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Integer and Nonlinear Optimization

Exercise 8

Problem 1

Consider the assignment problem with budget constraints: n workers are to be assigned to n jobs with the cost training worker i for job j being t_{ij} and profit of worker i doing job j being c_{ij}. The total budget for training is b.

$$
\begin{array}{lrcll}
\max & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} & & & \\
\text{s.t.} & \sum_{j=1}^n x_{ij} & = & 1 & \forall i = 1, \dots, n \\
& \sum_{i=1}^n x_{ij} & = & 1 & \forall j = 1, \dots, n \\
& \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij} & \leq & b & \\
& x_{ij} & \in & \{0,1\} & \forall i = 1, \dots, n,\ j = 1, \dots, n
\end{array}
$$

Find the four different Lagrangian relaxations resulting from taking only restriction (2), only restriction (4), restrictions (2) and (3) or restrictions (2) and (4) into the objective. Identify the type of problem you obtain in each of the four cases.

Problem 2

Prove that $z_{LR}(\lambda)$ is piecewise linear on the domain over which it is finite (cf. Corollary 5.18).

Problem 3

Consider Example 5.12.:

$$
\begin{array}{lrcl}
\max & 7x_1 + 2x_2 & & \\
\text{s.t.} & -x_1 + 2x_2 & \leq & 4 \\
& 5x_1 + x_2 & \leq & 20 \\
& -2x_1 - 2x_2 & \leq & -7 \\
& -x_1 & \leq & -2 \\
& x_2 & \leq & 4 \\
& \underline{x} & \in & \mathbb{Z}_+^2
\end{array}
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\\a\end{array}}\right\} \quad Q = \{\underline{x} \in \mathbb{Z}_+^2 : A^2\underline{x} \leq b^2\}
$$

and the Lagrangian relaxation wrt. $-x_1 + 2x_2 \leq 4$.

1) View $z(\lambda, \underline{x}) = (\underline{c} - \lambda A^1)\underline{x} + \lambda b^1$ as an affine function of $\underline{x}$ for fixed λ. Then $z_{LR}(\lambda)$ can be determined by solving the linear program

$$z_{LR}(\lambda) = \max\{z(\lambda, \underline{x}) : \underline{x} \in \operatorname{conv}(Q)\}.$$

Determine $z_{LR}(\lambda)$ for $\lambda = 0, 1, \frac{1}{9}$ and graph $z_{LR}(\lambda)$ for all $\lambda \geq 0$.

2) Determine $z_{LR}(\lambda)$ by maximization over the finite set Q, that ist

$$z_{LR}(\lambda) = \max_{\underline{x}^i \in Q} z(\lambda, \underline{x}^i).$$

Then $z(\lambda, \underline{x}^i) = \underline{c}\,\underline{x}^i + \lambda(b^1 - A^1\underline{x}^1)$ is an affine function of λ for fixed $\underline{x}^i$. Determine $z(\lambda, \underline{x}^i)$ for all $\underline{x}^i \in Q$ and give a graphical representation of the solutions.

3) What is the value of z_{LD}?

Problem 4

Consider a 0-1 knapsack problem

$$\begin{aligned} z = \max 10y_1 + 4y_2 + 14y_3 \\ 3y_1 + y_2 + 4y_3 \leq 4 \\ y_i \in \{0, 1\}, i = 1, 2, 3. \end{aligned}$$

Construct a Lagrangian dual by dualizing the knapsack constraint. What is the optimal value of the dual variable λ?

Suppose one runs the subgradient algorithm (Algorithm 5.20) using step size (b) in Theorem 5.21, starting with $u^0 = 0, \mu_0 = 1$ and $p = \frac{1}{2}$. Show that the subgradient algorithm does not reach the optimal dual solution.