mathbb{N}} \times \mathbb{A}_{\mathbb{N}} & \xrightarrow{g'} & S \times \mathbb{A}_{\mathbb{N}} \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S \times \mathrm{pt}_{\mathbb{N}} \times \mathbb{A}^1 & \xrightarrow{g} & S \times \mathbb{A}^1
\end{array}$$

of $\mathscr{S}$-schemes where g denotes the projection. Then by (8.2.10), the exchange transformation

$$g^*f_* \xrightarrow{Ex} f'_*g'^*$$

in $D_{log''',pw}(eSm, \Lambda)$ is an isomorphism, and by (8.2.12), the functor

$$g^* : \mathrm{D}_{log''',pw}(eSm/(S \times \mathbb{A}^1), \Lambda) \to \mathrm{D}_{log''',pw}(eSm/(S \times \mathrm{pt}_{\mathbb{N}} \times \mathbb{A}^1), \Lambda)$$

is conservative. Thus to show the support property for f, it suffices to show that the support property for f'.

By (8.3.4), we can use (5.5.5), and this proves the support property for f'. □

8.4.2. Let S be an $\mathscr{S}$-scheme with a trivial log structure. For any open subscheme X of $S \times \mathbb{A}_{\mathbb{N}}$ or $S \times \mathbb{A}^1$, we have $lSm/X = eSm/X$ by [Ogu14, I.4.5.3.5]. Thus for these X, we have

$$\mathrm{D}_{log''',pw}(eSm/X, \Lambda) = \mathrm{D}_{log,pw}(X, \Lambda).$$

Then (8.4.1) implies the axiom (ii) of (2.9.1) for $\mathrm{D}_{log,pw}(-, \Lambda)$. Therefore we have proved the following theorem.

Theorem 8.4.3. *The eSm-premotivic triangulated category*

$$\mathrm{D}_{log,pw}(-, \Lambda)$$

is a log motivic triangulated category.

Chapter 9

Premotivic triangulated prederivators

9.1 Axioms of premotivic triangulated prederivators

9.1.1. Through this section, fix a category $\mathscr{S}$ with fiber products and a class of morphisms $\mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks.

Definition 9.1.2. We will introduce several notations and terminology.

(1) An *$\mathscr{S}$-diagram* is a functor

$$\mathscr{X} : I \to \mathscr{S}$$

where I is a small category. The 2-category of $\mathscr{S}$-diagrams is denoted by $\mathscr{S}^{\mathrm{dia}}$. We often write $\mathscr{X} = (\mathscr{X}, I)$ for $\mathscr{X}$. The category I is called the *index category* of $\mathscr{X}$, and an object λ of I is called an *index* of $\mathscr{X}$.

(2) Let $f : I \to J$ be a functor of small categories, and let μ be an object of J. We denote by I_μ the full subcategory of I such that λ is an object of I_μ if and only if $u(\lambda)$ is isomorphic to μ in J.

We denote by I/μ the category where

(i) object is a pair $(\lambda \in \mathrm{ob}(I),\ a : f(\lambda) \to \mu)$,

(ii) morphism $(\lambda, a) \to (\lambda', a')$ is the data of commutative diagrams:

$$\lambda \xrightarrow{b} \lambda' \qquad \begin{array}{ccc} f(\lambda) & \xrightarrow{b} & f(\lambda') \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle a'} & \\ & \mu & \end{array}$$

(3) Let $f : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ be a 1-morphism of $\mathscr{S}$-diagrams. Abusing the notation, we denote by f the induced functor $I \to J$. For an object μ of J, we denote by $\mathscr{X}_\mu$ and $\mathscr{X}/\mu$ the $\mathscr{S}$-diagrams

$$I_\mu \longrightarrow I \xrightarrow{\mathscr{X}} \mathscr{S}, \quad I/\mu \longrightarrow I \xrightarrow{\mathscr{X}} \mathscr{S}$$

respectively where the first arrows are the induced functors. Then we denote be

$$\mu : \mathscr{X}_\mu \to \mathscr{X}, \quad \overline{\mu} : \mathscr{X}/\mu \to \mathscr{X}$$

the induced functors.

(4) Let $f : \mathscr{X} \to \mathscr{Y}$ be a 1-morphism of $\mathscr{S}$-diagrams, and let μ be an index of $\mathscr{Y}$. Consider the induced 2-diagrams

$$\begin{array}{ccc} \mathscr{X}_\mu & \xrightarrow{\mu} & \mathscr{X} \\ \downarrow & \nearrow\!\!\!\!\nearrow & \downarrow f \\ \mathscr{Y}_\mu & \xrightarrow[\mu]{} & \mathscr{Y} \end{array} \quad \begin{array}{ccc} \mathscr{X}/\mu & \xrightarrow{\overline{\mu}} & \mathscr{X} \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow f \\ \mathscr{Y}_\mu & \xrightarrow[\mu]{} & \mathscr{Y} \end{array} \quad \begin{array}{ccc} \mathscr{X}/\mu & \xrightarrow{\overline{\mu}} & \mathscr{X} \\ \downarrow & \nearrow\!\!\!\!\nearrow & \downarrow f \\ \mathscr{Y}/\mu & \xrightarrow[\overline{\mu}]{} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams. Here, the arrows $\Leftrightarrow$ and $\Rightarrow$ express the induced 2-morphisms. Then we denote by

$$f_\lambda : \mathscr{X}_\mu \to \mathscr{Y}_\mu, \quad f_{\overline{\mu}\mu} : \mathscr{X}/\mu \to \mathscr{Y}_\mu, \quad f_{\overline{\mu}} : \mathscr{X}/\mu \to \mathscr{Y}/\mu$$

the 1-morphisms in the above 2-diagrams.

Let λ be an index of $\mathscr{X}$ such that $f(\lambda)$ is isomorphic to μ. Then we denote by

$$f_{\lambda\mu} : \mathscr{X}_\lambda \to \mathscr{Y}_\mu$$

the induced 1-morphism.

(5) Let $f : \mathscr{X} \to \mathscr{Y}$ be a 1-morphism of $\mathscr{S}$-diagrams. For a property $\mathbf{P}$ of morphisms in $\mathscr{S}$, we say that f is a $\mathbf{P}$ *morphism* if for any index λ of $\mathscr{X}$, the morphism $f_{\lambda\mu} : \mathscr{X}_\lambda \to \mathscr{Y}_\mu$ where $\mu = f(\lambda)$ is a $\mathbf{P}$ morphism in $\mathscr{S}$.

(6) We denote by dia the 2-category of small categories.

(7) We denote by $Tri^\otimes$ the 2-category of triangulated symmetric monoidal categories.

(8) We denote by $\mathbf{e}$ the trivial category.

Definition 9.1.3. A *$\mathscr{P}$-premotivic triangulated prederivator $\mathscr{T}$* over $\mathscr{S}$ is a 1-contravariant and 2-contravariant 2-functor

$$\mathscr{T} : \mathscr{S}^{\mathrm{dia}} \longrightarrow \mathrm{Tri}^\otimes$$

with the following properties.

(PD–1) For any 1-morphism $f : \mathscr{X} \to \mathscr{Y}$ of $\mathscr{S}$-diagrmas, we denote by $f^* : \mathscr{T}(\mathscr{Y}) \to \mathscr{T}(\mathscr{X})$ the image of f under $\mathscr{T} : \mathscr{S}^{\mathrm{dia}} \longrightarrow \mathrm{Tri}^\otimes$. Then the functor f^* admits a right adjoint denoted by f^*.
For any 2-morphism $t : f \to g$ of 1-morphisms $f, g : \mathscr{X} \to \mathscr{Y}$ of $\mathscr{S}$-diagrams, we denote by $t^* : g^* \to f^*$ the image of t under $\mathscr{T}$.

(PD–2) For any $\mathscr{P}$-morphism $f : \mathscr{X} \to \mathscr{Y}$, the functor f^* admits a left adjoint denoted by $f_\sharp$.

(PD–3) For any $\mathscr{S}$-diagram $\mathscr{X} = (\mathscr{X}, I)$, if I is a discrete category, then the induced functor

$$\mathscr{T}(\mathscr{X}) \longrightarrow \prod_{\lambda \in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda)$$

is an equivalence of categories.

(PD–4) For any $\mathscr{S}$-diagram $\mathscr{X} = (\mathscr{X}, I)$, the family of functors λ^* for $\lambda \in \mathrm{ob}(I)$ is conservative.

(PD–5) For any object S of $\mathscr{S}$, the fibered category

$$\mathscr{T}(-, \mathbf{e})$$

is a $\mathscr{P}$-premotivic triangulated category.

(PD–6) For any morphism $f : \mathscr{X} \to \mathscr{Y}$ of $\mathscr{S}$-diagrams and any index μ of $\mathscr{Y}$, in the 2-diagram

$$\begin{array}{ccc} \mathscr{X}/\mu & \xrightarrow{\overline{\mu}} & \mathscr{X} \\ {\scriptstyle f_{\overline{\mu}\mu}}\downarrow & \swarrow\!\!\swarrow_{t} & \downarrow{\scriptstyle f} \\ \mathscr{Y}_\mu & \xrightarrow[\mu]{} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrmas, the exchange transformation

$$\mu^* f_* \xrightarrow{ad} f_{\overline{\mu}\mu *} f^*_{\overline{\mu}\mu} \mu^* f_* \xrightarrow{t^*} f_{\overline{\mu}\mu *} \overline{\mu}^* f^* f_* \xrightarrow{ad'} f_{\overline{\mu}\mu *} \overline{\mu}^*$$

is an isomorphism.

Remark 9.1.4. Our axioms are selected from [Ayo07, 2.4.16] and the axioms of *algebraic derivators* in [Ayo07, 2.4.12].

Definition 9.1.5. Let $\mathscr{T}$ be a $\mathscr{P}$-premotivic triangulated prederivator.

(1) A *cartesian section* of $\mathscr{T}$ is the data of an object $A_{\mathscr{X}}$ of $\mathscr{T}(\mathscr{X})$ for each $\mathscr{S}$-diagram $\mathscr{X}$ and of isomorphisms

$$f^*(A_{\mathscr{Y}}) \xrightarrow{\sim} A_{\mathscr{X}}$$

for each morphism $f : \mathscr{X} \to \mathscr{Y}$ of $\mathscr{S}$-diagrams, subject to following coherence conditions as in (1.1.5). The tensor product of two cartesian sections is defined termwise.

(2) A set of *twists* τ for $\mathscr{T}$ is a set of Cartesian sections of $\mathscr{T}$ stable by tensor product. For short, we say also that $\mathscr{T}$ is τ-twisted .

Proposition 9.1.6. *Let $\mathscr{T}$ be a $\mathscr{P}$-premotivic triangulated prederivator, and let $\mathscr{X}$ be an $\mathscr{S}$-diagram. Then $\lambda_\sharp K$ is compact for any index λ of $\mathscr{X}$ and any compact object K of $\mathscr{T}(\mathscr{X}_\lambda)$.*

Proof. We have the isomorphism

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(\lambda_\sharp K, -) \cong \mathrm{Hom}_{\mathscr{T}(\mathscr{X}_\lambda)}(K, \lambda^*(-)).$$

Using this, ther conclusion follows fom the fact that λ^* preserves small sums. □

Definition 9.1.7. Let $\mathscr{T}$ be a τ-twisted $\mathscr{P}$-premotivic triangulated prederivator. For any object S of $\mathscr{S}$, we denote by $\mathcal{F}_{\mathscr{P}/S}$ the family of motives of the form

$$M_S(X)\{i\}$$

for $\mathscr{P}$-morphism $X \to S$ and twist $i \in \tau$. Then for any $\mathscr{S}$-diagram $\mathscr{X}$, we denote by $\mathcal{F}_{\mathscr{P}/\mathscr{X}}$ the family of motives of the form

$$\lambda_\sharp K$$

for index λ of $\mathscr{X}$ and object K of $\mathcal{F}_{\mathscr{P}/\mathscr{X}_\lambda}$.

Proposition 9.1.8. *Let $\mathscr{T}$ be a τ-twisted $\mathscr{P}$-premotivic triangulated prederivator. Assume that $\mathscr{T}(-, \mathbf{e})$ is compactly generated by $\mathscr{P}$ and τ. Then $\mathcal{F}_{\mathscr{P}/\mathscr{X}}$ generates $\mathscr{T}(\mathscr{X})$.*

Proof. Let $K \to K'$ be a morphism in $\mathscr{T}(\mathscr{X})$ such that the homomorphism

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(\lambda_\sharp L, K) \to \mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(\lambda_\sharp L, K')$$

is an isomorphism for any index λ of $\mathscr{X}$ and any element L of $\mathcal{F}_{\mathscr{P}/\mathscr{X}_\lambda}$. We want to show that the morphism $K \to K'$ in $\mathscr{T}(\mathscr{X})$ is an isomorphism.

The homomorphism

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X}_\lambda)}(L, \lambda^* K) \to \mathrm{Hom}_{\mathscr{T}(\mathscr{X}_\lambda)}(L, \lambda^* K')$$

is an isomorphism, and $\mathcal{F}_{\mathscr{P}/\mathscr{X}_\lambda}$ generates $\mathscr{T}(\mathscr{X}_\lambda)$ by assumption. Thus the morphism $\lambda^* K \to \lambda^* K'$ in $\mathscr{T}(\mathscr{X}_\lambda)$ is an isomorphism. Then (PD–4) implies that the morphism $K \to K'$ in $\mathscr{T}(\mathscr{X})$ is an isomorphism, which completes the proof. □

9.2 Consequences of axioms

9.2.1. Throughout this section, fix a category $\mathscr{S}$ with fiber products and a class of morphisms $\mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks.

Definition 9.2.2. Let $f : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ be a 1-morphism of $\mathscr{S}$-diagrams.

(1) We say that f is *reduced* if the functor $f : I \to J$ is an equivalence.

(2) We say that f is *Cartesian* if f is reduced and for any morphism $\mu \to \mu'$ in J, the diagram

$$\begin{array}{ccc} \mathscr{X}_\mu & \longrightarrow & \mathscr{X}_{\mu'} \\ \downarrow & & \downarrow \\ \mathscr{Y}_\mu & \longrightarrow & \mathscr{Y}_{\mu'} \end{array}$$

in $\mathscr{S}$ is Cartesian.

9.2.3. Let $f : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ and $g : (\mathscr{Y}', J') \to (\mathscr{Y}, J)$ be 1-morphisms of $\mathscr{S}$-diagrams. Consider the category $J' \times_J I$. We have the functors

$$u_1 : J' \times_J I \xrightarrow{p_1} J' \xrightarrow{\mathscr{Y}'} \mathscr{S},$$

$$u_2 : J' \times_J I \xrightarrow{p_2} J' \xrightarrow{\mathscr{X}} \mathscr{S},$$

$$u : J' \times_J I \xrightarrow{p} J' \xrightarrow{\mathscr{Y}} \mathscr{S}$$

where p_1, p_2, and p denote the projections. Then we denote by

$$(\mathscr{Y}' \times_{\mathscr{Y}} \mathscr{X}, J' \times_J I)$$

the functor $J' \times_J I \to \mathscr{S}$ obtained by taking fiber products $u_1(\lambda) \times_{u(\lambda)} u_2(\lambda)$ for $\lambda \in \mathrm{ob}(J' \times_J I)$. Note that by [Ayo07, 2.4.10], the commutative diagram

$$\begin{array}{ccc} \mathscr{Y}' \times_{\mathscr{Y}} \mathscr{X} & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}' & \xrightarrow{g} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams is Cartesian where g' and f' denote the first and second projections respectively.

Proposition 9.2.4. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a Cartesian $\mathscr{P}$-morphism of $\mathscr{S}$-diagrams, and let μ be an index of $\mathscr{Y}$. Then in the Cartesian diagram*

$$\begin{array}{ccc} \mathscr{X}_\mu & \xrightarrow{\mu} & \mathscr{X} \\ \downarrow{\scriptstyle f_\mu} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}_\mu & \xrightarrow{\mu} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams, the exchange transformation

$$f^* \mu_* \xrightarrow{Ex} \mu_* f_\mu^*$$

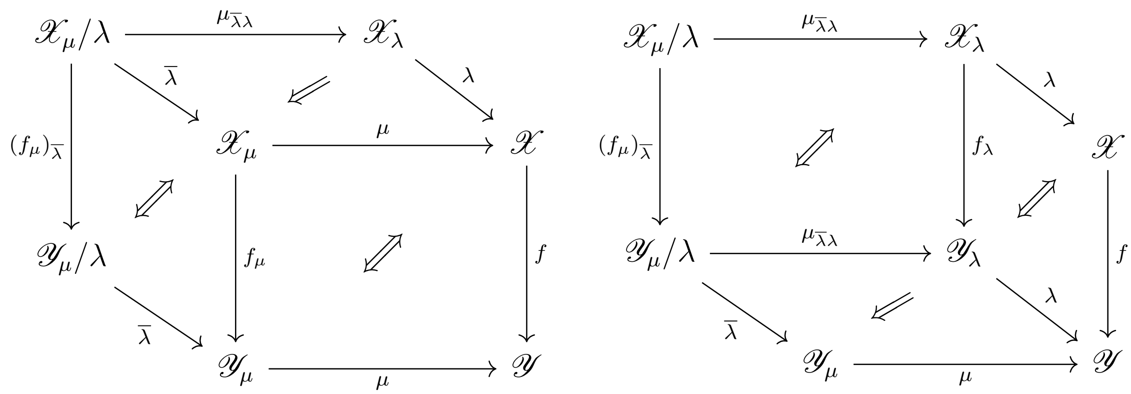
is an isomorphism.

Proof. Let λ be an index of $\mathscr{X}$ (so an index of $\mathscr{Y}$ since f is Cartesian). By (PD–4), it suffices to show that the natural transformation

$$\lambda^* f^* \mu_* \xrightarrow{Ex} \lambda^* \mu_* f_\mu^*$$

is an isomorphism.

Consider the 2-diagrams



of $\mathscr{S}$-diagrams. Then we have the commutative diagram

$$\begin{array}{ccccc}
\lambda^* f^* \mu_* & \xrightarrow{\sim} & f_\lambda^* \lambda^* \mu_* & \xrightarrow{Ex} & f_\lambda^* \mu_{\bar\lambda\lambda *} \overline{\lambda}^* \\
\big\downarrow{\scriptstyle Ex} & & & & \big\downarrow{\scriptstyle Ex} \\
\lambda^* \mu_* f_\mu^* & \xrightarrow{Ex} & \mu_{\bar\lambda\lambda *} \overline{\lambda}^* f_\mu^* & \xrightarrow{\sim} & \mu_{\bar\lambda\lambda *} (f_\mu)_{\bar\lambda}^* \overline{\lambda}^*
\end{array}$$

of functors. By (PD–6), the lower left horizontal and upper right horizontal arrows are isomorphisms. Thus it suffices to show that the right vertical arrow is an isomorphism. We have the identification

$$\mathscr{X}_\mu/\lambda = \mathscr{X}_\mu \times \mathrm{Hom}_J(\mu, \lambda), \quad \mathscr{Y}_\mu/\lambda = \mathscr{Y}_\mu \times \mathrm{Hom}_J(\mu, \lambda)$$

where J denotes the index category of $\mathscr{Y}$. Thus by (PD–3), it suffices to show that for any morphism $\mu \to \lambda$ in J, in the induced Cartesian diagram

$$\begin{array}{ccc}
\mathscr{X}_\mu & \xrightarrow{\mathrm{id}_{\mu\lambda}} & \mathscr{X}_\lambda \\
\big\downarrow{\scriptstyle f_\mu} & & \big\downarrow{\scriptstyle f_\lambda} \\
\mathscr{Y}_\mu & \xrightarrow{\mathrm{id}_{\mu\lambda}} & \mathscr{Y}_\lambda
\end{array}$$

in $\mathscr{S}$, the exchange transformation

$$f_\lambda^* \mathrm{id}_{\mu\lambda *} \xrightarrow{Ex} \mathrm{id}_{\mu\lambda *} f_\mu^*$$

is an isomorphism. This follows from (PD–5) and the assumption that f is a Cartesian $\mathscr{P}$-morphism. □

Proposition 9.2.5. *Consider a Cartesian diagram*

$$\begin{array}{ccc}
\mathscr{X}' & \xrightarrow{g'} & \mathscr{X} \\
\big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\
\mathscr{Y}' & \xrightarrow{g} & \mathscr{Y}
\end{array}$$

of $\mathscr{S}$-diagrams where f is a Cartesian $\mathscr{P}$-morphism. Then the exchange transformation

$$f'_{\sharp}g'^{*} \xrightarrow{Ex} g^{*}f_{\sharp}$$

is an isomorphism.

Proof. Note that f' is also a Cartesian $\mathscr{P}$-morphism. Let μ' be an index of $\mathscr{Y}'$. By (PD–4), it suffices to show that the natural transformation

$$\mu'^{*}f'_{\sharp}g'^{*} \xrightarrow{Ex} \mu'^{*}g^{*}f_{\sharp}$$

is an isomorphism. We put $\mu = g(\mu')$.

Consider the commutative diagrams

$$\begin{array}{ccccc} \mathscr{X}'_{\mu'} & \xrightarrow{\mu'} & \mathscr{X}' & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{\scriptstyle f_{\mu'}} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{\mu'} & \mathscr{Y}' & \xrightarrow{g} & \mathscr{Y} \end{array} \qquad \begin{array}{ccccc} \mathscr{X}'_{\mu'} & \xrightarrow{g'_{\mu'\mu}} & \mathscr{X}_{\mu} & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{\scriptstyle f_{\mu'}} & & \downarrow{\scriptstyle f_{\mu}} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{g_{\mu'\mu}} & \mathscr{Y}_{\mu} & \xrightarrow{g} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams. Then we have the commutative diagram

$$\begin{array}{ccccc} f'_{\mu'\sharp}\mu'^{*}g'^{*} & \xrightarrow{Ex} & \mu'^{*}f'_{\sharp}g'^{*} & \xrightarrow{Ex} & \mu'^{*}g^{*}f_{\sharp} \\ \downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim} \\ f'_{\mu'\sharp}g^{*}_{\mu'\mu}\mu^{*} & \xrightarrow{Ex} & g^{*}_{\mu'\mu}f_{\mu\sharp}\mu^{*} & \xrightarrow{Ex} & g^{*}_{\mu'\mu}\mu^{*}f_{\sharp} \end{array}$$

of functors. The upper left horizontal and lower right horizontal arrows are isomorphisms by (9.2.4), and the lower left horizontal arrow is an isomorphism by (PD–5) since the commutative diagram

$$\begin{array}{ccc} \mathscr{X}'_{\mu'} & \xrightarrow{g'_{\mu'\mu}} & \mathscr{X}_{\mu} \\ \downarrow{\scriptstyle f'_{\mu'}} & & \downarrow{\scriptstyle f_{\mu}} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{g_{\mu'\mu}} & \mathscr{Y}_{\mu} \end{array}$$

is Cartesian by assumption. Thus the upper right horizontal arrow is also an isomorphism. □

Proposition 9.2.6. *Let $\mathscr{X}$ be an $\mathscr{S}$-diagram. Assume that the index category of $\mathscr{X}$ has a terminal object λ. Consider the 1-morphisms*

$$\mathscr{X}_{\lambda} \xrightarrow{\lambda} \mathscr{X} \xrightarrow{f} \mathscr{X}_{\lambda}$$

where f denotes the morphism induced by the functor $I \to \mathbf{e}$ to the terminal object λ. Then the natural transformation

$$\lambda_{\sharp}\lambda^{*}f^{*} \xrightarrow{ad} f^{*}$$

is an isomorphism.

Proof. Let λ' be an index of $\mathscr{X}$. By (PD–4), it suffices to show that the natural transformation

$$\lambda'^*\lambda_\sharp\lambda^* f^* \xrightarrow{ad'} \lambda'^* f^*$$

is an isomorphism. We will show that its right adjoint

$$f_*\lambda'_* \xrightarrow{ad} f_*\lambda_*\lambda^*\lambda'_*$$

is an isomorphism.

Consider the diagram

$$\begin{array}{ccccc}
\mathscr{X}_{\lambda'} & \xrightarrow{\mathrm{id}} & \mathscr{X}_{\lambda'} & & \\
\downarrow{\scriptstyle \mathrm{id}_{\lambda'\lambda}} & & \downarrow{\scriptstyle \lambda'} & \searrow^{\mathrm{id}_{\lambda'\lambda}} & \\
\mathscr{X}_{\lambda} & \xrightarrow{\lambda} & \mathscr{X} & \xrightarrow{f} & \mathscr{X}_{\lambda}
\end{array}$$

of $\mathscr{S}$-diagrams. Then we have the commutative diagram

$$\begin{array}{ccc}
f_*\lambda'_* & & \\
\downarrow{\scriptstyle ad} & \searrow^{\sim} & \\
f_*\lambda_*\lambda^*\lambda'_* & \xrightarrow{Ex} & f_*\lambda_*\mathrm{id}_{\lambda'\lambda*}\mathrm{id}^*
\end{array}$$

of functors, so it suffices to show that the horizontal arrow is an isomorphism. This follows from (PD–6) since $\mathscr{X}_{\lambda'} = \mathscr{X}_{\lambda'}/\lambda$. ∎

Proposition 9.2.7. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a reduced morphism of $\mathscr{S}$-diagrams, and let μ be an index of $\mathscr{Y}$. Consider the Cartesian diagram*

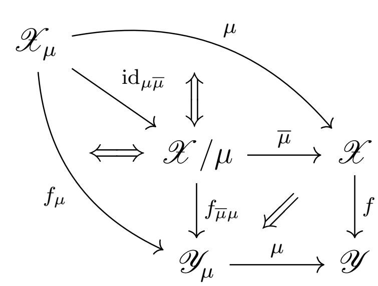
$$\begin{array}{ccc}
\mathscr{X}_\mu & \xrightarrow{\mu} & \mathscr{X} \\
\downarrow{\scriptstyle f_\mu} & & \downarrow{\scriptstyle f} \\
\mathscr{Y}_\mu & \xrightarrow{\mu} & \mathscr{Y}
\end{array}$$

of $\mathscr{S}$-diagrams. Then the exchange transformation

$$\mu^* f_* \xrightarrow{Ex} f_{\mu*}\mu^*$$

is an isomorphism.

Proof. Consider the 2-diagram



of $\mathscr{S}$-diagrams. Then the exchange transformation

$$\mu^* f_* \xrightarrow{Ex} f_{\mu *}\mu^*$$

has the decomposition

$$\mu^* f_* \xrightarrow{Ex} f_{\mu\mu *}\overline{\mu}^* \xrightarrow{ad} f_{\overline{\mu}\mu *}\mathrm{id}_{\mu\overline{\mu}*}\mathrm{id}^*_{\mu\overline{\mu}}\overline{\mu}^* \xrightarrow{\sim} f_{\mu *}\mu^*.$$

By (PD–6), the first arrow is an isomorphism. Thus it suffices to show that the second arrow is an isomorphism.

Consider the 1-morphisms

$$\mathscr{X}_\mu \overset{\mathrm{id}_{\mu\overline{\mu}}}{\to} \mathscr{X}/\mu \overset{\mathrm{id}_{\overline{\mu}\mu}}{\to} \mathscr{X}_\mu \overset{f_\mu}{\to} \mathscr{Y}_\mu$$

of $\mathscr{S}$-diagrams. Then it suffices to show that the natural transformation

$$\mathrm{id}_{\overline{\mu}\mu *}\mathrm{id}_{\mu\overline{\mu}*}\mathrm{id}^*_{\mu\overline{\mu}} \xrightarrow{ad} \mathrm{id}_{\overline{\mu}\mu *}$$

is an isomorphism, which follows from (9.2.6). □

Proposition 9.2.8. *Consider a Cartesian diagram*

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{f'} & & \downarrow{f} \\ \mathscr{Y}' & \xrightarrow{g} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams where

(i) f is reduced,

(ii) for any index μ' of $\mathscr{Y}'$, in the Cartesian diagram

$$\begin{array}{ccc} \mathscr{X}'_{\mu'} & \xrightarrow{g'_{\mu'\mu}} & \mathscr{X}_\mu \\ \downarrow{f'_{\mu'}} & & \downarrow{f_\mu} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{g_{\mu'\mu}} & \mathscr{Y}_\mu \end{array}$$

in $\mathscr{S}$ where $\mu = g(\mu')$, the exchange transformation

$$g^*_{\mu'\mu}f_{\mu *} \xrightarrow{Ex} f'_{\mu' *}g'^*_{\mu'\mu}$$

is an isomorphism.

Then the exchange transformation

$$g^* f_* \xrightarrow{Ex} f'_* g'^*$$

is an isomorphism.

Proof. Note that f' is also reduced. Let μ' be an index of $\mathscr{Y}'$. By (PD–4), it suffices to show that the natural transformation

$$\mu'^* g^* f_* \xrightarrow{Ex} \mu'^* f'_* g'^*$$

is an isomorphism. We put $\mu = g(\mu')$.

Consider the commutative diagrams

$$\begin{array}{ccccc} \mathscr{X}'_{\mu'} & \xrightarrow{\mu'} & \mathscr{X}' & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{\scriptstyle f_{\mu'}} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{\mu'} & \mathscr{Y}' & \xrightarrow{g} & \mathscr{Y} \end{array} \qquad \begin{array}{ccccc} \mathscr{X}'_{\mu'} & \xrightarrow{g'_{\mu'\mu}} & \mathscr{X}_{\mu} & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{\scriptstyle f_{\mu'}} & & \downarrow{\scriptstyle f_{\mu}} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{g_{\mu'\mu}} & \mathscr{Y}_{\mu} & \xrightarrow{g} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams. Then we have the commutative diagram

$$\begin{array}{ccccc} \mu'^* g^* f_* & \xrightarrow{Ex} & \mu'^* f'_* g'^* & \xrightarrow{Ex} & f_{\mu' *} \mu'^* g'^* \\ \downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim} \\ g^*_{\mu'\mu} \mu^* f_* & \xrightarrow{Ex} & g^*_{\mu'\mu} f_{\mu *} \mu^* & \xrightarrow{Ex} & g^*_{\mu'\mu} f'_{\mu' *} g'^* \end{array} \tag{9.2.8.1}$$

of functors. The upper right horizontal and lower left horizontal arrows are isomorphisms by (9.2.7), and the lower right horizontal arrow is an isomorphism by (PD–5) since the commutative diagram

$$\begin{array}{ccc} \mathscr{X}'_{\mu'} & \xrightarrow{g'_{\mu'\mu}} & \mathscr{X}_{\mu} \\ \downarrow{\scriptstyle f'_{\mu'}} & & \downarrow{\scriptstyle f_{\mu}} \\ \mathscr{Y}'_{\mu'} & \xrightarrow{g_{\mu'\mu}} & \mathscr{Y}_{\mu} \end{array}$$

is Cartesian by assumption. Thus the upper left horizontal arrow of (9.2.8.1) is also an isomorphism. □

9.2.9. Under the notations and hypotheses of (9.2.8), we will give two examples satisfying the conditions of (loc. cit).

(1) When f is reduced and g is a $\mathscr{P}$-morphism, the conditions are satisfied by ($\mathscr{P}$-BC).

(2) Assume that $\mathscr{T}(-, \mathbf{e})$ satisfies (Loc). Then the conditions are satisfied when f is a reduced strict closed immersion by (2.6.2).

9.2.10. Let $i : \mathscr{Z} \to \mathscr{X}$ be a Cartesian strict closed immersion of $\mathscr{S}$-diagrmas. Then for any morphism $\lambda \to \lambda'$ in the index category of $\mathscr{X}$, we have the commutative diagram

$$\begin{array}{ccccc} \mathscr{Z}_{\lambda} & \xrightarrow{i_{\lambda}} & \mathscr{X}_{\lambda} & \xleftarrow{j_{\lambda}} & \mathscr{U}_{\lambda} \\ \downarrow{\scriptstyle \mathrm{id}_{\lambda\lambda'}} & & \downarrow{\scriptstyle \mathrm{id}_{\lambda\lambda'}} & & \downarrow \\ \mathscr{Z}_{\lambda'} & \xrightarrow{i_{\lambda'}} & \mathscr{X}_{\lambda'} & \xleftarrow{j_{\lambda'}} & \mathscr{U}_{\lambda'} \end{array}$$

in $\mathscr{S}$ where each square is Cartesian and j_λ (resp. $j_{\lambda'}$) denotes the complement of i_λ (resp. $i_{\lambda'}$). From this, we obtain the Cartesian open immersion $j : \mathscr{U} \to \mathscr{X}$. It is called the *complement* of i.

9.2.11. We have assumed or proven the axioms DerAlg 0, DerAlg 1, DerAlg 2d, DerAlg 2g, DerAlg 3d, and DerAlg 3g in [Ayo07, 4.2.12]. With the additional assumption that $\mathscr{T}(-, \mathbf{e})$ satisfies (Loc), the following results are proved in [Ayo07, Section 2.4.3].

(1) Let $i : \mathscr{Z} \to \mathscr{X}$ be a Cartesian strict closed immersion, and let $j : \mathscr{U} \to \mathscr{X}$ denote its complement. Then the pair of functors (i^*, j^*) is conservative.

(2) Let $i : \mathscr{Z} \to \mathscr{X}$ be a strict closed immersion. Then the counit

$$i^* i_* \xrightarrow{ad'} \mathrm{id}$$

is an isomorphism.

(3) Consider a Cartesian diagram

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{g'} & \mathscr{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}' & \xrightarrow{g} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrams where f is a $\mathscr{P}$-morphism and g is a Cartesian strict closed immersion. Then the exchange transformation

$$f_\sharp g'_* \xrightarrow{Ex} g_* f'_\sharp$$

is an isomorphism.

9.2.12. The notion of $\mathscr{P}$-premotivic triangulated prederivators can be used to descent theory of $\mathscr{P}$-premotivic triangulated categories. Let t be a Grothendieck topology on $\mathscr{S}$. Recall from [CD12, 3.2.5] that $\mathscr{T}$ satisfies t-descent if the unit

$$\mathrm{id} \xrightarrow{ad} f_* f^*$$

is an isomorphism for any t-hypercover (see [CD12, 3.2.1] for the definition of t-hypercover) $f : \mathscr{X} \to \mathscr{Y}$ of $\mathscr{S}$-diagrams. In (9.5.1), we will construct a eSm-premotivic triangulated prederiator satisfying strict étale descent.

9.3 Localizing subcategories

9.3.1. Throughout this section, we fix a category $\mathscr{S}$ and classes of morphisms $\mathscr{P}' \subset \mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks. We fix also a τ-twisted $\mathscr{P}$-premotivic triangulated prederivator $\mathscr{T}$ such that $\mathscr{T}(-, \mathbf{e})$ is compactly generated by $\mathscr{P}$ and τ. Then by (9.1.8), $\mathscr{T}(\mathscr{X})$ is compactly generated for any $\mathscr{S}$-diagram $\mathscr{X}$.

For an $\mathscr{S}$-diagram $\mathscr{X}$, we denote by $\mathcal{F}_{\mathscr{P}/\mathscr{X}}$ the family of motives of the form

$$\lambda_\sharp K$$

for index λ of $\mathscr{X}$ and object K of $\mathcal{F}_{\mathscr{P}/\mathscr{X}_\lambda}$ (see (1.5.1) for the definition of $\mathcal{F}_{\mathscr{P}/\mathscr{X}_\lambda}$). Then we denote by $\mathscr{T}(\mathscr{P}'/\mathscr{X})$ the localizing subcategory of $\mathscr{T}(\mathscr{X})$ generated by $\mathcal{F}_{\mathscr{P}'/\mathscr{X}}$. Note that $\mathscr{T}(\mathscr{P}'/\mathscr{X})$ is compactly generated by (9.1.6).

We denote by $\mathscr{T}(\mathscr{P}')$ the collection of $\mathscr{T}(\mathscr{P}'/\mathscr{X})$ for object $\mathscr{X}$ of $\mathscr{S}$. The purpose of this section is to show that $\mathscr{T}(\mathscr{P}')$ has a structure of $\mathscr{P}'$-premotivic triangulated prederivator.

9.3.2. We denote by $\rho_\sharp$ the inclusion functor

$$\mathscr{T}(\mathscr{P}') \to \mathscr{T}.$$

Then the set of twists τ for $\mathscr{T}$ gives a set of twists for $\mathscr{T}(\mathscr{P}')$. It is denoted by τ again. By (9.3.1), $\mathscr{T}(\mathscr{P}')$ is compactly generated, so by [Nee01, 8.4.4], $\rho_\sharp$ has a right adjoint

$$\rho^* : \mathscr{T} \to \mathscr{T}(\mathscr{P}')$$

since $\rho_\sharp$ respects small sums. For any $\mathscr{S}$-diagram $\mathscr{X}$, we denote by

$$\rho_{\sharp,\mathscr{X}} : \mathscr{T}(\mathscr{P}'/\mathscr{X}) \rightleftarrows \mathscr{T}(\mathscr{X}) : \rho^*_{\mathscr{X}}$$

the specification of $\rho_\sharp$ and ρ^* to $\mathscr{X}$.

9.3.3. Let $\mathscr{X}$ and $\mathscr{Y}$ be $\mathscr{S}$-diagrams. Consider a diagram

$$\begin{array}{ccc}
\mathscr{T}(\mathscr{P}'/\mathscr{X}) & & \mathscr{T}(\mathscr{P}'/\mathscr{Y}) \\
{\scriptstyle \rho_{\sharp,\mathscr{X}}}\downarrow\uparrow{\scriptstyle \rho^*_{\mathscr{X}}} & & {\scriptstyle \rho_{\sharp,\mathscr{Y}}}\downarrow\uparrow{\scriptstyle \rho^*_{\mathscr{Y}}} \\
\mathscr{T}(\mathscr{X}) & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \mathscr{T}(\mathscr{Y})
\end{array}$$

such that α is left adjoint to β. Suppose that α maps $\mathcal{F}_{\mathscr{P}'/\mathscr{X}}$ into $\mathscr{T}(\mathscr{P}'/\mathscr{Y})$ and that α commutes with twists. Then as in (1.5.3), we define

$$\alpha_{\mathscr{P}'} : \mathscr{T}(\mathscr{P}'/\mathscr{X}) \to \mathscr{T}(\mathscr{P}'/\mathscr{Y}),$$

$$\beta_{\mathscr{P}'} : \mathscr{T}(\mathscr{P}'/\mathscr{Y}) \to \mathscr{T}(\mathscr{P}'/\mathscr{X})$$

as $\alpha_{\mathscr{P}'} = \rho^*_S \alpha \rho_{\sharp,\mathscr{X}}$ and $\beta_{\mathscr{P}'} = \rho^*_X \beta \rho_{\sharp,\mathscr{Y}}$. We often omit $\mathscr{P}'$ in $\alpha_{\mathscr{P}'}$ and $\beta_{\mathscr{P}'}$ for brevity. Then as in (1.5.4), α commutes with $\rho_\sharp$, and $\alpha_{\mathscr{P}'}$ is left adjoint to $\beta_{\mathscr{P}'}$. Note that β commutes with ρ^* in this case.

Proposition 9.3.4. *Let $\mathscr{X}$ be an $\mathscr{S}$-diagram. For any indices λ and λ' of $\mathscr{X}$ and any object K of $\mathscr{T}(\mathscr{X}_\lambda)$ in $\mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda)$, the object $\lambda'^*\lambda_\sharp K$ of $\mathscr{T}(\mathscr{X}_{\lambda'})$ is in $\mathscr{T}(\mathscr{P}'/\mathscr{X}_{\lambda'})$.*

Proof. Consider the 2-diagram

$$\begin{array}{ccc} \mathscr{X}_{\lambda'}/\lambda & \xrightarrow{\overline{\lambda}} & \mathscr{X}_{\lambda'} \\ {\scriptstyle \lambda'_{\overline{\lambda}\lambda}}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle \lambda'} \\ \mathscr{X}_{\lambda} & \xrightarrow[\lambda]{} & \mathscr{X} \end{array}$$

of $\mathscr{S}$-diagrams. Then the exchange transformation

$$\lambda'_{\overline{\lambda}\lambda\sharp}\overline{\lambda}^* \xrightarrow{Ex} \lambda'^*\lambda_\sharp$$

is an isomorphism by (PD–6), so we need to show that $\lambda'_{\overline{\lambda}\lambda\sharp}\overline{\lambda}^*K$ is in $\mathscr{T}(\mathscr{P}'/\mathscr{X})$. This follows from the identification

$$\mathscr{X}_{\lambda'}/\lambda = \mathscr{X}_{\lambda'} \times \mathrm{Hom}_I(\lambda', \lambda)$$

where I denotes the index category of $\mathscr{X}$. □

Proposition 9.3.5. *Let $\mathscr{X}$ be an $\mathscr{S}$-diagram, and let K be an object of $\mathscr{T}(\mathscr{X})$. If λ^*K is in $\mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda)$ for any index λ of $\mathscr{X}$, then K is in $\mathscr{T}(\mathscr{P}'/\mathscr{X})$.*

Proof. We denote by $\mathcal{T}'$ the full subcategory of $\mathscr{T}(\mathscr{X})$ consisting of objects K of $\mathscr{T}(\mathscr{X})$ such that λ^*K is in $\mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda)$ for any index λ of $\mathscr{X}$. Then $\mathcal{T}'$ is a triangulated subcategory of $\mathscr{T}(\mathscr{X})$. By (9.3.4), $\mathcal{F}_{\mathscr{P}'/\mathscr{X}}$ is in $\mathcal{T}'$. We will first show that the family $\mathcal{F}_{\mathscr{P}'/\mathscr{X}}$ generates $\mathcal{T}'$.

Let $K \to K'$ be a morphism in $\mathcal{T}'$ such that the homomorphism

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(\lambda_\sharp L, K) \to \mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(\lambda_\sharp L, K')$$

is an isomorphism for any index λ of $\mathscr{X}$ and any element L of $\mathcal{F}_{\mathscr{P}'/\mathscr{X}_\lambda}$. We want to show that the morphism $K \to K'$ in $\mathscr{T}(\mathscr{X})$ is an isomorphism.

The homomorphism

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X}_\lambda)}(L, \lambda^*K) \to \mathrm{Hom}_{\mathscr{T}(\mathscr{X}_\lambda)}(L, \lambda^*K')$$

is an isomorphism, and $\mathcal{F}_{\mathscr{P}'/\mathscr{X}_\lambda}$ generates $\mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda)$. Thus the morphism $\lambda^*K \to \lambda^*K'$ in $\mathscr{T}(\mathscr{X}_\lambda)$ is an isomorphism since λ^*K and λ^*K' are in $\mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda)$. Then (PD–4) implies that the morphism $K \to K'$ in $\mathscr{T}(\mathscr{X})$ is an isomorphism. Thus so far we have proven that the family $\mathcal{F}_{\mathscr{P}'/\mathscr{X}}$ generates $\mathcal{T}'$.

Since $\mathcal{F}_{\mathscr{P}'/\mathscr{X}_\lambda}$ consists of compact objects by (9.1.6), $\mathcal{T}$ is compactly generated. Thus by (1.4.4), $\mathcal{T}'$ is the localizing subcategory generated by $\mathcal{F}_{\mathscr{P}'/\mathscr{X}_\lambda}$, which is $\mathscr{T}(\mathscr{P}'/\mathscr{X})$ by definition. This completes the proof. □

9.3.6. Let $f : \mathscr{X} \to \mathscr{Y}$ be a 1-morphism of $\mathscr{S}$-diagrams. Then by (9.3.5), f^* maps $\mathcal{F}_{\mathscr{P}'/\mathscr{Y}}$ into $\mathscr{T}(\mathscr{P}'/\mathscr{X})$. Thus by (9.3.3), f^* commutes with $\rho_\sharp$, f_* commutes with ρ^*, and we have the adjunction:

$$f^*_{\mathscr{P}'} : \mathscr{T}(\mathscr{P}'/\mathscr{Y}) \rightleftarrows \mathscr{T}(\mathscr{P}'/\mathscr{X}) : f_{\mathscr{P}'*}$$

When f is a $\mathscr{P}$-morphism, $f_\sharp$ maps $\mathcal{F}_{\mathscr{P}'/\mathscr{X}}$ into $\mathscr{T}(\mathscr{P}'/\mathscr{Y})$ by definition. Thus by (9.3.3), $f_\sharp$ commutes with $\rho_\sharp$, f^* commutes with ρ^*, and we have the adjunction:

$$f_{\mathscr{P}'\sharp} : \mathscr{T}(\mathscr{P}'/\mathscr{X}) \rightleftarrows \mathscr{T}(\mathscr{P}'/\mathscr{Y}) : f^*_{\mathscr{P}'}$$

9.3.7. Now, we will verify the axioms of (9.1.3) for $\mathscr{T}(\mathscr{P}')$.

(1) As in (1.5.6), $\mathscr{T}(\mathscr{P}')$ satisfies (PD–1), (PD–2), and (PD–5).

(2) *Axiom* (PD–3). Let $(\mathscr{X}, I)$ be an $\mathscr{S}$-diagram such that I is a discrete category. Consider the diagram

$$\begin{array}{ccc} \mathscr{T}(\mathscr{P}'/\mathscr{X}) & \xrightarrow{\alpha'} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda) \\ \downarrow{\scriptstyle \rho_\sharp} & & \downarrow{\scriptstyle \rho_\sharp} \\ \mathscr{T}(\mathscr{X}) & \xrightarrow{\alpha} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda) \end{array} \tag{9.3.7.1}$$

where α and α' are the functors induced by $\lambda^* : \mathscr{T}(\mathscr{X}) \to \mathscr{T}(\mathscr{X}_\lambda)$ and $\lambda^* : \mathscr{T}(\mathscr{P}'/\mathscr{X}) \to \mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda)$ for $\lambda \in \mathrm{ob}(I)$ respectively. By (9.3.6), it commutes. The lower horizontal arrow is an equivalence by (PD–3) for $\mathscr{T}$, and the vertical arrows are fully faithful. Thus α' is fully faithful.

Then consider the diagram

$$\begin{array}{ccc} \mathscr{T}(\mathscr{P}'/\mathscr{X}) & \xleftarrow{\beta'} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda) \\ \downarrow{\scriptstyle \rho_\sharp} & & \downarrow{\scriptstyle \rho_\sharp} \\ \mathscr{T}(\mathscr{X}) & \xleftarrow{\beta} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda) \end{array}$$

where β and β' are the functors induced by

$$\lambda_\sharp : \mathscr{T}(\mathscr{X}_\lambda) \to \mathscr{T}(\mathscr{X}), \quad \lambda_\sharp : \mathscr{T}(\mathscr{P}'/\mathscr{X}_\lambda) \to \mathscr{T}(\mathscr{P}'/\mathscr{X})$$

for $\lambda \in \mathrm{ob}(I)$ respectively. By (9.3.6), it commutes. The lower horizontal arrow is an equivalence by (PD–3) for $\mathscr{T}$, and the vertical arrows are fully faithful. Thus β' is fully faithful. Then α' is an equivalence since both α' and β' are fully faithful and α' is left adjoint to β'. Thus $\mathscr{T}(\mathscr{P}')$ satisfies (PD–3).

(3) *Axiom* (PD–4). Let $(\mathscr{X}, I)$ be an $\mathscr{S}$-diagram. Consider the commutative diagram (9.3.7.1). The lower horizontal arrow of (loc. cit) is conservative by (PD–4) for $\mathscr{T}$, and the vertical arrows of (loc. cit) are fully faithful. Thus the upper horizontal arrow of (loc. cit) is conservative, so $\mathscr{T}(\mathscr{P}')$ satisfies (PD–4).

(4) *Axiom* (PD–6). Let $f : \mathscr{X} \to \mathscr{Y}$ be a 1-morphism of $\mathscr{S}$-diagrams, and let μ be an index of $\mathscr{Y}$. Consider the 2-diagram

$$\begin{array}{ccc} \mathscr{X}/\mu & \xrightarrow{\overline{\mu}} & \mathscr{X} \\ {\scriptstyle f_{\overline{\mu}\mu}}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle f} \\ \mathscr{Y}_\mu & \xrightarrow[\mu]{} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrmas. The horizontal arrows are $\mathscr{P}'$-morphisms, so μ^* and $\overline{\mu}^*$ commutes with ρ^* by (9.3.6). Then we can apply the technique of (1.5.5) to conclude that (PD–6) for $\mathscr{T}$ implies (PD–6) for $\mathscr{T}(\mathscr{P}')$.

9.3.8. Thus by (9.1.6) and (9.3.7), we have proven that

(i) $\mathscr{T}(\mathscr{P}')$ is a τ-twisted $\mathscr{P}$-premotivic triangulated prederivator,

(ii) $\mathscr{T}(\mathscr{P}')$ is compactly generated.

9.4 Bousfield localization

9.4.1. Throughout this section, we fix a category $\mathscr{S}$ and a class of morphisms $\mathscr{P}$ of $\mathscr{S}$ containing all isomorphisms and stable by compositions and pullbacks. We fix also a τ-twisted $\mathscr{P}$-premotivic triangulated prederivator $\mathscr{T}$ such that $\mathscr{T}(-,\mathbf{e})$ is compactly generated by $\mathscr{P}$ and τ. Then by (9.1.8), $\mathscr{T}(\mathscr{X})$ is compactly generated for any $\mathscr{S}$-diagram $\mathscr{X}$. For any object S of $\mathscr{S}$, we also fix an essentially small family of morphisms $\mathscr{W}_S$ in $\mathscr{T}(S)$ stable by twists in τ, $f_\sharp$ for $\mathscr{P}$-morphism f in $\mathscr{S}$, and f^* for morphism in $\mathscr{S}$. Assume that any cone of $\mathscr{W}_S$ is compact in $\mathscr{T}(S)$.

Definition 9.4.2. Let $\mathscr{X}$ be an $\mathscr{S}$-diagram.

(1) For an $\mathscr{S}$-diagram $\mathscr{X}$, we denote by $\mathscr{W}_\mathscr{X}$ the family of morphisms of the form

$$\lambda_\sharp K \to \lambda_\sharp K'$$

for index λ of $\mathscr{X}$ and morphism $K \to K'$ in $\mathscr{W}_{\mathscr{X}_\lambda}$ (see (1.6.2) for the definition of $\mathscr{W}_{\mathscr{X}_\lambda}$).

(2) We denote by $\mathcal{T}_{\mathscr{W},\mathscr{X}}$ the localizing subcategory of $\mathscr{T}(\mathscr{X})$ generated by the cones of $\mathscr{W}_\mathscr{X}$. Note that $\mathcal{T}_{\mathscr{W},\mathscr{X}}$ is compactly generated since any cone of $\mathscr{W}_\mathscr{X}$ consists of compact objects by (9.1.6).

(3) We denote by $\mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]$ the Verdier Quotient $\mathscr{T}(\mathscr{X})/\mathcal{T}_{\mathscr{W},\mathscr{W}}$. Then we denote by $\mathscr{T}[\mathscr{W}^{-1}]$ the collection of $\mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]$ for $\mathscr{S}$-diagrams $\mathscr{X}$.

(4) We say that an object L of $\mathscr{T}(\mathscr{X})$ is *$\mathscr{W}$-local* if

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(K,L)=0$$

for any object K of $\mathscr{T}(\mathscr{X})$ which is the cone of a morphism in $\mathscr{W}$. Equivalently,

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(K,L)=0$$

for any object K of $\mathcal{T}_{\mathscr{W},\mathscr{X}}$.

(5) We say that a morphism $K \to K'$ in $\mathscr{T}(\mathscr{X})$ is a *$\mathscr{W}$-weak equivalence* if the cone of the morphism is in $\mathcal{T}_{\mathscr{W},\mathscr{X}}$. Equivalently, the induced homomorphism

$$\mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(K',L) \to \mathrm{Hom}_{\mathscr{T}(\mathscr{X})}(K,L)$$

is an isomorphism for any $\mathscr{W}$-local object L of $\mathscr{T}(\mathscr{X})$. This equivalence follows from [Nee01, 9.1.14].

9.4.3. The purpose of this section is to show that $\mathscr{T}[\mathscr{W}^{-1}]$ has a structure of $\mathscr{P}$-premotivic triangulated prederivator. The set of twists τ for $\mathscr{T}$ gives a set of twists for $\mathscr{T}[\mathscr{W}^{-1}]$. It is denoted by τ again. By [Nee01, Introduction 1.16], $\mathscr{T}(S)[\mathscr{W}^{-1}]$ is well generated, so by [Nee01, 9.1.19], we have the adjunction

$$\pi_{\mathscr{X}} : \mathscr{T}(\mathscr{X}) \rightleftarrows \mathscr{T}(S)[\mathscr{W}^{-1}] : \mathcal{O}_{\mathscr{X}}$$

of triangulated categories where $\pi_{\mathscr{X}}$ denotes the Verdier quotient functor and $\mathcal{O}_{\mathscr{X}}$ denotes the Bousfield localization functor. Note that by [Nee01, 9.1.16], the functor $\mathcal{O}_{\mathscr{X}}$ is fully faithful, and its essential images are exactly $\mathscr{W}$-local objects of $\mathscr{T}(\mathscr{X})$. We denote by π and $\mathcal{O}$ the collections of $\pi_{\mathscr{X}}$ and $\mathcal{O}_{\mathscr{X}}$ for $\mathscr{S}$-diagrams $\mathscr{X}$ respectively.

9.4.4. Let $\mathscr{X}$ and $\mathscr{Y}$ be $\mathscr{S}$-diagrams. Consider a diagram

$$\begin{array}{ccc}
\mathscr{T}(\mathscr{X}) & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \mathscr{T}(\mathscr{Y}) \\
{\scriptstyle \pi_{\mathscr{X}}}\downarrow\uparrow{\scriptstyle \mathcal{O}_{\mathscr{X}}} & & {\scriptstyle \pi_{\mathscr{X}}}\downarrow\uparrow{\scriptstyle \mathcal{O}_{\mathscr{Y}}} \\
\mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] & & \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]
\end{array}$$

such that α is left adjoint to β. Suppose that α maps the cones of $\mathscr{W}_{\mathscr{X}}$ into $\mathscr{T}_{\mathscr{W},\mathscr{Y}}$ and commutes with twists. Then as in (1.6.4), we define

$$\alpha_{\mathscr{W}} : \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] \to \mathscr{T}(\mathscr{Y})[\mathscr{W}^{-1}],$$
$$\beta_{\mathscr{W}} : \mathscr{T}(\mathscr{Y})[\mathscr{W}^{-1}] \to \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]$$

as $\alpha_{\mathscr{W}} = \pi_{\mathscr{Y}}\alpha\mathcal{O}_{\mathscr{X}}$ and $\beta_{\mathscr{W}} = \pi_{\mathscr{X}}\beta\mathcal{O}_{\mathscr{Y}}$. We often omit $\mathscr{W}$ in $\alpha_{\mathscr{W}}$ and $\beta_{\mathscr{W}}$ for brevity. Then as in (1.6.5), α commutes with π, and $\alpha_{\mathscr{W}}$ is left adjoint to $\beta_{\mathscr{W}}$. Note that β commutes with $\mathcal{O}$ in this case.

Proposition 9.4.5. *Let $\mathscr{X}$ be an $\mathscr{S}$-diagram. For any indices λ and λ' of $\mathscr{X}$ and any object K of $\mathcal{T}_{\mathscr{W},\mathscr{X}_\lambda}$ in $\mathscr{T}(\mathscr{X}_\lambda)[\mathscr{W}^{-1}]$, the object $\lambda'^*\lambda_\sharp K$ of $\mathscr{T}(\mathscr{X}_{\lambda'})$ is in $\mathcal{T}_{\mathscr{W},\mathscr{X}_{\lambda'}}$.*

Proof. The proof is parallel to the proof of (9.3.4). □

Proposition 9.4.6. *Let $\mathscr{X}$ be an $\mathscr{S}$-diagram, and let K be an object of $\mathscr{T}(\mathscr{X})$. If λ^*K is in $\mathscr{T}(\mathscr{X}_\lambda)[\mathscr{W}^{-1}]$ for any index λ of $\mathscr{X}$, then K is in $\mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]$.*

Proof. The proof is parallel to the proof of (9.3.5). □

9.4.7. Let $f : \mathscr{X} \to \mathscr{Y}$ be a 1-morphism of $\mathscr{S}$-diagrams. Then by (9.4.6), f^* maps the cones of $\mathscr{W}_{\mathscr{Y}}$ into $\mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]$. Thus by (9.4.4), f^* commutes with π, f_* commutes with $\mathcal{O}$, and we have the adjunction:

$$f^*_{\mathscr{W}} : \mathscr{T}(\mathscr{Y})[\mathscr{W}^{-1}] \rightleftarrows \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] : f_{\mathscr{W}*}$$

When f is a $\mathscr{P}$-morphism, $f_\sharp$ maps the cones of $\mathscr{W}_{\mathscr{X}}$ into $\mathscr{T}(\mathscr{Y})[\mathscr{W}^{-1}]$ by definition. Thus by (9.4.4), $f_\sharp$ commutes with π, f^* commutes with $\mathcal{O}$, and we have the adjunction:

$$f_{\mathscr{W}\sharp} : \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] \rightleftarrows \mathscr{T}(\mathscr{Y})[\mathscr{W}^{-1}] : f^*_{\mathscr{W}}$$

9.4.8. Now, we will verify the axioms of (9.1.3) for $\mathscr{T}[\mathscr{W}^{-1}]$.

(1) As in (1.6.7), $\mathscr{T}[\mathscr{W}^{-1}]$ satisfies (PD–1), (PD–2), and (PD–5).

(2) *Axiom* (PD–3). Let $(\mathscr{X}, I)$ be an $\mathscr{S}$-diagram such that I is a discrete category. Consider the diagram

$$\begin{array}{ccc} \mathscr{T}(\mathscr{X}) & \xrightarrow{\alpha} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda) \\ \uparrow{\scriptstyle \mathcal{O}} & & \uparrow{\scriptstyle \mathcal{O}} \\ \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] & \xrightarrow{\alpha'} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda)[\mathscr{W}^{-1}] \end{array} \tag{9.4.8.1}$$

where α and α' are the functors induced by $\lambda^* : \mathscr{T}(\mathscr{X}) \to \mathscr{T}(\mathscr{X}_\lambda)$ and $\lambda^* : \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] \to \mathscr{T}(\mathscr{X}_\lambda)[\mathscr{W}^{-1}]$ for $\lambda \in \mathrm{ob}(I)$ respectively. By (9.4.7), it commutes. The upper horizontal arrow is an equivalence by (PD–3) for $\mathscr{T}$, and the vertical arrows are fully faithful. Thus α' is fully faithful.

Then consider the diagram

$$\begin{array}{ccc} \mathscr{T}(\mathscr{X}) & \xleftarrow{\beta} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda) \\ \uparrow{\scriptstyle \mathcal{O}} & & \uparrow{\scriptstyle \mathcal{O}} \\ \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}] & \xleftarrow{\beta'} & \prod_{\lambda\in \mathrm{ob}(I)} \mathscr{T}(\mathscr{X}_\lambda)[\mathscr{W}^{-1}] \end{array}$$

where β and β' are the functors induced by

$$\lambda_* : \mathscr{T}(\mathscr{X}_\lambda) \to \mathscr{T}(\mathscr{X})\, quad \lambda_* : \mathscr{T}(\mathscr{X}_\lambda)[\mathscr{W}^{-1}] \to \mathscr{T}(\mathscr{X})[\mathscr{W}^{-1}]$$

for $\lambda \in \mathrm{ob}(I)$ respectively. By (9.4.7), it commutes. The upper horizontal arrow is an equivalence by (PD–3) for $\mathscr{T}$, and the vertical arrows are fully faithful. Thus the β' is fully faithful. Then α' is an equivalence since both α' and β' are fully faithful and α' is left adjoint to β'. Thus $\mathscr{T}[\mathscr{W}^{-1}]$ satisfies (PD–3).

(3) *Axiom* (PD–4). Let $(\mathscr{X}, I)$ be an $\mathscr{S}$-diagram. Consider the commutative diagram (9.4.8.1). The upper horizontal arrow of (loc. cit) is conservative by (PD–4) for $\mathscr{T}$, and the vertical arrows of (loc. cit) are fully faithful. Thus the lower horizontal arrow of (loc. cit) is conservative, so $\mathscr{T}[\mathscr{W}^{-1}]$ satisfies (PD–4).

(4) *Axiom* (PD–6). Let $f : \mathscr{X} \to \mathscr{Y}$ be a 1-morphism of $\mathscr{S}$-diagrams, and let μ be an index of $\mathscr{Y}$. Consider the 2-diagram

$$\begin{array}{ccc} \mathscr{X}/\mu & \xrightarrow{\overline{\mu}} & \mathscr{X} \\ {\scriptstyle f_{\overline{\mu}\mu}}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ \mathscr{Y}_\mu & \xrightarrow[\mu]{} & \mathscr{Y} \end{array}$$

of $\mathscr{S}$-diagrmas. The horizontal arrows are $\mathscr{P}'$-morphisms, so μ^* and $\overline{\mu}^*$ commutes with $\mathcal{O}$ by (9.3.6). Then we can apply the technique of (1.6.6) to conclude that (PD–6) for $\mathscr{T}$ implies (PD–6) for $\mathscr{T}[\mathscr{W}^{-1}]$.

9.4.9. Thus we have proven that $\mathscr{T}[\mathscr{W}^{-1}]$ is a τ-twisted $\mathscr{P}$-premotivic triangulated prederivator,

9.5 Construction

9.5.1. Consider the ft-premotivic triangulated category

$$\mathrm{D}_{\mathbb{A}^1,qw}(ft,\Lambda).$$

It can be extend to $\mathscr{S}$-diagrmas by [CD12, 5.2.7]. Then it satisfies (PD–5) by construction, and it satisfies (PD–1) and (PD–2) by [CD12, 3.1.11]. It also satisfies (PD–3) and (PD–4) by [CD12, 3.1.10] and [CD12, 3.1.6] respectively. Finally, it satisfies (PD–6) by [CD12, 3.1.15, 3.1.16]. We denote by τ the set of twists generated by (1) and [1]. Then $\mathrm{D}_{\mathbb{A}^1}(ft,\Lambda)$ is a ft-premotivic triangulated prederivator.

9.5.2. By (1.7.5), the ft-premotivic triangulated category

$$\mathrm{D}_{\mathbb{A}^1,qw}(ft/(-,\mathbf{e}),\Lambda)$$

restricted to $\mathscr{S}$-schemes is compactly generated by ft and τ. Consider $\mathscr{W}_{log}$ defined in (1.7.2). Then every cone of $\mathscr{W}_{log,S}$ is compact for any $\mathscr{S}$-scheme S since $\mathrm{D}_{\mathbb{A}^1}(ft/S,\Lambda)$ is compactly generated by $\mathscr{P}$ and τ, so the conditions of (9.4.1) are satisfied. Thus by (9.4.9), we obtain the ft-premotivic triangulated prederivator $\mathrm{D}_{\mathbb{A}^1,qw}(ft,\Lambda)[\mathscr{W}_{log}^{-1}]$. It is also denoted by

$$\mathrm{D}_{log,qw}(ft,\Lambda).$$

9.5.3. By (1.7.5), the ft-premotivic triangulated category

$$\mathrm{D}_{log,qw}(ft/(-,\mathbf{e}),\Lambda)$$

is compactly generated by ft and τ. Then the conditions of (9.3.1) are satisfied for $eSm \subset ft$. Thus by (9.3.8), we obtain the eSm-premotivic triangulated prederivator

$$\mathrm{D}_{log,qw}(eSm,\Lambda).$$

It is also denoted by $\mathrm{D}_{log,qw}(-,\Lambda)$. Note that for any $\mathscr{S}$-scheme S, we have the equivalence

$$\mathrm{D}_{log,qw}(S,\Lambda) \cong \mathrm{D}_{log,pw}(S,\Lambda).$$

by (1.7.8). Thus the restriction of $\mathrm{D}_{log,qw}(-,\Lambda)$ to $\mathscr{S}$-schemes is a log motivic category by (2.9.4). Note that it satisfies strict étale descent by [CD12, 5.2.10].

Chapter 10

Poincaré duality

10.0.1. Throughout this chapter, we fix a full subcategory $\mathscr{S}$ of the category of fs log schemes satisfying the conditions of (2.0.1). We also fix a log motivic triangulated category $\mathscr{T}$ over $\mathscr{S}$. In Sections 6 and 7, we further assume that $\mathscr{T}$ can be extended to an eSm-premotivic triangulated prederivator satisfying strict étale descent.

10.1 Compactified exactifications

10.1.1. *Compactification via toric geometry.* Let $\theta : P \to Q$ be a homomorphism of fs monoids such that $\theta^{\rm gp}$ is an isomorphism. Choose a fan Σ of the dual lattice $(\overline{P}^{\rm gp})^\vee$ whose support is $(\overline{P})^\vee$ and containing $(Q/\theta(P^*))^\vee$ as a cone. This fan induces a factorization

$$\operatorname{spec}(Q/\theta(P^*)) \to M \to \operatorname{spec} \overline{P}$$

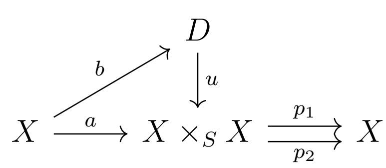
of the morphism $\operatorname{spec}(Q/\theta(P^*)) \to \operatorname{spec} \overline{P}$ for some fs monoscheme M. Consider the open immersions $\operatorname{spec} P_i \to M$ of fs monoschemes induced by the fan, and, we denote by P_i' the preimage of P_i via the homomorphism $P^{\rm gp} \to \overline{P}^{\rm gp}$. Then the family of P_i' forms a fs monoscheme M' with the factorization

$$\operatorname{spec} Q \to M' \to \operatorname{spec} P$$

of the morphism $\operatorname{spec} Q \to \operatorname{spec} P$. Here, the first arrow is an open immersion, and the second arrow is a proper log étale monomorphism.

We will sometimes use this construction later.

Definition 10.1.2. Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes, let $a : X \to X \times_S X$ denote the diagonal morphism, and let $p_1, p_2 : X \times_S X \rightrightarrows X$ denote the first and second projections respectively. A *compactified exactification* of the diagram $X \to X \times_S X \rightrightarrows X$ is a commutative diagram



of $\mathscr{S}$-schemes such that

(i) there is an open immersion $v : I \to D$ of fs log schemes such that the compositions p_1uv and p_2uv are strict,

(ii) b is a strict closed immersion and factors through I,

(iii) u is a proper and log étale monomorphism of fs log schemes.

We often say that $u : D \to X$ is a compactified exactification of a if no confusion seems likely to arise. We also call I an *interior* of E. Then p_1uv and p_2uv are strict log smooth, and the morphism $X \to I$ of $\mathscr{S}$-schemes induced by b is a strict regular embedding. Note also that the natural transformation

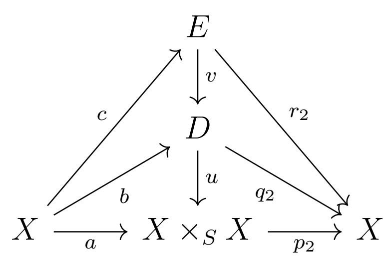
$$\mathrm{id} \xrightarrow{ad} u_*u^*$$

is an isomorphism by (Htp–4) and that the natural transformation

$$\Omega_{f,I} \xrightarrow{T_{D,I}} \Omega_{f,D}$$

given in (4.2.2) is an isomorphism by construction.

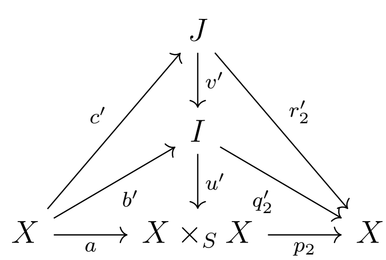
10.1.3. Under the notations and hypotheses of (10.1.2), let $\mathcal{CE}_a$ denote the category whose objects consist of compactified exactifications of a and morphisms consist of commutative diagrams



of $\mathscr{S}$-schemes. Note that v is a proper log étale monomorphism. For such a morphism in $\mathcal{CE}_a$, we associate the natural transformation

$$T_{D,E} : \Omega_{f,E} \longrightarrow \Omega_{f,D}$$

given in (4.2.2). We will show that it is an isomorphism. Let I be an interior of D, and let J be an interior of E contained in $I \times_D E$. Consider the induced commutative diagram



of $\mathscr{S}$-schemes. Then v' is a strict étale monomorphism, so it is an open immersion by [EGA, IV.17.9.1]. Consider the diagram

$$\begin{array}{ccc} \Omega_{f,J} & \xrightarrow{T_{E,J}} & \Omega_{f,E} \\ \downarrow{\scriptstyle T_{I,J}} & & \downarrow{\scriptstyle T_{D,E}} \\ \Omega_{f,I} & \xrightarrow{T_{D,I}} & \Omega_{f,D} \end{array}$$

of functors. It commutes by (4.2.13), and the horizontal arrows are isomorphisms by (10.1.2). The left vertical arrow is also an isomorphism since v' is an open immersion, so $T_{D,E}$ is an isomorphism.

Definition 10.1.4. Let $\theta : P \to Q$ be a homomorphism of fs monoids. Then the submonoid of P^{gp} consisting of elements $p \in P^{\mathrm{gp}}$ such that $n\theta^{\mathrm{gp}}(p) \in Q$ for some $n \in \mathbb{N}^+$ is called the *fs exactification* of θ. It is the fs version of [Ogu14, I.4.2.12].

10.1.5. Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ of exact log smooth type, and let Q_1 denote the fs exactification of the summation homomorphism of $Q^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q^{\mathrm{gp}}$. Applying (10.1.1), we obtain the morphisms

$$\operatorname{spec} Q_1 \to M \to \operatorname{spec}(Q \oplus_P Q)$$

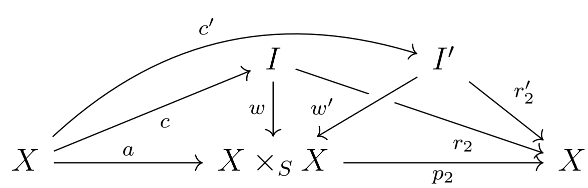
of fs monoschemes. If we put

$$I = (X \times_S X) \times_{\mathbb{A}_{Q \oplus_P Q}} \mathbb{A}_{Q_1}, \quad D = (X \times_S X) \times_{\mathbb{A}_{Q \oplus_P Q}} \mathbb{A}_M,$$

then the projection $u : D \to X \times_S X$ is a compactified exactification of the diagonal morphism $a : X \to X \times_S X$ with an interior I. In particular, a has a compactified exactification.

Proposition 10.1.6. *Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes, and let $a : X \to X \times_S X$ denote the diagonal morphism. For any compactified exactifications $u : D \to X \times_S X$ and $u' : D \to X \times_S X$, the morphism $D \times_{X \times_S X} D' \to X \times_S X$ is a compactified exactification.*

Proof. Consider the induced commutative diagram



of $\mathscr{S}$-schemes where I (resp. I') is an interior of D (resp. D'). To show the claim, it suffices to construct an open immersion $I'' \to I \times_{X \times_S X} I'$ such that a factors through I'' and that the morphisms $I'' \rightrightarrows X$ induced by p_1 and p_2 are strict.

We have the induced morphisms

$$X \xrightarrow{\alpha} I \times_{X \times_S X} I' \xrightarrow{\beta} I \times_{r_2, X, r_2'} I'$$

of $\mathscr{S}$-schemes. Let $x \in X$ be a point. Consider the associated homomorphisms

$$\overline{\mathcal{M}}_{I \times_{r_2, X, r_2'} I', \overline{\beta\alpha(x)}} \xrightarrow{\lambda} \overline{\mathcal{M}}_{I \times_{X \times_S X} I', \overline{\alpha(x)}} \xrightarrow{\eta} \overline{\mathcal{M}}_{X, \overline{x}}$$

of fs monoids. Then $\eta\lambda$ is an isomorphism since r_2 and r_2' are strict. In particular, λ is injective. Since β is a pullback of the diagonal morphism $X \times_S X \to (X \times_S X) \times_{p_2, X, p_2} (X \times_S X)$ that is a closed immersion, λ is a pushout of a $\mathbb{Q}$-surjective homomorphism. Thus λ is $\mathbb{Q}$-surjective, so λ is Kummer. Then by (3.3.6), η is an isomorphism, i.e., α is strict. Thus the conclusion follows from (3.3.3). □

Corollary 10.1.7. *Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes, and let $a : X \to X \times_S X$ denote the diagonal morphism. Then the category $\mathcal{CE}_a$ is connected.*

Proof. It is a direct consequence of (10.1.6). □

10.2 Functoriality of purity transformations

10.2.1. Let $h : X \to Y$ and $g : Y \to S$ be exact log smooth separated morphisms of $\mathscr{S}$-schemes. We put $f = gh$. Consider the commutative diagram

$$\begin{array}{ccccc}
X & & & & \\
\downarrow{\scriptstyle a'} & \searrow^{a} & & & \\
X \times_Y X & \xrightarrow{\varphi} & X \times_S X & & \\
\downarrow{\scriptstyle p_2'} & & \downarrow{\scriptstyle \varphi'} & \searrow^{p_2} & \\
X & \xrightarrow{a''} & Y \times_S X & \xrightarrow{p_2''} & X \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi''} & & \downarrow{\scriptstyle h} \\
Y & \xrightarrow{a'''} & Y \times_S Y & \xrightarrow{p_2'''} & Y
\end{array}$$

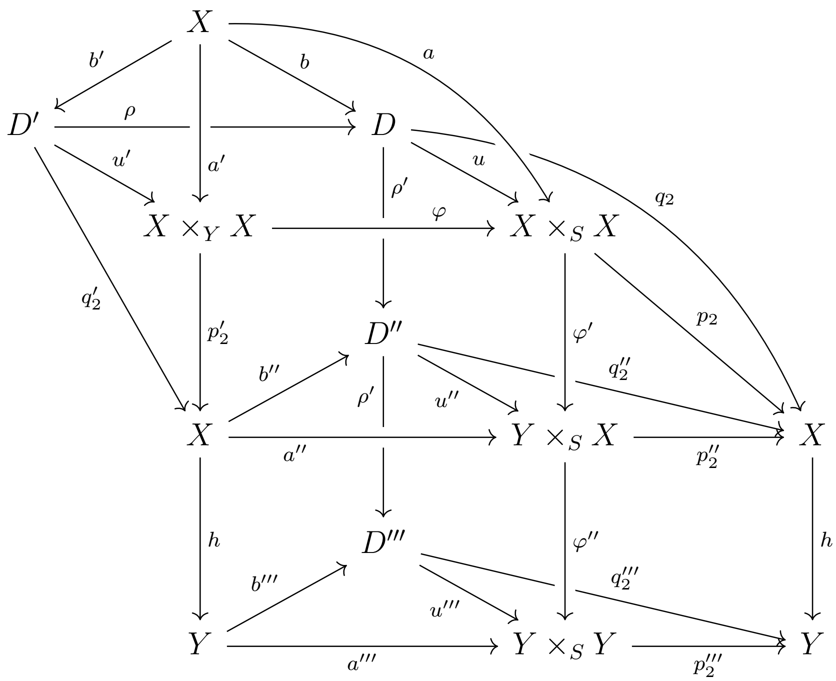
of $\mathscr{S}$-schemes where

(i) a, a', and a''' denote the diagonal morphisms,

(ii) p_2, p_2', and p_2''' denote the second projections,

(iii) each small square is Cartesian.

10.2.2. Under the notations and hypotheses of (10.2.1), assume that we have a commutative diagram



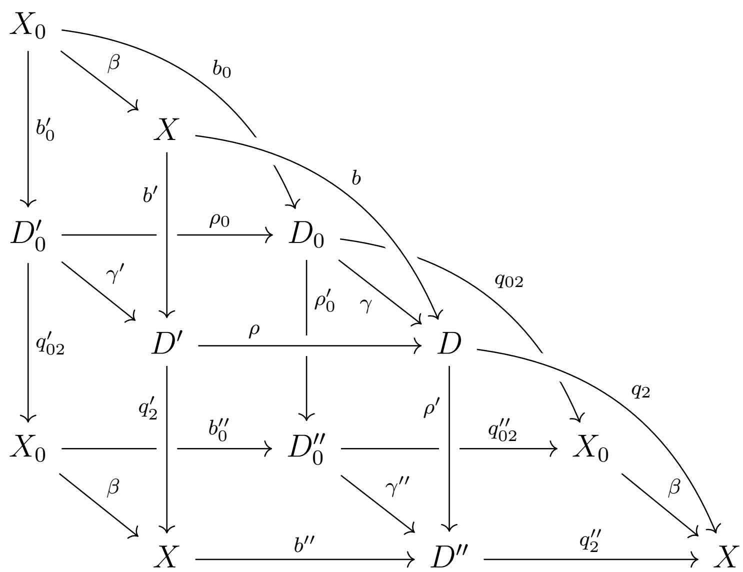
(10.2.2.1)

of $\mathscr{S}$-schemes where each small square is Cartesian and u (resp. u', resp. u''') is a compactified exactification of a (resp. a', resp. a''').

We will use these notations and hypotheses later.

10.2.3. Under the notations and hypotheses of (10.2.2), let $\alpha : S_0 \to S$ be a morphism of $\mathscr{S}$-schemes, and consider the commutative diagrams

$$\begin{array}{ccccc} X_0 & \xrightarrow{h_0} & Y_0 & \xrightarrow{g_0} & S_0 \\ \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{h} & Y & \xrightarrow{g} & S \end{array}$$



of $\mathscr{S}$-schemes where each small square is Cartesian. We put $f_0 = g_0 h_0$. Then the diagram

$$\begin{array}{ccc}
\beta^*\Omega_{h,D'}\Omega_{g,f,D''} & \xrightarrow{C} & \beta^*\Omega_{f,D} \\
\downarrow{\scriptstyle Ex} & & \\
\Omega_{h_0,D'_0}\beta^*\Omega_{g,f,D''} & & \downarrow{\scriptstyle Ex} \\
\downarrow{\scriptstyle Ex} & & \\
\Omega_{h_0,D'_0}\Omega_{g_0,f_0,D''_0}\beta^* & \xrightarrow{C} & \Omega_{f_0,D_0}\beta^*
\end{array}$$

of functors commutes since it is the big outside diagram of the commutative diagram

$$\begin{array}{ccccc}
\beta^* b'^! q_2'^* b''^! q_2''^* & \xrightarrow{Ex} & \beta^* b'^! \rho^! \rho'^* q_2''^* & \xrightarrow{\sim} & \beta^* b^! q_2^* \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \\
b_0'^! \gamma'^* q_2'^* b''^! q_2''^* & \xrightarrow{Ex} & b_0'^! \gamma'^* \rho^! \rho'^* q_2''^* & & \downarrow{\scriptstyle Ex} \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle Ex} & & \\
b_0'^! q_{02}'^* \beta^* b''^! q_2''^* & & b_0'^! \rho_0^! \gamma^* \rho'^* q_2''^* & \xrightarrow{\sim} & b_0^! \gamma^* q_2^* \\
\downarrow{\scriptstyle Ex} & & & & \\
b_0'^! q_{02}'^* b_0''^! \gamma''^* q_2''^* & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
\downarrow{\scriptstyle \sim} & & & & \\
b_0'^! q_{02}'^* b_0''^! q_{02}''^* \alpha^* & \xrightarrow{Ex} & b_0'^! \rho_0^! \rho_0'^* q_2''^* \beta^* & \xrightarrow{\sim} & b_0^! q_{02}^* \beta^*
\end{array}$$

of functors.

We will use these notations and hypotheses later.

10.2.4. Under the notations and hypotheses of (10.2.2), we denote by $\mathcal{I}$ (resp. $\mathcal{I}'$, resp. $\mathcal{I}''$) the ideal sheaf of X on D (resp. D', resp. D'''). Then by [Ogu14, IV.3.2.2], the morphisms

$$\mathcal{I}/\mathcal{I}^2 \to b^*\Omega^1_{D/X}, \quad \mathcal{I}'/\mathcal{I}'^2 \to b'^*\Omega^1_{D'/X}, \quad \mathcal{I}''/\mathcal{I}''^2 \to b''^*\Omega^1_{D''/X}$$

of quasi-coherent sheaves on $\underline{X}$ are isomorphisms, and by [Ogu14, IV.3.2.4, IV.1.3.1], the morphisms

$$\begin{aligned}\Omega^1_{D/X} &\to u^*\Omega^1_{X\times_S X/X} \to u^*p_1^*\Omega^1_{X/S},\\ \Omega^1_{D'/X} &\to u'^*\Omega^1_{X\times_Y X/X} \to u'^*p_1'^*\Omega^1_{X/Y},\\ \Omega^1_{D''/X} &\to u''^*\Omega^1_{Y\times_S X/X} \to u''^*p_1''^*\Omega^1_{Y/S}\end{aligned}$$

of quasi-coherent sheaves on $\underline{X}$ are isomorphisms where

$$p_1 : X\times_S X \to X, \quad p_1' : X\times_Y X \to X, \quad p_1'' : Y\times_S X \to Y$$

denote the first projections. Then from the exact sequence

$$0 \longrightarrow h^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

of quasi-coherent sheaves on $\underline{X}$ given in [Ogu14, IV.3.2.3], we have the exact sequence

$$0 \longrightarrow \mathcal{I}''/\mathcal{I}''^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}'/\mathcal{I}'^2 \longrightarrow 0$$

of quasi-coherent sheaves on $\underline{X}$. This shows that the induced diagram

$$\begin{array}{ccc} N_XD' & \xrightarrow{\chi} & N_XD \\ \downarrow{\scriptstyle t_2'} & & \downarrow{\scriptstyle \chi'} \\ X & \xrightarrow{e_2} & N_XD'' \end{array} \tag{10.2.4.1}$$

of $\mathscr{S}$-schemes is Cartesian. Thus the induced diagram

$$\begin{array}{ccc} D_XD' & \longrightarrow & D_XD \\ \downarrow & & \downarrow \\ X\times\mathbb{A}^1 & \longrightarrow & D_XD'' \end{array}$$

of $\mathscr{S}$-schemes is also Cartesian. Then as in (4.3.1), we have the natural transformations

$$\Omega^d_{h,D'}\Omega^d_{g,f,D''} \xrightarrow{C} \Omega^d_{f,D},$$

$$\Omega^n_{h,D'}\Omega^n_{g,f,D''} \xrightarrow{C} \Omega^n_{f,D} \tag{10.2.4.2}$$

These are called again *composition transformations.* Note that the left adjoint versions are

$$\Sigma^d_{f,D} \xrightarrow{C} \Sigma^d_{h,D'}\Sigma^d_{g,f,D''},$$

$$\Sigma^n_{f,D} \xrightarrow{C} \Sigma^n_{h,D'}\Sigma^n_{g,f,D''}.$$

In the Cartesian diagram (10.2.4.1), the morphisms e_2, t'_2, χ, and χ' are strict, χ' are strict smooth, and e_2 is a strict closed immersion. Thus by (2.5.10) and (4.3.1), the natural transformation (10.2.4.2) is an isomorphism.

Applying (10.2.3) to the cases when the diagram

$$\begin{array}{ccccc}
X_0 & \xrightarrow{b''_0} & D_0 & \xrightarrow{q_{02}} & X_0 \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\
X & \xrightarrow{b''} & D & \xrightarrow{q_2} & X
\end{array}$$

is equal to one of the diagrams in (4.1.2.1) and similar things are true for D' and D'', we have the commutative diagram

$$\begin{array}{ccc}
\Omega^n_{h,D'}\Omega^n_{g,f,D''} & \xrightarrow{C} & \Omega^n_{f,D} \\
\downarrow{\scriptstyle T^nT^n} & & \downarrow{\scriptstyle T^n} \\
\Omega^d_{h,D'}\Omega^d_{g,f,D''} & \xrightarrow{C} & \Omega^d_{f,D} \\
\downarrow{\scriptstyle T^dT^d} & & \downarrow{\scriptstyle T^d} \\
\Omega_{h,D'}\Omega_{g,f,D''} & \xrightarrow{C} & \Omega_{f,D}
\end{array}$$

of functors.

10.2.5. Under the notations and hypotheses of (10.2.2), consider the diagram

$$\begin{array}{ccc}
f_\sharp & \xrightarrow{\mathfrak{p}^n_{f,D}} & f_!\Sigma^n_{f,D} \\
 & & \downarrow{\scriptstyle C} \\
 & & f_!\Sigma^n_{g,f,D''}\Sigma^n_{h,D'} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
 & & g_!h_!\Sigma^n_{g,f,D''}\Sigma^n_{h,D'} \\
 & & \uparrow{\scriptstyle Ex} \\
g_\sharp h_\sharp & \xrightarrow{\mathfrak{p}^n_{g,D'''}\mathfrak{p}^n_{h,D'}} & g_!\Sigma^n_{g,D'''}h_!\Sigma^n_{h,D'}
\end{array} \tag{10.2.5.1}$$

of functors. We will show that it commutes. Its right adjoint is the big outside diagram of the diagram

$$
\begin{array}{ccccccccc}
\Omega^n_{h,D'}h^!\Omega^n_{g,D'''}g^! & \xrightarrow{T^nT^n} & \Omega^d_{h,D'}h^!\Omega^d_{g,D'''}g^! & \xrightarrow{T^dT^d} & \Omega_{h,D'}h^!\Omega_{g,D'''}g^! & \xrightarrow{T_{D'}T_{D'''}} & \Omega_h h^!\Omega_g g^! & \xrightarrow{\mathfrak{q}_h\mathfrak{q}_g} & h^*g^* \\
\uparrow{\scriptstyle Ex} & & \uparrow{\scriptstyle Ex} & & \uparrow{\scriptstyle Ex} & & \uparrow{\scriptstyle Ex} & & \Big\downarrow \\
\Omega^n_{h,D'}\Omega^n_{g,f,D''}h^!g^! & \xrightarrow{T^nT^n} & \Omega^d_{h,D'}\Omega^d_{g,f,D''}h^!g^! & \xrightarrow{T^dT^d} & \Omega_{h,D'}\Omega_{g,f,D''}h^!g^! & \xrightarrow{T_{D'}T_{D''}} & \Omega_h\Omega_{g,f}h^!g^! & & \Big\downarrow{\scriptstyle \sim} \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \Big\downarrow \\
\Omega^n_{h,D'}\Omega^n_{g,f,D''}f^! & \xrightarrow{T^nT^n} & \Omega^d_{h,D'}\Omega^d_{g,f,D''}f^! & \xrightarrow{T^dT^d} & \Omega_{h,D'}\Omega_{g,f,D''}f^! & \xrightarrow{T_{D'}T_{D''}} & \Omega_h\Omega_{g,f}f^! & & \Big\downarrow \\
\downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C} & & \Big\downarrow \\
\Omega^n_{f,D}f^! & \xrightarrow{T^n} & \Omega^d_{f,D}f^! & \xrightarrow{T^d} & \Omega_{f,D}f^! & \xrightarrow{T_D} & \Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & f^*
\end{array}
$$

of functors. It commutes by (4.2.10), (4.3.2), (4.4.3), and (10.2.4). Thus (10.2.5.1) also commutes.

Note also that the right vertical top arrow of (10.2.5.1) is an isomorphism by (10.2.4) and that the right vertical bottom arrow of (10.2.5.1) is an isomorphism by (4.2.8).

10.2.6. Let $h : X \to Y$ and $g : Y \to S$ be exact log smooth separated morphisms of $\mathscr{S}$-schemes. Assume that f (resp. g) has a fs chart $\theta : Q \to R$ (resp. $\eta : R \to P$) of exact log smooth type. In this setting, we will construct the diagram (10.2.2.1) and verify the hypotheses of (10.2.2).

We denote by T and T''' the fs exactification of the summation homomorphisms

$$Q \oplus_P Q \to Q, \quad R \oplus_P R \to R$$

respectively. Then we put

$$T' = T \oplus_{Q\oplus_P Q} (Q \oplus_R Q).$$

By (10.1.1), we have the factorization

$$\operatorname{spec} T''' \to M''' \to \operatorname{spec} R \oplus_P R$$

such that the first arrow is an open immersion of fs monoschemes and the second arrow is a proper log étale monomorphism of fs monoschemes. We put

$$M'' = M''' \times_{\operatorname{spec}(R\oplus_P R)} \operatorname{spec}(R \oplus_P Q).$$

Consider the induced morphism

$$\operatorname{spec} T \to M'' \times_{\operatorname{spec}(R\oplus_P Q)} \operatorname{spec}(Q \oplus_P Q).$$

By the method of (10.1.1), it has a factorization

$$\operatorname{spec} T \to M \to M'' \times_{\operatorname{spec}(R\oplus_P Q)} \operatorname{spec}(Q \oplus_P Q)$$

where the first arrow is an open immersion of fs monoschemes and the second arrow is a proper log étale monomorphism of fs monoschemes. We put

$$M' = M \times_{\mathrm{spec}(Q \oplus_P Q)} \mathrm{spec}(Q \oplus_R Q),$$

and we put

$$I = (X \times_S X) \times_{\mathbb{A}_{Q \oplus_P Q}} \mathbb{A}_T, \quad I' = (X \times_Y X) \times_{\mathbb{A}_{Q \oplus_R Q}} \mathbb{A}_{T'}, \quad I''' = (Y \times_S Y) \times_{\mathbb{A}_{R \oplus_P R}} \mathbb{A}_{T'''},$$

$$D = (X \times_S X) \times_{\mathbb{A}_{Q \oplus_P Q}} \mathbb{A}_M, \quad D' = (X \times_Y X) \times_{\mathbb{A}_{Q \oplus_R Q}} \mathbb{A}_{M'}, \quad D''' = (Y \times_S Y) \times_{\mathbb{A}_{R \oplus_P R}} \mathbb{A}_{M'''}.$$

Then we have the commutative diagram (10.2.2.1). By construction, D (resp. D', resp. D''') are compactified exactifications of the diagonal morphism $a : X \to X \times_S X$ (resp. $a' : X \to X \times_Y X$, resp. $a''' : Y \to Y \times_S Y$) with an interior I (resp. I', resp. I''').

Proposition 10.2.7. *Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes, let D be a compactified exactification of the diagonal morphism $a : X \to X \times_S X$, and let $g : S' \to S$ be a morphism of $\mathscr{S}$-schemes. We put*

$$X' = X \times_S S', \quad D' = D \times_{X \times_S X} (X' \times_{S'} X').$$

Then the diagram

$$\begin{array}{ccc} f'_\sharp g'^* & \xrightarrow{\mathfrak{p}^n_{f'}} & f'_! \Sigma^n_{f',D'} g'^! \\ & & \big\downarrow{\scriptstyle Ex} \\ \big\downarrow{\scriptstyle Ex} & & f'_! g'^* \Sigma^n_{f,D} \\ & & \big\uparrow{\scriptstyle Ex} \\ g^* f_\sharp & \xrightarrow{\mathfrak{p}^n_f} & g^* f_! \Sigma^n_{f,D} \end{array} \tag{10.2.7.1}$$

of functors commutes.

Proof. The right adjoint of (10.2.7.1) is the big outside diagram of the diagram

$$\begin{array}{ccccccc} \Omega^n_f f^! g_* & \xrightarrow{T^n} & \Omega^d_f f^! g_* & \xrightarrow{T^d} & \Omega_f f^! g_* & \xrightarrow{\mathfrak{q}_f} & f^* g_* \\ \big\uparrow{\scriptstyle Ex} & & \big\uparrow{\scriptstyle Ex} & & \big\uparrow{\scriptstyle Ex} & & \\ \Omega^n_f g'_* f'^! & \xrightarrow{T^n} & \Omega^d_f g'_* f'^! & \xrightarrow{T^d} & \Omega_f g'_* f'^! & & \big\downarrow{\scriptstyle Ex} \\ \big\downarrow{\scriptstyle Ex} & & \big\downarrow{\scriptstyle Ex} & & \big\downarrow{\scriptstyle Ex} & & \\ g'_* \Omega^n_{f'} f'^! & \xrightarrow{T^n} & g'_* \Omega^d_{f'} f'^! & \xrightarrow{T^d} & g'_* \Omega_{f'} f'^! & \xrightarrow{\mathfrak{q}_{f'}} & g'_* f'^* \end{array}$$

of functors. By (4.2.10) and (4.4.5), each small diagram commutes. The conclusion follows from this. $\square$

10.3 Poincaré duality for Kummer log smooth separated morphisms

Proposition 10.3.1. *Let $f : X \to S$ be a strict smooth separated morphism of $\mathscr{S}$-schemes. Then the natural transformation*

$$\mathfrak{p}_f^n : f_\sharp \longrightarrow f_! \Sigma_f^n$$

is an isomorphism.

Proof. It follows from (2.5.9) and (4.2.9). □

10.3.2. Let $f : X \to S$ be a Kummer log smooth separated morphism of $\mathscr{S}$-schemes. Then the diagonal morphism $a : X \to X \times_S X$ is a strict regular embedding by (3.3.5). In particular, we can use the notation Σ_f^n.

Proposition 10.3.3. *Let $f : X \to S$ be a Kummer log smooth separated morphism of $\mathscr{S}$-schemes. Then the natural transformation*

$$\mathfrak{p}_f^n : f_\sharp \longrightarrow f_! \Sigma_f^n$$

is an isomorphism.

Proof. By (5.2.2), there is a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

of $\mathscr{S}$-schemes such that

(i) g is Kummer log smooth,

(ii) g^* is conservative,

(iii) f' is strict.

Hence we reduce to showing that the natural transformation

$$g^* f_\sharp \xrightarrow{\mathfrak{p}_f^n} g^* f_! \Sigma_f^n$$

is an isomorphism.

Consider the commutative diagram (10.2.7.1). The left vertical arrow and right lower vertical arrow are isomorphisms since f and g are exact log smooth. The right upper vertical arrow is an isomorphism by (4.2.4). Thus we reduce to showing that the upper horizontal arrow is an isomorphism. It follows from (10.3.1). □

10.4 Poincaré duality for $\mathbb{A}_Q \to \mathbb{A}_P$

Definition 10.4.1. In this section, we will consider the following conditions:

($\mathrm{PD}_{f,D}$) Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes, and let D be a compactified exactification of the diagonal morphism a. We denote by ($\mathrm{PD}_{f,D}$) the condition that the natural transformation

$$\mathfrak{p}_f^n : f_\sharp \longrightarrow f_! \Sigma_{f,D}^n$$

is an isomorphism.

(PD_f) Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes, and let $a : X \to X \times_S X$ denote the diagonal morphism. We denote by (PD_f) the conditions that

(i) there is a compactified exactification of a,

(ii) for any compactified exactification D of a, ($\mathrm{PD}_{f,D}$) is satisfied.

(PD^m) We denote by (PD^m) the condition that (PD_f) is satisfied for any *vertical* exact log smooth separated morphism $f : X \to S$ with a fs chart $\theta : P \to Q$ such that θ is a vertical homomorphism of exact log smooth type and

$$\max_{x \in X} \operatorname{rk} \overline{\mathcal{M}}_{X,\overline{x}}^{\mathrm{gp}} + \max_{s \in S} \operatorname{rk} \overline{\mathcal{M}}_{S,\overline{s}}^{\mathrm{gp}} \leq m.$$

Note that by (2.8.2), we get equivalent conditions if we use $\Sigma_{f,D}^o$ instead of $\Sigma_{f,D}^n$

Proposition 10.4.2. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes, and let $E \to D$ be a morphism in $\mathcal{CE}_a$ where $a : X \to X \times_S X$ denotes the diagonal morphism. Then* ($\mathrm{PD}_{f,D}$) *is equivalent to* ($\mathrm{PD}_{f,E}$)*.*

Proof. The diagram

$$\begin{array}{ccccccccc}
\Omega_{g,f,E}^n & \xrightarrow{T^n} & \Omega_{g,f,E}^d & \xrightarrow{T^d} & \Omega_{g,f,E} & \xrightarrow{T_E} & \Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & f^* \\
\downarrow{\scriptstyle T_{D,E}} & & \downarrow{\scriptstyle T_{D,E}} & & \downarrow{\scriptstyle T_{D,E}} & & \| & & \| \\
\Omega_{g,f,D}^n & \xrightarrow{T^n} & \Omega_{g,f,D}^d & \xrightarrow{T^d} & \Omega_{g,f,D} & \xrightarrow{T_D} & \Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & f^*
\end{array}$$

of functors commutes by (4.2.11) and (4.2.13). The left vertical arrow is an isomorphism because the normal cones $N_X D$ and $N_X E$ are isomorphic to the vector bundle associated to the sheaf $\Omega_{X/S}^1$. Then the conclusion follows from the fact that the composition of row arrows are $\mathfrak{q}_{f,E}^n$ and $\mathfrak{q}_{f,D}^n$ respectively. □

Corollary 10.4.3. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes such that there is a compactified exactification D of the diagonal morphism $a : X \to X \times_S X$. Then* (PD_f) *is equivalent to* ($\mathrm{PD}_{f,D}$)*.*

Proof. Since $\mathcal{CE}_a$ is connected by (10.1.6), the conclusion follows from (10.4.2). □

Proposition 10.4.4. *Under the notations and hypotheses of (10.2.2), if* $(\mathrm{PD}_{g,D'''})$ *and* $(\mathrm{PD}_{h,D'})$ *are satisfied, then* $(\mathrm{PD}_{f,D})$ *is satisfied.*

Proof. By (10.2.6), we can use (10.2.5). Then by (loc. cit), in the commutative diagram (10.2.5.1), the upper horizontal arrow is an isomorphism if and only if the lower horizontal arrow is an isomorphism. The conclusion follows from this. □

Proposition 10.4.5. *Let* $f : X \to S$ *be an exact log smooth separated morphism of* $\mathscr{S}$*-schemes, and let* D *be a compactified exactification of the diagonal morphism* $a : X \to X \times_S X$*. Then* (PD_f) *is strict étale local on* X*.*

Proof. Let $\{u_i : X_i \to X\}_{i\in I}$ be a strict étale cover of X. We put

$$f_i = fu_i, \quad D_i = D \times_{X\times_S X} (X_i \times_S X_i), \quad D_i'' = D \times_{X\times_S X} (X_i \times_S X_i)$$

Then D_i is a compactified exactification of the diagonal morphism $a_i : X_i \to X_i \times_S X_i$. Hence by (10.4.3), it suffices to show that $(\mathrm{PD}_{f,D})$ is satisfied if and only if (PD_{f_i,D_i}) is satisfied for all i. Note that by (2.1.3), $(\mathrm{PD}_{f,D})$ is equivalent to the condition that the natural transformation

$$f_\sharp u_{i\sharp} \xrightarrow{\mathfrak{p}^n_{f,D}} f_! \Sigma^n_{f,D} u_{i\sharp}$$

is an isomorphism for any $i \in I$.

By (10.2.6), we can use (10.2.5) for $u_i : X_i \to X$ and $X \to S$. Then by (loc. cit), in the commutative diagram

$$\begin{array}{ccccc}
f_{i\sharp} & & \xrightarrow{\qquad\mathfrak{p}^n_{f_i,D_i}\qquad} & & f_{i!}\Sigma^n_{f_i,D_i} \\
 & & & & \downarrow C \\
 & & & & f_{i!}\Sigma^n_{u_i,f_i,D_i''}\Sigma^n_{u_i} \\
\downarrow \wr & & & & \downarrow \wr \\
 & & & & f_! u_{i!}\Sigma^n_{u_i,f_i,D_i''}\Sigma^n_{u_i} \\
 & & & & \uparrow Ex \\
f_\sharp u_{i\sharp} & \xrightarrow{\mathfrak{p}^n_{f,D}} & f_!\Sigma^n_{f,D}u_{i\sharp} & \xrightarrow{\mathfrak{p}^n_{u_i}} & f_!\Sigma^n_{f,D}u_!\Sigma^n_{u_i}
\end{array}$$

of functors, the right vertical top arrow and the right vertical bottom arrow are isomorphisms. The lower horizontal right arrow is also an isomorphism by (10.3.3). Thus the upper horizontal arrow is an isomorphism if and only if the lower horizontal left arrow is an isomorphism, which is what we want to prove. □

10.4.6. Let S be an $\mathscr{S}$-scheme with a fs chart P that is exact at some point $s \in S$, and let $\theta : P \to Q$ be a locally exact, injective, logarithmic, and vertical homomorphism of fs monoids such that the cokernel of θ^{gp} is torsion free. We put

$$X = S \times_{\mathbb{A}_P} \mathbb{A}_Q, \quad m = \dim P + \dim Q,$$

and assume $m > 0$. By (10.1.1), there is a compactified exactification D of the diagonal morphism $a : X \to X \times_S X$.

Proposition 10.4.7. *Under the notations and hypotheses of (10.4.6), the natural transformation*

$$f_\sharp f^* \xrightarrow{\mathfrak{p}^n_{f,D}} f_! \Sigma^n_{f,D} f^*$$

is an isomorphism.

Proof. Let G be a maximal θ-critical face of Q, and we put

$$U = S \times_{\mathbb{A}_P} \mathbb{A}_{Q_F}, \quad D' = D \times_{X \times_S X} (U \times_S U).$$

We denote by $j : U \to X$ the induced open immersion. Then the diagram

$$\begin{array}{ccccccc}
(fj)_\sharp j^* f^* & \xrightarrow{\qquad\qquad\sim\qquad\qquad} & & & f_\sharp j_\sharp j^* f^* & \xrightarrow{ad'} & f_\sharp f^* \\
\downarrow{\scriptstyle \mathfrak{p}^n_{fj,D'}} & & & & \downarrow{\scriptstyle \mathfrak{p}^n_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^n_{f,D}} \\
(fj)_! \Sigma^n_{fj,D'} f^* f^* & \xrightarrow{\sim} & f_! j_\sharp \Sigma^n_{fj,D'} j^* f^* & \xrightarrow{Ex} & f_! \Sigma^n_{f,D} j_\sharp j^* f^* & \xrightarrow{ad'} & f_! \Sigma^n_{f,D} f^*
\end{array}$$

of functors commutes by (10.2.5) and (10.2.6). By (Htp–3) and (Htp–7), the upper and lower right side horizontal arrows are isomorphisms, and the lower middle horizontal arrow is an isomorphism by (4.2.7). The composition $fj : U \to S$ is Kummer log smooth and separated, so the left vertical arrow is an isomorphism by (10.3.3). Thus the right vertical arrow is also an isomorphism. □

Proposition 10.4.8. *Under the notations and hypotheses of (10.4.6),* (PD^{m-1}) *implies* (PD_f).

Proof. Assume (PD^{m-1}). By (10.4.3), it suffices to show ($\mathrm{PD}_{f,D}$). We put

$$d = \mathrm{rk}\, Q^{\mathrm{gp}} - \mathrm{rk}\, P^{\mathrm{gp}}, \quad \tau = (d)[2d].$$

Then it suffices to show that the natural transformation

$$\mathfrak{p}^o_{f,D} : f_\sharp \longrightarrow f_! \tau$$

is an isomorphism. Guided by a method of [CD12, 2.4.42], we will construct its left inverse ϕ_2 as follows:

$$\phi_2 : f_! \tau \xrightarrow{ad} f_! \tau f^* f_\sharp \xrightarrow{(\mathfrak{p}^o_{f,D})^{-1}} f_\sharp f^* f_\sharp \longrightarrow f_\sharp$$

Here, the second arrow is defined and an isomorphism by (10.4.7). We have $\phi_2 \circ \mathfrak{p}^o_{f,D} = \mathrm{id}$ as in the proof of (5.4.7).

To construct a right inverse of $\mathfrak{p}^o_{f,D}$, consider the Cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & \mathbb{A}_Q - \mathbb{A}_{(Q,Q^+)} \\ \downarrow{\scriptstyle i} & & \downarrow \\ X & \longrightarrow & \mathbb{A}_Q \end{array}$$

of $\mathscr{S}$-schemes, and we put $g = fi$. Note that the morphism $\underline{g} : \underline{Z} \to \underline{S}$ of underlying schemes is an isomorphism by assumption on θ. Consider the commutative diagram

$$\begin{array}{ccccccc} f_\sharp j_\sharp j^* & \xrightarrow{ad'} & f_\sharp & \xrightarrow{ad} & f_\sharp i_* i^* & \xrightarrow{\partial_i} & f_\sharp j_\sharp j^*[1] \\ \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} \\ f_! \tau j_\sharp j^* & \xrightarrow{ad'} & f_! \tau & \xrightarrow{ad} & f_! \tau i_* i^* & \xrightarrow{\partial_i} & f_! \tau j_\sharp j^*[1] \end{array}$$

of functors where j denotes the complement of i. The two rows are distinguished triangles by (Loc). The first vertical arrow is an isomorphism by (PD^{m-1}), and ϕ_2 induces the left inverse to the third vertical arrow. If we show that the third vertical arrow is an isomorphism, then the second vertical arrow is also an isomorphism. Hence it suffices to construct a right inverse of the natural transformation

$$f_\sharp i_* \xrightarrow{\mathfrak{p}^o_{f,D}} f_! \tau i_*.$$

Consider also the commutative diagram

$$\begin{array}{ccccccc} f_\sharp j_\sharp j^* f^* & \xrightarrow{ad'} & f_\sharp f^* & \xrightarrow{ad} & f_\sharp i_* i^* f^* & \xrightarrow{\partial_i} & f_\sharp j_\sharp j^* f^*[1] \\ \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} & & \downarrow{\scriptstyle \mathfrak{p}^o_{f,D}} \\ f_! \tau j_\sharp j^* f^* & \xrightarrow{ad'} & f_! \tau f^* & \xrightarrow{ad} & f_! \tau i_* i^* f^* & \xrightarrow{\partial_i} & f_! \tau j_\sharp j^* f^*[1] \end{array}$$

of functors. The two rows are distinguished triangles by (Loc), and the first vertical arrow is an isomorphism by (PD^{m-1}). Since the second vertical arrow is an isomorphism by (10.4.7), the third vertical arrow is also an isomorphism.

Then a right inverse of $f_\sharp i_* \xrightarrow{\mathfrak{p}^o_{f,D}} f_! \tau i_*$ is constructed by

$$\begin{aligned} \phi_1' : f_! \tau i_* &\xrightarrow{Ex} f_! i_* \tau \xrightarrow{\sim} g_* \tau \xrightarrow{ad} g_* g^* g_* \tau \xrightarrow{Ex^{-1}} g_* g^* \tau g_* \\ &\xrightarrow{\sim} f_! i_* \tau g^* g_* \xrightarrow{Ex^{-1}} f_! \tau i_* g^* g_* \xrightarrow{(\mathfrak{p}^o_{f,D})^{-1}} f_\sharp i_* g^* g_* \xrightarrow{ad'} f_\sharp i_*. \end{aligned}$$

Here, the fourth and sixth natural transformations are defined and isomorphisms by (Stab), and the seventh natural transformation is an isomorphism by the above paragraph. To show that ϕ'_1 is a right inverse of $f_\sharp i_* \xrightarrow{\mathfrak{p}^o_{f,D}} f_!\tau i_*$, it suffices to check that the composition of the outer cycle of the diagram

$$\begin{array}{ccccccc}
f_\sharp i_* & \xrightarrow{\mathfrak{p}^o_{f,D}} & f_!\tau i_* & \xrightarrow{\sim} & g_*\tau & \xrightarrow{ad} & g_*g^*g_*\tau \\
\uparrow{\scriptstyle ad'} & & \uparrow{\scriptstyle ad'} & \searrow\!\!= & f_!\tau i_* \xrightarrow{\sim} g_*\tau & \swarrow{\scriptstyle ad'} \;\; = & \downarrow{\scriptstyle Ex^{-1}} \\
 & & & \nearrow{\scriptstyle ad'} & & & \\
f_\sharp i_* g^* g_* & \xleftarrow[(\mathfrak{p}^o_{f,D})^{-1}]{} & f_!\tau i_* g^* g_* & \xleftarrow[Ex^{-1}]{} & f_! i_* \tau g^* g_* & \xleftarrow[\sim]{} & g_*g^*\tau g_*
\end{array}$$

of functors is the identity. It is true since each small diagram commutes. □

Proposition 10.4.9. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ where θ is a vertical homomorphism of exact log smooth type. Then* (PD^{m-1}) *implies* (PD_f).

Proof. By (10.4.5) and [Ogu14, II.2.3.2], we may assume that f has a factorization

$$X \xrightarrow{u} S_0 \xrightarrow{v} S$$

such that

(i) v strict étale,

(ii) the fs chart $S_0 \to \mathbb{A}_P$ is exact at some point of $s_0 \in S_0$,

(iii) s_0 is in the image of u.

By (10.2.6), we can use (10.4.4) for the morphisms $X \to S_0$ and $S_0 \to S$, and by (10.3.3), (PD_v) is satisfied. Hence replacing S by S_0, we may assume that the fs chart $S \to \mathbb{A}_P$ is exact at some point of $s \in S$ and that s is in the image of f.

By assumption, the induced morphism

$$X \to S \times_{\mathbb{A}_P} \mathbb{A}_Q$$

is strict étale and separated. We denote by P' the submonoid of Q consisting of elements $q \in Q$ such that $nq \in P + Q^*$ for some $n \in \mathbb{N}^+$. Then the induced homomorphism $\theta' : P' \to Q$ is locally exact, injective, logarithmic, and vertical, and the cokernel of θ^{gp} is torsion free. In particular, the induced morphism

$$S \times_{\mathbb{A}_P} \mathbb{A}_Q \to S \times_{\mathbb{A}_P} \mathbb{A}_{P'}$$

is exact log smooth, so it is an open morphism by [Nak09, 5.7].

We denote by S' the image of X via the composition

$$X \to S \times_{\mathbb{A}_P} \mathbb{A}_Q \to S \times_{\mathbb{A}_P} \mathbb{A}_{P'}.$$

we consider S' as an open subscheme of $S \times_{\mathbb{A}_P} \mathbb{A}_{P'}$. Then the induced morphism $g : S' \to S$ has the fs chart $\theta' : P \to Q'$ of Kummer log smooth type. Consider the factorization

$$X \xrightarrow{g_3} S' \times_{\mathbb{A}_{P'}} \mathbb{A}_Q \xrightarrow{g_2} S' \xrightarrow{g_1} S$$

of $f : X \to S$. Then (PD_{g_1}) and (PD_{g_3}) are satisfied by (10.3.3) since g_1 and g_3 are Kummer log smooth and separated. The set $g_1^{-1}(a)$ is nonempty since s is in the image of f, and the chart $S' \to \mathbb{A}_{P'}$ is exact at a point in $g_1^{-1}(s)$. Thus (PD_{g_2}) is satisfied by (10.4.8), so (PD_f) is satisfied by (10.2.6) and (10.4.4). □

Theorem 10.4.10. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ where θ is a vertical homomorphism of exact log smooth type. Then (PD_f) is satisfied.*

Proof. It suffices to show (PD^m) for any m. By (10.3.3), (PD^0) is satisfied. Then the conclusion follows from (10.4.9) and induction on m. □

Corollary 10.4.11. *Under the notations and hypotheses of (10.4.10), let D be a compactified exactification of the diagonal morphism $a : X \to X \times_S X$. Then the composition*

$$\Omega_{f,D} f^! \xrightarrow{T_D} \Omega_f f^! \xrightarrow{\mathfrak{q}_f} f^*$$

is an isomorphism.

Proof. By (10.4.10), the composition

$$\Omega_{f,D}^n f^! \xrightarrow{T^n} \Omega_{f,D}^d f^! \xrightarrow{T^d} \Omega_{f,D} f^! \xrightarrow{T_D} \Omega_f f^! \xrightarrow{\mathfrak{q}_f} f^*$$

is an isomorphism. By (4.2.9), the first and second arrows are isomorphisms, so the composition of the third and fourth arrows are also an isomorphism. □

10.5 Purity

Proposition 10.5.1. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ where θ is a vertical homomorphism of exact log smooth type. Consider a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of $\mathscr{S}$-schemes. Then the exchange transformation

$$Ex : g^* f_! \longrightarrow f'_! g'^*$$

is an isomorphism.

Proof. By (10.1.5), there is a compactified exactification D of the diagonal morphism $a : X \to X \times_S X$. Then by (10.2.7), the diagram

$$\begin{array}{ccc} f'_\sharp g'^* & \xrightarrow{\mathfrak{p}^n_{f'}} & f'_! \Sigma^n_{f',D'} g'^! \\ & & \downarrow Ex \\ \downarrow Ex & & f'_! g'^* \Sigma^n_{f,D} \\ & & \uparrow Ex \\ g^* f_\sharp & \xrightarrow{\mathfrak{p}^n_f} & g^* f_! \Sigma^n_{f,D} \end{array}$$

of functors commutes. The left vertical arrow is an isomorphism by (eSm-BC), and the right upper vertical arrow is an isomorphism by (4.2.8). The horizontal arrows are also isomorphisms by (10.4.10). Thus the right lower vertical arrow is an isomorphism. Then the conclusion follows from the fact that the functor

$$\Sigma^n_{f,D} \cong \Sigma^o_{f,D}$$

is an equivalence of categories. □

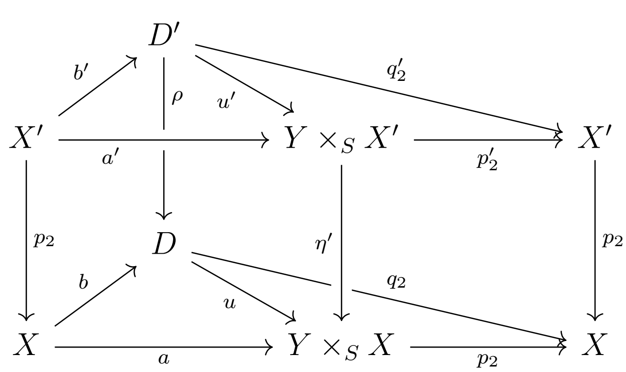
10.5.2. Under the notations and hypotheses of (4.2.3), note that by (10.5.1), the condition ($CE^!$) is satisfied when η in (loc. cit) is a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes with a fs chart a fs chart $\theta : P \to Q$ where θ is a vertical homomorphism of exact log smooth type.

Proposition 10.5.3. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes with a fs chart $\theta : P \to Q$ where θ is a vertical homomorphism of exact log smooth type, and let D be a compactified exactification of the diagonal morphism $a : X \to X \times_S X$. Then the transition transformation*

$$T_D : \Omega_{f,D} \longrightarrow \Omega_f$$

is an isomorphism.

Proof. Let $v : I \to D$ be an interior of D. Consider the commutative diagram



of $\mathscr{S}$-schemes where p_2 denotes the second projection and each square is Cartesian. We denote by $v' : I' \to D'$ the pullback of $v : I \to D$. Then I' is also an interior of D'. By (10.5.2) and (4.2.3), we have the exchange transformations

$$\Omega_{p_2} p_2^! \xrightarrow{Ex} p_2^! \Omega_f, \quad \Omega_{p_2,D'} p_2^! \xrightarrow{Ex} p_2^! \Omega_{f,D}, \quad \Omega_{p_2,I'} p_2^! \xrightarrow{Ex} p_2^! \Omega_{f,I},$$

and we have the commutative diagram

$$\begin{array}{ccccccc}
a^! \Omega_{f',I'} p_2^! & \xrightarrow{T_{I,D}} & a^! \Omega_{p_2,D'} p_2^! & \xrightarrow{T'_D} & a^! \Omega_{p_2} p_2^! & & \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} & \searrow{\scriptstyle \mathfrak{q}_f} & \\
a^! p_2^! \Omega_{f,I} & \xrightarrow{T_{I,D}} & a^! p_2^! \Omega_{f,D} & \xrightarrow{T_D} & a^! p_2^! \Omega_f & & \\
& & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \\
& & \Omega_{f,D} & \xrightarrow{T_D} & \Omega_f & = & a^! p_2^*
\end{array} \tag{10.5.3.1}$$

of functors. The natural transformations

$$a^! p_2^! \Omega_{f,I} \xrightarrow{T_{I,D}} a^! p_2^! \Omega_{f,D}, \quad a^! \Omega_{f',I'} p_2^! \xrightarrow{T_{I,D}} a^! \Omega_{f',D'} p_2^!$$

are isomorphisms by (4.2.7), and the natural transformation

$$a^! \Omega_{f',I'} p_2^! \xrightarrow{Ex} a^! p_2^! \Omega_{f,I}$$

is an isomorphism by (4.2.6) since $r_2 = q_2 v$ is strict by definition of interior. The composition

$$a^! \Omega^n_{p_2,D'} p_2^! \xrightarrow{T^n} a^! \Omega^d_{p_2,D'} p_2^! \xrightarrow{T^d} a^! \Omega_{p_2,D'} p_2^! \xrightarrow{T_{D'}} a^! \Omega_{p_2} p_2^! \xrightarrow{\mathfrak{q}_f} a^! p_2^*$$

is also an isomorphism by (10.4.11). Applying these to (10.5.3.1), we conclude that the natural transformation

$$T_D : \Omega_{f,D} \longrightarrow \Omega_f$$

is an isomorphism. □

Proposition 10.5.4. *Let $f : X \to S$ be a vertical exact log smooth separated morphism of $\mathscr{S}$-schemes. Then* (Pur_f) *is satisfied.*

Proof. We want to show that the purity transformation

$$\mathfrak{q}_f : \Omega_f f^! \longrightarrow f^*$$

is an isomorphism. It is equivalent to showing that the natural transformation

$$\mathfrak{p}_f : f_\sharp \longrightarrow f_! \Sigma_f$$

is an isomorphism.

(I) *Locality on S.* Let $\{u_i : S_i \to S\}_{i\in I}$ be a strict étale separated cover of S. Consider the Cartesian diagram

$$\begin{array}{ccc} X_i & \xrightarrow{u_i'} & X \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f} \\ S_i & \xrightarrow{u_i} & S \end{array}$$

of $\mathscr{S}$-schemes. Then by (4.4.5), the diagram

$$\begin{array}{ccc} f_{i\sharp}u_i'^* & \xrightarrow{\mathfrak{p}_{f_i}} & f_{i!}\Sigma_{f_i}u_i'^* \\ & & \downarrow{\scriptstyle Ex} \\ \downarrow{\scriptstyle Ex} & & f_{i!}u_i'^*\Sigma_f \\ & & \uparrow{\scriptstyle Ex} \\ u_i^*f_\sharp & \xrightarrow{\mathfrak{p}_f} & u_i^*f_!\Sigma_f \end{array}$$

of functors commutes. The left vertical arrow is an isomorphism by (eSm-BC), and the right lower vertical arrow is an isomorphism since u_i is exact log smooth. The right upper vertical arrow is also an isomorphism by (2.5.10). Thus the upper horizontal arrow is an isomorphism if and only if the lower horizontal arrow is an isomorphism.

Since the family of functors $\{u_i^*\}_{i\in I}$ is conservative by (két-sep), the lower horizontal arrow is an isomorphism if and only if the natural transformation

$$\mathfrak{p}_f : f_\sharp \longrightarrow f_!\Sigma_f$$

is an isomorphism. Thus we have proven that the question is strict étale separated local on S.

(II) *Locality on X.* Let $\{v_i : X_i \to X\}_{i\in J}$ be a strict étale separated cover of X. By (4.4.3), we have the commutative diagram

$$\begin{array}{ccccc} \Omega_{v_i}v_i^!\Omega_f f^! & \xrightarrow{\mathfrak{q}_{v_i}} & v_i^*\Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & v_i^*f^* \\ \uparrow{\scriptstyle Ex} & & & & \\ \Omega_{v_i}\Omega_{f,fv_i}v_i^!f^! & & & & \\ \downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim} \\ \Omega_{v_i}\Omega_{f,fv_i}(fv_i)^! & & & & \\ \downarrow{\scriptstyle C} & & & & \\ \Omega_{fv_i}(fv_i)^! & & \xrightarrow{\mathfrak{q}_{fv_i}} & & (fv_i)^* \end{array}$$

of functors. The left top vertical arrow is an isomorphism by (2.5.10), and the left bottom vertical arrow is an isomorphism by (4.3.1). The upper left horizontal arrow is also an isomorphism by (2.5.9), so the upper right horizontal arrow is an isomorphism if and only if the lower horizontal arrow is an isomorphism.

Since the family of functors $\{v_i^*\}_{i \in J}$ is conservative, the lower horizontal arrow is an isomorphism if and only if the natural transformation

$$\mathfrak{q}_f : \Omega_f f^! \longrightarrow f^*$$

is an isomorphism. Thus we have proven that the question is strict étale separated local on X.

(III) *Final step of the proof.* Since the question is strict étale separated local on X and S, we may assume that $f : X \to S$ has a fs chart $\theta : P \to Q$ of exact log smooth type by (3.1.4) (in (loc. cit), if we localize X and S further so that $\underline{X}$ and $\underline{S}$ are affine, then the argument is strict étale separated local instead of strict étale local). Localizing Q further, since f is vertical, we may assume that θ is vertical.

By (10.1.5), there is a compactified exactification D of the diagonal morphism $a : X \to X \times_S X$. Then we have the natural transformation

$$\Omega_{f,D} f^! \xrightarrow{T_D} \Omega_f f^! \xrightarrow{\mathfrak{q}_f} f^!.$$

The composition is an isomorphism by (10.4.11), and the first arrow is an isomorphism by (10.5.3). Thus the second arrow is an isomorphism. □

Theorem 10.5.5. *Let $f : X \to S$ be an exact log smooth separated morphism of $\mathscr{S}$-schemes. Then* (Pur_f) *is satisfied.*

Proof. Let $j : U \to X$ denote the verticalization of f. By (4.4.3), the diagram

$$\begin{array}{ccccc}
\Omega_j j^! \Omega_f f^! & \xrightarrow{\mathfrak{q}_j} & j^* \Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & j^* f^* \\
\uparrow{\scriptstyle Ex} & & & & \\
\Omega_j \Omega_{f,fj} j^! f^! & & & & \big\downarrow{\scriptstyle \sim} \\
\downarrow{\scriptstyle \sim} & & & & \\
\Omega_j \Omega_{f,fj} (fj)^! & & & & \\
\downarrow{\scriptstyle C} & & & & \\
\Omega_{fj} (fj)^! & & \xrightarrow{\qquad \mathfrak{q}_f \qquad} & & (fj)^*
\end{array}$$

of functors commutes. The left top vertical arrow is an isomorphism by (??), and the left bottom vertical arrow is an isomorphism by (4.3.1). The upper left horizontal arrow is an isomorphism by (2.5.9), and the lower horizontal arrow is an isomorphism by (10.5.4). Thus the upper right horizontal arrow is an isomorphism.

Then consider the commutative diagram

$$\begin{array}{ccc}
\Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & f^* \\
\downarrow{\scriptstyle ad} & & \downarrow{\scriptstyle ad} \\
j_*j^*\Omega_f f^! & \xrightarrow{\mathfrak{q}_f} & j_*j^*f^*
\end{array}$$

of functors. We have shown that the lower horizontal arrow is an isomorphism. Since the right vertical arrow is an isomorphism by (Htp–2), the remaining is to show that the left vertical arrow is an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{j} & X \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\
U \times_S X & \xrightarrow{j'} & X \times_S X \\
\downarrow{\scriptstyle p_2'} & & \downarrow{\scriptstyle p_2} \\
U & \xrightarrow{j} & X
\end{array}$$

of $\mathscr{S}$-schemes where

(i) p_2 denotes the second projection.

(ii) a denotes the diagonal morphism,

(iii) each square is Cartesian.

Then j' is the verticalization of p_2, so by (Htp–2), the natural transformation

$$p_2^* \xrightarrow{ad} j'_*j'^*p_2^*$$

is an isomorphism. Consider the natural transformations

$$\Omega_f \xrightarrow{\sim} a^!p_2^* \xrightarrow{ad} a^!j'_*j'^*p_2^* \xleftarrow{Ex} j_*a'^!j'^*p_2^* \xleftarrow{Ex} j_*j^*a^!p_2^* \xleftarrow{\sim} j_*j^*\Omega_f.$$

We have shown that the second arrow is an isomorphism. The third arrow is an isomorphism by (eSm-BC), and the fourth arrow is an isomorphism by (Supp). This completes the proof. □

10.6 Purity transformations

10.6.1. Throughout this section, assume that $\mathscr{T}$ can be extended to an eSm-premotivic triangulated prederivator satisfying strict étale descent.

Definition 10.6.2. Let $i : (\mathscr{X}, I) \to (\mathscr{Y}, I)$ be a Cartesian strict regular embedding of $\mathscr{S}$-diagrams. For any morphism $\lambda \to \mu$ in I, there are induced morphisms

$$D_{\mathscr{X}_\lambda}\mathscr{Y}_\lambda \to D_{\mathscr{X}_\mu}\mathscr{Y}_\mu, \quad N_{\mathscr{X}_\lambda}\mathscr{Y}_\lambda \to N_{\mathscr{X}_\mu}\mathscr{Y}_\mu$$

of $\mathscr{S}$-schemes. Using these, we have the following $\mathscr{S}$-schemes.

(1) $D_{\mathscr{X}}\mathscr{Y}$ denotes the $\mathscr{S}$-diagram constructed by $D_{\mathscr{X}_\lambda}\mathscr{Y}_\lambda$ for $\lambda \in I$,

(2) $N_{\mathscr{X}}\mathscr{Y}$ denotes the $\mathscr{S}$-diagram constructed by $N_{\mathscr{X}_\lambda}\mathscr{Y}_\lambda$ for $\lambda \in I$.

Note that if the induced morphism $\underline{\mathscr{Y}_\lambda} \to \underline{\mathscr{Y}_\mu}$ is flat for any morphism $\lambda \to \mu$ in I, then the induced morphisms $\mathscr{X} \to D_{\mathscr{X}}\mathscr{Y}$ and $\mathscr{X} \to N_{\mathscr{X}}\mathscr{Y}$ are Cartesian strict regular embeddings by [Ful98, B.7.4].

Definition 10.6.3. Let $f : X \to S$ be an exact log smooth morphism of $\mathscr{S}$-schemes, and let $h : \mathscr{X} \to X$ be a morphism of $\mathscr{S}$-diagrams. Then we denote by

$$N_{\mathscr{X}}(X \times_S \mathscr{X})$$

the vector bundle of $\mathscr{X}$ associated to the dual free sheaf $(h^*\Omega_{X/S})^\vee$. Note that when the induced morphism $\mathscr{X} \to X \times_S \mathscr{X}$ is a Cartesian strict regular embedding, this definition is equivalent to the definition in (10.6.2).

10.6.4. Let $f : X \to S$ be a separated vertical exact log smooth morphism of $\mathscr{S}$-schemes. We will construct several $\mathscr{S}$-diagrams and their morphisms.

(1) *Construction of* $\mathscr{X}$. Let $\{h_\lambda : \mathscr{X}_\lambda \to X\}_{\lambda \in I_0}$ be a strict étale cover such that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{X}_\lambda & \xrightarrow{h_\lambda} & X \\ \downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle f} \\ S_\lambda & \xrightarrow{l_\lambda} & S \end{array}$$

of $\mathscr{S}$-schemes where

(i) $\mathscr{X}_\lambda$ and S_λ are affine,

(ii) f_λ has a fs chart $\theta_\lambda : P_\lambda \to Q_\lambda$ of exact log smooth type,

(iii) l_λ is strict étale.

Then we denote by $\mathscr{X} = (\mathscr{X}, I)$ the Čech hypercover associated to $\{h_\lambda : \mathscr{X}_\lambda \to X\}_{\lambda \in I_0}$.

(2) *Construction of* $\mathscr{D}$. For $\lambda \in I_0$, we denote by h'_λ the induced morphism

$$X \times_S \mathscr{X}_\lambda \to X \times_S X$$

of $\mathscr{S}$-schemes, and let U_λ denote the open subscheme

$$X \times_S \mathscr{X}_\lambda - (h'_\lambda)^{-1}(a(X)).$$

Then we denote by $\mathscr{D}_\lambda$ the Čech hypercover of $X \times_S \mathscr{X}_\lambda$ associated to

$$\{\mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda \to X \times_S \mathscr{X}_\lambda,\ U_\lambda \to X \times_S \mathscr{X}_\lambda\},$$

and we denote by $\mathscr{D} = (\mathscr{D}, J)$ the Čech hypercover of $X \times_S X$ associated to

$$\{\mathscr{D}_\lambda \to X \times_S X\}_{\lambda \in I_0}.$$

Note that from our construction, we have the morphism

$$u : \mathscr{D} \to X \times_S \mathscr{X}$$

of $\mathscr{S}$-diagrams.

(3) *Construction of $\mathscr{E}$.* For $\lambda \in I_0$, we put $Y_\lambda = \mathscr{X}_\lambda \times_{S_\lambda \times_S X} \mathscr{X}_\lambda$. Then the composition

$$Y_\lambda \to \mathscr{X}_\lambda \to \mathbb{A}_{Q_\lambda}$$

where the first arrow is the first projection gives a fs chart of Y_λ. The induced morphism $Y_\lambda \to S_\lambda$ also has a fs chart $P_\lambda \to Q_\lambda$. As in (10.1.5), choose a proper birational morphism

$$M_\lambda \to \operatorname{spec}(Q_\lambda \oplus_{P_\lambda} Q_\lambda)$$

of fs monoschemes, and we put

$$E_\lambda = (\mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda) \times_{\mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}} \mathbb{A}_{M_\lambda},$$

and let $u''_\lambda : E_\lambda \to \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda$ denote the projection. Note that the diagonal morphism $\mathscr{X}_\lambda \to \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda$ factors through E_λ by construction in (loc. cit). Let $b''_\lambda : \mathscr{X} \to E_\lambda$ denote the factorization. We will show that the projection

$$Y_\lambda \times_{\mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda} E_\lambda \to Y_\lambda$$

is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & E'_\lambda & \longrightarrow & E_\lambda & \longrightarrow & \mathbb{A}_{M_\lambda} \\
 & & \downarrow & & \downarrow & & \downarrow \\
Y_\lambda & \longrightarrow & Y_\lambda \times_{S_\lambda} Y_\lambda & \longrightarrow & \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda & \longrightarrow & \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda} \\
\downarrow & & \downarrow{\scriptstyle \iota_1} & & & & \\
\mathbb{A}_{M_\lambda} & \longrightarrow & \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda} & & & &
\end{array}$$

of $\mathscr{S}$-diagrams where

(i) ι_1 denotes the fs chart induced by the fs charts $P_\lambda \to Q_\lambda$ of $Y_\lambda \to S_\lambda$ defined above,

(ii) the arrow $Y_\lambda \times_{S_\lambda} Y_\lambda \to \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda$ is the morphism induced by the first projection $Y_\lambda \to \mathscr{X}_\lambda$, the identity $S_\lambda \to S_\lambda$, and the second projection $Y_\lambda \to \mathscr{X}_\lambda$,

(iii) $E'_\lambda = (Y_\lambda \times_{S_\lambda} Y_\lambda) \times_{\mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda} E_\lambda$.

By (3.2.3), we have an isomorphism

$$(Y_\lambda \times_{S_\lambda} Y_\lambda) \times_{\iota_1, \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}} \mathbb{A}_M \cong E'_\lambda,$$

and this shows the assertion since the morphism $\mathbb{A}_{M_\lambda} \to \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}$ is a monomorphism of fs log schemes.

Now, we denote by $\mathscr{E}_\lambda$ the Čech hypercover of $X \times_S \mathscr{X}_\lambda$ associated to

$$\{E_\lambda \to X \times_S \mathscr{X}_\lambda,\ U_\lambda \to X \times_S \mathscr{X}_\lambda\},$$

and we denote by $\mathscr{E} = (\mathscr{E}, J)$ the Čech hypercover of $X \times_S X$ associated to

$$\{\mathscr{E}_\lambda \to X \times_S X\}_{\lambda \in I_0}.$$

Note that from our construction, we have the morphism

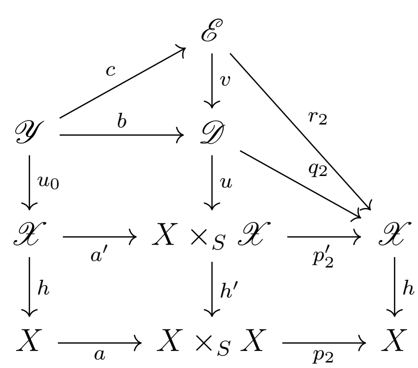
$$v : \mathscr{E} \to \mathscr{D}$$

of $\mathscr{S}$-diagrams. We put

$$\mathscr{Y} = \mathscr{X} \times_{X \times_S \mathscr{X}} \mathscr{D}.$$

Then the assertion in the above paragraph shows that the projection $\mathscr{Y} \times_{\mathscr{D}} \mathscr{E} \to \mathscr{Y}$ is an isomorphism, so the projection $b : \mathscr{Y} \to \mathscr{D}$ factors through $c : \mathscr{Y} \to \mathscr{E}$.

(4) *Commutative diagrams.* Now, we have the commutative diagram



(10.6.4.1)

of $\mathscr{S}$-diagrams where

(i) each small square is Cartesian,

(ii) u, v, and c are the morphisms constructed above.

As in (4.1.2), we also have the commutative diagrams

$$\begin{array}{ccccc} \mathscr{Y} & \xrightarrow{c} & \mathscr{E} & \xrightarrow{r_2} & \mathscr{X} \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \alpha_1} \\ \mathscr{Y}\times\mathbb{A}^1 & \xrightarrow{d} & D_{\mathscr{Y}}\mathscr{E} & \xrightarrow{s_2} & \mathscr{X}\times\mathbb{A}^1 \\ \downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \pi} \\ \mathscr{Y} & & & & \mathscr{X} \end{array} \qquad \begin{array}{ccccc} \mathscr{Y} & \xrightarrow{e} & \mathscr{E} & \xrightarrow{t_2} & \mathscr{X} \\ \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \alpha_0} \\ \mathscr{Y}\times\mathbb{A}^1 & \xrightarrow{d} & D_{\mathscr{Y}}\mathscr{E} & \xrightarrow{s_2} & \mathscr{X}\times\mathbb{A}^1 \\ \downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \pi} \\ \mathscr{Y} & & & & \mathscr{X} \end{array}$$

of $\mathscr{S}$-diagrams where

(i) each square is Cartesian,

(ii) α_0 denotes the 0-section, and α_1 denotes the 1-section,

(iii) d and s_2 are the morphisms constructed as in (4.1.2.1),

(iv) π and ϕ denotes the projections.

Then we have the commutative diagram

$$\begin{array}{ccccc} \mathscr{Y} & \xrightarrow{e} & N_{\mathscr{Y}}\mathscr{E} & \overset{t_2}{\searrow} & \\ \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u_1} & & \\ \mathscr{X} & \xrightarrow{e'} & N_{\mathscr{X}}(X\times_S\mathscr{X}) & \xrightarrow{t'_2} & \mathscr{X} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{e''} & N_X(X\times_S X) & \xrightarrow{t''_2} & X \end{array}$$

of $\mathscr{S}$-diagrams where

(i) each small square is Cartesian,

(ii) e'' denotes the 0-section, and t''_2 denotes the projection.

For $\lambda \in I$, we also have the corresponding commutative diagrams

$$\begin{array}{ccccc} & \overset{c_\lambda}{\nearrow} & \mathscr{E}_\lambda & \overset{r_{2\lambda}}{\searrow} & \\ & & \downarrow{\scriptstyle v_\lambda} & & \\ \mathscr{Y}_\lambda & \xrightarrow{b_\lambda} & \mathscr{D}_\lambda & \overset{q_{2\lambda}}{\searrow} & \\ \downarrow{\scriptstyle u_{0\lambda}} & & \downarrow{\scriptstyle u_\lambda} & & \\ \mathscr{X}_\lambda & \xrightarrow{a'_\lambda} & X\times_S\mathscr{X}_\lambda & \xrightarrow{p'_{2\lambda}} & \mathscr{X}_\lambda \end{array}$$

$$\begin{CD} \mathscr{Y}_\lambda @>{c_\lambda}>> \mathscr{E}_\lambda @>{r_{2\lambda}}>> \mathscr{X}_\lambda \\ @V{\gamma_{1\lambda}}VV @V{\beta_{1\lambda}}VV @V{\alpha_{1\lambda}}VV \\ \mathscr{Y}_\lambda \times \mathbb{A}^1 @>{d_\lambda}>> D_{\mathscr{Y}_\lambda}\mathscr{E}_\lambda @>{s_{2\lambda}}>> \mathscr{X}_\lambda \times \mathbb{A}^1 \\ @V{\phi_\lambda}VV @. @V{\pi_\lambda}VV \\ \mathscr{Y}_\lambda @. @. \mathscr{X}_\lambda \end{CD} \qquad \begin{CD} \mathscr{Y}_\lambda @>{e_\lambda}>> N_{\mathscr{Y}_\lambda}\mathscr{E}_\lambda @>{t_{2\lambda}}>> \mathscr{X}_\lambda \\ @V{\gamma_{0\lambda}}VV @V{\beta_{0\lambda}}VV @V{\alpha_{0\lambda}}VV \\ \mathscr{Y}_\lambda \times \mathbb{A}^1 @>{d_\lambda}>> D_{\mathscr{Y}_\lambda}\mathscr{E}_\lambda @>{s_{2\lambda}}>> \mathscr{X}_\lambda \times \mathbb{A}^1 \\ @V{\phi_\lambda}VV @. @V{\pi_\lambda}VV \\ \mathscr{Y}_\lambda @. @. \mathscr{X}_\lambda \end{CD}$$

$$\begin{CD} \mathscr{Y}_\lambda @>{e_\lambda}>> N_{\mathscr{Y}_\lambda}\mathscr{E}_\lambda @>{t_{2\lambda}}>> \mathscr{X}_\lambda \\ @V{u_{0\lambda}}VV @V{u_{1\lambda}}VV @| \\ \mathscr{X}_\lambda @>{e'_\lambda}>> N_{\mathscr{X}_\lambda}(X \times_S \mathscr{X}_\lambda) @>{t'_{2\lambda}}>> \mathscr{X}_\lambda \end{CD}$$

of $\mathscr{S}$-schemes. We also put

$$g = fh, \quad g_\lambda = fh_\lambda,$$

10.6.5. Under the notations and hypotheses of (10.6.4), we have an isomorphism $N_{\mathscr{Y}}\mathscr{E} \cong N_{\mathscr{X}}(X \times_S \mathscr{X}) \times_{\mathscr{X}} \mathscr{Y}$ by [Ogu14, IV.1.3.1]. In particular, the morphism $u_1 : N_{\mathscr{Y}}\mathscr{E} \to N_{\mathscr{X}}(X \times_S \mathscr{X})$ is a strict étale hypercover. We have the natural transformation

$$\Omega^n_{f,g} \xrightarrow{T^{n'}} \Omega^n_{f,g,\mathscr{E}}$$

given by

$$e'^! t_2'^* \xrightarrow{ad} e'^! u_{1*} u_1^* t_2'^* \xrightarrow{\sim} e'^! u_{1*} t_2^* \xrightarrow{Ex^{-1}} u_{0*} e^! t_2^*.$$

Here, the first arrow is an isomorphism since $\mathscr{T}$ satisfies strict étale descent, and the third arrow is defined and an isomorphism by (9.2.9) since e' is a Cartesian strict closed immersion. Thus the composition is an isomorphism.

We similarly have the natural transformation

$$\Omega^n_{f,g_\lambda} \xrightarrow{T^{n'}} \Omega^n_{f,g_\lambda,\mathscr{E}_\lambda},$$

which is also an isomorphism.

10.6.6. Under the notations and hypotheses of (10.6.4), for $\lambda \in I_0$, we temporary put

$$A_\lambda = X \times_S \mathscr{X}_\lambda, \quad B_\lambda = \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda$$

for simplicity. We had the Cartesian diagram

$$\begin{CD} E_\lambda @>>> \mathbb{A}_{M_\lambda} \\ @VVV @VVV \\ B_\lambda @>>> \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda} \end{CD}$$

of $\mathscr{S}$-schemes. Consider the commutative diagram

$$\begin{CD}
E_\lambda \times_{A_\lambda} E_\lambda @>{\zeta_2''}>> E_\lambda \times_{A_\lambda} B_\lambda @>>> \mathbb{A}_{M_\lambda} \\
@VVV @VV{\zeta_1'}V @VVV \\
B_\lambda \times_{A_\lambda} E_\lambda @>{\zeta_2'}>> B_\lambda \times_{A_\lambda} B_\lambda @>{\zeta_2}>> \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda} \\
@VVV @VV{\zeta_1}V \\
\mathbb{A}_{M_\lambda} @>>> \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}
\end{CD}$$

of $\mathscr{S}$-schemes where

(i) each square is Cartesian,

(ii) ζ_1 denotes the composition $B_\lambda \times_{A_\lambda} B_\lambda \to B_\lambda \to \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}$ where the first arrow is the first projection,

(iii) ζ_2 denotes the composition $B_\lambda \times_{A_\lambda} B_\lambda \to B_\lambda \to \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}$ where the first arrow is the second projection.

By (3.2.3), we have isomorphisms

$$B_\lambda \times_{A_\lambda} E_\lambda \cong B_\lambda \times_{\zeta_1, \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}} \mathbb{A}_M \cong B_\lambda \times_{\zeta_2, \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}} \mathbb{A}_M \cong E_\lambda \times_{A_\lambda} B_\lambda,$$

so using this, we have the Cartesian diagram

$$\begin{CD}
E_\lambda \times_{A_\lambda} E_\lambda @>{\zeta_2''}>> E_\lambda \times_{A_\lambda} B_\lambda \\
@VVV @VV{\zeta_1'}V \\
E_\lambda \times_{A_\lambda} B_\lambda @>{\zeta_1'}>> B_\lambda \times_{A_\lambda} B_\lambda
\end{CD}$$

of $\mathscr{S}$-schemes. Since ζ_1' is a pullback of $\mathbb{A}_M \to \mathbb{A}_{Q_\lambda \oplus_{P_\lambda} Q_\lambda}$ that is a monomorphism, the morphism ζ_1'' is an isomorphism. From this, we conclude that the induced morphism

$$E_\lambda \times_{X \times_S \mathscr{X}_\lambda} \mathscr{E}_\lambda \to E_\lambda \times_{X \times_S \mathscr{X}_\lambda} \mathscr{D}_\lambda$$

is an isomorphism for $\lambda \in I_0$.

Now, for $\lambda \in I$ instead of $\lambda \in I_0$, if $D_\lambda = D_{\lambda_1} \times_{X \times_S X} \cdots \times_{X \times_S X} D_{\lambda_r}$ for some $\lambda_1, \ldots, \lambda_r \in I_0$, we put

$$E_\lambda = E_{\lambda_1} \times_{X \times_S X} \cdots \times_{X \times_S X} E_{\lambda_r}.$$

From the result in the above paragraph, we see that the induced morphism

$$E_\lambda \times_{X \times_S \mathscr{X}_\lambda} \mathscr{E}_\lambda \to E_\lambda \times_{X \times_S \mathscr{X}_\lambda} \mathscr{D}_\lambda$$

is an isomorphism. In particular, the the first projection

$$E_\lambda \times_{X \times_S \mathscr{X}_\lambda} \mathscr{E}_\lambda \to E_\lambda$$

is a strict étale Čech hypercover.

10.6.7. Under the notations and hypotheses of (10.6.6), consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & \mathscr{E}_\lambda & & & & \\
 & & & \overset{w''_\lambda}{\nearrow} & & \overset{v_\lambda}{\searrow} & & & \\
\mathscr{Y}_\lambda & \xrightarrow{c''_\lambda} & \mathscr{E}'_\lambda & \xrightarrow{v'_\lambda} & \mathscr{D}'_\lambda & \xrightarrow{w'_\lambda} & \mathscr{D}_\lambda & & \\
\downarrow{\scriptstyle u_{0\lambda}} & & \downarrow{\scriptstyle v''_\lambda} & & \downarrow{\scriptstyle u'_\lambda} & & \downarrow{\scriptstyle u_\lambda} & \overset{q_{2\lambda}}{\searrow} & \\
\mathscr{X}_\lambda & \xrightarrow{b''_\lambda} & E_\lambda & \xrightarrow{u''_\lambda} & \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda & \xrightarrow{w_\lambda} & X \times_S \mathscr{X}_\lambda & \xrightarrow{p'_{2\lambda}} & \mathscr{X}_\lambda
\end{array}
$$

of $\mathscr{S}$-diagrams where each small square is Cartesian and w_λ denotes the induced morphism. Then we have the commutative diagram

$$
\begin{array}{ccccccccc}
\Omega^n_{f,g_\lambda,\mathscr{E}'_\lambda} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{f,g,\mathscr{E}'_\lambda} & \xrightarrow{T^d} & \Omega_{f,g,\mathscr{E}'_\lambda} & \xrightarrow{T_{\mathscr{D}'_\lambda,\mathscr{E}'_\lambda}} & \Omega_{f,g_\lambda,\mathscr{D}'_\lambda} & \xrightarrow{T_{\mathscr{D}_\lambda,\mathscr{D}'_\lambda}} & \Omega_{f,g_\lambda,\mathscr{D}_\lambda} \\
\downarrow{\scriptstyle T_{E_\lambda,\mathscr{E}'_\lambda}} & & \downarrow{\scriptstyle T_{E_\lambda,\mathscr{E}'_\lambda}} & & \downarrow{\scriptstyle T_{E_\lambda,\mathscr{E}'_\lambda}} & & \downarrow{\scriptstyle T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda,D'_\lambda}} & & \downarrow{\scriptstyle T_{\mathscr{D}_\lambda}} \\
\Omega^n_{f,g_\lambda,E_\lambda} & \underset{(T^n)^{-1}}{\longleftarrow} & \Omega^d_{f,g,E_\lambda} & \underset{T^d}{\longrightarrow} & \Omega_{f,g,E_\lambda} & \underset{T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda,E_\lambda}}{\longrightarrow} & \Omega_{f,g_\lambda,\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda} & \underset{T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda}}{\longrightarrow} & \Omega_{f,g_\lambda}
\end{array}
\tag{10.6.7.1}
$$

of functors. Here, the arrows are defined by the $\mathscr{S}$-diagram versions of (4.2.2), (4.2.9), and (4.2.11). Since u_λ is an exact log smooth morphism and a'_λ is reduced, the exchange transformation

$$u_{0\lambda*}b^!_\lambda \xrightarrow{Ex} a'^!_\lambda u_{\lambda*}$$

for the commutative diagram

$$
\begin{array}{ccc}
\mathscr{Y}_\lambda & \xrightarrow{b_\lambda} & \mathscr{D}_\lambda \\
\downarrow{\scriptstyle u_{0\lambda}} & & \downarrow{\scriptstyle u_\lambda} \\
\mathscr{X}_\lambda & \xrightarrow{a'_\lambda} & X \times_S \mathscr{X}
\end{array}
$$

is an isomorphism by (9.2.9). The unit $\mathrm{id} \xrightarrow{ad} u_{\lambda*}u^*_\lambda$ is also an isomorphism since $\mathscr{T}$ satisfies strict étale descent. Thus by construction in (4.2.2(ii)), the transition transformation $T_{\mathscr{D}_\lambda}$ is an isomorphism. Similarly, the other vertical arrows of (10.6.7.1) are isomorphisms.

By construction in (10.6.4) using (10.1.5), the conditions of (4.1.3) are satisfied, so by (4.2.12), the lower horizontal arrows of (10.6.7.1) denoted by $(T^n)^{-1}$ and T^d are isomorphisms. The lower horizontal arrow of (10.6.7.1) denoted by $T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda,E_\lambda}$ is an isomorphism by (10.5.3), and the lower horizontal arrow of (10.6.7.1) denoted by $T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda}$ is an isomorphism by construction (4.2.2(iii)). Thus we have shown that the lower horizontal arrows of (10.6.7.1) are all isomorphisms, so the upper horizontal arrows of (10.6.7.1) are also isomorphisms.

Now, consider the commutative diagram

$$\begin{array}{ccccccccc}
\Omega^n_{f,g_\lambda,\mathscr{E}'_\lambda} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{f_\lambda,g,\mathscr{E}'_\lambda} & \xrightarrow{T^d} & \Omega_{f_\lambda,g,\mathscr{E}'_\lambda} & \xrightarrow{T_{\mathscr{D}'_\lambda,\mathscr{E}'_\lambda}} & \Omega_{f,g_\lambda,\mathscr{D}'_\lambda} & \xrightarrow{T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda,\mathscr{D}'_\lambda}} & \Omega_{f,g_\lambda,\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda} \\
\downarrow{\scriptstyle T_{\mathscr{E}_\lambda,\mathscr{E}'_\lambda}} & & \downarrow{\scriptstyle T_{\mathscr{E}_\lambda,\mathscr{E}'_\lambda}} & & \downarrow{\scriptstyle T_{\mathscr{E}_\lambda,\mathscr{E}'_\lambda}} & & \downarrow{\scriptstyle T_{\mathscr{D}_\lambda,\mathscr{D}'_\lambda}} & & \downarrow{\scriptstyle T_{\mathscr{X}_\lambda\times_{S_\lambda}\mathscr{X}_\lambda}} \\
\Omega^n_{f,g_\lambda,\mathscr{E}_\lambda} & \xleftarrow{(T^n)^{-1}} & \Omega^d_{f_\lambda,g,\mathscr{E}_\lambda} & \xrightarrow{T^d} & \Omega_{f_\lambda,g,\mathscr{E}_\lambda} & \xrightarrow{T_{\mathscr{D}_\lambda,\mathscr{E}_\lambda}} & \Omega_{f,g_\lambda,\mathscr{D}_\lambda} & \xrightarrow{T_{\mathscr{D}_\lambda}} & \Omega_{f,g_\lambda}
\end{array}$$

of functors. Here, the arrows are defined by the $\mathscr{S}$-diagram versions of (4.2.2), (4.2.9), and (4.2.11). We have shown that the upper horizontal arrows and the right side vertical arrow are isomorphisms, and the other vertical arrows are also isomorphisms by (4.2.2) and (4.2.11). The lower horizontal arrows are isomorphisms. In particular, the natural transformation

$$\Omega^n_{f,g_\lambda,\mathscr{E}_\lambda} \xleftarrow{(T^n)^{-1}} \Omega^d_{f_\lambda,g,\mathscr{E}_\lambda}$$

is an isomorphism. Let T^n denote its inverse.

10.6.8. Under the notations and hypotheses of (10.6.7), as in (4.2.3), we have several exchange transformations (or inverse exchange transformations) as follows.

(1) We put $\Omega_{f,g,\lambda} = a'^! \lambda_* p^*_{2\lambda'}$. Then we have the natural transformations

$$\lambda_*\Omega_{f,g_\lambda} \xrightarrow{Ex} \Omega_{f,g,\lambda} \xrightarrow{Ex} \Omega_{f,g}\lambda_*$$

given by

$$\lambda_* a'^!_\lambda p'^*_{2\lambda} \xrightarrow{Ex} a'^! \lambda_* p'^*_{2\lambda} \xrightarrow{Ex^{-1}} a'^! p'^*_2 \lambda_*.$$

Here, the first arrow is an isomorphism by (9.2.7), and the second arrow is defined and an isomorphism by (9.2.5) since p'_2 is Cartesian exact log smooth.

(2) We put $\Omega_{f,g,\mathscr{D},\lambda} = u_{0*} b^! \lambda_* q^*_{2\lambda}$. Then we have the natural transformations

$$\lambda_*\Omega_{f,g_\lambda,\mathscr{D}_\lambda} \xrightarrow{Ex} \Omega_{f,g,\mathscr{D},\lambda} \xleftarrow{Ex^{-1}} \Omega_{f,g,\mathscr{D}}\lambda_*$$

given by

$$\lambda_* u_{0\lambda*} b^!_\lambda q^*_{2\lambda} \xrightarrow{\sim} u_{0*}\lambda_* b^!_\lambda q^*_{2\lambda} \xrightarrow{Ex} u_{0*} b^! \lambda_* q^*_{2\lambda} \xleftarrow{Ex} u_{0*} b^! q^*_2 \lambda_*.$$

(3) We have the *inverse* exchange transformation

$$\Omega_{f,g,\mathscr{E}}\lambda_* \xrightarrow{Ex^{-1}} \lambda_*\Omega_{f,g_\lambda,\mathscr{E}_\lambda}$$

given by

$$u_{0*} c^! r^*_2 \lambda_* \xrightarrow{Ex} u_{0*} c^! \lambda_* r^*_{2\lambda} \xrightarrow{Ex^{-1}} u_{0*} \lambda_* c^!_\lambda r^*_{2\lambda} \xrightarrow{\sim} \lambda_* u_{0\lambda*} c^!_\lambda r^*_{2\lambda}.$$

Here, the second arrow is defined and an isomorphism by (9.2.9) since c is a Cartesian strict closed immersion.

(4) We have the *inverse* exchange transformation

$$\Omega^d_{f,g,\mathscr{E}}\lambda_* \xrightarrow{Ex^{-1}} \lambda_*\Omega^d_{f,g_\lambda,\mathscr{E}_\lambda}$$

given by

$$u_{0*}\phi_* d^! s_2^* \pi^* \lambda_* \xrightarrow{Ex} u_{0*}\phi_* d^! s_2^* \lambda_* \pi_\lambda^* \xrightarrow{Ex} u_{0*}\phi_* d^! \lambda_* s_{2\lambda}^* \pi_\lambda^*$$
$$\xrightarrow{Ex^{-1}} u_{0*}\phi_*\lambda_* d_\lambda^! s_{2\lambda}^* \pi_\lambda^* \xrightarrow{\sim} \lambda_* u_{0\lambda*}\phi_{\lambda*} d_\lambda^! s_{2\lambda}^* \pi_\lambda^*.$$

Here, the third arrow is defined and an isomorphism by (9.2.9) since d is a Cartesian strict closed immersion.

(5) We have the *inverse* exchange transformation

$$\Omega^n_{f,g,\mathscr{E}}\lambda_* \xrightarrow{Ex^{-1}} \lambda_*\Omega^n_{f,g_\lambda,\mathscr{E}_\lambda}$$

given by

$$u_{0*}e^! t_2^* \lambda_* \xrightarrow{Ex} u_{0*}e^! \lambda_* t_{2\lambda}^* \xrightarrow{Ex^{-1}} u_{0*}\lambda_* e_\lambda^! t_{2\lambda}^* \xrightarrow{\sim} \lambda_* u_{0\lambda*} e_\lambda^! t_{2\lambda}^*.$$

Here, the second arrow is defined and an isomorphism by (9.2.9) since e is a Cartesian strict closed immersion.

(6) We have the *inverse* exchange transformation

$$\Omega^n_{f,g}\lambda_* \xrightarrow{Ex^{-1}} \lambda_*\Omega^n_{f,g_\lambda}$$

given by

$$e'^! t_2'^* \lambda_* \xrightarrow{Ex} e'^! \lambda_* t_{2\lambda}'^* \xrightarrow{Ex^{-1}} \lambda_* e_\lambda'^! t_{2\lambda}'^*.$$

Here, the first arrow is an isomorphism by (9.2.5) since t_2' is Cartesian exact log smooth, and the second arrow is defined and an isomorphism by (9.2.9) since e' is a Cartesian strict closed immersion. Thus the composition is also an isomorphism.

(7) We have the exchange transformation

$$\lambda^*\Omega^d_{f,g,\mathscr{E}} \xrightarrow{Ex} \Omega^d_{f,g_\lambda,\mathscr{E}_\lambda}\lambda^*$$

given by

$$\lambda^* u_{0*}\phi_* d^! s_2^* \pi^* \xrightarrow{Ex} u_{0\lambda*}\lambda^*\phi_* d^! s_2^* \pi^* \xrightarrow{Ex} u_{0\lambda*}\phi_{\lambda*}\lambda^* d^! s_2^* \pi^*$$
$$\xrightarrow{Ex} u_{0\lambda*}\phi_{\lambda*} d_\lambda^! \lambda^* s_2^* \pi^* \xrightarrow{\sim} u_{0\lambda*}\phi_{\lambda*} d_\lambda^! s_{2\lambda}^* \pi_\lambda^* \lambda^*.$$

Here, the first and second arrows are isomorphisms by (9.2.8), and the third arrow is defined and an isomorphism by (9.2.11) since d is a Cartesian strict closed immersion. Thus the composition is an isomorphism.

(8) We have the exchange transformation

$$\lambda^*\Omega^n_{f,g,\mathscr{E}} \xrightarrow{Ex} \Omega^n_{f,g_\lambda,\mathscr{E}_\lambda}\lambda^*$$

given by

$$\lambda^* u_{0*} e^! t_2^* \pi^* \xrightarrow{Ex} u_{0\lambda*}\lambda^* e^! t_2^* \pi^* \xrightarrow{Ex} u_{0\lambda*} e_\lambda^! \lambda^* t_2^* \pi^* \xrightarrow{\sim} w_{\lambda*} e_\lambda^! t_{2\lambda}^* \pi_\lambda^* \lambda^*.$$

Here, the first arrow is an isomorphism by (9.2.8), and the second arrow is defined and an isomorphism by (9.2.9) since e is a Cartesian strict closed immersion. Thus the composition is an isomorphism.

We also have the natural transformation

$$\Omega_{f,g,\mathscr{D},\lambda} \xrightarrow{T_{\mathscr{D}}} \Omega_{f,g,\lambda}$$

given by

$$u_{0*} b^! \lambda_* q_{2\lambda}^* \xrightarrow{Ex} a'^! u_* \lambda_* q_{2\lambda}^* \xrightarrow{\sim} a'^! \lambda_* u_{\lambda*} u_\lambda^* p_{2\lambda}^* \xrightarrow{ad^{-1}} a'^! \lambda_* p_{2\lambda}^*.$$

Here, the third arrow is defined and an isomorphism since $\mathscr{T}$ satisfies strict étale descent.

10.6.9. Under the notations and hypotheses of (10.6.8), for $\lambda \in I$, we have the commutative diagram

$$\begin{array}{ccc}
\lambda^*\Omega^n_{f,g,\mathscr{E}} & \xrightarrow{(T^n)^{-1}} & \lambda^*\Omega^d_{f,g,\mathscr{E}} \\
\downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
\Omega^n_{f,g_\lambda,\mathscr{E}_\lambda}\lambda^* & \xrightarrow{(T^n)^{-1}} & \Omega^d_{f,g_\lambda,\mathscr{E}_\lambda}\lambda^*
\end{array}$$

of functors. By (loc. cit), the vertical arrows are isomorphisms, and by (10.6.7), the lower horizontal arrow is an isomorphism. Thus the upper horizontal arrow is also an isomorphism. Then by (PD–4), the natural transformation

$$\Omega^n_{f,g,\mathscr{E}} \xleftarrow{(T^n)^{-1}} \Omega^d_{f,g,\mathscr{E}}$$

is an isomorphism. Let T^n denote its inverse.

Now, consider the commutative diagram

$$\begin{array}{ccccccccccc}
\lambda_*\Omega^n_{f,g_\lambda} & \xrightarrow{T^{n'}} & \lambda_*\Omega^n_{f,g_\lambda,\mathscr{E}_\lambda} & \xrightarrow{T^n} & \lambda_*\Omega^d_{f,g_\lambda,\mathscr{E}_\lambda} & \xrightarrow{T^d} & \lambda_*\Omega_{f,g_\lambda,\mathscr{E}_\lambda} & \xrightarrow{T_{\mathscr{D}_\lambda,\mathscr{E}_\lambda}} & \lambda_*\Omega_{f,g_\lambda,\mathscr{D}_\lambda} & \xrightarrow{T_{\mathscr{D}_\lambda}} & \lambda_*\Omega_{f,g_\lambda} \\
\uparrow{\scriptstyle Ex^{-1}} & & \uparrow{\scriptstyle Ex^{-1}} & & \uparrow{\scriptstyle Ex^{-1}} & & \uparrow{\scriptstyle Ex^{-1}} & & \downarrow{\scriptstyle Ex} & & \downarrow{\scriptstyle Ex} \\
 & & & & & & & & \Omega_{f,g,\lambda} & \xrightarrow{T_{\mathscr{D}}} & \Omega_{f,g,\lambda} \\
 & & & & & & & & \uparrow{\scriptstyle Ex^{-1}} & & \downarrow{\scriptstyle Ex} \\
\Omega^n_{f,g}\lambda_* & \xrightarrow{T^{n'}} & \Omega^n_{f,g,\mathscr{E}}\lambda_* & \xrightarrow{T^n} & \Omega^d_{f,g,\mathscr{E}}\lambda_* & \xrightarrow{T^d} & \Omega_{f,g,\mathscr{E}}\lambda_* & \xrightarrow{T_{\mathscr{D},\mathscr{E}}} & \Omega_{f,g,\mathscr{D}}\lambda_* & \xrightarrow{T_{\mathscr{D}}} & \Omega_{f,g}\lambda_*
\end{array}$$

of functors. Here, the arrows are constructed in (10.6.8), (10.6.5), and the $\mathscr{S}$-diagram version of (4.2.2). The top horizontal arrows are isomorphisms by (10.6.5) and (10.6.8), and we have shown in (loc. cit) that the left side vertical and the right side vertical arrows are isomorphisms. Thus the composition of the five lower horizontal arrows

$$\Omega^n_{f,g}\lambda_* \longrightarrow \Omega_{f,g}\lambda_*$$

is an isomorphism. Then its left adjoint

$$\lambda^*\Sigma_{f,g} \longrightarrow \Sigma^n_{f,g}\lambda^*$$

is also an isomorphism where

$$\Sigma_{f,g} = p'_{2\sharp}a'_*, \quad \Sigma^n_{f,g} = t'_{2\sharp}e'_*.$$

We also denote by $T^n_{\mathscr{E}}$ the composition

$$\Omega^n_{f,g} \xrightarrow{T^{n'}} \Omega^n_{f,g,\mathscr{E}} \xrightarrow{T^n} \Omega^d_{f,g,\mathscr{E}} \xrightarrow{T^d} \Omega_{f,g,\mathscr{E}} \xrightarrow{T_{\mathscr{D},\mathscr{E}}} \Omega_{f,g,\mathscr{D}} \xrightarrow{T_{\mathscr{D}}} \Omega_{f,g}.$$

It is called again a *transition transformation.* Then its left adjoint

$$\Sigma_{f,g} \longrightarrow \Sigma^n_{f,g}$$

is an isomorphism by (PD–4) and the above paragraph. Therefore, we have proven the following theorem.

Theorem 10.6.10. *Under the notations and hypotheses of (10.6.8), the transition transformation*

$$\Omega^n_{f,g} \xrightarrow{T^n_{\mathscr{E}}} \Omega_{f,g}$$

is an isomorphism.

10.6.11. Under the notations and hypotheses of (10.6.8), we put

$$\Omega_{f,X\times_S\mathscr{X}} = h_*\Omega_{f,g}h^*, \quad \Omega^n_f = h_*\Omega^n_{f,g}h^*.$$

Then the natural transformation

$$\Omega^n_f = h_*\Omega^n_{f,g}h^* \xrightarrow{T^n_{\mathscr{E}}} h_*\Omega_{f,g}h^* = \Omega_{f,X\times_S\mathscr{X}}$$

is an isomorphism by (10.6.10). We also have the natural transformation

$$T_{X\times_S\mathscr{X}} : \Omega_{f,X\times_S\mathscr{X}} \longrightarrow \Omega_f$$

given by

$$h_*a'^!p'^*_2h^* \xrightarrow{Ex} a^!h'_*p'^*_2h^* \xrightarrow{\sim} a^!h'_*h'^*p^*_2 \xrightarrow{ad^{-1}} a^!p^*_2.$$

Here, the first arrow is an isomorphism by (9.2.9), and the third arrow is defined and an isomorphism since $\mathscr{T}$ satisfies strict étale descent. Thus the composition is also an isomorphism.

Now, consider the natural transformations

$$\Omega_f^n \xrightarrow{T_{\mathscr{E}}^n} \Omega_{f,X\times_S\mathscr{X}} \xrightarrow{T_{X\times_S\mathscr{X}}} \Omega_f.$$

The composition is also denoted by $T_{\mathscr{E}}^n$. It is an isomorphism by (10.6.10) and the above paragraph.

Then consider the natural transformations

$$\Omega_f^n f^! \xrightarrow{T_{\mathscr{E}}^n} \Omega_f f^! \xrightarrow{\mathfrak{q}_f} f^*.$$

The composition is denoted by $\mathfrak{q}_{f,\mathscr{E}}^n$. By (10.5.5) and the above paragraph, we have proven the following theorem.

Theorem 10.6.12. *Under the notations and hypotheses of (10.6.5), the natural transformation*

$$\Omega_f^n f^! \xrightarrow{\mathfrak{q}_{f,\mathscr{E}}^n} f^*.$$

is an isomorphism.

10.7 Canonical version of purity transformations

10.7.1. Assume that $\mathscr{T}$ can be extended to an eSm-premotivic triangulated prederivator satisfying strict étale descent. Let $f : X \to S$ be a separated vertical exact log smooth morphism of $\mathscr{S}$-schemes. The category of localized compactified exactifications of the diagonal morphism $a : X \to X \times_S X$, denoted by $\mathcal{LCE}_a$, is the category whose object is the data of $v_\lambda : \mathscr{E}_i \to \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda$ and commutative diagram

$$\begin{array}{ccc} \mathscr{X}_\lambda & \xrightarrow{h_\lambda} & X \\ \downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle f} \\ S_\lambda & \xrightarrow{l_\lambda} & S \end{array}$$

for $\lambda \in I$ where

1. I is a set, and the diagram commutes,
2. f_λ and l_λ are strict étale,
3. v_i is a compactified exactification of the diagonal morphism $\mathscr{X}_\lambda \to \mathscr{X}_\lambda \times_{S_\lambda} \mathscr{X}_\lambda$.

Morphism is the data of

$$S'_\lambda \to S_\lambda, \quad \mathscr{X}'_\lambda \to \mathscr{X}_\lambda, \quad \mathscr{E}'_\lambda \to \mathscr{E}_\lambda$$

compatible with the morphisms in (10.6.4.1).

Then $\mathcal{LCE}_a$ is not empty by (10.6.4), and as in (10.1.7), it is connected since we can take the fiber products of $(\mathscr{X}_\lambda, S_\lambda, \mathscr{E}_\lambda)_{\lambda \in I}$ and $(\mathscr{X}'_\lambda, S'_\lambda, \mathscr{E}'_\lambda)_{\lambda \in I}$. For any object ω of $\mathcal{LCE}_a$, as in (10.6.10), we can associate the natural transformation

$$\Omega_f^n \xrightarrow{T_\omega^n} \Omega_f.$$

Then as in (4.2.13), we have the compatibility, i.e., this defines the functor

$$T^n : \mathcal{LCE}_a \to \mathrm{Hom}(\Omega_f^n, \Omega_f).$$

To make various natural transformations T_ω^n canonical, we take the limit

$$\varprojlim_{\omega} T_(\omega)^n.$$

It is denoted by $T^n : \Omega_f^n \to \Omega_f$. Now, the definition of the *purity transformation* is the composition

$$\Omega_f^n f^! \xrightarrow{T^n} \Omega_f f^! \xrightarrow{\mathfrak{q}_f} f^*,$$

and it is denoted by $\mathfrak{q}_f^n$. By (10.6.12), we have the following theorem.

Theorem 10.7.2. *Let $f : X \to S$ be a separated vertical exact log smooth morphism. Then the purity transformation*

$$\Omega_f^n f^! \xrightarrow{\mathfrak{q}_f^n} f^*.$$

is an isomorphism.

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