Lie algebras

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I Basic concepts

1 Algebras

(1.1) Non-associative algebras. a) Let $R \neq \{0\}$ be a commutative ring. An R-module $\mathfrak{A}$ together with an R-bilinear **multiplication** $\mu\colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, that is $\mu(a+b,c) = \mu(a,c) + \mu(b,c)$ and $\mu(a,b+c) = \mu(a,b) + \mu(a,c)$ and $\mu(ra,b) = r\mu(a,b) = \mu(a,rb)$, for all $a,b,c \in \mathfrak{A}$ and $r \in R$, is called a **non-associative** (that is not necessarily associative) R**-algebra**.

An R-submodule $\mathfrak{B} \leq_R \mathfrak{A}$ such that $\mu(\mathfrak{B},\mathfrak{B}) \subseteq \mathfrak{B}$ is called an R**-subalgebra**. Then $\mathfrak{B}$ becomes a non-associative R-algebra with respect to the restricted multiplication. If the multiplication in $\mathfrak{A}$ fulfills certain identities, for example those in (1.2) or (1.3), then these are automatically fulfilled in $\mathfrak{B}$ as well.

An R-submodule $\mathfrak{I} \leq_R \mathfrak{A}$ such that $\mu(\mathfrak{A},\mathfrak{I}) \subseteq \mathfrak{I}$ and $\mu(\mathfrak{I},\mathfrak{A}) \subseteq \mathfrak{I}$ is called an **ideal** of $\mathfrak{A}$; we write $\mathfrak{I} \trianglelefteq \mathfrak{A}$. Note that any ideal also is an R-subalgebra. We always have $\{0\} \trianglelefteq \mathfrak{A}$ and $\mathfrak{A} \trianglelefteq \mathfrak{A}$, and if $\mathfrak{I},\mathfrak{J} \trianglelefteq \mathfrak{A}$ are ideals, then so are their **sum** $\mathfrak{I}+\mathfrak{J} := \{x+y \in \mathfrak{A}; x \in \mathfrak{I}, y \in \mathfrak{J}\} \trianglelefteq \mathfrak{A}$ and their intersection $\mathfrak{I} \cap \mathfrak{J} \trianglelefteq \mathfrak{A}$.

Then the quotient R-module $\mathfrak{A}/\mathfrak{I} := \{a+\mathfrak{I} \subseteq \mathfrak{A}; a \in \mathfrak{A}\}$ consisting of the additive cosets of $\mathfrak{I}$ in $\mathfrak{A}$ becomes a non-associative R-algebra, being called the associated **quotient** R**-algebra**, with respect to the induced multiplication $\overline{\mu}\colon \mathfrak{A}/\mathfrak{I} \times \mathfrak{A}/\mathfrak{I} \to \mathfrak{A}/\mathfrak{I}$ defined by $\overline{\mu}(a+\mathfrak{I}, b+\mathfrak{I}) := \mu(a,b)+\mathfrak{I}$, for all $a,b \in \mathfrak{A}$: We have $\mu(a+c,b+d) = \mu(a,b)+\mu(a,d)+\mu(c,b)+\mu(c,d) \in \mu(a,b)+\mathfrak{I}$, for all $c,d \in \mathfrak{I}$, hence $\overline{\mu}$ is well-defined. If the multiplication in $\mathfrak{A}$ fulfills certain identities, for example those in (1.2) or (1.3), then these are automatically fulfilled in $\mathfrak{A}/\mathfrak{I}$.

b) If $\mathfrak{B}$ also is a non-associative R-algebra, then a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ of R-modules such that $\varphi(\mu_{\mathfrak{A}}(a,b)) = \mu_{\mathfrak{B}}(\varphi(a),\varphi(b))$, for all $a,b \in \mathfrak{A}$, is called a **homomorphism** of non-associative R-algebras; similarly, we have **monomorphisms**, **epimorphisms**, **isomorphisms**, **endomorphisms** and **automorphisms**. In particular, if φ is bijective, then $\varphi^{-1}\colon \mathfrak{B} \to \mathfrak{A}$ is a homomorphism of R-algebras as well; we write $\mathfrak{A} \cong \mathfrak{B}$.

The image $\mathrm{im}(\varphi) = \varphi(\mathfrak{A}) \subseteq \mathfrak{B}$ is an R-subalgebra. Moreover, the **kernel** $\ker(\varphi) := \{a \in \mathfrak{A}; \varphi(a) = 0\} \leq_R \mathfrak{A}$ of φ is an ideal of $\mathfrak{A}$: We have $\varphi(\mu_{\mathfrak{A}}(a,x)) = \mu_{\mathfrak{B}}(\varphi(a),\varphi(x)) = \mu_{\mathfrak{B}}(\varphi(a),0) = 0$ as well as $\varphi(\mu_{\mathfrak{A}}(x,a)) = \mu_{\mathfrak{B}}(\varphi(x),\varphi(a)) = \mu_{\mathfrak{B}}(0,\varphi(a)) = 0$, for all $x \in \ker(\varphi)$ and $a \in \mathfrak{A}$.

If $\mathfrak{I} \trianglelefteq \mathfrak{A}$ is an ideal, then we have the **natural** epimorphism $\nu_{\mathfrak{I}}\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}\colon a \mapsto a+\mathfrak{I}$, where $\ker(\nu_{\mathfrak{I}}) = \mathfrak{I}$. This leads to the **homomorphism principle**:

Assume that $\mathfrak{I} \subseteq \ker(\varphi)$. Then there is a unique homomorphism $\varphi^{\mathfrak{I}}\colon \mathfrak{A}/\mathfrak{I} \to \mathfrak{B}\colon a+\mathfrak{I} \mapsto \varphi(a)$ giving rise to a factorization $\varphi = \varphi^{\mathfrak{I}}\nu_{\mathfrak{I}}\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to \mathfrak{B}$. We have $\mathrm{im}(\varphi^{\mathfrak{I}}) = \mathrm{im}(\varphi) \subseteq \mathfrak{B}$ and $\ker(\varphi^{\mathfrak{I}}) = \ker(\varphi)/\mathfrak{I} = \{x+\mathfrak{I} \in \mathfrak{A}/\mathfrak{I}; x \in \ker(\varphi)\} \trianglelefteq \mathfrak{A}/\mathfrak{I}$. In particular, $\varphi^{\mathfrak{I}}$ is injective if and only if $\mathfrak{I} = \ker(\varphi)$, and we have an isomorphism $\overline{\varphi} := \varphi^{\ker(\varphi)}\colon \mathfrak{A}/\ker(\varphi) \to \mathrm{im}(\varphi)$.

c) Let $\mathfrak{A}$ be a finitely generated R-free R-module, and let $d := \mathrm{rk}_R(\mathfrak{A}) \in \mathbb{N}_0$. In particular, this happens if $R = K$ is a field, in which case $\mathfrak{A}$ is a K-vector space, and if $\mathfrak{A}$ is finitely generated as such, in which case we have $d = \dim_K(\mathfrak{A})$.

Now, we may choose an R-basis $\{c_1, \ldots, c_d\} \subseteq \mathfrak{A}$. Then μ is uniquely defined by the associated **structure constants** $\gamma_{ij}^k \in R$ given by $\mu(c_i, c_j) = \sum_{k=1}^d \gamma_{ij}^k c_k \in \mathfrak{A}$, for all $i, j \in \{1, \ldots, d\}$: Indeed, for elements $a = \sum_{i=1}^d \alpha_i c_i \in \mathfrak{A}$ and $b = \sum_{j=1}^d \beta_j c_j \in \mathfrak{A}$, where $\alpha_i, \beta_j \in R$, by R-bilinearity we have $\mu(a, b) = \sum_{i=1}^d \sum_{j=1}^d \alpha_i \beta_j \mu(c_i, c_j) = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d \alpha_i \beta_j \gamma_{ij}^k c_k \in \mathfrak{A}$.

(1.2) Associative algebras. a) Let $R \neq \{0\}$ be a commutative ring. A non-associative R-algebra $\mathfrak{A}$ is called **associative**, if associativity $\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$ holds, for all $a, b, c \in \mathfrak{A}$. In this case, we write $ab = a \cdot b := \mu(a, b)$, thus associativity becomes $(ab)c = a(bc)$.

An associative R-algebra $\mathfrak{A}$ is called **unital**, if there is a (necessarily unique) multiplicatively neutral element $1 \in \mathfrak{A}$, that is $1 \cdot a = a = a \cdot 1$, for all $a \in \mathfrak{A}$. An associative R-algebra $\mathfrak{A}$ is called **abelian** or **commutative**, if commutativity $ab = ba$ holds, for all $a, b \in \mathfrak{A}$.

b) If $\mathfrak{I}, \mathfrak{J} \trianglelefteq \mathfrak{A}$ are ideals, then apart from taking their sum and their intersection there is another available construction, being called their **product**: Let $\mathfrak{I}\mathfrak{J} := \langle xy \in \mathfrak{L}; x \in \mathfrak{I}, y \in \mathfrak{J} \rangle_R \leq_R \mathfrak{A}$. Then $a(xy) = (ax)y \in \mathfrak{I}\mathfrak{J}$ and $(xy)a = x(ya) \in \mathfrak{I}\mathfrak{J}$, for all $x \in \mathfrak{I}$, $y \in \mathfrak{J}$ and $a \in \mathfrak{A}$, shows that $\mathfrak{I}\mathfrak{J} \trianglelefteq \mathfrak{A}$ indeed is an ideal.

Example. Here are basic but important examples:

i) If V is an R-module, then $\mathrm{End}_R(V) = \mathrm{Hom}_R(V, V) := \{\alpha \colon V \to V \text{ } R\text{-linear}\}$ is a unital associative R-algebra, with respect to pointwise addition and R-scalar multiplication, that is $\alpha + \beta \colon V \to V \colon x \mapsto \alpha(x) + \beta(x)$ and $r\alpha \colon V \to V \colon x \mapsto r\alpha(x)$, for all $\alpha, \beta \in \mathrm{End}_R(V)$ and $r \in R$, and to composition of maps as multiplication, that is $\alpha\beta \colon V \to V \colon x \mapsto \alpha(\beta(x))$, for all $\alpha, \beta \in \mathrm{End}_R(V)$; the multiplicatively neutral element is given by $\mathrm{id}_V \in \mathrm{End}_R(V)$:

It is immediate that $\mathrm{End}_R(V)$ becomes an R-module. Using general properties of maps we have $(\alpha\beta)\gamma = \alpha(\beta\gamma) \colon V \to V \colon x \mapsto \alpha(\beta(\gamma(x)))$, for all $\alpha, \beta, \gamma \in \mathrm{End}_R(V)$; and we have $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma \colon V \to V \colon x \mapsto \alpha(\gamma(x)) + \beta(\gamma(x))$ and $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \colon V \to V \colon x \mapsto \alpha(\beta(x)) + \alpha(\gamma(x))$ and $r\alpha \cdot \beta = r \cdot \alpha\beta = \alpha \cdot r\beta \colon V \to V \colon x \mapsto r\alpha(\beta(x))$, for all $\alpha, \beta, \gamma \in \mathrm{End}_R(V)$ and $r \in R$.

ii) Similarly, for $n \in \mathbb{N}_0$, the set $R^{n \times n}$ of all $(n \times n)$-matrices with entries in R becomes a unital associative R-algebra, with respect to addition of matrices, R-scalar multiplication, and multiplication of matrices; the multiplicatively neutral element is given by the identity matrix $E_n \in R^{n \times n}$, and $R^{n \times n}$ is commutative if and only if $n \leq 1$:

Indeed, letting $R^{n \times 1}$ be the R-module of all column n-tuples with entries in R, then by using the standard R-basis of $R^{n \times 1}$, we get a natural R-module isomorphism $\mathrm{End}_R(R^{n \times 1}) \cong R^{n \times n}$, which translates composition of maps in

$\mathrm{End}_R(R^{n\times 1})$ into multiplication of matrices in $R^{n\times n}$. Thus $R^{n\times n}$ becomes an associative R-algebra, which is isomorphic to $\mathrm{End}_R(R^{n\times 1})$. Finally, we have

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\neq\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Letting $\{E_{11},\ldots,E_{nn}\}\subseteq R^{n\times n}$ be the standard R-basis, where, for all $i,j\in\{1,\ldots,n\}$, the entries of the matrix unit E_{ij} are given by the Kronecker function $\delta_{ij}\colon [k,l]\mapsto\begin{cases}1, & \text{if } k=i, l=j,\\ 0, & \text{otherwise.}\end{cases}$ Then we have $E_{ij}E_{kl}=\delta_{jk}E_{il}\in R^{n\times n}$, hence the associated structure constants are $\gamma_{ij,kl}^{st}=\delta_{jk}\cdot\delta_{is}\delta_{lt}\in\{0,1\}\subseteq R$.

(1.3) Lie algebras. a) Let $R\neq\{0\}$ be a commutative ring. A non-associative R-algebra $\mathfrak{L}$ whose multiplication fulfills the following properties is called a **Lie R-algebra**; in this case, we write $[xy]=[x,y]:=\mu(x,y)$, for all $x,y\in\mathfrak{L}$: **i)** We have $[x,x]=0$, for all $x\in\mathfrak{L}$; **ii)** we have the **Jacobi identity** $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$, for all $x,y,z\in\mathfrak{L}$.

Condition i) implies $0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]=[x,y]+[y,x]$, thus we have anti-commutativity $[y,x]=-[x,y]$, for all $x,y\in\mathfrak{L}$. Conversely, if $2\in R^*$, then for all $x\in\mathfrak{L}$ the latter condition entails $[x,x]=-[x,x]$, thus $2[x,x]=0$, hence we recover condition i).

For example, the zero multiplication given by $[x,y]:=0$, for all $x,y\in\mathfrak{L}$, is a Lie R-algebra structure on $\mathfrak{L}$; in this case $\mathfrak{L}$ is called **abelian** or **commutative**.

b) If $\mathfrak{I},\mathfrak{J}\trianglelefteq\mathfrak{L}$ are ideals, then apart from taking their sum and their intersection there is another available construction, being called their **product**: Let $[\mathfrak{I},\mathfrak{J}]:=\langle[x,y]\in\mathfrak{L}; x\in\mathfrak{I}, y\in\mathfrak{J}\rangle_R\leq_R\mathfrak{L}$. Then we have $[a,[x,y]]=-[x,[y,a]]-[y,[a,x]]=[x,[a,y]]+[[a,x],y]\in\mathfrak{I}\mathfrak{J}$, for all $x\in\mathfrak{I}$, $y\in\mathfrak{J}$ and $a\in\mathfrak{L}$, showing that $[\mathfrak{I},\mathfrak{J}]\trianglelefteq\mathfrak{L}$ is an ideal. Note that due to anti-commutativity an R-submodule $\mathfrak{I}\leq_R\mathfrak{L}$ is an ideal if and only if $\mu(\mathfrak{L},\mathfrak{I})\subseteq\mathfrak{I}$, if and only if $\mu(\mathfrak{L},\mathfrak{I})\subseteq\mathfrak{I}$.

Example. Interesting examples are constructed from associative algebras:

Let $\mathfrak{A}$ be an associative R-algebra, and let a multiplication be defined by the **commutator** or **Lie bracket** $[x,y]:=xy-yx$, for all $x,y\in\mathfrak{A}$. Then $[\cdot,\cdot]\colon\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ is R-bilinear; we have $[x,x]:=xx-yx=0$, for all $x,y\in\mathfrak{A}$; and using associativity we have the Jacobi identity $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=[x,yz-zy]+[y,zx-xz]+[z,xy-yx]=\big(x(yz-zy)-(yz-zy)x\big)+\big(y(zx-xz)-(zx-xz)y\big)+\big(z(xy-yx)-(xy-yx)z\big)=(xyz-xzy-yzx+zyx)+(yzx-yxz-zxy+xzy)+(zxy-zyx-xyz+yxz)=0$, for all $x,y,z\in\mathfrak{A}$. Thus $\mathfrak{A}$ is a Lie R-algebra with respect to $[\cdot,\cdot]$, being called **the Lie algebra** of $\mathfrak{A}$.

In particular, continuing the examples in (1.2), for any R-module V, the Lie R-algebra of the associative R-algebra $\mathrm{End}_R(V)$ it is denoted by $\mathfrak{gl}(V)$. (The notation is reminiscent of the close relationship of $\mathfrak{gl}(V)$ to the general linear group $\mathrm{GL}(V)$; similarly, the Lie algebras exhibited in (2.4) are related to various groups occurring in geometric algebra.)

Similarly, for $n \in \mathbb{N}_0$ the Lie R-algebra of the associative R-algebra $R^{n\times n}$ is denoted by $\mathfrak{gl}_n(R)$; it is isomorphic to $\mathfrak{gl}(R^{n\times 1})$, and being called the associated **general linear** Lie algebra. In terms of the standard R-basis $\{E_{11}, \ldots, E_{nn}\} \subseteq R^{n\times n}$ we get $[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{il}E_{jk} \in R^{n\times n}$, hence the associated structure constants are given as $\gamma_{ij,kl}^{st} = \delta_{jk} \cdot \delta_{is}\delta_{lt} - \delta_{il} \cdot \delta_{js}\delta_{kt} \in \{0, \pm 1\} \subseteq R$. Any Lie R-subalgebra of $\mathfrak{gl}_n(R)$ is called a **linear** Lie algebra.

Theorem. [**Ado, 1935** for $\mathrm{char}(K) = 0$**; Iwasawa, 1948** for $\mathrm{char}(K) > 0$] Let K be a field. Then any finite-dimensional Lie K-Algebra $\mathfrak{L}$ is isomorphic to a linear Lie K-algebra.

Proof. Omitted; see [4, Ch.6.2, 6.3]. $\sharp$

2 Lie algebras

(2.1) Centralisers and normalisers. a) Let $R \neq \{0\}$ be a commutative ring, and let $\mathfrak{L}$ be a Lie R-algebra. Let $Z(\mathfrak{L}) := \{x \in \mathfrak{L}; [x, a] = 0 \text{ for all } a \in \mathfrak{L}\} \leq_R \mathfrak{L}$ be the **center** of $\mathfrak{L}$; in other words we have $Z(\mathfrak{L}) = \ker(\mathrm{ad}_{\mathfrak{L}})$. Hence in particular $Z(\mathfrak{L}) \trianglelefteq \mathfrak{L}$ is an ideal, which is also verified directly as follows: We have $[[a, x], b] = -[b, [a, x]] = [a, [x, b]] + [x, [b, a]] = 0$, for all $a, b \in \mathfrak{L}$ and $x \in Z(\mathfrak{L})$. We have $Z(\mathfrak{L}) = \mathfrak{L}$ if and only if $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) = \{0\}$, that is $\mathfrak{L}$ is commutative.

More generally, if $M \subseteq \mathfrak{L}$ is a subset, then let $C_{\mathfrak{L}}(M) := \{x \in \mathfrak{L}; [x, a] = 0 \text{ for all } a \in M\} \leq_R \mathfrak{L}$ be the **centraliser** of M in $\mathfrak{L}$. Then $C_{\mathfrak{L}}(M)$ is a Lie R-subalgebra of $\mathfrak{L}$: We have $[[x, y], a] = -[a, [x, y]] = [x, [y, a]] + [y, [a, x]] = [x, [y, a]] - [y, [x, a]] = 0$, for all $a \in M$ and $x, y \in C_{\mathfrak{L}}(M)$. Note that $C_{\mathfrak{L}}(M) = C_{\mathfrak{L}}(\langle M \rangle_R)$, and in particular we have $Z(\mathfrak{L}) = C_{\mathfrak{L}}(\mathfrak{L})$.

b) Similarly, if $M \leq_R \mathfrak{L}$ is an R-submodule, then let $N_{\mathfrak{L}}(M) := \{x \in \mathfrak{L}; [x, a] \in M \text{ for all } a \in M\} \leq_R \mathfrak{L}$ be the **normaliser** of M in $\mathfrak{L}$. Then $N_{\mathfrak{L}}(M)$ is a Lie R-subalgebra of $\mathfrak{L}$: We have $[[x, y], a] = -[a, [x, y]] = [x, [y, a]] + [y, [a, x]] = [x, [y, a]] - [y, [x, a]] \in M$, for all $a \in M$ and $x, y \in N_{\mathfrak{L}}(M)$.

Then $C_{\mathfrak{L}}(M) \leq_R N_{\mathfrak{L}}(M)$, where indeed $C_{\mathfrak{L}}(M) \trianglelefteq N_{\mathfrak{L}}(M)$ is an ideal: We have $[x, a] = 0$ and $[y, a] \in M$, for all $x \in C_{\mathfrak{L}}(M)$, $y \in N_{\mathfrak{L}}(M)$ and $a \in M$, hence $[[y, x], a] = -[a, [y, x]] = [y, [x, a]] + [x, [a, y]] = 0$ shows that $[y, x] \in C_{\mathfrak{L}}(M)$.

If $\mathfrak{K} \subseteq \mathfrak{L}$ is a Lie R-subalgebra, then we have $\mathfrak{K} \subseteq N_{\mathfrak{L}}(\mathfrak{K})$, hence $\mathfrak{K} \trianglelefteq N_{\mathfrak{L}}(\mathfrak{K})$, and since any Lie R-subalgebra of $\mathfrak{M} \subseteq \mathfrak{L}$ such that $\mathfrak{K} \trianglelefteq \mathfrak{M}$ is contained in $N_{\mathfrak{L}}(\mathfrak{K})$ we conclude that $N_{\mathfrak{L}}(\mathfrak{K})$ is the largest Lie R-subalgebra of $\mathfrak{L}$ containing $\mathfrak{K}$ as an ideal. In particular, if $\mathfrak{K} = N_{\mathfrak{L}}(\mathfrak{K})$ then $\mathfrak{K}$ is called **self-normalising**.

Example. For $n \in \mathbb{N}$ we have $Z(\mathfrak{gl}_n(R)) = \mathfrak{z}_n(R) := R \cdot E_n$, the set of all **scalar** matrices; note that $\mathfrak{z}_n(R)$ is an associative R-subalgebra of $R^{n\times n}$:

We may assume that $n \geq 2$. We have $[\mathfrak{z}_n(R), \mathfrak{gl}_n(R)] = \{0\}$, hence $\mathfrak{z}_n(R) \subseteq Z(\mathfrak{gl}_n(R))$. Conversely, let $A = [a_{ij}]_{ij} \in Z(\mathfrak{gl}_n(R))$, and let $k \neq l \in \{1, \ldots, n\}$. Hence we get $E_{kl}A = AE_{kl}$, whose left hand side has non-zero entries in row

k only, these being $[a_{l1}, a_{l2}, \ldots, a_{ln}] \in R^n$, while whose right hand side has non-zero entries in column l only, these being $[a_{1k}, a_{2k}, \ldots, a_{nk}]^{\mathrm{tr}} \in R^{n\times 1}$. This shows that $a_{ij} = 0$ whenever $i \neq j$, and $a_{kk} = a_{ll}$, implying that $A \in \mathfrak{z}_n(R)$. $\sharp$

(2.2) Special linear Lie algebras. Let K be a field, and let $V := K^{n\times 1}$ for some $n \in \mathbb{N}$. Then let $\mathfrak{sl}(V) := \{A \in \mathfrak{gl}(V); \mathrm{Tr}(A) = 0\}$; recall that the trace map $\mathrm{Tr}\colon \mathfrak{gl}(V) \to K$ is independent of a choice of a K-basis of V. Since Tr is K-linear, and $\mathrm{Tr}([A, B]) = \mathrm{Tr}(AB - BA) = 0$, for all $A, B \in \mathfrak{gl}(V)$, we conclude that $\mathfrak{sl}(V) \trianglelefteq \mathfrak{gl}(V)$ is an ideal; in particular it is a Lie K-subalgebra, being called the associated **special linear** Lie K-algebra.

Identifying $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(K)$, and letting $\{E_{11}, \ldots, E_{nn}\} \subseteq \mathfrak{gl}_n(K)$ be the standard K-basis, we get the **standard** K-basis

$$\big\{E_{ii} - E_{i+1,i+1}; i \in \{1, \ldots, n-1\}\big\} \,\dot{\cup}\, \big\{E_{ij}; i \neq j \in \{1, \ldots, n\}\big\} \subseteq \mathfrak{sl}_n(K),$$

having cardinality $(n-1)+n(n-1) = n^2-1 = \dim_K(\ker(\mathrm{Tr})) = \dim_K(\mathfrak{sl}_n(K))$.

Lemma. We have $Z(\mathfrak{sl}_n(K)) = \mathfrak{z}_n(K) \cap \mathfrak{sl}_n(K)$; hence $Z(\mathfrak{sl}_n(K)) = \{0\}$ if $\mathrm{char}(K) \nmid n$, and $Z(\mathfrak{sl}_n(K)) = \mathfrak{z}_n(K)$ if $\mathrm{char}(K) \mid n$:

Proof. We may assume that $n \geq 2$. We have $\mathfrak{z}_n(K) = Z(\mathfrak{gl}_n(K))$, hence $\mathfrak{z}_n(K) \cap \mathfrak{sl}_n(K) \subseteq Z(\mathfrak{sl}_n(K))$. Conversely, let $A = [a_{ij}]_{ij} \in Z(\mathfrak{sl}_n(K))$, and let $k \neq l \in \{1, \ldots, n\}$, thus $E_{kl} \in \mathfrak{sl}_n(K)$. Hence we get $E_{kl}A = AE_{kl}$, which as in the case of the general linear Lie algebra implies that $A \in \mathfrak{z}_n(K)$. $\sharp$

A finite-dimensional Lie K-algebra which is non-commutative and does not have any proper non-zero ideals, is called **simple**. We will show later that if $\mathrm{char}(K) = 0$ then $\mathfrak{sl}_n(K)$ is simple for $n \geq 2$. For the time being, we are content with the following example:

Example: The special linear algebra of degree 2. Let K be a field and $\mathfrak{L} := \mathfrak{sl}_2(K)$, and let $\{E, H, F\} \subseteq \mathfrak{L}$ be the standard K-basis, that is $E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then we have

$$\begin{aligned}
[E, F] &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= H, \\
[H, E] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} &= 2E, \\
[H, F] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} &= -2F.
\end{aligned}$$

If $\mathrm{char}(K) \neq 2$ then $\mathfrak{L}$ is simple: Let $\{0\} \neq \mathfrak{I} \trianglelefteq \mathfrak{L}$ and let $0 \neq A := aE + cH + bF \in \mathfrak{I}$, for some $a, b, c \in K$. If $a \neq 0$ then we get $[F, [F, A]] = [F, -aH + 2cF] =$

$-2aF \in \mathfrak{I}$, hence $F \in \mathfrak{I}$, thus $H = [E, F] \in \mathfrak{I}$, and $2E = [H, E] \in \mathfrak{I}$. If $b \neq 0$ then we get $[E, [E, A]] = [E, -2cE + bH] = -2bE$, hence $E \in \mathfrak{I}$, thus $H = [E, F] \in \mathfrak{I}$, and $-2F = [H, F] \in \mathfrak{I}$. If both $a = 0 = b$, then $cH \in \mathfrak{I}$, hence $H \in \mathfrak{I}$, thus $2E = [H, E] \in \mathfrak{I}$ and $-2F = [H, F] \in \mathfrak{I}$. Hence we have $\mathfrak{I} = \mathfrak{L}$.

If $\mathrm{char}(K) = 2$ then we have $[\mathfrak{L}, \mathfrak{L}] = Z(\mathfrak{L}) = \langle H \rangle_K$: We have $[E, F] = H$ and $[H, E] = 0 = [H, F]$, hence $[\mathfrak{L}, \mathfrak{L}] = \langle H \rangle_K$. Moreover, this implies $H \in Z(\mathfrak{L})$. Conversely, it suffices to consider $A := aE + bF \in Z(\mathfrak{L}) \cap \langle E, F \rangle_K$, for $a, b \in K$, then $0 = [E, A] = \beta H$ and $0 = [F, A] = aH$ implies that both $a = 0 = b$. ♯

(2.3) Classical Lie algebras. Let K be a field such that $\mathrm{char}(K) \neq 2$. Moreover, let $\alpha\colon K \to K$ be a field automorphism such that $\alpha^2 = \mathrm{id}_K$, and let $K' := \mathrm{Fix}_K(\alpha) \subseteq K$ be the fixed field of α. For example, we always have $\alpha := \mathrm{id}_K$, in which case $K' = K$; and we have complex conjugation $\alpha := \bar{\ }\colon \mathbb{C} \to \mathbb{C}$, in which case $\mathbb{C}' = \mathbb{R}$. Finally, let $V := K^{n \times 1}$ for some $n \in \mathbb{N}$, and let $\Phi\colon V \times V \to K$ be a non-degenerate K-sesquilinear form.

Then let $\mathfrak{L}(\Phi) := \{A \in \mathfrak{gl}(V); A + A^* = 0\}$, where $A^* \in \mathfrak{gl}(V)$ denotes the adjoint map of $A \in \mathfrak{gl}(V)$ with respect to Φ. Then, since $\mathfrak{gl}(V) \to \mathfrak{gl}(V)\colon A \mapsto A^*$ is K'-linear, from $[A, B]^* = (AB - BA)^* = B^*A^* - A^*B^* = (-B)(-A) - (-A)(-B) = BA - AB = -[A, B]$, for all $A, B \in \mathfrak{L}(\Phi)$, we conclude that $\mathfrak{L}(\Phi)$ is a Lie K'-subalgebra of $\mathfrak{gl}(V)$, called the **classical** Lie K'-algebra associated with Φ.

In the sequel we borrow some facts from geometric algebra, for which we refer to [5, 11]. In particular, recall that if $\alpha = \mathrm{id}_K$ and Φ is symmetric then $\varphi\colon V \to K\colon v \mapsto \frac{1}{2}\Phi(v, v)$ is the associated quadratic form; conversely, given φ then Φ is recovered by polarisation $\Phi(v, w) = \varphi(v + w) - \varphi(v) - \varphi(w)$, for all $v, w \in V$.

(2.4) Orthogonal and symplectic Lie algebras. We keep the setting of (2.3), and let $\mathrm{char}(K) \neq 2$ and $\alpha = \mathrm{id}_K$.

a) Let Φ be skew-symmetric. Then we have $n = 2l$, for some $l \in \mathbb{N}$, and V is an orthogonal direct sum of hyperbolic planes. Thus, adjusting indices suitably, we may let $\{e_1, e_2, \ldots, e_l, e_{-1}, e_{-2}, \ldots, e_{-l}\} \subseteq V$ be the standard K-basis, such that the K-subspaces $\langle e_i, e_{-i} \rangle_K \leq_K V$, for all $i \in \{1, \ldots, l\}$, form mutually orthogonal hyperbolic planes. Hence the associated Gram matrix of Φ equals

$$J := \left[\begin{array}{c|c} 0 & E_l \\ \hline -E_l & 0 \end{array}\right] \in K^{2l \times 2l}.$$

Then we have $A^* = (JAJ^{-1})^{\mathrm{tr}} = -JA^{\mathrm{tr}}J$, thus the associated Lie K-algebra becomes $\mathfrak{sp}_{2l}(K) := \{A \in \mathfrak{gl}_{2l}(K); JA^{\mathrm{tr}}J = A\}$; it is called the associated **symplectic** Lie algebra. From $\mathrm{Tr}(A) = \mathrm{Tr}(JA^{\mathrm{tr}}J) = \mathrm{Tr}(J^2A^{\mathrm{tr}}) = -\mathrm{Tr}(A)$, for all $A \in \mathfrak{sp}_{2l}(K)$, we conclude $\mathfrak{sp}_{2l}(K) \subseteq \mathfrak{sl}_{2l}(K)$.

Writing $A := \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \in \mathfrak{gl}_{2l}(K)$, where $A_{ij} \in K^{l \times l}$, for $i, j \in \{1, 2\}$, we have $A \in \mathfrak{sp}_{2l}(K)$ if and only if

$$\left[\begin{array}{c|c} 0 & -E_l \\ \hline E_l & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}^{\mathrm{tr}} & A_{21}^{\mathrm{tr}} \\ \hline A_{12}^{\mathrm{tr}} & A_{22}^{\mathrm{tr}} \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & E_n \\ \hline -E_n & 0 \end{array}\right],$$

that is

$$\left[\begin{array}{c|c} -A_{21} & -A_{22} \\ \hline A_{11} & A_{12} \end{array}\right] = \left[\begin{array}{c|c} -A_{21}^{\mathrm{tr}} & A_{11}^{\mathrm{tr}} \\ \hline -A_{22}^{\mathrm{tr}} & A_{12}^{\mathrm{tr}} \end{array}\right],$$

in other words if and only if $A_{22} = -A_{11}^{\mathrm{tr}}$ and $A_{12}^{\mathrm{tr}} = A_{12}$ and $A_{21}^{\mathrm{tr}} = A_{21}$. Hence we get the following **standard** K-basis of $\mathfrak{sp}_{2l}(K)$

$$\begin{array}{rl} & \{E_{ij} - E_{-j,-i}; i,j \in \{1,\ldots,l\}\} \\ \dot\cup & \{E_{i,-j} + E_{-j,i}; i \neq j \in \{1,\ldots,l\}\} \\ \dot\cup & \{E_{i,-i}; i \in \{\pm 1,\ldots,\pm l\}\}; \end{array}$$

in particular we have $\dim_K(\mathfrak{sp}_{2l}(K)) = l^2 + l(l-1) + 2l = 2l^2 + l = \frac{1}{2}n(n+1)$.

b) Let Φ be symmetric of maximal Witt index, that is the maximal isotropic K-subspaces of V have K-dimension $\lfloor \frac{n}{2} \rfloor$. Assume first that $n = 2l$, for some $l \in \mathbb{N}$, hence Φ has Witt index l, and V is an orthogonal direct sum of hyperbolic planes. We let $\{e_1, e_2, \ldots, e_l, e_{-1}, e_{-2}, \ldots, e_{-l}\} \subseteq V$ be the standard K-basis, where the K-subspaces $\langle e_i, e_{-i}\rangle_K \leq_K V$, for all $i \in \{1,\ldots,l\}$, form mutually orthogonal hyperbolic planes; in other words, the quadratic form associated with Φ is given as $\varphi(x_1,\ldots,x_{-l}) := \sum_{i=1}^{l} x_i x_{-i}$. Hence the associated Gram matrix of Φ is given as

$$J := \left[\begin{array}{c|c} 0 & E_l \\ \hline E_l & 0 \end{array}\right] \in K^{2l \times 2l}.$$

Then we have $A^* = (JAJ^{-1})^{\mathrm{tr}} = JA^{\mathrm{tr}}J$, thus the associated Lie K-algebra becomes $\mathfrak{o}_{2l}(K) := \{A \in \mathfrak{gl}_{2l}(K); JA^{\mathrm{tr}}J = -A\}$; it is called the associated **even degree orthogonal** Lie algebra. From $-\mathrm{Tr}(A) = \mathrm{Tr}(JA^{\mathrm{tr}}J) = \mathrm{Tr}(J^2A^{\mathrm{tr}}) = \mathrm{Tr}(A)$, for all $A \in \mathfrak{o}_{2l}(K)$, we conclude $\mathfrak{o}_{2l}(K) \subseteq \mathfrak{sl}_{2l}(K)$.

Writing $A := \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \in \mathfrak{gl}_{2l}(K)$, where $A_{ij} \in K^{l \times l}$, for $i,j \in \{1,2\}$, we have $A \in \mathfrak{o}_{2l}(K)$ if and only if

$$\left[\begin{array}{c|c} 0 & -E_l \\ \hline -E_l & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}^{\mathrm{tr}} & A_{21}^{\mathrm{tr}} \\ \hline A_{12}^{\mathrm{tr}} & A_{22}^{\mathrm{tr}} \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & E_n \\ \hline E_n & 0 \end{array}\right],$$

that is

$$\left[\begin{array}{c|c} -A_{21} & -A_{22} \\ \hline -A_{11} & -A_{12} \end{array}\right] = \left[\begin{array}{c|c} A_{21}^{\mathrm{tr}} & A_{11}^{\mathrm{tr}} \\ \hline A_{22}^{\mathrm{tr}} & A_{12}^{\mathrm{tr}} \end{array}\right],$$

in other words if and only if $A_{22} = -A_{11}^{\mathrm{tr}}$ and $A_{12}^{\mathrm{tr}} = -A_{12}$ and $A_{21}^{\mathrm{tr}} = -A_{21}$. Hence we get the following **standard** K-basis of $\mathfrak{o}_{2l}(K)$

$$\begin{array}{rl} & \{E_{ij} - E_{-j,-i}; i,j \in \{1,\ldots,l\}\} \\ \dot\cup & \{E_{i,-j} + E_{-j,i}; i \neq j \in \{1,\ldots,l\}\}; \end{array}$$

in particular we have $\dim_K(\mathfrak{o}_{2l}(K)) = l^2 + l(l-1) = 2l^2 - l = \frac{1}{2}n(n-1)$.

c) Assume now that $n = 2l+1$, for some $l \in \mathbb{N}$, hence Φ has Witt index l, and V is an orthogonal direct sum of hyperbolic planes and a non-degenerate line. We let $\{e_0, e_1, e_2, \ldots, e_l, e_{-l}, e_{-2}, \ldots, e_{-1}\} \subseteq V$ be the standard K-basis, where the K-subspaces $\langle e_i, e_{-i}\rangle_K \leq_K V$, for all $i \in \{1, \ldots, l\}$, form mutually orthogonal hyperbolic planes, being orthogonal to the non-isotropic line $\langle e_0\rangle_K \leq_K V$. Thus we have $\epsilon := \Phi(e_0, e_0) \neq 0$, where we additionally assume that $\epsilon \in K$ is a square. (If $\epsilon \in K$ is a non-square, this yields an inequivalent symmetric form, which has an isomorphic classical Lie algebra associated with it.)

Thus we may assume that $\epsilon = 1$; in other words, the quadratic form associated with Φ is given as $\varphi(x_1, \ldots, x_{-1}) := \frac{1}{2}x_0^2 + \sum_{i=1}^{l} x_i x_{-i}$. Hence the associated Gram matrix of Φ is given as

$$J := \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & E_l \\ \hline 0 & E_l & 0 \end{array}\right] \in K^{(2l+1)\times(2l+1)}.$$

Then we have $A^* = (JAJ^{-1})^{\mathrm{tr}} = JA^{\mathrm{tr}}J$, thus the associated Lie K-algebra becomes $\mathfrak{o}_{2l+1}(K) := \{A \in \mathfrak{gl}_{2l+1}(K); JA^{\mathrm{tr}}J = -A\}$; it is called the associated **odd degree orthogonal** Lie algebra. From $-\mathrm{Tr}(A) = \mathrm{Tr}(JA^{\mathrm{tr}}J) = \mathrm{Tr}(J^2A^{\mathrm{tr}}) = \mathrm{Tr}(A)$, for all $A \in \mathfrak{o}_{2l+1}(K)$, we conclude $\mathfrak{o}_{2l+1}(K) \subseteq \mathfrak{sl}_{2l+1}(K)$.

Writing $A := \left[\begin{array}{c|c|c} A_{00} & A_{01} & A_{02} \\ \hline A_{10} & A_{11} & A_{12} \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right] \in \mathfrak{gl}_{2l+1}(K)$, where $A_{i,j} \in K$ and $A_{0,j} \in K^{1\times l}$ and $A_{i,0} \in K^{l\times 1}$ and $A_{i,j} \in K^{l\times l}$, for $i,j \in \{1,2\}$, we have $A \in \mathfrak{o}_{2l+1}(K)$ if and only if

$$\left[\begin{array}{c|c|c} -A_{00} & -A_{01} & -A_{02} \\ \hline -A_{20} & -A_{21} & -A_{22} \\ \hline -A_{10} & -A_{11} & -A_{12} \end{array}\right] = \left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 0 & -E_l \\ \hline 0 & -E_l & 0 \end{array}\right] \cdot \left[\begin{array}{c|c|c} A_{00} & A_{01} & A_{02} \\ \hline A_{10} & A_{11} & A_{12} \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} A_{00} & A_{10}^{\mathrm{tr}} & A_{20}^{\mathrm{tr}} \\ \hline A_{01}^{\mathrm{tr}} & A_{11}^{\mathrm{tr}} & A_{21}^{\mathrm{tr}} \\ \hline A_{02}^{\mathrm{tr}} & A_{12}^{\mathrm{tr}} & A_{22}^{\mathrm{tr}} \end{array}\right] \cdot \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & E_l \\ \hline 0 & E_l & 0 \end{array}\right] = \left[\begin{array}{c|c|c} A_{00} & A_{20}^{\mathrm{tr}} & A_{10}^{\mathrm{tr}} \\ \hline A_{01}^{\mathrm{tr}} & A_{21}^{\mathrm{tr}} & A_{11}^{\mathrm{tr}} \\ \hline A_{02}^{\mathrm{tr}} & A_{22}^{\mathrm{tr}} & A_{12}^{\mathrm{tr}} \end{array}\right],$$

in other words if and only if $A_{00} = 0$ and $A_{02} = -A_{10}^{\mathrm{tr}}$ and $A_{01} = -A_{20}^{\mathrm{tr}}$ and $A_{22} = -A_{11}^{\mathrm{tr}}$ and $A_{12}^{\mathrm{tr}} = -A_{12}$ and $A_{21}^{\mathrm{tr}} = -A_{21}$. Hence we get the following **standard** K-basis of $\mathfrak{o}_{2l+1}(K)$

$$\begin{array}{cl} & \{E_{0i} - E_{-i,0}; i \in \{\pm 1, \ldots, \pm l\}\} \\ \dot\cup & \{E_{ij} - E_{-j,-i}; i,j \in \{1, \ldots, l\}\} \\ \dot\cup & \{E_{i,-j} + E_{-j,i}; i \neq j \in \{1, \ldots, l\}\}; \end{array}$$

in particular we have $\dim_K(\mathfrak{o}_{2l+1}(K)) = 2l + l^2 + l(l-1) = 2l^2 + l = \frac{1}{2}n(n-1)$.

(2.5) Real forms. We keep the setting of (2.3), and let $n \geq 2$.

a) Let $K := \mathbb{R}$ and $\alpha = \mathrm{id}_{\mathbb{R}}$, and let Φ^+ be symmetric and positive definite. Then we may assume that the standard $\mathbb{R}$-basis $\{e_1, \ldots, e_n\} \subseteq V$ is orthonormal; in other words, the quadratic form associated with Φ^+ is given as $\varphi^+(x_1, \ldots, x_n) := \frac{1}{2} \cdot \sum_{i=1}^n x_i^2$. Hence the associated Gram matrix of Φ^+ is just the identity matrix E_n.

Then we have $A^* = A^{\mathrm{tr}}$, thus the associated Lie $\mathbb{R}$-algebra $\mathfrak{o}_n^+(\mathbb{R}) := \{A \in \mathfrak{gl}_n(\mathbb{R}); A^{\mathrm{tr}} = -A\}$ coincides with the set of all skew-symmetric matrices; it is called the associated **orthogonal** Lie algebra. We have $\mathfrak{o}_n^+(\mathbb{R}) \subseteq \mathfrak{sl}_n(\mathbb{R})$, and the following **standard** $\mathbb{R}$-basis, of cardinality $\dim_{\mathbb{R}}(\mathfrak{o}_n^+(\mathbb{R})) = \frac{1}{2}n(n-1)$,

$$\{E_{ij} - E_{ji}; i < j \in \{1, \ldots, n\}\} \subseteq \mathfrak{o}_n^+(\mathbb{R}).$$

We aim at comparing the Lie $\mathbb{R}$-algebra $\mathfrak{o}_n^+(\mathbb{R})$ with the Lie $\mathbb{R}$-algebra $\mathfrak{o}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}); JA^{\mathrm{tr}}J = -A\}$ associated with the form Φ in (2.4). To do so, we use the complexifications $\mathfrak{o}_n(\mathbb{R})^{\mathbb{C}} := \mathfrak{o}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{o}_n^+(\mathbb{R})^{\mathbb{C}} := \mathfrak{o}_n^+(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, which are Lie $\mathbb{C}$-algebras. Since the standard $\mathbb{R}$-bases of $\mathfrak{o}_n(\mathbb{R}) \subseteq \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{o}_n^+(\mathbb{R}) \subseteq \mathfrak{gl}_n(\mathbb{C})$ are $\mathbb{C}$-linearly independent, we may identify $\mathfrak{o}_n(\mathbb{R})^{\mathbb{C}}$ with $\mathfrak{o}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}); JA^{\mathrm{tr}}J = -A\}$, and $\mathfrak{o}_n^+(\mathbb{R})^{\mathbb{C}}$ with $\mathfrak{o}_n^+(\mathbb{C}) := \{A \in \mathfrak{gl}_n(\mathbb{C}); A^{\mathrm{tr}} = -A\}$.

Let first $n = 2$. Applying the matrix $U := \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \in \mathrm{GU}_2(\mathbb{C})$ to the Gram matrix of Φ^+ yields $U^{\mathrm{tr}}E_2U = U^{\mathrm{tr}}U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = J$, which is the Gram matrix of Φ. Hence we conclude that, for arbitrary $n \in \mathbb{N}$, there is a unitary matrix $U \in \mathrm{GU}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}); \overline{M}^{\mathrm{tr}} = M^{-1}\}$ such that $U^{\mathrm{tr}}U = J$; note that $U^{\mathrm{tr}}U = J = J^{-1} = U^{-1}\overline{U}$. Then, for $A \in \mathfrak{o}_n(\mathbb{C})$ we have $U^{-1}\overline{U}A^{\mathrm{tr}}U^{\mathrm{tr}}U = JA^{\mathrm{tr}}J = -A$, implying $(UAU^{-1})^{\mathrm{tr}} = \overline{U}A^{\mathrm{tr}}U^{\mathrm{tr}} = -UAU^{-1}$, thus $UAU^{-1} \in \mathfrak{o}_n^+(\mathbb{C})$. Conversely, for $A \in \mathfrak{o}_n^+(\mathbb{C})$ we have $A^{\mathrm{tr}} = -A$, implying $J(U^{-1}AU)^{\mathrm{tr}}J = U^{-1}\overline{U} \cdot U^{\mathrm{tr}}A^{\mathrm{tr}}\overline{U} \cdot U^{\mathrm{tr}}U = -U^{-1}AU$, thus $U^{-1}AU \in \mathfrak{o}_n(\mathbb{C})$.

This shows that U induces an isomorphism $\mathfrak{o}_n(\mathbb{C}) \cong \mathfrak{o}_n^+(\mathbb{C})$ of Lie $\mathbb{C}$-algebras. Hence we conclude that both $\mathfrak{o}_n(\mathbb{R})$ and $\mathfrak{o}_n^+(\mathbb{R})$ are $\mathbb{R}$-forms of $\mathfrak{o}_n(\mathbb{C}) \cong \mathfrak{o}_n^+(\mathbb{C})$. Actually, $\mathfrak{o}_n(\mathbb{R})$ and $\mathfrak{o}_n^+(\mathbb{R})$ are non-isomorphic as Lie $\mathbb{R}$-algebras, being called 'split' and 'non-split' forms, respectively, but we do not prove this here.

b) Let $K := \mathbb{C}$ and let $\alpha = \overline{}$ be complex conjugation, hence we have $K' = \mathbb{R}$. Let Φ^+ be hermitian and positive definite. Then we may assume that the standard $\mathbb{C}$-basis $\{e_1, \ldots, e_n\} \subseteq V$ is orthonormal; in other words, the quadratic form associated with Φ^+ is given as $\varphi^+(x_1, \ldots, x_n) := \frac{1}{2} \cdot \sum_{i=1}^n |x_i|^2$. Hence the associated Gram matrix of Φ^+ is just the identity matrix E_n.

Then $A^* = \overline{A}^{\mathrm{tr}}$, thus the associated Lie $\mathbb{R}$-algebra $\mathfrak{gu}_n^+(\mathbb{C}) := \{A \in \mathfrak{gl}_n(\mathbb{C}); \overline{A}^{\mathrm{tr}} = -A\}$ coincides with the set of all skew-hermitian matrices; it is called the associated **general unitary** Lie algebra. The Lie $\mathbb{R}$-algebra $\mathfrak{su}_n^+(\mathbb{C}) := \mathfrak{gu}_n^+(\mathbb{C}) \cap \mathfrak{sl}_n(\mathbb{C})$ is called the associated **special unitary** Lie algebra. We have the following **standard** $\mathbb{R}$-bases of $\mathfrak{gu}_n^+(\mathbb{R})$ and $\mathfrak{su}_n^+(\mathbb{R})$

$$\begin{array}{rl} & \{iE_{jj}; j \in \{1,\dots,n\}\} \\ \dot\cup & \{E_{jk}-E_{kj}, i(E_{jk}+E_{kj}); j<k \in \{1,\dots,n\}\} \end{array}$$

and

$$\begin{array}{rl} & \{i(E_{jj}-E_{j+1,j+1}); j \in \{1,\dots,n-1\}\} \\ \dot\cup & \{E_{jk}-E_{kj}, i(E_{jk}+E_{kj}); j<k \in \{1,\dots,n\}\}, \end{array}$$

respectively; in particular we have $\dim_{\mathbb{R}}(\mathfrak{gu}_n^+(\mathbb{R})) = n + 2\cdot\frac{1}{2}n(n-1) = n^2$ and $\dim_{\mathbb{R}}(\mathfrak{su}_n^+(\mathbb{R})) = (n-1) + 2\cdot\frac{1}{2}n(n-1) = n^2-1$.

Complexification yields $\mathfrak{gu}_n^+(\mathbb{C})^{\mathbb{C}} := \mathfrak{gu}_n^+(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{su}_n^+(\mathbb{C})^{\mathbb{C}} := \mathfrak{su}_n^+(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$, as well as $\mathfrak{gl}_n(\mathbb{R})^{\mathbb{C}} := \mathfrak{gl}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{sl}_n(\mathbb{R})^{\mathbb{C}} := \mathfrak{sl}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, which are Lie $\mathbb{C}$-algebras. Since the standard $\mathbb{R}$-bases of $\mathfrak{gu}_n^+(\mathbb{C}) \subseteq \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{R}) \subseteq \mathfrak{gl}_n(\mathbb{C})$, as well as of $\mathfrak{su}_n^+(\mathbb{C}) \subseteq \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{R}) \subseteq \mathfrak{sl}_n(\mathbb{C})$ are $\mathbb{C}$-linearly independent, we may identify both $\mathfrak{gu}_n^+(\mathbb{C})^{\mathbb{C}}$ and $\mathfrak{gl}_n(\mathbb{R})^{\mathbb{C}}$ with $\mathfrak{gl}_n(\mathbb{C})$, and both $\mathfrak{su}_n^+(\mathbb{C})^{\mathbb{C}}$ and $\mathfrak{sl}_n(\mathbb{R})^{\mathbb{C}}$ with $\mathfrak{sl}_n(\mathbb{C})$. Hence we conclude that both $\mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{gu}_n^+(\mathbb{C})$ are $\mathbb{R}$-forms of $\mathfrak{gl}_n(\mathbb{C})$, and that both $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{su}_n^+(\mathbb{C})$ are $\mathbb{R}$-forms of $\mathfrak{sl}_n(\mathbb{C})$. Actually, $\mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{gu}_n^+(\mathbb{C})$ are non-isomorphic as Lie $\mathbb{R}$-algebras, and neither are $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{su}_n^+(\mathbb{C})$, being called 'split' and 'non-split' forms, respectively, but we do not prove this here.

3 Representations

(3.1) Representations of associative algebras. Let $R \neq \{0\}$ be a commutative ring, and let $\mathfrak{A}$ be a unital associative R-algebra. A homomorphism $\varphi\colon \mathfrak{A} \to \mathrm{End}_R(V)$, where V is an R-module, and such that $\varphi(1) = \mathrm{id}_V$, is called a **representation** of $\mathfrak{A}$; in this case V is called an **$\mathfrak{A}$-module**. Identifying $\mathrm{End}_R(R^{n\times 1}) \cong R^{n\times n}$, for some $n \in \mathbb{N}_0$, we obtain a representation $\varphi\colon \mathfrak{A} \to R^{n\times n}$ of **degree** n; in particular we have $\varphi(1) = E_n$.

We also write $a\cdot v := \varphi(a)(v) \in V$, for all $a \in \mathfrak{A}$ and $v \in V$. Moreover, if $U \leq_R V$ is an R-submodule such that $\mathfrak{A} \cdot U \leq_R U$, then φ induces a representation of $\mathfrak{A}$ on U; in this case U is called an **$\mathfrak{A}$-submodule** of V, and we write $U \leq_{\mathfrak{A}} V$. Moreover, φ induces a representation of $\mathfrak{A}$ on V/U via the **natural** epimorphism $\nu_U\colon V \to V/U$ of R-modules; then V/U is called a **quotient $\mathfrak{A}$-module** of V.

If W also is an $\mathfrak{A}$-module, then a homomorphism $\varphi\colon V \to W$ of R-modules such that $\varphi(av) = a\varphi(v)$, for all $a \in \mathfrak{A}$ and $v \in V$, is called a **homomorphism** of $\mathfrak{A}$-modules; similarly, we have **monomorphisms**, **epimorphisms**, **isomorphisms**, **endomorphisms** and **automorphisms**. If φ is bijective, then $\varphi^{-1}\colon W \to V$ is a homomorphism of $\mathfrak{A}$-modules as well; we write $V \cong W$, and the representations afforded by V and W, respectively, are called **equivalent**.

The image $\mathrm{im}(\varphi) = \varphi(V) \leq_{\mathfrak{A}} W$ and the kernel $\ker(\varphi) := \{v \in V; \varphi(v) = 0\} \leq_{\mathfrak{A}} V$ are $\mathfrak{A}$-submodules. Then the **homomorphism principle** for R-modules extends straightforwardly: Given $U \leq_{\mathfrak{A}} \ker(\varphi)$, there is a unique homomorphism $\varphi^U\colon V/U \to W\colon v + U \mapsto \varphi(v)$ of $\mathfrak{A}$-modules giving rise to a factorization

$\varphi = \varphi^U \nu_U \colon V \to V/U \to W$. We have $\mathrm{im}(\varphi^U) = \mathrm{im}(\varphi) \leq_{\mathfrak{A}} W$ and $\ker(\varphi^U) = \ker(\varphi)/U = \{v + U \in V/U; v \in \ker(\varphi)\} \leq_{\mathfrak{A}} V/U$. In particular, φ^U is injective if and only if $U = \ker(\varphi)$, and we have an isomorphism of $\mathfrak{A}$-modules $\overline{\varphi} := \varphi^{\ker(\varphi)} \colon V/\ker(\varphi) \to \mathrm{im}(\varphi)$.

Example. Given $x \in \mathfrak{A}$, then we have the associated **left** and **right multiplication maps** $\lambda_{\mathfrak{A}}(x) \colon \mathfrak{A} \to \mathfrak{A} \colon a \mapsto xa$ and $\rho_{\mathfrak{A}}(x) \colon \mathfrak{A} \to \mathfrak{A} \colon a \mapsto ax$, respectively. Then we have $\lambda_{\mathfrak{A}}(x)(a+b) = x(a+b) = xa+xb$ and $\lambda_{\mathfrak{A}}(x)(ra) = x(ra) = r(xa)$, as well as $\rho_{\mathfrak{A}}(x)(a+b) = (a+b)x = ax+bx$ and $\rho_{\mathfrak{A}}(x)(ra) = (ra)x = r(ax)$, for all $a, b \in \mathfrak{A}$ and $r \in R$, using R-bilinearity of multiplication in $\mathfrak{A}$. Hence we indeed have $\lambda_{\mathfrak{A}}(x), \rho_{\mathfrak{A}}(x) \in \mathrm{End}_R(\mathfrak{A})$. For later use we already note the following commutativity property: By associativity we have $\lambda_{\mathfrak{A}}(x)\rho_{\mathfrak{A}}(y) = \rho_{\mathfrak{A}}(y)\lambda_{\mathfrak{A}}(x) \colon a \mapsto x(ay) = (xa)y$, for all $x, y, a \in \mathfrak{A}$.

Next we have $\lambda_{\mathfrak{A}}(x+y) = \lambda_{\mathfrak{A}}(x) + \lambda_{\mathfrak{A}}(y) \colon a \mapsto (x+y)a = xa+ya$ and $\lambda_{\mathfrak{A}}(rx) = r\lambda_{\mathfrak{A}}(x) \colon a \mapsto (rx)a = r(xa)$, as well as $\rho_{\mathfrak{A}}(x+y) = \rho_{\mathfrak{A}}(x) + \rho_{\mathfrak{A}}(y) \colon a \mapsto a(x+y) = ax+ay$ and $\rho_{\mathfrak{A}}(rx) = r\rho_{\mathfrak{A}}(x) \colon a \mapsto a(rx) = r(ax)$, for all $x, y, a \in \mathfrak{A}$, again using R-bilinearity of multiplication in $\mathfrak{A}$. Hence both maps $\lambda_{\mathfrak{A}}, \rho_{\mathfrak{A}} \colon \mathfrak{A} \to \mathrm{End}_R(\mathfrak{A})$ are indeed R-linear.

Moreover, by associativity we have $\lambda_{\mathfrak{A}}(xy) = \lambda_{\mathfrak{A}}(x)\lambda_{\mathfrak{A}}(y) \colon a \mapsto (xy)a = x(ya)$, for all $x, y, a \in \mathfrak{A}$, thus $\lambda_{\mathfrak{A}}$ is a representation of $\mathfrak{A}$, being called the **regular** representation; note that if $\mathfrak{A}$ is unital then we have $\lambda_{\mathfrak{A}}(1) = \mathrm{id}_{\mathfrak{A}}$. But from $\rho_{\mathfrak{A}}(xy) \colon a \mapsto a(xy) = (ax)y$ and $\rho_{\mathfrak{A}}(x)\rho_{\mathfrak{A}}(y) \colon a \mapsto (ay)x$, for all $x, y, a \in \mathfrak{A}$, we infer that we have the **right regular** representation $\rho_{\mathfrak{A}} \colon \mathfrak{A} \to \mathrm{End}_R(\mathfrak{A})^{\mathrm{opp}}$, where $\mathrm{End}_R(\mathfrak{A})^{\mathrm{opp}}$ is the **opposite** ring of $\mathrm{End}_R(\mathfrak{A})$, whose multiplication is given by $\mu(\alpha, \beta) := \beta\alpha$, for all $\alpha, \beta \in \mathrm{End}_R(\mathfrak{A})$.

(3.2) Simple modules for associative algebras. Let $R \neq \{0\}$ be a commutative ring, let $\mathfrak{A}$ be a unital associative R-algebra, and let V be an $\mathfrak{A}$-module. If $V \neq \{0\}$ does not have any proper non-zero $\mathfrak{A}$-submodules, then V is called **simple** or **irreducible**; likewise the representation of $\mathfrak{A}$ afforded by V is called **irreducible** or **simple**.

Theorem: Schur's Lemma. Let V and W be simple $\mathfrak{A}$-modules.

i) Then any non-zero $\mathfrak{A}$-homomorphism from V to W is an isomorphism.

ii) If $R = K$ is an algebraically closed field, and V is finite-dimensional as K-vector space, then we have $\mathrm{End}_{\mathfrak{A}}(V) = K \cdot \mathrm{id}_V$.

Proof. i) Let $0 \neq \varphi \colon V \to W$ be an $\mathfrak{A}$-homomorphism; recall that both $V \neq \{0\} \neq W$. Then $\{0\} \neq \varphi(V) \leq_{\mathfrak{A}} W$ implies that φ is surjective; and $\ker(\varphi) <_{\mathfrak{A}} V$ implies that $\ker(\varphi) = \{0\}$, hence φ is injective.

ii) Let $\varphi \colon V \to V$ be an $\mathfrak{A}$-endomorphism. Hence φ is K-linear, thus K being algebraically closed implies that φ has an eigenvalue $\lambda \in K$. Since $\mathrm{End}_{\mathfrak{A}}(V)$

is a unital associative K-algebra, we infer that $\varphi - \lambda \cdot \mathrm{id}_V \in \mathrm{End}_{\mathfrak{A}}(V)$. Since $\{0\} \neq \ker(\varphi - \lambda \cdot \mathrm{id}_V) \leq_{\mathfrak{A}} V$ we conclude that $\varphi - \lambda \cdot \mathrm{id}_V = 0 \in \mathrm{End}_{\mathfrak{A}}(V)$. ♯

(3.3) Semisimple modules for associative algebras. Let $R \neq \{0\}$ be a commutative ring, and let $\mathfrak{A}$ be a unital associative R-algebra. If $\{V_i; i \in \mathcal{I}\}$ are $\mathfrak{A}$-modules, where $\mathcal{I}$ is an index set, then the direct sum $\bigoplus_{i\in\mathcal{I}} V_i$ of R-modules becomes an $\mathfrak{A}$-module with respect to diagonal $\mathfrak{A}$-action, that is $a \cdot [v_i; i \in \mathcal{I}] = [av_i; i \in \mathcal{I}]$, for all $a \in \mathfrak{A}$; recall that the elements of $\bigoplus_{i\in\mathcal{I}} V_i$ are precisely the $\mathcal{I}$-tuples $[v_i \in V_i; i \in \mathcal{I}] \in \prod_{i\in\mathcal{I}} V_i$, such that $v_i = 0 \in V_i$ for almost all $i \in \mathcal{I}$.

Let V be an $\mathfrak{A}$-module. If $V = \bigoplus_{i\in\mathcal{I}} S_i$ is the direct sum of simple $\mathfrak{A}$-submodules $S_i \leq_{\mathfrak{A}} V$, where $\mathcal{I}$ is an index set, then V is called **semisimple**; likewise the representation of $\mathfrak{A}$ afforded by V is called **semisimple**.

Theorem. An $\mathfrak{A}$-module V is semisimple if and only if any $\mathfrak{A}$-submodule $U \leq_{\mathfrak{A}} V$ has a **complement**, that is there is $W \leq_{\mathfrak{A}} V$ such that $V = U \oplus W$.

Proof. i) Let V be semisimple, such that $V = \bigoplus_{i\in\mathcal{I}} S_i$, where $\mathcal{I}$ is an index set and the $S_i \leq_{\mathfrak{A}} V$ are simple. Then let $\mathcal{M} := \{\mathcal{J} \subseteq \mathcal{I}; U \cap \bigoplus_{j\in\mathcal{J}} S_j = \{0\}\}$, being partially ordered by set-theoretic inclusion. Then we have $\emptyset \in \mathcal{M}$. Moreover, if $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots$ is a chain in $\mathcal{M}$, then we have $\widehat{\mathcal{J}} := \bigcup_{i\in\mathbb{N}} \mathcal{J}_i \in \mathcal{M}$ as well: Assume to the contrary that there is $0 \neq u \in U \cap \bigoplus_{j\in\widehat{\mathcal{J}}} S_j$, then there is $i \in \mathbb{N}$ such that $u \in U \cap \bigoplus_{j\in\mathcal{J}_i} S_j$, a contradiction. Hence $\widehat{\mathcal{J}}$ is an upper bound of the given chain in $\mathcal{M}$.

Thus by Zorn's Lemma we conclude that there is a maximal element $\mathcal{J} \in \mathcal{M}$, and we let $W := \bigoplus_{j\in\mathcal{J}} S_j \leq_{\mathfrak{A}} V$. We show that $U \oplus W = V$: Assume to the contrary that $U \oplus W <_{\mathfrak{A}} V$, then there is $i \in \mathcal{I} \setminus \mathcal{J}$ such that $S_i \not\leq_{\mathfrak{A}} U \oplus W$, hence since S_i is simple we have $S_i \cap (U \oplus W) = \{0\}$. Now by maximality of $\mathcal{J}$ there is $0 \neq u \in U \cap (S_i \oplus W)$, which since $U \cap W = \{0\}$ implies that there is $0 \neq v \in S_i \cap (U \oplus W)$, a contradiction.

ii) Let V be such that any $\mathfrak{A}$-submodule has a complement. We first show that, whenever $\widetilde{U} \leq_{\mathfrak{A}} U \leq_{\mathfrak{A}} V$, then $\widetilde{U}$ has a complement in U: Indeed, let $W \leq_{\mathfrak{A}} V$ such that $V = \widetilde{U} \oplus W$, then we have $U = V \cap U = (\widetilde{U} \oplus W) \cap U = \widetilde{U} \oplus (W \cap U)$.

Next, let $\{0\} \neq U \leq_{\mathfrak{A}} V$ be any non-zero $\mathfrak{A}$-submodule. Then we show that U has a simple $\mathfrak{A}$-submodule: To this end, we may assume that $U = u\mathfrak{A}$, for some $0 \neq u \in U$. Let $\mathcal{M}$ be the set of all proper $\mathfrak{A}$-submodules of U, being partially ordered by set-theoretic inclusion. Then we have $\{0\} \in \mathcal{M}$. Moreover, if $U_1 \leq_{\mathfrak{A}} U_2 \leq_{\mathfrak{A}} \cdots <_{\mathfrak{A}} U$ is a chain in $\mathcal{M}$, then we have $u \notin U_i$, for all $i \in \mathbb{N}$, hence $u \notin \widehat{U} := \bigcup_{i\in\mathbb{N}} U_i$ as well, implying that $\widehat{U} <_{\mathfrak{A}} U$ is an upper bound of the given chain in $\mathcal{M}$. Thus by Zorn's Lemma we conclude that there is a maximal element $\widetilde{U} \in \mathcal{M}$, that is $\widetilde{U} <_{\mathfrak{A}} U$ is a maximal proper $\mathfrak{A}$-submodule. Letting $S \leq_{\mathfrak{A}} U$ such that $U = \widetilde{U} \oplus S$, we conclude that $S \cong U/\widetilde{U}$ is simple.

Now, let $U \leq_{\mathfrak{A}} V$ be the sum of all simple $\mathfrak{A}$-submodules of V. Then we have to show that $U = V$: Assume to the contrary that $U <_{\mathfrak{A}} V$, and let $\{0\} \neq W \leq_{\mathfrak{A}} V$

such that $V = U \oplus W$. Then there is a simple $\mathfrak{A}$-submodule $S \leq_{\mathfrak{A}} W$, hence $S \leq_{\mathfrak{A}} U \cap W$, a contradiction.

Finally, it remains to be shown that V can be written as a direct sum of certain simple $\mathfrak{A}$-submodules: Let $V = \sum_{i \in \mathcal{I}} S_i$, where $\mathcal{I}$ is an index set and the $S_i \leq_{\mathfrak{A}} V$ are simple. Then let $\mathcal{M} := \{\mathcal{J} \subseteq \mathcal{I}; \sum_{j \in \mathcal{J}} S_j \leq_{\mathfrak{A}} V \text{ is direct}\}$, being partially ordered by set-theoretic inclusion. Then we have $\emptyset \in \mathcal{M}$. Moreover, if $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots$ is a chain in $\mathcal{M}$, then we have $\widehat{\mathcal{J}} := \bigcup_{i \in \mathbb{N}} \mathcal{J}_i \in \mathcal{M}$ as well: Let $\sum_{j \in \widehat{\mathcal{J}}} v_j = 0$, where $v_j \in S_j$. Since $v_j = 0 \in S_j$ for almost all $j \in \widehat{\mathcal{J}}$, there is $i \in \mathbb{N}$ such that all non-zero summands occurring are elements of $\bigoplus_{j \in \mathcal{J}_i} S_j$, a contradiction. Hence $\widehat{\mathcal{J}}$ is an upper bound of the given chain in $\mathcal{M}$. Thus by Zorn's Lemma we conclude that there is a maximal element $\mathcal{J} \in \mathcal{M}$, and we let $U := \bigoplus_{j \in \mathcal{J}} S_j$. We show that $U = V$: Assume to the contrary that $U <_{\mathfrak{A}} V$, then there is $i \in \mathcal{I} \setminus \mathcal{J}$ such that $S_i \not\leq_{\mathfrak{A}} U$, hence since S_i is simple we have $S_i \cap U = \{0\}$, thus $S_i + U$ is direct, hence $\{i\} \mathbin{\dot\cup} \mathcal{J} \in \mathcal{M}$, a contradiction. $\sharp$

(3.4) Representations of Lie algebras. Let $R \neq \{0\}$ be a commutative ring, and let $\mathfrak{L}$ be a Lie R-algebra. A homomorphism $\varphi\colon \mathfrak{L} \to \mathfrak{gl}(V)$, where V is an R-module, is called a **representation** of $\mathfrak{L}$; then V is called an **$\mathfrak{L}$-module**.

In particular, identifying $\mathrm{End}_R(R^{n \times 1}) \cong R^{n \times n}$, for some $n \in \mathbb{N}_0$, we obtain a representation $\varphi\colon \mathfrak{L} \to \mathfrak{gl}_n(R)$ of **degree** n. For example, for any R-module V we have the **trivial** representation, where all elements of $\mathfrak{L}$ act by the zero map.

If the Lie structure of $\mathfrak{L}$ is given as the commutator of a unital associative R-algebra, then any representation $\mathfrak{L} \to \mathrm{End}_R(V)$ as associative algebras also is a representation $\mathfrak{L} \to \mathfrak{gl}(V)$ of Lie R-algebras. In particular, this holds for the tautological representation of $\mathrm{End}_R(V) = \mathfrak{gl}(V)$.

Conversely, any representation $\varphi\colon \mathfrak{L} \to \mathfrak{gl}(V)$ of a Lie R-algebra can be considered as the tautological representation of the unital associative R-subalgebra of $\mathfrak{gl}(V)$ generated by the Lie R-subalgebra $\varphi(\mathfrak{L}) \subseteq \mathfrak{gl}(V)$. Hence the above comments on modules and their homomorphisms for associative algebras hold verbally for Lie algebras.

Example. Given $x \in \mathfrak{L}$, we have the associated **adjoint map** $\mathrm{ad}_{\mathfrak{L}}(x)\colon \mathfrak{L} \to \mathfrak{L}\colon a \mapsto [x, a]$; note that formally $\mathrm{ad}_{\mathfrak{L}}$ coincides with the left multiplication map of $\mathfrak{L}$ as a non-associative R-algebra. Then we have $\mathrm{ad}_{\mathfrak{L}}(x)(a + b) = [x, a + b] = [x, a] + [x, b]$ and $\mathrm{ad}_{\mathfrak{L}}(x)(ra) = [x, ra] = r[x, a]$, for all $a, b \in \mathfrak{L}$ and $r \in R$, using R-bilinearity of multiplication in $\mathfrak{L}$. Hence we indeed have $\mathrm{ad}_{\mathfrak{L}}(x) \in \mathrm{End}_R(\mathfrak{L})$.

Next we have $\mathrm{ad}_{\mathfrak{L}}(x + y) = \mathrm{ad}_{\mathfrak{L}}(x) + \mathrm{ad}_{\mathfrak{L}}(y)$ and $\mathrm{ad}_{\mathfrak{L}}(rx) = r \cdot \mathrm{ad}_{\mathfrak{L}}(x)$, for all $x, y \in \mathfrak{L}$, again using R-bilinearity of multiplication in $\mathfrak{L}$. Hence the map $\mathrm{ad}_{\mathfrak{L}}\colon \mathfrak{L} \to \mathrm{End}_R(\mathfrak{L})$ is indeed R-linear. Moreover, using the Jacobi identity we obtain $\mathrm{ad}_{\mathfrak{L}}([x, y])(a) = [[x, y], a] = -[a, [x, y]] = [x, [y, a]] + [y, [a, x]] = [x, [y, a]] - [y, [x, a]] = \big(\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(y) - \mathrm{ad}_{\mathfrak{L}}(y)\mathrm{ad}_{\mathfrak{L}}(x)\big)(a) = [\mathrm{ad}_{\mathfrak{L}}(x), \mathrm{ad}_{\mathfrak{L}}(y)](a)$, for all $x, y, a \in \mathfrak{L}$.

Hence we conclude that $\mathrm{ad}_{\mathfrak{L}}\colon \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ is a representation, being called the **adjoint** representation. We have $\ker(\mathrm{ad}_{\mathfrak{L}}) = \{x \in \mathfrak{L}; [x,y] = 0 \text{ for all } y \in \mathfrak{L}\} = Z(\mathfrak{L})$. In particular, if $Z(\mathfrak{L}) = \{0\}$ then the adjoint representation is injective, so that if $R = K$ is a field then $\mathfrak{L}$ is isomorphic to a linear Lie K-algebra, which thus is a trivial case of the Ado–Iwasawa Theorem.

Moreover, the $\mathfrak{L}$-submodules of $\mathfrak{L}$ with respect to the adjoint representation are precisely the ideals of $\mathfrak{L}$; thus, if $R = K$ is a field and $\mathfrak{L}$ is a simple Lie K-algebra, then $\mathfrak{L}$ is a simple $\mathfrak{L}$-module with respect to the adjoint representation.

(3.5) Modules for Lie algebras. a) Let $R \neq \{0\}$ be a commutative ring, let $\mathfrak{L}$ be a Lie R-algebra, and let V and W be $\mathfrak{L}$-modules. We describe a couple of constructions producing new $\mathfrak{L}$-modules from the given ones:

i) Let $V^* := \mathrm{Hom}_R(V,R)$ be the **dual** R-module of V. Then V^* becomes an $\mathfrak{L}$-module by letting $x\alpha\colon V \to R\colon v \mapsto \alpha(-xv)$, for all $x \in \mathfrak{L}$ and $\alpha \in V^*$, being called the **contragredient** $\mathfrak{L}$-module of V:

We have $(x\alpha)(av+w) = \alpha(-x(av+w)) = a\alpha(-xv) + \alpha(-xw) = a(x\alpha)(v) + (x\alpha)(w)$, for all $v,w \in V$ and $a \in R$, implying that $x\alpha \in V^*$ indeed. Moreover, we have $(ax+y)\alpha = a(x\alpha) + y\alpha\colon v \mapsto \alpha(-(ax+y)v) = a\alpha(-xv) + \alpha(-yv)$, for all $x,y \in \mathfrak{L}$, hence this defines an R-linear map $\mathfrak{L} \to \mathrm{End}_R(V^*)$; and finally we get $[x,y]\alpha = x(y\alpha) - y(x\alpha)\colon v \mapsto \alpha(-[x,y]v) = \alpha(-xyv + yxv) = -\alpha(xyv) + \alpha(yxv) = (x\alpha)(yv) - (y\alpha)(xv) = -(y(x\alpha))(v) + (x(y\alpha))(v)$, saying that the map $\mathfrak{L} \to \mathrm{End}_R(V^*)$ is a homomorphism of Lie R-algebras. $\sharp$

This construction is universal in the following sense: The map $S\colon \mathfrak{L} \to \mathfrak{L}\colon x \mapsto -x$ is an **antiautomorphism** of the Lie R-algebra $\mathfrak{L}$, also being called the **antipode** of $\mathfrak{L}$: We have $S \in \mathrm{End}_R(\mathfrak{L})$, and $S([x,y]) = -[x,y] = [y,x] = [-y,-x] = [S(y),S(x)]$, for all $x,y \in \mathfrak{L}$; and from $S^2 = \mathrm{id}_{\mathfrak{L}}$ we infer that S is bijective. Hence the $\mathfrak{L}$-action on the contragredient module V^* is given as $x\alpha\colon v \mapsto \alpha(S(x)\cdot v)$, for all $x \in \mathfrak{L}$ and $\alpha \in V^*$, where the reversing property of S ensures that V^* is a (left) $\mathfrak{L}$-module again rather than a right $\mathfrak{L}$-module.

ii) The tensor product $V \otimes_R W$ becomes an $\mathfrak{L}$-module by letting $x(v \otimes w) := xv \otimes w + v \otimes xw$, for all $x \in \mathfrak{L}$ and $v \in V$ and $w \in W$:

Let first $\widehat{x}\colon V \times W \to V \otimes_R W\colon (v,w) \mapsto xv \otimes w + v \otimes xw$. Then we get $(v+v',w) \mapsto x(v+v')\otimes w + (v+v')\otimes xw = (xv\otimes w + v\otimes xw) + (xv'\otimes w + v'\otimes xw) = x(v,w) + x(v',w)$ and $(v,w+w') \mapsto xv\otimes(w+w') + v\otimes x(w+w') = (xv\otimes w + v\otimes xw) + (xv\otimes w' + v\otimes xw') = x(v,w) + x(v,w')$, for all $v,v' \in V$ and $w,w' \in W$, as well as $(av,w),(v,aw) \mapsto x(av)\otimes w + (av)\otimes xw = a(xv\otimes w + v\otimes xw) = xv\otimes(aw) + v\otimes x(aw)$, for all $a \in R$, showing that $\widehat{x}$ is R-bilinear. Thus there is a well-defined and unique R-endomorphism of $V \otimes_R W$ as claimed.

Moreover, we have $(ax+y)(v\otimes w) = (ax+y)v\otimes w + v\otimes(ax+y)w = a(xv\otimes w + v\otimes xw) + (yv\otimes w + v\otimes yw) = ax(v\otimes w) + y(v\otimes w)$, for all $x,y \in \mathfrak{L}$, hence this defines an R-linear map $\mathfrak{L} \to \mathrm{End}_R(V\otimes_R W)$; and finally we get $[x,y](v\otimes w) = [x,y]v\otimes w + v\otimes[x,y]w = (xyv - yxv)\otimes w + v\otimes(xyw - yxw) =$

$xyv\otimes w - yxv\otimes w + v\otimes xyw - v\otimes yxw = (xyv\otimes w + v\otimes xyw) + (xv\otimes yw + yv\otimes xw) - (yxv\otimes w + v\otimes yxw) - (xv\otimes yw + yv\otimes xw) = x(yv\otimes w + v\otimes yw) - y(xv\otimes w + v\otimes xw) = x(y(v\otimes w)) - y(x(v\otimes w))$, saying that the map $\mathfrak{L}\to \mathrm{End}_R(V\otimes_R W)$ is a homomorphism of Lie R-algebras. $\sharp$

Similarly, we would like to have a universal construction giving rise to the $\mathfrak{L}$-module structure on tensor products. But in order to write down the action of $\mathfrak{L}$ we need a unital algebra, bringing us into the realm of associative algebras. Actually, there is the notion of the **universal enveloping algebra** $\mathfrak{U}(\mathfrak{L})$ of $\mathfrak{L}$, for which we have an embedding of Lie algebras $\mathfrak{L}\to\mathfrak{U}(\mathfrak{L})$. Moreover, we have $\mathfrak{U}(\mathfrak{L}\oplus\mathfrak{L})\cong\mathfrak{U}(\mathfrak{L})\otimes_R\mathfrak{U}(\mathfrak{L})$, so that the diagonal embedding of Lie algebras $\mathfrak{L}\to\mathfrak{L}\oplus\mathfrak{L}$ gives rise to an embedding of Lie algebras $\Delta\colon\mathfrak{L}\to\mathfrak{U}(\mathfrak{L})\otimes_R\mathfrak{U}(\mathfrak{L})\colon x\mapsto x\otimes 1+1\otimes x$, whose extension to $fU(\mathfrak{L})$ is called the **comultiplication**. Now the $\mathfrak{L}$-module V becomes a $\mathfrak{U}(\mathfrak{L})$-module, hence $V\otimes_R W$ becomes a $\mathfrak{U}(\mathfrak{L})\otimes_R\mathfrak{U}(\mathfrak{L})$-module, and by restriction along Δ the latter becomes an $\mathfrak{L}$-module as desired. Finally, the antipode, extended to $\mathfrak{U}(\mathfrak{L})$, and the comultiplication fulfill certain compatibility rules, so that $\mathfrak{U}(\mathfrak{L})$ actually becomes a **Hopf algebra**. We do not present any details on this here.

b) Combining the above constructions we in particular get the following: Let both V and W be finitely generated R-free R-modules. Then we have the R-isomorphism $\eta\colon W\otimes_R V^*\to\mathrm{Hom}_R(V,W)\colon w\otimes\alpha\mapsto\big(V\to W\colon v\mapsto\alpha(v)\cdot w\big)$:

Note first that both V^* and $V\otimes_R W$ are finitely generated R-free R-modules as well. More precisely, letting $\mathcal{B}:=\{v_1,\ldots,v_n\}\subseteq V$ and $\mathcal{C}:=\{w_1,\ldots,w_m\}\subseteq W$ be R-bases, where $n:=\mathrm{rk}_R(V)\in\mathbb{N}_0$ and $m:=\mathrm{rk}_R(W)\in\mathbb{N}_0$, then the **dual** R-basis $\mathcal{B}^*:=\{v_1^*,\ldots,v_n^*\}\subseteq V^*$ is defined by $v_i^*(v_j)=\delta_{ij}$, for all $i,j\in\{1,\ldots,n\}$, and we have the R-basis $\{v_1\otimes w_1,\ldots,v_1\otimes w_m,v_2\otimes w_1,\ldots,v_n\otimes w_m\}\subseteq V\otimes_R W$.

For the map $\widehat{\eta}\colon W\times V^*\to\mathrm{Hom}_R(V,W)\colon(w,\alpha)\mapsto\big(V\to W\colon v\mapsto\alpha(v)w\big)$ we have $\widehat{\eta}(w+w',\alpha)=\widehat{\eta}(w,\alpha)+\widehat{\eta}(w',\alpha)=\big(v\mapsto\alpha(v)(w+w')=\alpha(v)w+\alpha(v)w'\big)$, for all $w,w'\in W$, and $\widehat{\eta}(w,\alpha+\beta)=\widehat{\eta}(w,\alpha)+\widehat{\eta}(w,\beta)=\big(v\mapsto(\alpha(v)+\beta(v))w=\alpha(v)w+\beta(v)w\big)$, for all $\alpha,\beta\in V^*$, as well as $\widehat{\eta}(aw,\alpha)=\widehat{\eta}(w,a\alpha)\mapsto\big(v\mapsto\alpha(v)(aw)=a\alpha(v)w\big)$, for all $a\in R$. Hence $\widehat{\eta}$ is R-bilinear, thus giving rise to a well-defined and unique R-homomorphism η as claimed.

It remains to be shown that η is bijective: To this end, for $\varphi\in\mathrm{Hom}_R(V,W)$ let $\varphi(v_i)=\sum_{j=1}^m a_{ji}w_j$, for all $i\in\{1,\ldots,n\}$, where ${}_{\mathcal{C}}\varphi_{\mathcal{B}}=[a_{ji}]_{ji}\in R^{m\times n}$ is the matrix of φ with respect to the R-bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Then letting $\Phi:=\sum_{i=1}^n\sum_{j=1}^m a_{ji}\cdot w_j\otimes v_i^*\in W\otimes_R V^*$ we have $\eta(\Phi)\colon v_k\mapsto\sum_{j=1}^m\sum_{i=1}^n a_{ji}\cdot v_i^*(v_k)\cdot w_j=\sum_{j=1}^m a_{jk}w_j=\varphi(v_k)$, for all $k\in\{1,\ldots,n\}$, thus $\eta(\Phi)=\varphi$, showing that η is surjective.

Similarly, if $\Phi:=\sum_{i=1}^n\sum_{j=1}^m a_{ji}\cdot w_j\otimes v_i^*\in W\otimes_R V^*$, where $a_{ji}\in R$, such that $\eta(\Phi)=0$, then we have $\sum_{j=1}^m\sum_{i=1}^n a_{ji}\cdot v_i^*(v_k)\cdot w_j=\sum_{j=1}^m a_{jk}w_j=0\in W$, for all $k\in\{1,\ldots,n\}$, hence $a_{jk}=0\in R$, for all $j\in\{1,\ldots,m\}$, thus $\Phi=0$, showing that η is injective. $\sharp$

Hence $\mathrm{Hom}_R(V,W) \cong W \otimes_R V^*$ becomes an $\mathfrak{L}$-module, where the action of $\mathfrak{L}$ is given as follows: For $x \in \mathfrak{L}$ we have $x(w \otimes \alpha) = xw \otimes \alpha + w \otimes x\alpha = xw \otimes \alpha - w \otimes \alpha(x\cdot ?)$, for all $w \in W$ and $\alpha \in V^*$, where $\alpha(x\cdot ?)$ denotes the R-linear form on V obtained from α by pre-composing with the action of x. Hence for $\varphi_{w,\alpha} := \eta(w \otimes \alpha) \colon V \to W \colon v \mapsto \alpha(v) \cdot w$ we get $x\varphi_{w,\alpha} \colon V \to W \colon v \mapsto \alpha(v) \cdot xw - \alpha(xv) \cdot w = x(\varphi_{w,\alpha}(v)) - \varphi_{w,\alpha}(xv)$. Thus by R-linearity we get $x\varphi \colon V \to W \colon v \mapsto x(\varphi(v)) - \varphi(xv)$, for all $\varphi \in \mathrm{Hom}_R(V,W)$.

In terms of matrices this reads as follows: Let ${}_{\mathcal{B}}x_{\mathcal{B}} \in R^{n\times n}$ and ${}_{\mathcal{C}}x_{\mathcal{C}} \in R^{m\times m}$ be the representing matrices of the action of $x \in \mathfrak{L}$ on V and W, respectively, with respect to the R-bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for the representing matrices of $\varphi \in \mathrm{Hom}_R(V,W)$ and $x\varphi \in \mathrm{Hom}_R(V,W)$, with respect to the R-bases $\mathcal{B}$ and $\mathcal{C}$, we get ${}_{\mathcal{C}}(x\varphi)_{\mathcal{B}} = {}_{\mathcal{C}}x_{\mathcal{C}} \cdot {}_{\mathcal{C}}\varphi_{\mathcal{B}} - {}_{\mathcal{C}}\varphi_{\mathcal{B}} \cdot {}_{\mathcal{B}}x_{\mathcal{B}} \in R^{m\times n}$.

4 Derivations

(4.1) Derivations. a) Let $R \neq \{0\}$ be a commutative ring, and let $\mathfrak{A}$ be a non-associative R-algebra with multiplication μ; then $\mathrm{End}_R(\mathfrak{A})$ is an associative R-algebra, becoming the Lie R-algebra $\mathfrak{gl}(\mathfrak{A})$ with respect to the associated commutator. An element $\partial \in \mathrm{End}_R(\mathfrak{A})$ is called a **derivation** of $\mathfrak{A}$ if the **product rule** $\partial\mu(a,b) = \mu(a,\partial b) + \mu(\partial a, b)$ holds, for all $a,b \in \mathfrak{A}$.

Let $\mathrm{Der}_R(\mathfrak{A}) \subseteq \mathrm{End}_R(\mathfrak{A})$ be the set of all derivations of $\mathfrak{A}$. Since $(\partial+\partial')\mu(a,b) = \partial\mu(a,b)+\partial'\mu(a,b) = \mu(a,\partial b)+\mu(\partial a,b)+\mu(a,\partial' b)+\mu(\partial' a,b) = \mu(a,(\partial+\partial')(b))+\mu((\partial+\partial')(a),b)$ as well as $(r\partial)\mu(a,b) = r\mu(a,\partial b) + r\mu(\partial a,b) = \mu(a,(r\partial)(b)) + \mu((r\partial)(a),b)$, for all $\partial,\partial' \in \mathrm{Der}_R(\mathfrak{A})$ and $a,b \in \mathfrak{A}$ and $r \in R$, we conclude that $\mathrm{Der}_R(\mathfrak{A})$ is an R-submodule of $\mathrm{End}_R(\mathfrak{A})$.

Let $\partial,\partial' \in \mathrm{Der}_R(\mathfrak{A})$. Then we have $\partial\partial'\mu(a,b) = \partial\big(\mu(a,\partial' b) + \mu(\partial' a,b)\big) = \partial\mu(a,\partial' b) + \partial\mu(\partial' a,b) = \mu(a,\partial\partial' b) + \mu(\partial a,\partial' b) + \mu(\partial' a,\partial b) + \mu(\partial\partial' a,b)$, for all $a,b \in \mathfrak{A}$. Hence we get $(\partial\partial'-\partial'\partial)\mu(a,b) = \mu(a,\partial\partial' b)+\mu(\partial a,\partial' b)+\mu(\partial' a,\partial b)+\mu(\partial\partial' a,b) - \mu(a,\partial'\partial b) - \mu(\partial' a,\partial b) - \mu(\partial a,\partial' b) - \mu(\partial'\partial a,b) = \mu(a,\partial\partial' b) + \mu(\partial\partial' a,b)-\mu(a,\partial'\partial b)-\mu(\partial'\partial a,b) = \mu(a,(\partial\partial'-\partial'\partial)(b))+\mu((\partial\partial'-\partial'\partial)(a),b)$, in other words we have $[\partial,\partial']\mu(a,b) = \mu(a,[\partial,\partial'](b))+\mu([\partial,\partial'](a),b)$, showing that $[\partial,\partial'] \in \mathrm{Der}_R(\mathfrak{A}) \subseteq \mathrm{End}_R(\mathfrak{A})$. Thus $\mathrm{Der}_R(\mathfrak{A}) \subseteq \mathfrak{gl}(\mathfrak{A})$ is a Lie R-subalgebra, called the **Lie algebra of derivations** of $\mathfrak{A}$.

b) The powers of $\partial \in \mathrm{Der}_R(\mathfrak{A})$ are given by the **Leibniz rule** $\partial^n\mu(a,b) = \sum_{i=0}^n \binom{n}{i}\mu(\partial^i a,\partial^{n-i} b) \in \mathfrak{A}$, for all $n \in \mathbb{N}_0$ and $a,b \in \mathfrak{A}$; here we let $\partial^0 := \mathrm{id}_{\mathfrak{A}}$:

We use induction on $n \in \mathbb{N}_0$; the case $n = 0$ being trivial, let $n \geq 1$. Then we have $\partial^n\mu(a,b) = \sum_{i=0}^{n-1}\binom{n-1}{i}\partial\mu(\partial^i a,\partial^{n-i-1} b) = \sum_{i=0}^{n-1}\binom{n-1}{i}\big(\mu(\partial^i a,\partial^{n-i} b) + \mu(\partial^{i+1} a,\partial^{n-i-1} b)\big) = \sum_{i=0}^{n-1}\binom{n-1}{i}\mu(\partial^i a,\partial^{n-i} b) + \sum_{i=1}^{n}\binom{n-1}{i-1}\mu(\partial^i a,\partial^{n-i} b) = \mu(\partial^0 a,\partial^n b) + \mu(\partial^n a,\partial^0 b) + \sum_{i=1}^{n-1}\big(\binom{n-1}{i-1} + \binom{n-1}{i}\big)\mu(\partial^i a,\partial^{n-i} b)$, where $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$, for all $i \in \{1,\ldots,n-1\}$, and $\binom{n}{0} = 1 = \binom{n}{n}$ implies the claim. $\sharp$

In particular, for $n = 2$ we have $\partial^2\mu(a,b) = \mu(a,\partial^2 b) + 2\mu(\partial a,\partial b) + \mu(\partial^2 a,b)$, which shows that ∂^2 in general is not a derivation, thus composition of maps in general does not induce an associative R-algebra structure on $\mathrm{Der}_R(\mathfrak{A})$.

(4.2) Inner derivations. Let $R \neq \{0\}$ be a commutative ring. For associative or Lie algebras there are certain derivations arising naturally:

a) Let $\mathfrak{A}$ be an associative R-algebra. Given $x \in \mathfrak{A}$, let $\mathrm{ad}_{\mathfrak{A}}(x) := \lambda_{\mathfrak{A}}(x) - \rho_{\mathfrak{A}}(x) \in \mathrm{End}_R(\mathfrak{A})$ be the associated **adjoint map**, that is $\mathrm{ad}_{\mathfrak{A}}(x)\colon a \mapsto xa - ax$, for all $a \in \mathfrak{A}$. The map $\mathrm{ad}_{\mathfrak{A}}\colon \mathfrak{A} \to \mathrm{End}_R(\mathfrak{A})$ is R-linear, but it is not a representation of $\mathfrak{A}$. But we have $\mathrm{ad}_{\mathfrak{A}}(x)(ab) = xab - abx = xab - axb + axb - abx = a(xb - bx) + (xa - ax)b = a \cdot \mathrm{ad}_{\mathfrak{A}}(x)(b) + \mathrm{ad}_{\mathfrak{A}}(x)(a) \cdot b$, for all $a, b \in \mathfrak{A}$. Thus we have $\mathrm{ad}_{\mathfrak{A}}(x) \in \mathrm{Der}_R(\mathfrak{A})$, called the associated **inner** derivation of $\mathfrak{A}$.

Let $\mathrm{ad}_{\mathfrak{A}}(\mathfrak{A}) \leq_R \mathrm{Der}_R(\mathfrak{A})$ be the set of all inner derivations of $\mathfrak{A}$. Note that we have $\mathrm{ad}_{\mathfrak{A}}(\mathfrak{A}) = \{0\}$ if and only if $\mathfrak{A}$ is commutative.

Then $\mathrm{ad}_{\mathfrak{A}}(\mathfrak{A}) \trianglelefteq \mathrm{Der}_R(\mathfrak{A})$ turns out to be an ideal: For $\partial \in \mathrm{Der}_R(\mathfrak{A})$, we have $[\partial, \mathrm{ad}_{\mathfrak{A}}(x)](a) = \partial \mathrm{ad}_{\mathfrak{A}}(x)(a) - \mathrm{ad}_{\mathfrak{A}}(x)(\partial a) = \partial(xa - ax) - \big(x(\partial a) - (\partial a)x\big) = x(\partial a) + (\partial x)a - a(\partial x) - (\partial a)x - x(\partial a) + (\partial a)x = (\partial x)a - a(\partial x) = \mathrm{ad}_{\mathfrak{A}}(\partial x)(a)$, for all $a \in \mathfrak{A}$; hence we have $[\partial, \mathrm{ad}_{\mathfrak{A}}(x)] = \mathrm{ad}_{\mathfrak{A}}(\partial x)$, for all $x \in \mathfrak{A}$.

b) Let $\mathfrak{L}$ be a Lie R-algebra. For $x \in \mathfrak{L}$ let $\mathrm{ad}_{\mathfrak{L}}(x) \in \mathrm{End}_R(\mathfrak{L})$ be the associated adjoint map. Then using the Jacobi identity we have $\mathrm{ad}_{\mathfrak{L}}(x)[a,b] = [x,[a,b]] = -[a,[b,x]] - [b,[x,a]] = -[a,[b,x]] - [b,[x,a]] = [a,[x,b]] + [[x,a],b] = [a, \mathrm{ad}_{\mathfrak{L}}(x)(b)] + [\mathrm{ad}_{\mathfrak{L}}(x)(a), b]$, for all $a, b \in \mathfrak{L}$. This shows that $\mathrm{ad}_{\mathfrak{A}}(x) \in \mathrm{Der}_R(\mathfrak{L})$, being called the associated **inner** derivation of $\mathfrak{L}$.

Let $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \leq_R \mathrm{Der}_R(\mathfrak{L})$ be the set of all inner derivations of $\mathfrak{L}$. Then $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \trianglelefteq \mathrm{Der}_R(\mathfrak{L})$ turns out to be an ideal: For $\partial \in \mathrm{Der}_R(\mathfrak{L})$, we have $[\partial, \mathrm{ad}_{\mathfrak{L}}(x)](a) = \partial \mathrm{ad}_{\mathfrak{L}}(x)(a) - \mathrm{ad}_{\mathfrak{L}}(x)(\partial a) = \partial([x,a]) - [x, \partial a] = [x, \partial a] + [\partial x, a] - [x, \partial a] = [\partial x, a] = \mathrm{ad}_{\mathfrak{L}}(\partial x)(a)$, for all $a \in \mathfrak{L}$; hence $[\partial, \mathrm{ad}_{\mathfrak{L}}(x)] = \mathrm{ad}_{\mathfrak{L}}(\partial x)$, for all $x \in \mathfrak{L}$.

Note that we have $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) = \{0\}$ if and only if $\mathfrak{L}$ is commutative; and if the Lie structure of $\mathfrak{L}$ is given by the commutator of an associative R-algebra, then the two notions of adjoint maps coincide.

(4.3) Automorphisms. a) Let K be a field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{A}$ be a unital associative K-algebra. Let $x \in \mathfrak{A}$ be nilpotent, that is there is $l \in \mathbb{N}$ such that $x^l = 0$, where $x^0 := 1 \in \mathfrak{A}$. Let the associated **divided powers** be defined as $x^{[n]} := \frac{x^n}{n!} \in \mathfrak{A}$, for all $n \in \mathbb{N}_0$. Let $\exp(x) = \sum_{n \geq 0} x^{[n]} = \sum_{n \geq 0} \frac{x^n}{n!} := \sum_{n=0}^{l-1} \frac{x^n}{n!} \in \mathfrak{A}$ denote the **exponential map** associated with x.

If $x, y \in \mathfrak{A}$ are nilpotent such that $x^l = 0 = y^l$ and $xy = yx$, then $(x+y)^{2l-1} = \sum_{i=0}^{2l-1} \binom{2l-1}{i} x^i y^{2l-1-i} = 0$, hence $x + y \in \mathfrak{A}$ is nilpotent as well, although possibly a larger exponent is needed. Then we get $\exp(x+y) = \sum_{n=0}^{l-1} \frac{(x+y)^n}{n!} = \sum_{n=0}^{l-1} \frac{1}{n!} \big(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \big) = \sum_{n=0}^{l-1} \big(\sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} \big) = \sum_{n \geq 0} \big(\sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} \big) = (\sum_{i \geq 0} \frac{x^i}{i!}) \cdot (\sum_{j \geq 0} \frac{y^j}{j!}) = \exp(x) \cdot \exp(y) = \exp(y) \cdot \exp(x) \in \mathfrak{A}$.

In particular, we have $\exp(x) \cdot \exp(-x) = \exp(x + (-x)) = \exp(0) = 1$, hence we conclude that $\exp(x) \in \mathfrak{A}$ is a unit, where $\exp(x)^{-1} = \exp(-x) \in \mathfrak{A}$.

b) Now let $\mathfrak{A}$ be a non-associative K-algebra, and let $\partial \in \mathrm{Der}_K(\mathfrak{A}) \leq_K \mathrm{End}_K(\mathfrak{A})$ be a derivation. Then, for all $n \in \mathbb{N}_0$, the Leibniz rule becomes $\partial^{[n]}\mu(a,b) = \frac{\partial^n}{n!}\mu(a,b) = \sum_{k=0}^n \mu(\frac{\partial^k}{k!}a, \frac{\partial^{n-k}}{(n-k)!}b) = \sum_{k=0}^n \mu(\partial^{[k]}a, \partial^{[n-k]}b) \in \mathfrak{A}$, for all $a, b \in \mathfrak{A}$.

If ∂ is nilpotent, then for $\exp(\partial) \in \mathrm{End}_K(\mathfrak{A})$ we obtain $\mu(\exp(\partial)a, \exp(\partial)b) = \mu(\sum_{i\geq 0}\frac{\partial^i}{i!}a, \sum_{j\geq 0}\frac{\partial^j}{j!}b) = \sum_{n\geq 0}\big(\sum_{k=0}^n \mu(\frac{\partial^k}{k!}a, \frac{\partial^{n-k}}{(n-k)!}b)\big) = \sum_{n\geq 0}\frac{\partial^n}{n!}\mu(a,b) = \exp(\partial)\mu(a,b)$, for all $a, b \in \mathfrak{A}$, hence $\exp(\partial)$ is a K-algebra automorphism of $\mathfrak{A}$.

c) Let $\mathfrak{L}$ be a Lie K-algebra, and let $x \in \mathfrak{L}$ such that $\mathrm{ad}_{\mathfrak{L}}(x) \in \mathrm{Der}_K(\mathfrak{L}) \leq_K \mathrm{End}_K(\mathfrak{L})$ is nilpotent. This gives rise to the **adjoint** or **inner** Lie K-algebra automorphism $\mathrm{Ad}_{\mathfrak{L}}(x) := \exp(\mathrm{ad}_{\mathfrak{L}}(x))$ of $\mathfrak{L}$. Let $\mathrm{Aut}(\mathfrak{L}) \leq \mathrm{GL}(\mathfrak{L})$ be the group of all Lie K-algebra automorphisms of $\mathfrak{L}$, and let $\mathrm{Inn}(\mathfrak{L}) := \langle \mathrm{Ad}_{\mathfrak{L}}(x) \in \mathrm{Aut}(\mathfrak{L}); x \in \mathfrak{L}, \mathrm{ad}_{\mathfrak{L}}(x) \text{ nilpotent}\rangle$ be the subgroup generated by all inner automorphisms.

Then $\mathrm{Inn}(\mathfrak{L}) \trianglelefteq \mathrm{Aut}(\mathfrak{L})$ is a normal subgroup: For any $x \in \mathfrak{L}$ such that $\mathrm{ad}_{\mathfrak{L}}(x)$ is nilpotent, and any $\alpha \in \mathrm{Aut}(\mathfrak{L})$, we have $\alpha \cdot \mathrm{ad}_{\mathfrak{L}}(x) \cdot \alpha^{-1} = \mathrm{ad}_{\mathfrak{L}}(\alpha(x)) \colon a \mapsto \alpha([x, \alpha^{-1}(a)]) = [\alpha(x), a]$, for all $a \in \mathfrak{L}$; hence $\mathrm{ad}_{\mathfrak{L}}(\alpha(x))$ is nilpotent as well, and we have $\alpha \cdot \mathrm{Ad}_{\mathfrak{L}}(x) \cdot \alpha^{-1} = \mathrm{Ad}_{\mathfrak{L}}(\alpha(x))$.

d) Let $\mathfrak{A}$ be an associative K-algebra, and let $x \in \mathfrak{A}$ be nilpotent, that is there is $l \in \mathbb{N}$ such that $x^l = 0$. Then we have $\lambda_{\mathfrak{A}}(x)^l = \lambda_{\mathfrak{A}}(x^l) = 0$ and $\rho_{\mathfrak{A}}(x)^l = \rho_{\mathfrak{A}}(x^l) = 0$, thus both $\lambda_{\mathfrak{A}}(x), \rho_{\mathfrak{A}}(x) \in \mathrm{End}_K(\mathfrak{A})$ are nilpotent; note that we have $\exp(\lambda_{\mathfrak{A}}(x)) = \lambda_{\mathfrak{A}}(\exp(x))$ and $\exp(\rho_{\mathfrak{A}}(x)) = \rho_{\mathfrak{A}}(\exp(x))$.

Since $\mathfrak{A}$ is associative, we have $\lambda_{\mathfrak{A}}(x)\rho_{\mathfrak{A}}(x) = \rho_{\mathfrak{A}}(x)\lambda_{\mathfrak{A}}(x)$, hence $\mathrm{ad}_{\mathfrak{A}}(x) = \lambda_{\mathfrak{A}}(x) + \rho_{\mathfrak{A}}(-x) \in \mathrm{End}_K(\mathfrak{A})$ is nilpotent as well. Note that the argument showing the nilpotence of $\mathrm{ad}_{\mathfrak{A}}(x) \in \mathrm{End}_K(\mathfrak{A})$ holds for fields of arbitrary characteristic, and that $\mathrm{ad}_{\mathfrak{A}}(x) \in \mathrm{End}_K(\mathfrak{A})$ being nilpotent does not imply that $x \in \mathfrak{A}$ is nilpotent, as for example the identity element in the unital case shows.

Hence the nilpotent element $x \in \mathfrak{A}$ gives rise to the **adjoint** K-algebra automorphism $\mathrm{Ad}_{\mathfrak{A}}(x) := \exp(\mathrm{ad}_{\mathfrak{A}}(x)) = \exp(\lambda_{\mathfrak{A}}(x) + \rho_{\mathfrak{A}}(-x)) = \exp(\lambda_{\mathfrak{A}}(x)) \cdot \exp(\rho_{\mathfrak{A}}(-x)) = \lambda_{\mathfrak{A}}(\exp(x)) \cdot \rho_{\mathfrak{A}}(\exp(-x)) = \lambda_{\mathfrak{A}}(\exp(x)) \cdot \rho_{\mathfrak{A}}(\exp(x)^{-1})$ of $\mathfrak{A}$. In other words we have $\mathrm{Ad}_{\mathfrak{A}}(x) = \kappa_{\mathfrak{A}}(\exp(x)) \colon a \mapsto \exp(x) \cdot a \cdot \exp(x)^{-1}$, for all $a \in \mathfrak{A}$, that is $\mathrm{Ad}_{\mathfrak{A}}(x)$ coincides with the inner K-algebra automorphism $\kappa_{\mathfrak{A}}(\exp(x))$ of $\mathfrak{A}$ associated with $\exp(x) \in \mathfrak{A}$.

For any K-algebra automorphism α of $\mathfrak{A}$ we have $\alpha(\exp(x)) = \exp(\alpha(x)) \in \mathfrak{A}$. Hence we get $\alpha \cdot \mathrm{Ad}_{\mathfrak{A}}(x) \cdot \alpha^{-1} = \mathrm{Ad}_{\mathfrak{A}}(\alpha(x)) \colon a \mapsto \alpha\big(\exp(x) \cdot \alpha^{-1}(a) \cdot \exp(x)^{-1}\big) = \exp(\alpha(x)) \cdot a \cdot \exp(\alpha(x))^{-1}$, for all $a \in \mathfrak{A}$; alternatively, using the Lie K-algebra structure of $\mathfrak{A}$ given by commutators, this follows directly from part c).

(4.4) Example: The special linear algebra of degree 2. a) Let K be a field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L} := \mathfrak{sl}_2(K)$. Letting $\{E, H, F\} \subseteq \mathfrak{L}$ be the standard K-basis, by (2.2) the adjoint representation $\mathrm{ad}_{\mathfrak{L}}\colon \mathfrak{L} \to \mathfrak{gl}_3(K)$ equals

$$\mathrm{ad}_{\mathfrak{L}}(E) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_{\mathfrak{L}}(H) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathrm{ad}_{\mathfrak{L}}(F) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Note that the standard K-basis consists of eigenvectors of $\mathrm{ad}_{\mathfrak{L}}(H)$.

The matrices $\mathrm{ad}_{\mathfrak{L}}(E), \mathrm{ad}_{\mathfrak{L}}(F) \in \mathfrak{gl}_3(K)$ are nilpotent, where

$$\mathrm{ad}_{\mathfrak{L}}(E)^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{ad}_{\mathfrak{L}}(F)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix},$$

and thus $\mathrm{ad}_{\mathfrak{L}}(E)^3 = 0 = \mathrm{ad}_{\mathfrak{L}}(F)^3$. This yields the Lie K-algebra automorphisms

$$\mathrm{Ad}_{\mathfrak{L}}(aE) = E_3 + a \cdot \mathrm{ad}_{\mathfrak{L}}(E) + \frac{a^2}{2} \cdot \mathrm{ad}_{\mathfrak{L}}(E)^2 = \begin{bmatrix} 1 & -2a & -a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}_3(K),$$

$$\mathrm{Ad}_{\mathfrak{L}}(bF) = E_3 + b \cdot \mathrm{ad}_{\mathfrak{L}}(F) + \frac{b^2}{2} \cdot \mathrm{ad}_{\mathfrak{L}}(F)^2 = \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ -b^2 & 2b & 1 \end{bmatrix} \in \mathrm{GL}_3(K),$$

for all $a, b \in K$. In particular, we get

$$\mathrm{Ad}_{\mathfrak{L}}(E) \cdot \mathrm{Ad}_{\mathfrak{L}}(-F) \cdot \mathrm{Ad}_{\mathfrak{L}}(E) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}_3(K),$$

that is the Lie K-algebra automorphism $\sigma \in \mathrm{Inn}(\mathcal{L})$ given by $\sigma(E) = -F$ and $\sigma(F) = -E$ and $\sigma(H) = -H$.

b) We consider $\mathfrak{L} = \mathfrak{sl}_2(K) \subseteq \mathfrak{gl}_2(K) =: \widehat{\mathfrak{L}}$. Now $\widehat{\mathfrak{L}}$ carries the structure of an associative K-algebra, hence the adjoint automorphisms of $\widehat{\mathfrak{L}}$ as an associative K-algebra are inner automorphisms of $\widehat{\mathfrak{L}}$ as a Lie K-algebra. More specifically: The elements $E, F \in \widehat{\mathfrak{L}}$ are nilpotent such that $E^2 = 0 = F^2$. Hence we have $\exp(aE) = E_2 + aE = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(K) \leq \mathrm{GL}_2(K)$, and similarly $\exp(bF) = E_2 + bF = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \in \mathrm{SL}_2(K) \leq \mathrm{GL}_2(K)$, for all $a, b \in K$.

This givies rise to inner automorphisms $\mathrm{Ad}_{\widehat{\mathfrak{L}}}(?) = \exp(\mathrm{ad}_{\widehat{\mathfrak{L}}}(?)) = \kappa_{\widehat{\mathfrak{L}}}(\exp(?))$ of $\widehat{\mathfrak{L}}$, and since the elements under consideration are in $\mathfrak{L}$, the latter automorphisms restrict to $\mathfrak{L}$, giving the automorphisms $\mathrm{Ad}_{\widehat{\mathfrak{L}}}(?)|_{\mathfrak{L}} = \exp(\mathrm{ad}_{\widehat{\mathfrak{L}}}(?))|_{\mathfrak{L}} = \exp(\mathrm{ad}_{\widehat{\mathfrak{L}}}(?)|_{\mathfrak{L}}) = \exp(\mathrm{ad}_{\mathfrak{L}}(?)) = \mathrm{Ad}_{\mathfrak{L}}(?)$. Indeed we get explicitly:

$$\mathrm{Ad}_{\widehat{\mathfrak{L}}}(aE)\colon E \mapsto \exp(aE) \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \exp(-aE) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E,$$

$$\mathrm{Ad}_{\widehat{\mathfrak{L}}}(aE)\colon H \mapsto \exp(aE)\cdot\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\cdot\exp(-aE) = \begin{bmatrix}1 & -2a\\ 0 & -1\end{bmatrix} = -2aE + H,$$

$$\mathrm{Ad}_{\widehat{\mathfrak{L}}}(aE)\colon F \mapsto \exp(aE)\cdot\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\cdot\exp(-aE) = \begin{bmatrix}a & -a^2\\ 1 & -a\end{bmatrix} = -a^2E + aH + F,$$

$$\mathrm{Ad}_{\widehat{\mathfrak{L}}}(bF)\colon E \mapsto \exp(bF)\cdot\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\cdot\exp(-bF) = \begin{bmatrix}-b & 1\\ -b^2 & b\end{bmatrix} = E - bH - b^2F,$$

$$\mathrm{Ad}_{\widehat{\mathfrak{L}}}(bF)\colon H \mapsto \exp(bF)\cdot\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\cdot\exp(-bF) = \begin{bmatrix}1 & 0\\ 2b & -1\end{bmatrix} = H + 2bF,$$

$$\mathrm{Ad}_{\widehat{\mathfrak{L}}}(bF)\colon F \mapsto \exp(bF)\cdot\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\cdot\exp(-bF) = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} = F.$$

c) Now let K be a field of arbitrary characteristic, and let still $\mathfrak{L} = \mathfrak{sl}_2(K) \subseteq \mathfrak{gl}_2(K) = \widehat{\mathfrak{L}}$. For $A \in \mathrm{GL}_2(K)$ let $\kappa_{\widehat{\mathfrak{L}}}(A)\colon \widehat{\mathfrak{L}} \to \widehat{\mathfrak{L}}\colon M \mapsto AMA^{-1}$ be the inner automorphism of the associative K-algebra $\widehat{\mathfrak{L}}$ associated with A. Hence $\kappa_{\widehat{\mathfrak{L}}}(A)$ also is an automorphism of $\widehat{\mathfrak{L}}$ as Lie K-algebras, and thus we get a group homomorphism $\kappa_{\widehat{\mathfrak{L}}}\colon \mathrm{GL}_2(K) \to \mathrm{Aut}(\widehat{\mathfrak{L}}) \leq \mathrm{GL}(\widehat{\mathfrak{L}}) \cong \mathrm{GL}_4(K)$. Moreover, since $\mathrm{Tr}(AMA^{-1}) = \mathrm{Tr}(M)$, for all $M \in \widehat{\mathfrak{L}}$ and all $A \in \mathrm{GL}_2(K)$, this restricts to a group homomorphism $\kappa_{\widehat{\mathfrak{L}}}|_{\mathfrak{L}}\colon \mathrm{GL}_2(K) \to \mathrm{Aut}(\mathfrak{L}) \leq \mathrm{GL}(\mathfrak{L}) \cong \mathrm{GL}_3(K)$.

Lemma. We have $\ker(\kappa_{\widehat{\mathfrak{L}}}) = \ker(\kappa_{\widehat{\mathfrak{L}}}|_{\mathfrak{L}}) = K^* \cdot E_2$.

Proof. For $t \in K^*$ we have $(tE_2)\cdot B\cdot(t^{-1}E_2) = B$, for all $B \in \widehat{\mathfrak{L}}$, hence $K^*E_2 \leq \ker(\kappa_{\widehat{\mathfrak{L}}})$. If $A := \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \ker(\kappa_{\widehat{\mathfrak{L}}}|_{\mathfrak{L}})$ then $A\cdot B\cdot A^{-1} = B$, for all $B \in \mathfrak{L}$, yields $\begin{bmatrix}0 & a\\ 0 & c\end{bmatrix} = AE = EA = \begin{bmatrix}c & d\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}a & -b\\ c & -d\end{bmatrix} = AH = HA = \begin{bmatrix}a & b\\ -c & -d\end{bmatrix}$ and $\begin{bmatrix}b & 0\\ d & 0\end{bmatrix} = AF = FA = \begin{bmatrix}0 & 0\\ a & b\end{bmatrix}$, thus $b = 0 = c$ and $a = d$, hence $A = aE_2$, where $0 \neq \det(A) = a^2$ yields $a \in K^*$. ♯

Hence we have $\kappa_{\widehat{\mathfrak{L}}}(\mathrm{GL}_2(K)) \cong \mathrm{GL}_2(K)/(K^*\cdot E_2) =: \mathrm{PGL}_2(K)$, the **projective general linear group** of degree 2. Similarly, restricting to $\mathrm{SL}_2(K) \leq \mathrm{GL}_2(K)$ we get $\kappa_{\widehat{\mathfrak{L}}}(\mathrm{SL}_2(K)) \cong \mathrm{SL}_2(K)/\langle -E_2\rangle =: \mathrm{PSL}_2(K)$, the **projective special linear group** of degree 2, where $\ker(\kappa_{\widehat{\mathfrak{L}}}) \cap \mathrm{SL}_2(K) = \langle -E_2\rangle$.

Lemma. We have $\left\langle \begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}; a, b \in K \right\rangle = \mathrm{SL}_2(K)$.

Proof. Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(K)$, where $a, b, c, d \in K$ are such that $\det(A) = ad - bc = 1$. Then we get: If $c \neq 0$ we have $A = \begin{bmatrix} 1 & \frac{a-1}{c} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{d-1}{c} \\ & 1 \end{bmatrix}$; if $b \neq 0$ then we have $A = \begin{bmatrix} 1 & 0 \\ \frac{d-1}{b} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{a-1}{b} & 1 \end{bmatrix}$; if both $b = 0 = c$ then we have $A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{a} - 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ a-1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{a} \\ 0 & 1 \end{bmatrix}$. ♯

To relate this to the above observations on exponentials, letting $\mathrm{char}(K) = 0$, the group $\langle \mathrm{Ad}_{\mathfrak{L}}(aE), \mathrm{Ad}_{\mathfrak{L}}(bF); a, b \in K \rangle = \langle \mathrm{Ad}_{\widehat{\mathfrak{L}}}(aE)|_{\mathfrak{L}}, \mathrm{Ad}_{\widehat{\mathfrak{L}}}(bF)|_{\mathfrak{L}}; a, b \in K \rangle = \langle \kappa_{\widehat{\mathfrak{L}}}(\exp(aE))|_{\mathfrak{L}}, \kappa_{\widehat{\mathfrak{L}}}(\exp(bF))|_{\mathfrak{L}}; a, b \in K \rangle = \Big\langle \kappa_{\widehat{\mathfrak{L}}}\Big(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \Big)|_{\mathfrak{L}}, \kappa_{\widehat{\mathfrak{L}}}\Big(\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \Big)|_{\mathfrak{L}}; a, b \in K \Big\rangle = \kappa_{\widehat{\mathfrak{L}}}\Big(\Big\langle \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}; a, b \in K \Big\rangle\Big)|_{\mathfrak{L}} = \kappa_{\widehat{\mathfrak{L}}}(\mathrm{SL}_2(K))|_{\mathfrak{L}} \leq \mathrm{Inn}(\mathfrak{L})$ is isomorphic to $\mathrm{PSL}_2(K)$.

Here is a couple of particularly interesting automorphisms of $\mathfrak{L}$ thus arising:

i) Letting $S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SL}_2(K)$, we get

$$aE + cH + bF = \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \mapsto S \cdot \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \cdot (-S) = \begin{bmatrix} -c & -b \\ -a & c \end{bmatrix} = -bE - cH - aF,$$

for all $a, b, c \in K$. Thus we obtain the Lie K-algebra automorphism $\sigma \in \mathrm{Aut}(\mathfrak{L})$ given by $\sigma(E) = -F$ and $\sigma(F) = -E$ and $\sigma(H) = -H$; in other words, we have $\sigma(A) = -A^{\mathrm{tr}}$, for all $A \in \mathfrak{L}$. If $\mathrm{char}(K) = 0$, then we have

$$S = \exp(E) \cdot \exp(-F) \cdot \exp(E) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and $\sigma \in \mathrm{Inn}(\mathfrak{L})$ coincides with the inner automorphism encountered earlier.

ii) For all $t \in K^*$ let $T(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \in \mathrm{SL}_2(K)$. Then $T := T(t)$ induces

$$aE + cH + bF = \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \mapsto T \cdot \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \cdot T^{-1} = \begin{bmatrix} c & t^2 a \\ \frac{b}{t^2} & -c \end{bmatrix} = t^2 aE + cH + \frac{b}{t^2} F,$$

for all $a, b, c \in K$. Thus we get the Lie K-algebra automorphism $\eta(t) \in \mathrm{Aut}(\mathfrak{L})$ given by $\eta(t) \colon E \mapsto t^2 E$ and $\eta(t) \colon H \mapsto H = t^0 H$ and $\eta(t) \colon F \mapsto t^{-2} F$. Note that the standard K-basis of $\mathfrak{L}$ consists of eigenvectors of $\eta(t)$, and that the corresponding eigenvalues are given as powers of $t \in K^*$, where he exponents are related to the eigenvalues of $\mathrm{ad}_{\mathfrak{L}}(H)$ occurring in the case $\mathrm{char}(K) = 0$.

Recall that we have noted earlier that

$$T = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{t} - 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ t-1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{t} \\ 0 & 1 \end{bmatrix},$$

which expresses T as a word in the chosen generators of $\mathrm{SL}_2(K)$.

5 Solvable algebras I

(5.1) Descending series. a) Let $R \neq \{0\}$ be a commutative ring, and let $\mathfrak{L}$ be a Lie R-algebra. Let $\mathfrak{L}^{(1)} := [\mathfrak{L}, \mathfrak{L}] = \langle [a,b] \in \mathfrak{L}; a, b \in \mathfrak{L} \rangle_R \trianglelefteq \mathfrak{L}$, being called the **derived subalgebra** of $\mathfrak{L}$. Then $\mathfrak{L}$ is called **perfect** if $\mathfrak{L}^{(1)} = \mathfrak{L}$, and $\mathfrak{L}^{(1)} = \{0\}$ if and only if $\mathfrak{L}$ is commutative.

Iterating this yields the **derived series** of $\mathfrak{L}$ defined by $\mathfrak{L}^{(i+1)} = [\mathfrak{L}^{(i)}, \mathfrak{L}^{(i)}]$, for all $i \in \mathbb{N}$; we let $\mathfrak{L}^{(0)} := \mathfrak{L}$. Then we have $\mathfrak{L}^{(i)} \trianglelefteq \mathfrak{L}$, for all $i \in \mathbb{N}_0$. The Lie algebra $\mathfrak{L}$ is called **solvable** if there is $l \in \mathbb{N}_0$ such that $\mathfrak{L}^{(l)} = \{0\}$; if l is chosen minimal then $\mathfrak{L}$ is said to have **derived length** l. In particular, if $\mathfrak{L} \neq \{0\}$, then $\mathfrak{L}$ is solvable of derived length 1 if and only if $\mathfrak{L}$ is commutative; while for perfect $\mathfrak{L}$ we have $\mathfrak{L}^{(i)} = \mathfrak{L}$, for all $i \in \mathbb{N}_0$, hence $\mathfrak{L}$ is not solvable.

Proposition. i) If $\mathfrak{L}$ is solvable, then so are all R-subalgebras and quotients.
ii) Conversely, if $\mathfrak{I} \trianglelefteq \mathfrak{L}$ such that both $\mathfrak{I}$ and $\mathfrak{L}/\mathfrak{I}$ are solvable, then so is $\mathfrak{L}$.
iii) If $\mathfrak{I}, \mathfrak{J} \trianglelefteq \mathfrak{L}$ are solvable, then so is $\mathfrak{I} + \mathfrak{J}$.

Proof. i) If $\mathfrak{K} \subseteq \mathfrak{L}$ is a Lie R-subalgebra, then we have $\mathfrak{K}^{(i)} \subseteq \mathfrak{L}^{(i)}$, for all $i \in \mathbb{N}_0$. Similarly, if $\varphi \colon \mathfrak{L} \to \mathfrak{K}$ is an epimorphism of Lie R-algebras, then we have $\varphi(\mathfrak{L}^{(i)}) = \mathfrak{K}^{(i)}$, for all $i \in \mathbb{N}_0$: We have $\varphi(\mathfrak{L}^{(0)}) = \mathfrak{K}^{(0)}$, and $\mathfrak{K}^{(i+1)} = [\mathfrak{K}^{(i)}, \mathfrak{K}^{(i)}] = [\varphi(\mathfrak{L}^{(i)})), \varphi(\mathfrak{L}^{(i)})] = \varphi([\mathfrak{L}^{(i)}, \mathfrak{L}^{(i)}]) = \mathfrak{L}^{(i+1)}$.

ii) We consider the natural map $\nu_{\mathfrak{I}} \colon \mathfrak{L} \to \mathfrak{L}/\mathfrak{I}$. From $(\mathfrak{L}/\mathfrak{I})^{(l)} = \{0\}$, for some $l \in \mathbb{N}_0$, we get $\nu_{\mathfrak{I}}(\mathfrak{L}^{(l)}) = \nu_{\mathfrak{I}}(\mathfrak{L})^{(l)} = \{0\}$, thus $\mathfrak{L}^{(l)} \subseteq \ker(\nu_{\mathfrak{I}}) = \mathfrak{I}$. Then, from $\mathfrak{I}^{(m)} = \{0\}$, for some $m \in \mathbb{N}_0$, we get $\mathfrak{L}^{(l+m)} = (\mathfrak{L}^{(l)})^{(m)} \subseteq \mathfrak{I}^{(m)} = \{0\}$.

iii) From the homomorphism principle, applied to the restriction to $\mathfrak{I}$ of the natural map $\nu_{\mathfrak{J}} \colon \mathfrak{L} \to \mathfrak{L}/\mathfrak{J}$, we get $\mathfrak{I}/(\mathfrak{I} \cap \mathfrak{J}) \cong (\mathfrak{I} + \mathfrak{J})/\mathfrak{J}$. Thus from $\mathfrak{I}$ being solvable, we infer that $(\mathfrak{I}+\mathfrak{J})/\mathfrak{J}$ is solvable as well, hence $\mathfrak{J}$ being solvable entails that $\mathfrak{I} + \mathfrak{J}$ is solvable. $\sharp$

b) Iterating this in a different way yields the **descending** or **lower central series** of $\mathfrak{L}$ defined by letting $\mathfrak{L}^{[0]} := \mathfrak{L}$ and $\mathfrak{L}^{[1]} := [\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}^{(1)}$, as well as $\mathfrak{L}^{[i+1]} = [\mathfrak{L}, \mathfrak{L}^{[i]}]$, for all $i \in \mathbb{N}$. Then we have $\mathfrak{L}^{[i]} \trianglelefteq \mathfrak{L}$, for all $i \in \mathbb{N}_0$. The Lie algebra $\mathfrak{L}$ is called **nilpotent** if there is $l \in \mathbb{N}_0$ such that $\mathfrak{L}^{[l]} = \{0\}$; if l is chosen minimal then $\mathfrak{L}$ is said to have **nilpotency length** l. In particular, if $\mathfrak{L} \neq \{0\}$, then $\mathfrak{L}$ is nilpotent of nilpotency length 1 if and only if $\mathfrak{L}$ is commutative; while for perfect $\mathfrak{L}$ we have $\mathfrak{L}^{[i]} = \mathfrak{L}$, for all $i \in \mathbb{N}_0$, hence $\mathfrak{L}$ is not nilpotent.

Then we have $[\mathfrak{L}^{[i]}, \mathfrak{L}^{[i]}] \subseteq \mathfrak{L}^{[i+j+1]}$, for all $i, j \in \mathbb{N}_0$: Proceeding by induction on $i \in \mathbb{N}_0$, for $i = 0$ we have $[\mathfrak{L}, \mathfrak{L}^{[j]}] = \mathfrak{L}^{[j+1]}$, for all $j \in \mathbb{N}_0$, and $[\mathfrak{L}^{[i+1]}, \mathfrak{L}^{[j]}] = [[\mathfrak{L}, \mathfrak{L}^{[i]}], \mathfrak{L}^{[j]}] \subseteq [\mathfrak{L}, [\mathfrak{L}^{[i]}, \mathfrak{L}^{[j]}]] + [\mathfrak{L}^{[i]}, [\mathfrak{L}, \mathfrak{L}^{[j]}]] \subseteq [\mathfrak{L}, \mathfrak{L}^{[i+j+1]}] + [\mathfrak{L}^{[i]}, \mathfrak{L}^{[j+1]}] \subseteq \mathfrak{L}^{[i+j+2]}$ In particular, this implies that any iterated product of $i \in \mathbb{N}$ elements of $\mathfrak{L}$ is contained in $\mathfrak{L}^{[i-1]}$.

In particular, since any element of $\mathfrak{L}^{(i)}$ is a sum of products of 2^i elements of $\mathfrak{L}$, we conclude that $\mathfrak{L}^{(i)} \subseteq \mathfrak{L}^{[2^i-1]}$, for all $i \in \mathbb{N}_0$. Hence any nilpotent Lie algebra is solvable, and its derived length $k \in \mathbb{N}_0$ is bounded above by $k \leq \lceil \log_2(l+1) \rceil$; this bound is actually sharp, as is shown by the examples given below.

Proposition. i) If $\mathfrak{L}$ is nilpotent, then so are all R-subalgebras and quotients; moreover, if $\mathfrak{L} \neq \{0\}$ then we have $Z(\mathfrak{L}) \neq \{0\}$.
ii) If $\mathfrak{L}/Z(\mathfrak{L})$ is nilpotent, then so is $\mathfrak{L}$.
iii) If $\mathfrak{I}, \mathfrak{J} \trianglelefteq \mathfrak{L}$ are nilpotent, then so is $\mathfrak{I} + \mathfrak{J}$.

Proof. i) If $\mathfrak{K} \subseteq \mathfrak{L}$ is a Lie R-subalgebra, then we have $\mathfrak{K}^{[i]} \subseteq \mathfrak{L}^{[i]}$, for all $i \in \mathbb{N}_0$. Similarly, if $\varphi\colon \mathfrak{L} \to \mathfrak{K}$ is an epimorphism of Lie R-algebras, then we have $\varphi(\mathfrak{L}^{[i]}) = \mathfrak{K}^{[i]}$, for all $i \in \mathbb{N}_0$: We have $\varphi(\mathfrak{L}^{[0]}) = \mathfrak{K}^{[0]}$, and $\mathfrak{K}^{[i+1]} = [\mathfrak{K}, \mathfrak{K}^{[i]}] = [\varphi(\mathfrak{L}), \varphi(\mathfrak{L}^{[i]})] = \varphi([\mathfrak{L}, \mathfrak{L}^{[i]}]) = \mathfrak{L}^{[i+1]}$. Finally, if $l \geq 1$ is minimal such that $\mathfrak{L}^{[l]} = \{0\}$, we get $[\mathfrak{L}, \mathfrak{L}^{[l-1]}] = \{0\}$, hence $\{0\} \neq \mathfrak{L}^{[l-1]} \subseteq Z(\mathfrak{L})$.

ii) We consider the natural map $\nu_{Z(\mathfrak{L})}\colon \mathfrak{L} \to \mathfrak{L}/Z(\mathfrak{L})$. From $(\mathfrak{L}/Z(\mathfrak{L}))^{[l]} = \{0\}$, for some $l \in \mathbb{N}_0$, we get $\nu_{Z(\mathfrak{L})}(\mathfrak{L}^{[l]}) = \nu_{Z(\mathfrak{L})}(\mathfrak{L})^{[l]} = \{0\}$, thus we have $\mathfrak{L}^{[l]} \subseteq \ker(\nu_{Z(\mathfrak{L})}) = Z(\mathfrak{L})$. Hence we obtain $\mathfrak{L}^{[l+1]} = [\mathfrak{L}, \mathfrak{L}^{[l]}] \subseteq [\mathfrak{L}, Z(\mathfrak{L})] = \{0\}$.

iii) Given any ideal $\mathfrak{K} \trianglelefteq \mathfrak{L}$, we first show that any iterated product of $i \in \mathbb{N}$ elements of $\mathfrak{L}$ of which at least $j \in \{0, \ldots, i\}$ belong to $\mathfrak{K}$ is actually contained on $\mathfrak{K}^{[j-1]}$, where we let $\mathfrak{K}^{[-1]} := \mathfrak{L}$:

We proceed by induction on $i \in \mathbb{N}$; the cases $i \leq 2$ being clear, let $i \geq 3$. Then let $x, y, z \in \mathfrak{L}$, where z is an iterated product of $i-2$ elements. If $x, y \notin \mathfrak{K}$, then at least j of the factors of z belong to $\mathfrak{K}$ and $[x, [y, z]] \in \mathfrak{K}^{[j-1]}$; if $x \notin \mathfrak{K}$ and $y \in \mathfrak{K}$, then at least $j-1$ of the factors of z belong to $\mathfrak{K}$ and $[y, z] \in [\mathfrak{K}, \mathfrak{K}^{[j-2]}] = \mathfrak{K}^{[j-1]}$; if $x \in \mathfrak{K}$ and $y \notin \mathfrak{K}$, then at least $j-1$ of the factors of $[y, z]$ belong to $\mathfrak{K}$ and $[x, [y, z]] \in [\mathfrak{K}, \mathfrak{K}^{[j-2]}] = \mathfrak{K}^{[j-1]}$; if $x, y \in \mathfrak{K}$, then at least $j-2$ of the factors of z belong to $\mathfrak{K}$ and $[x, [y, z]] \in [\mathfrak{K}, [\mathfrak{K}, \mathfrak{K}^{[j-3]}]] = \mathfrak{K}^{[j-1]}$.

Now let $l \in \mathbb{N}$ such that $\mathfrak{I}^{[l]} = \{0\} = \mathfrak{J}^{[l]}$. Then $(\mathfrak{I} + \mathfrak{J})^{[2l]} \trianglelefteq \mathfrak{L}$ consists of sums of iterated products of $2l$ elements of $\mathfrak{L}$ of which at least l belong to $\mathfrak{I}$ or at least l belong to $\mathfrak{J}$. Hence we conclude that $(\mathfrak{I} + \mathfrak{J})^{[2l]} \subseteq \mathfrak{I}^{[l]} + \mathfrak{J}^{[l]} = \{0\}$. $\sharp$

(5.2) Example: Triangular matrices. Let $R \neq \{0\}$ be a commutative ring. We consider various associative R-subalgebras of $R^{n \times n}$, for $n \in \mathbb{N}_0$. Going over to commutators we obtain associated Lie R-subalgebras of $\mathfrak{gl}_n(R)$:

Let $\mathfrak{b}_n(R) := \{A = [a_{ij}]_{ij} \in R^{n \times n}; a_{ij} = 0 \text{ for } i > j\}$ be the **Borel subalgebra** of **upper triangular** matrices, let $\mathfrak{n}_n(R) := \{A = [a_{ij}]_{ij} \in R^{n \times n}; a_{ij} = 0 \text{ for } i \geq j\}$ be the **nilpotent subalgebra** of **strictly upper triangular** matrices, and let $\mathfrak{t}_n(R) := \{A = [a_{ij}]_{ij} \in R^{n \times n}; a_{ij} = 0 \text{ for } i \neq j\}$ be the **toral subalgebra** of **diagonal** matrices.

Then for all $\mathfrak{A} \in \{\mathfrak{b}_n(R), \mathfrak{n}_n(R), \mathfrak{t}_n(R)\}$ we have $\mathfrak{A} \leq_R R^{n \times n}$, as well as $AB \in \mathfrak{A}$, for all $A, B \in \mathfrak{A}$, thus these indeed are associative R-subalgebras of $R^{n \times n}$. Moreover, we have $\mathfrak{b}_n(R) = \mathfrak{n}_n(R) \oplus \mathfrak{t}_n(R)$ as R-modules, where $\mathfrak{n}_n(R)$ is R-free of rank $\frac{1}{2}n(n-1)$ with standard R-basis $\{E_{ij} \in \mathfrak{n}_n(R); i < j \in \{1, \ldots, n\}\}$, and $\mathfrak{t}_n(R)$ is R-free of rank n with standard R-basis $\{E_{ii} \in \mathfrak{t}_n(R); i \in \{1, \ldots, n\}\}$.

i) The Lie R-algebra $\mathfrak{t}_n(R)$ is commutative. Moreover, we have $N_{\mathfrak{gl}_n(R)}(\mathfrak{t}_n(R)) = C_{\mathfrak{gl}_n(R)}(\mathfrak{t}_n(R)) = \mathfrak{t}_n(R)$: If $A \in N_{\mathfrak{gl}_n(R)}(\mathfrak{t}_n(R))$, then we have $[A, E_{ii}] = AE_{ii} - E_{ii}A \in \mathfrak{t}_n(R)$, for all $i \in \{1, \ldots, n\}$, hence A is a diagonal matrix.

ii) We have $[E_{kl}, E_{ij}] = \delta_{li}E_{kj} - \delta_{jk}E_{il}$, for all $i, j, k, l \in \{1, \ldots, n\}$ such that $i < j$ and $k < l$. Hence for the descending central series we have $\mathfrak{n}_n(R)^{[c]} = \langle E_{ij} \in \mathfrak{n}_n(R); j - i > c\rangle_R$, for all $c \in \mathbb{N}_0$: From $\mathfrak{n}_n(R)^{[0]} = \langle E_{ij} \in \mathfrak{n}_n(R); j - i > 0\rangle_R$, by induction we get $\mathfrak{n}_n(R)^{[c+1]} = [\mathfrak{n}_n(R), \mathfrak{n}_n(R)^{[c]}] = \langle E_{ij} \in \mathfrak{n}_n(R); j - i > c + 1\rangle_R$. Thus we conclude that $\mathfrak{n}_n(R)$ is nilpotent of nilpotency length $n - 1$.

For the derived series we have $\mathfrak{n}_n(R)^{(c)} = \langle E_{ij} \in \mathfrak{n}_n(R); j - i \geq 2^c\rangle_R$, for all $c \in \mathbb{N}_0$: From $\mathfrak{n}_n(R)^{(0)} = \langle E_{ij} \in \mathfrak{n}_n(R); j - i \geq 1\rangle_R$, by induction we get $\mathfrak{n}_n(R)^{(c+1)} = [\mathfrak{n}_n(R)^{(c)}, \mathfrak{n}_n(R)^{(c)}] = \langle E_{ij} \in \mathfrak{n}_n(R); j - i \geq 2^c + 2^c = 2^{c+1}\rangle_R$. Thus we conclude that $\mathfrak{n}_n(R)$ is solvable of derived length $\lceil \log_2(n) \rceil$.

iii) We have $[E_{kk}, E_{ij}] = \delta_{ki}E_{ij} - \delta_{kj}E_{ij}$, for all $i, j, k \in \{1, \ldots, n\}$ such that $i < j$. Hence we have $[\mathfrak{t}_n(R), \mathfrak{n}_n(R)] = \mathfrak{n}_n(R)$. For the descending central series this implies $\mathfrak{b}_n(R)^{[c]} = \mathfrak{n}_n(R)$, for all $c \in \mathbb{N}$; in particular we have $\mathfrak{n}_n(R) \trianglelefteq \mathfrak{b}_n(R)$: From $\mathfrak{b}_n(R)^{[1]} = [\mathfrak{b}_n(R), \mathfrak{b}_n(R)] = \mathfrak{n}_n(R) \trianglelefteq \mathfrak{b}_n(R)$, by induction we get $\mathfrak{b}_n(R)^{[c+1]} = [\mathfrak{b}_n(R), \mathfrak{b}_n(R)^{[c]}] = [\mathfrak{b}_n(R), \mathfrak{n}_n(R)] = \mathfrak{n}_n(R)$. Thus we conclude that $\mathfrak{b}_n(R)$ is not nilpotent.

But the restriction to $\mathfrak{t}_n(R)$ of the natural map $\nu_{\mathfrak{n}_n(R)} \colon \mathfrak{b}_n(R) \to \mathfrak{b}_n(R)/\mathfrak{n}_n(R)$ yields an isomorphism $\mathfrak{b}_n(R)/\mathfrak{n}_n(R) \cong \mathfrak{t}_n(R)$, which is commutative and thus solvable. Hence since $\mathfrak{n}_n(R) \trianglelefteq \mathfrak{b}_n(R)$ is a solvable ideal we conclude that $\mathfrak{b}_n(R)$ is solvable. For the derived series we have $\mathfrak{b}_n(R)^{(c)} = \mathfrak{n}_n(R)^{(c-1)}$, for all $c \in \mathbb{N}$, hence the derived length of $\mathfrak{b}_n(R)$ exceeds the derived length of $\mathfrak{n}_n(R)$ by 1, that is equals $1 + \lceil \log_2(n) \rceil$.

Finally, we have $N_{\mathfrak{gl}_n(R)}(\mathfrak{b}_n(R)) = \mathfrak{b}_n(R)$ and $C_{\mathfrak{gl}_n(R)}(\mathfrak{b}_n(R)) = \mathfrak{z}_n(R)$, as well as $N_{\mathfrak{gl}_n(R)}(\mathfrak{n}_n(R)) = \mathfrak{b}_n(R)$ and $C_{\mathfrak{gl}_n(R)}(\mathfrak{n}_n(R)) = \mathfrak{z}_n(R)$: If $A \in \mathfrak{gl}_n(R)$, then $[A, E_{ij}] = AE_{ij} - E_{ij}A \in \mathfrak{b}_n(R)$ for all $i < j \in \{1, \ldots, n\}$, shows that A is an upper triangular matrix, and $[A, E_{ij}] = AE_{ij} - E_{ij}A = 0$, for all $i < j \in \{1, \ldots, n\}$, shows that A is a scalar matrix. Note that we have $C_{\mathfrak{gl}_n(R)}(\mathfrak{b}_n(R)) \subseteq C_{\mathfrak{gl}_n(R)}(\mathfrak{t}_n(R)) = \mathfrak{t}_n(R)$ anyway.

(5.3) Nilpotent Lie algebras. Let K be a field, and $\mathfrak{L}$ be a finite-dimensional Lie K-algebra. An element $x \in \mathfrak{L}$ is called **ad-nilpotent** if $\mathrm{ad}_{\mathfrak{L}}(x) \in \mathfrak{gl}(\mathfrak{L})$ is a nilpotent K-endomorphism, that is there is $l \in \mathbb{N}_0$ such that $\mathrm{ad}_{\mathfrak{L}}(x)^l = 0$.

If $\mathfrak{L}$ is nilpotent of nilpotency length $l \in \mathbb{N}_0$, then $\mathfrak{L}^{[l]} = [\mathfrak{L}, [\mathfrak{L}, [\ldots, [\mathfrak{L}, \mathfrak{L}]]]] = \{0\}$ says that $\mathrm{ad}_{\mathfrak{L}}(x_1)\mathrm{ad}_{\mathfrak{L}}(x_2) \cdots \mathrm{ad}_{\mathfrak{L}}(x_l) = 0 \in \mathfrak{gl}(\mathfrak{L})$, for all $x_1, \ldots, x_l \in \mathfrak{L}$; hence we have $\mathrm{ad}_{\mathfrak{L}}(x)^l = 0 \in \mathfrak{gl}(\mathfrak{L})$, for all $x \in \mathfrak{L}$, thus $x \in \mathfrak{L}$ is ad-nilpotent.

Nicely enough, conversely to this observation, it turns out that complete ad-nilpotency already implies nilpotency. We now proceed to prove this:

Theorem. Let $n \in \mathbb{N}$, and let $\mathfrak{L} \subseteq \mathfrak{gl}_n(K)$ be a Lie K-subalgebra all of whose elements are nilpotent. Then we have $\bigcap_{A \in \mathfrak{L}} \ker(A) \neq \{0\}$, that is there is $0 \neq v \in K^{n \times 1}$ such that $\mathfrak{L} \cdot v = \{0\}$.

Proof. We proceed by induction on $d := \dim_K(\mathcal{L}) \in \mathbb{N}_0$. The case $d = 0$ being trivial, let $d \geq 1$. Note that the case $d = 1$ actually is well-known: In this case we have $\mathfrak{L} := \langle A \rangle_K$, where $0 \neq A \in \mathfrak{gl}_n(K)$ is nilpotent, then its minimum polynomial equals $X^l \in K[X]$, for some $l \in \{1, \ldots, n\}$, hence its characteristic polynomial equals $X^n \in K[X]$, showing that A has an eigenvector in $0 \neq v \in K^{n \times 1}$ with respect to the eigenvalue 0.

Let first $\mathfrak{K} \subset \mathfrak{L}$ be a proper Lie K-subalgebra. Since for $\mathrm{ad}_{\mathfrak{L}} \colon \mathfrak{K} \to \mathfrak{gl}(\mathfrak{L})$ we have $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{K}) \cdot \mathfrak{K} = [\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$, we have an induced representation $\rho \colon \mathfrak{K} \to \mathfrak{gl}(\mathfrak{L}/\mathfrak{K})$. Since $\dim_K(\mathfrak{K}) < d$ there is $v \in \mathfrak{L} \setminus \mathfrak{K}$ such that its natural image $\overline{v} \in \mathfrak{L}/\mathfrak{K}$ fulfills $\rho(\mathfrak{K}) \cdot \overline{v} = \{0\}$, thus $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{K}) \cdot v \subseteq \mathfrak{K}$, showing that $v \in N_{\mathfrak{L}}(\mathfrak{K}) \setminus \mathfrak{K}$.

Let now $\mathfrak{K} \subset \mathfrak{L}$ be a maximal proper Lie K-subalgebra, and let $A \in \mathfrak{L} \setminus \mathfrak{K}$. Then the above argument shows that $N_{\mathfrak{L}}(\mathfrak{K}) = \mathfrak{L}$, that is $\mathfrak{K} \trianglelefteq \mathfrak{L}$. Moreover, $\mathfrak{K} \subset \mathfrak{K} + \langle A \rangle_K \subseteq \mathfrak{L}$ is a Lie K-subalgebra, entailing that $\mathfrak{K} + \langle A \rangle_K = \mathfrak{L}$.

Since $\dim_K(\mathfrak{K}) < d$ again, we have $\{0\} \neq U := \bigcap_{B \in \mathfrak{K}} \ker(B) \leq_K K^{n \times 1}$. Then we have $B(Au) = A(Bu) - [A, B]u = 0$, for all $B \in \mathfrak{K}$ and $u \in U$. Hence we have $A \cdot U \leq_K U$. Thus A induces a nilpotent K-endomorphism of U, which has an eigenvector within $0 \neq v \in U$ with respect to the eigenvalue 0. Hence in conclusion we have $\mathfrak{L} \cdot v = (\mathfrak{K} + \langle A \rangle_K) \cdot v = \{0\}$. ♯

Corollary. There exists a **flag** $\{0\} = V_0 < V_1 < \cdots < V_n = V := K^{n \times 1}$, that is we have $\dim_K(V_i) = i$, such that $\mathcal{L} \cdot V_i \leq_K V_{i-1}$, for all $i \in \{1, \ldots, n\}$.

Thus, choosing an adjusted K-basis of V by proceeding through $V_1, V_2, \ldots, V_n$, yields a matrix $A \in \mathrm{GL}_n(K)$ such that $A^{-1} \cdot \mathfrak{L} \cdot A \subseteq \mathfrak{n}_n(K) \subseteq \mathfrak{gl}_n(K)$; hence $\mathfrak{L}$ is isomorphic to a Lie K-subalgebra of $\mathfrak{n}_n(K)$, in particular is nilpotent.

Proof. We proceed by induction on $n \in \mathbb{N}$; the case $n = 1$ being trivial, let $n \geq 2$. By the theorem, there is $0 \neq v \in V$ such that $\mathfrak{L} \cdot v = \{0\}$. Hence let $V_1 := \langle v \rangle_K$ and $W := V/V_1 \cong K^{(n-1) \times 1}$. Then $\mathfrak{L}$ acts with nilpotent K-endomorphisms on W. Hence by induction there is a flag $\{0\} = W_0 < W_1 < \cdots < W_{n-1} = W$ such that $\mathcal{L} \cdot W_i \leq_K W_{i-1}$, for all $i \in \{1, \ldots, n-1\}$. Letting $V_i \leq_K V$ be the preimage of $W_{i-1} \leq_K W$ with respect to the natural map $V \to V/V_1 = W$, for all $i \in \{1, \ldots, n\}$, yields a flag as desired. ♯

Corollary. Let $\mathfrak{L}$ be a nilpotent finite-dimensional Lie K-algebra. Then for any proper Lie K-subalgebra $\mathfrak{K} \subset \mathfrak{L}$ we have $\mathfrak{K} \neq N_{\mathfrak{L}}(\mathfrak{K})$.

Proof. The adjoint representation of $\mathfrak{L}$ induces a representation $\mathfrak{K} \to \mathfrak{gl}(\mathfrak{L}/\mathfrak{K})$, whose image by the nilpotency of $\mathfrak{L}$ consists of nilpotent maps. Hence there is $v \in \mathfrak{L} \setminus \mathfrak{K}$ such that $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{K}) \cdot v \subseteq \mathfrak{K}$, thus $v \in N_{\mathfrak{L}}(\mathfrak{K}) \setminus \mathfrak{K}$. ♯

Theorem: Engel. Let $\mathfrak{L}$ be a finite-dimensional Lie K-algebra all of whose elements are ad-nilpotent. Then $\mathfrak{L}$ is nilpotent.

Proof. Let $\mathrm{ad}_{\mathfrak{L}}\colon \mathfrak{L} \to \mathfrak{gl}(\mathcal{L})$ be the adjoint representation, where $\ker(\mathrm{ad}_{\mathfrak{L}}) = Z(\mathfrak{L})$. Hence it suffices to show that $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \cong \mathfrak{L}/Z(\mathfrak{L})$ is nilpotent. By assumption $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \subseteq \mathfrak{gl}(\mathcal{L})$ is a Lie K-subalgebra all of whose elements are nilpotent, hence by the above corollary is nilpotent. ♯

(5.4) Solvable Lie algebras. We proceed to generalise the above observations for nilpotent Lie algebras to solvable ones. It turns out that we need further, fairly strong assumptions on the underlying field K, which cannot be dispensed of, inasmuch the theorem to follow elsewise does not hold in general.

Theorem. Let $n \in \mathbb{N}$, let K be an algebraically closed field such that $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > n$, and let $\mathfrak{L} \subseteq \mathfrak{gl}_n(K)$ be a solvable Lie K-subalgebra. Then there is $0 \neq v \in K^{n\times 1}$ such that $\mathfrak{L}\cdot v \leq_K \langle v\rangle_K$, that is v is a common eigenvector for all elements of $\mathfrak{L}$.

Proof. We proceed by induction ob $d := \dim_K(\mathfrak{L}) \in \mathbb{N}_0$; the case $d = 0$ being trivial, let $d \geq 1$. Then $\mathfrak{L}$ being solvable we have $[\mathfrak{L}, \mathfrak{L}] \lhd \mathfrak{L}$. Thus $\mathfrak{L}/[\mathfrak{L}, \mathfrak{L}] \neq \{0\}$ is commutative, hence any K-subspace of $\mathfrak{L}/[\mathfrak{L}, \mathfrak{L}]$ is an ideal. Taking a preimage of a maximal proper K-subspace of $\mathfrak{L}/[\mathfrak{L}, \mathfrak{L}]$, with respect to the natural map $\mathfrak{L} \to \mathfrak{L}/[\mathfrak{L}, \mathfrak{L}]$, yields an ideal $[\mathfrak{L}, \mathfrak{L}] \subseteq \mathfrak{K} \lhd \mathfrak{L}$ such that $\dim_K(\mathfrak{K}) = d - 1$. Hence letting $C \in \mathfrak{L} \setminus \mathfrak{K}$, then we have $\mathfrak{L} = \mathfrak{K} + \langle C\rangle_K$.

Hence by induction there is a common eigenvector $0 \neq w \in K^{n\times 1}$ for all elements of $\mathfrak{K}$, in other words there is a K-linear map $\lambda\colon \mathfrak{K} \to K\colon B \mapsto \lambda_B$ such that $Bw = \lambda_B w$, for all $B \in \mathfrak{K}$. Let $U := \{u \in K^{n\times 1}; Bu = \lambda_B u, \text{ for all } B \in \mathfrak{K}\} \leq_K K^{n\times 1}$; thus we have $0 \neq w \in U$. It now suffices to show that $U \leq_K K^{n\times 1}$ is an $\mathfrak{L}$-submodule; then, since K is algebraically closed, C has an eigenvector $0 \neq v \in U$, which hence is a common eigenvector for all elements of $\mathfrak{L}$.

It remains to show that $Au \in U$, for all $A \in \mathfrak{L}$ and $u \in U$. To do so, we have to show that $B(Au) = \lambda_B Au$, for all $B \in \mathfrak{K}$. Since $B(Au) = A(Bu) - [A, B]u = \lambda_B Au - \lambda_{[A,B]}u$, this amounts to showing that $\lambda_{[A,B]} = 0$:

For $i \in \mathbb{N}$ let $W_i := \langle u, Au, A^2u, \ldots, A^{i-1}u\rangle_K \leq_K K^{n\times 1}$, hence we have $\{0\} =: W_0 < W_1 < \cdots < W_l = W_{l+1} = \cdots$, where $\dim_K(W_i) = i$, for all $i \in \{0, \ldots, l\}$, and $l \in \{0, \ldots, n\}$ is minimal such that $\{u, Au, A^2u, \ldots, A^l u\} \subseteq K^{n\times 1}$ is K-linearly dependent. Hence we have $AW_i \leq_K W_{i+1} \leq_K W_l$, for all $i \in \mathbb{N}_0$.

Next, by induction on $i \in \mathbb{N}_0$ we show that $BA^iu \equiv \lambda_B A^iu \pmod{W_i}$, in particular implying that $BA^iu \in W_{i+1}$: For $i = 0$ we have $Bu = \lambda_B u$; hence let $i \geq 1$. Then we have $BA^iu = BAA^{i-1}u = (AB - [A, B])A^{i-1}u$, where by induction we have $BA^{i-1}u \equiv \lambda_B A^{i-1}u \pmod{W_{i-1}}$ and $[A, B]A^{i-1}u \equiv \lambda_{[A,B]}A^{i-1}u \pmod{W_{i-1}}$. Hence, using $AW_{i-1} \leq_K W_i$, we infer that $BA^iu \equiv \lambda_B A^iu - \lambda_{[A,B]}A^{i-1}u \equiv \lambda_B A^iu \pmod{W_i}$.

This shows that $BW_i \leq_K W_{i+1} \leq_K W_l$, for all $i \in \mathbb{N}_0$, hence the matrix of the action of B on W_l, with respect to the K-basis $\{u, Au, A^2u, \ldots, A^{l-1}u\} \subseteq W_l$, is upper triangular with all diagonal entries being equal to λ_B. Thus we have $\mathrm{Tr}_{W_l}(B) = l\lambda_B$, for all $B \in \mathfrak{K}$. In particular, since $AW_l \leq_K W_l$ as well, the element $[A, B] \in \mathfrak{K}$ acts as the commutator of two K-endomorphisms of W_l, thus we have $0 = \mathrm{Tr}_{W_l}([A, B]) = l\lambda_{[A,B]}$. Finally, due to the assumption on $\mathrm{char}(K)$ we have $l \in K^*$, hence this entails $\lambda_{[A,B]} = 0$. ♯

Corollary: Lie's Theorem. There exists a flag $\{0\} = V_0 < V_1 < \cdots < V_n = V := K^{n\times 1}$ such that $\mathcal{L} \cdot V_i \leq_K V_i$, for all $i \in \{0, \ldots, n\}$.

Thus, choosing an adjusted K-basis of V by proceeding through $V_1, V_2, \ldots, V_n$, yields a matrix $A \in \mathrm{GL}_n(K)$ such that $A^{-1} \cdot \mathfrak{L} \cdot A \subseteq \mathfrak{b}_n(K) \subseteq \mathfrak{gl}_n(K)$; hence $\mathfrak{L}$ is isomorphic to a Lie K-subalgebra of $\mathfrak{b}_n(K)$.

Proof. We proceed by induction on $n \in \mathbb{N}$; the case $n = 1$ being trivial, let $n \geq 2$. By the theorem, there is $0 \neq v \in V$ such that $\mathfrak{L} \cdot v \leq_K \langle v \rangle_K$. Hence let $V_1 := \langle v \rangle_K$ and $W := V/V_1 \cong K^{(n-1)\times 1}$, with associated representation $\varphi\colon \mathfrak{L} \to \mathfrak{gl}(W)$. Then $\varphi(\mathfrak{L}) \cong \mathfrak{L}/\ker(\varphi)$ is a solvable Lie K-subalgebra of $\mathfrak{gl}(W)$. Hence by induction there is a flag $\{0\} = W_0 < W_1 < \cdots < W_{n-1} = W$ such that $\mathcal{L} \cdot W_i \leq_K W_i$, for all $i \in \{1, \ldots, n-1\}$. Letting $V_i \leq_K V$ be the preimage of $W_{i-1} \leq_K W$ with respect to the natural map $V \to V/V_1 = W$, for all $i \in \{1, \ldots, n\}$, yields a flag as desired. ♯

Theorem. Let K be a field such $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a solvable finite-dimensional Lie K-algebra, where $d := \dim_K(\mathfrak{L}) \in \mathbb{N}_0$.

a) Let K be algebraically closed. Then there exists a chain of ideals $\{0\} = \mathfrak{I}_0 < \mathfrak{I}_1 < \cdots < \mathfrak{I}_d := \mathfrak{L}$ of $\mathfrak{L}$, such that $\dim_K(\mathfrak{I}_i) = i$, for all $i \in \{0, \ldots, d\}$.

b) The derived subalgebra $[\mathfrak{L}, \mathfrak{L}]$ is nilpotent.

Proof. a) We consider the adjoint representation $\mathrm{ad}_{\mathfrak{L}}\colon \mathfrak{L} \to \mathfrak{gl}(\mathcal{L})$. Then $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \cong \mathfrak{L}/Z(\mathfrak{L})$ is a solvable Lie K-subalgebra of $\mathfrak{gl}(\mathcal{L})$, hence there is a flag $\{0\} = \mathfrak{I}_0 < \mathfrak{I}_1 < \cdots < \mathfrak{I}_d := \mathfrak{L}$ such that $[\mathfrak{L}, \mathfrak{I}_i] = \mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \cdot \mathfrak{I}_i \leq_K \mathfrak{I}_i$, thus $\mathfrak{I}_i \trianglelefteq \mathfrak{L}$ is an ideal, for all $i \in \{0, \ldots, d\}$.

b) Let $\overline{K}$ be an algebraic closure of K, and let $\mathfrak{L}^{\overline{K}} := \mathfrak{L} \otimes_K \overline{K}$. Then we have $(\mathfrak{L}^{\overline{K}})^{[k]} = (\mathfrak{L}^{[k]})^{\overline{K}}$, for all $k \in \mathbb{N}_0$. Hence $[\mathfrak{L}, \mathfrak{L}]$ is nilpotent if and only if $[\mathfrak{L}^{\overline{K}}, \mathfrak{L}^{\overline{K}}]$ is. Thus we may assume that $K = \overline{K}$ is algebraically closed.

Now, by Engel's Theorem it suffices to show that any element of $[\mathfrak{L}, \mathfrak{L}]$ is $\mathrm{ad}_{[\mathfrak{L},\mathfrak{L}]}$-nilpotent. To this end, let $\{0\} = \mathfrak{I}_0 < \mathfrak{I}_1 < \cdots < \mathfrak{I}_d := \mathfrak{L}$ be a chain of ideals such that $\dim_K(\mathfrak{I}_i) = i$, and let $\{x_1, \ldots, x_d\} \subseteq \mathfrak{L}$ be a K-basis such that $\langle x_1, \ldots, x_i \rangle_K = \mathfrak{I}_i$, for all $i \in \{0, \ldots, d\}$. Then, with respect to this K-basis, we have $\mathrm{ad}_{\mathfrak{L}}(x) \in \mathfrak{b}_d(K)$, implying that $\mathrm{ad}_{\mathfrak{L}}([x, y]) = [\mathrm{ad}_{\mathfrak{L}}(x), \mathrm{ad}_{\mathfrak{L}}(y)] \in [\mathfrak{b}_d(K), \mathfrak{b}_d(K)] = \mathfrak{n}_d(K)$, for all $x, y \in \mathfrak{L}$. Hence we infer that $\mathrm{ad}_{\mathfrak{L}}([\mathfrak{L}, \mathfrak{L}])$ consists of nilpotent K-endomorphisms, thus $\mathrm{ad}_{[\mathfrak{L},\mathfrak{L}]}([\mathfrak{L}, \mathfrak{L}])$ does so as well. ♯

6 Solvable algebras II

(6.1) Semisimple endomorphisms. Let K be an algebraically closed field, and let V be a finite-dimensional K-vector space. A K-endomorphism $\varphi \in \mathrm{End}_K(V)$ is called **semisimple**, if its minimum polynomial $\mu_\varphi \in K[X]$ is multiplicity-free, that is all its roots are simple.

This is equivalent to saying that φ is diagonalisable, or still in other words that $V = \bigoplus_{a\in K} T_a(\varphi)$, where $T_a(\varphi) := \ker(\varphi - a\cdot \mathrm{id}_V) = \{v \in V; \varphi(v) = av\} \leq_K V$; note that $T_a(\varphi) \neq \{0\}$ if and only if $a \in K$ is an eigenvalue of φ, in which case $T_a(\varphi)$ is the associated eigenspace.

If $W \leq_K V$ is a φ-invariant subspace, then $\mu_{\varphi|_W} \mid \mu_\varphi \in K[X]$ implies that $\varphi|_W \in \mathrm{End}_K(W)$ is semisimple again.

Lemma. Let $\mathcal{M} := \{\varphi_i \in \mathrm{End}_K(V); i \in \mathcal{I}\}$, where $\mathcal{I} \neq \emptyset$ is an index set, be a set of pairwise commuting semisimple K-endomorphisms. Then $\mathcal{M}$ is **simultaneously diagonalisable**, that is there is a K-basis of V consisting of eigenvectors for all elements of $\mathcal{M}$.

Proof. We proceed by induction on $\dim_K(V) \in \mathbb{N}_0$, the cases $\dim_K(V) \leq 1$ being trivial. Letting $\varphi \in \mathcal{M}$ we have $V = \bigoplus_{a\in K} T_a(\varphi)$. Then, for all $a \in K$ and $\psi \in \mathcal{M}$, the K-subspace $T_a(\varphi) \leq_K V$ is ψ-invariant: For $v \in T_a(\varphi)$ we have $\varphi\psi(v) = \psi\varphi(v) = a\psi(v)$, thus $\psi(v) \in T_a(\varphi)$. Hence it suffices to consider the eigenspaces $W \leq_K V$ of φ, and the set $\mathcal{M}_W := \{\varphi_i|_W \in \mathrm{End}_K(W); i \in \mathcal{I}\}$ in turn. If there is $\varphi \in \mathcal{M}$ having two distinct eigenvalues, then we are done by induction. Otherwise we have $T_{a_\varphi}(\varphi) = V$, for all $\varphi \in \mathcal{M}$ and certain $a_\varphi \in K$, thus all non-zero elements of V are eigenvectors for all elements of $\mathcal{M}$. ♯

We now consider arbitrary K-endomorphisms, and show that these can be naturally decomposed additively into semisimple and nilpotent parts:

(6.2) Theorem. Let K be an algebraically closed field, let V be a finite-dimensional K-vector space, and let $\varphi \in \mathrm{End}_K(V)$.

i) Then there are a unique **semisimple part** $\varphi_s \in \mathrm{End}_K(V)$ and a unique **nilpotent part** $\varphi_n \in \mathrm{End}_K(V)$ such that $\varphi_s\varphi_n = \varphi_n\varphi_s$ and we have the (additive) **Jordan-Chevalley decomposition** $\varphi = \varphi_s + \varphi_n$.

ii) There are polynomials $f_s, f_n \in K[X]$ where $f_s(0) = 0 = f_n(0)$, such that $\varphi_s = f_s(\varphi) \in \mathrm{End}_K(V)$ and $\varphi_n = f_n(\varphi) \in \mathrm{End}_K(V)$. In particular, both φ_s and φ_n commute with all K-endomorphisms of V commuting with φ.

Proof. We first show existence: Let $\chi_\varphi = \prod_{i=1}^k (X - a_i)^{m_i} \in K[X]$ be the characteristic polynomial of φ, where $a_1, \ldots, a_k \in K$ are the distinct eigenvalues of φ, and $m_1, \ldots, m_k \in \mathbb{N}$ are the associated multiplicities. Then we have $V = \bigoplus_{i=1}^k V_i$, where $V_i := T_{(X-a_i)^{m_i}}(\varphi) \leq_K V$, where we let $T_f(\varphi) := \ker(f(\varphi)) \leq_K V$ be the generalised eigenspace of φ with respect to $f \in K[X]$. Then $V_i \leq_K V$

is a φ-invariant K-subspace, where $\varphi_i := \varphi|_{V_i}$ has characteristic polynomial $\chi_{\varphi_i} = (X - a_i)^{m_i} \in K[X]$, for all $i \in \{1, \ldots, k\}$.

We consider the congruences $f \equiv a_i \pmod{\chi_{\varphi_i}}$, for all $i \in \{1, \ldots, k\}$, and $f \equiv 0 \pmod{X}$ in $K[X]$. The moduli χ_{φ_i} are pairwise coprime. If $0 \in K$ is an eigenvalue of φ, then the last congruence is a consequence of $f \equiv 0 \pmod{X^m}$, where $m \in \mathbb{N}$ is the associated algebraic multiplicity, hence is redundant and will be ignored. Otherwise the modulus X is coprime to χ_{φ_i}, for all $i \in \{1, \ldots, k\}$.

Hence by the Chinese Remainder Theorem there is $f_s \in K[X]$ simultaneously fulfilling all the above congruences. Let $f_n := X - f_s(X) \in K[X]$. From $f_s \equiv 0 \pmod{X}$ we infer $f_s(0) = 0$, and $f_n(0) = 0$. Let now $\varphi_s := f_s(\varphi) \in \mathrm{End}_K(V)$ and $\varphi_n := f_n(\varphi) = \varphi - \varphi_s \in \mathrm{End}_K(V)$. Hence we have $\varphi = \varphi_s + \varphi_n$, and since φ_s and φ_n are polynomials in φ, we infer that φ_s and φ_n commute, and commute with all K-endomorphisms commuting with φ.

Now $V_i \leq_K V$ is φ_s-invariant and φ_n-invariant, for all $i \in \{1, \ldots, k\}$, and we have $\varphi_s|_{V_i} = f_s(\varphi_i)$ and $\varphi_n|_{V_i} = f_n(\varphi_i)$. Since $f_s \equiv a_i \pmod{\chi_{\varphi_i}}$, the Cayley-Hamilton Theorem says that $f_s(\varphi_i) = a_i \cdot \mathrm{id}_{V_i}$. This implies that φ_s is semisimple. Next, we have $f_n(\varphi_i) = \varphi_i - f_s(\varphi_i) = \varphi_i - a_i \cdot \mathrm{id}_{V_i}$, thus $f_n(\varphi_i)^{m_i} = (\varphi_i - a_i \cdot \mathrm{id}_{V_i})^{m_i} = \chi_{\varphi_i}(\varphi_i) = 0$. This implies that φ_n is nilpotent.

It remains to prove uniqueness: Let $\varphi'_s \in \mathrm{End}_K(V)$ be semisimple and $\varphi'_n \in \mathrm{End}_K(V)$ be nilpotent such that $\varphi'_s\varphi'_n = \varphi'_n\varphi'_s$ and $\varphi = \varphi'_s + \varphi'_n$. Then both φ'_s and φ'_n commute with φ, hence commute with φ_s and φ_n. Now $\varphi_s + \varphi_n = \varphi = \varphi'_s + \varphi'_n$ implies $\varphi_s - \varphi'_s = \varphi'_n - \varphi_n$. The left hand side, being a sum of commuting semisimple K-endomorphisms, is semisimple again. Similarly, the right hand side, being a sum of commuting nilpotent K-endomorphisms, is nilpotent again. Now, the only K-endomorphism which is both semisimple and nilpotent is the zero map, hence $\varphi_s = \varphi'_s$ and $\varphi'_n - \varphi_n$. ♯

(6.3) Theorem. Let K be an algebraically closed field.

a) Let V be a finite-dimensional K-vector space, and let $A \in \mathfrak{gl}(V)$ with Jordan-Chevalley decomposition $A = A_s + A_n \in \mathfrak{gl}(V)$. Then we have the Jordan-Chevalley decomposition $\mathrm{ad}_{\mathfrak{gl}(V)}(A) = \mathrm{ad}_{\mathfrak{gl}(V)}(A_s) + \mathrm{ad}_{\mathfrak{gl}(V)}(A_n) \in \mathfrak{gl}(\mathfrak{gl}(V))$.

b) Let $\mathfrak{A}$ be a finite-dimensional non-associative K-algebra. Then $\mathrm{Der}_K(\mathfrak{A}) \leq_K \mathrm{End}_K(\mathfrak{A})$ has Jordan-Chevalley decompositions, that is for any $\partial \in \mathrm{Der}_K(\mathfrak{A})$ the semisimple and nilpotent parts $\partial_s \in \mathrm{End}_K(\mathfrak{A})$ and $\partial_n \in \mathrm{End}_K(\mathfrak{A})$, respectively, are derivations as well.

Proof. a) If $A \in \mathfrak{gl}(V)$ is nilpotent, then $\lambda_{\mathfrak{gl}(V)}(A)$ and $\rho_{\mathfrak{gl}(V)}(A)$ are so as well, and since $\lambda_{\mathfrak{gl}(V)}\rho_{\mathfrak{gl}(V)} = \rho_{\mathfrak{gl}(V)}\lambda_{\mathfrak{gl}(V)}$ we infer that $\mathrm{ad}_{\mathfrak{gl}(V)}(A) = \lambda_{\mathfrak{gl}(V)}(A) - \rho_{\mathfrak{gl}(V)}(A)$ is nilpotent, too; note that we have seen this argument in (4.3) already.

If $A \in \mathfrak{gl}(V)$ is semisimple, then by choosing a suitable K-basis of V we may assume that $A = \sum_{i=1}^n a_i E_{ii}$ is a diagonal matrix, where $n := \dim_K(V) \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in K$ are the eigenvalues of A. Letting $\{E_{11}, \ldots, E_{nn}\} \subseteq \mathfrak{gl}(V)$ be the standard K-basis, we have $\mathrm{ad}_{\mathfrak{gl}(V)}(A)(E_{ij}) = \sum_{i=1}^n a_i(E_{ii}E_{ij} - E_{ij}E_{ii}) = (a_i - a_j)E_{ij}$, for all $i, j \in \{1, \ldots, n\}$. Hence the standard K-basis of $\mathfrak{gl}(V)$ consists of eigenvectors for $\mathrm{ad}_{\mathfrak{gl}(V)}(A)$.

Thus, if $A = A_s + A_n \in \mathfrak{gl}(V)$ is a Jordan-Chevalley decomposition, then $\mathrm{ad}_{\mathfrak{gl}(V)}(A_s)$ is semisimple, and $\mathrm{ad}_{\mathfrak{gl}(V)}(A_n)$ is nilpotent. Finally, we obtain $[\mathrm{ad}_{\mathfrak{gl}(V)}(A_s), \mathrm{ad}_{\mathfrak{gl}(V)}(A_n)] = \mathrm{ad}_{\mathfrak{gl}(V)}([A_s, A_n]) = 0$, saying that $\mathrm{ad}_{\mathfrak{gl}(V)}(A_s)$ and $\mathrm{ad}_{\mathfrak{gl}(V)}(A_n)$ commute. Hence we conclude that $\mathrm{ad}_{\mathfrak{gl}(V)}(A_s)$ and $\mathrm{ad}_{\mathfrak{gl}(V)}(A_n)$ are the semisimple and nilpotent parts of $\mathrm{ad}_{\mathfrak{gl}(V)}(A) \in \mathfrak{gl}(\mathfrak{gl}(V))$.

b) Given $\partial \in \mathrm{Der}_K(\mathfrak{A})$, it suffices to show that $\partial_s \in \mathrm{End}_K(\mathfrak{A})$ is a derivation: To this end, for $a \in K$ let $\mathfrak{A}_a := T_{(X-a)^n}(\partial) := \{x \in \mathfrak{A}; (\partial - a \cdot \mathrm{id}_{\mathfrak{A}})^n(x) = 0\}$ be the generalised eigenspace of ∂ with respect to the polynomial $(X-a)^n \in K[X]$, where $n := \dim_K(\mathfrak{A})$; then $\mathfrak{A} = \bigoplus_{a \in K} \mathfrak{A}_a$, and ∂_s acts as $a \cdot \mathrm{id}_{\mathfrak{A}_a}$ on $\mathfrak{A}_a$.

We have $(\partial - (a+b) \cdot \mathrm{id}_{\mathfrak{A}})^k \mu(x,y) = \sum_{i=0}^k \binom{k}{i} \mu\big((\partial - a \cdot \mathrm{id}_{\mathfrak{A}})^i x, (\partial - b \cdot \mathrm{id}_{\mathfrak{A}})^{k-i} y\big)$, for all $x, y \in \mathfrak{A}$ and $a, b \in K$ and $k \in \mathbb{N}_0$: We proceed by induction on $k \in \mathbb{N}_0$; the case $k = 0$ being trivial, let $k \geq 1$. Then we have $(\partial - (a+b) \cdot \mathrm{id}_{\mathfrak{A}})^k \mu(x,y) = \sum_{i=0}^{k-1} \binom{k-1}{i} (\partial - (a+b) \cdot \mathrm{id}_{\mathfrak{A}}) \mu\big((\partial - a \cdot \mathrm{id}_{\mathfrak{A}})^i x, (\partial - b \cdot \mathrm{id}_{\mathfrak{A}})^{k-i-1} y\big)$. The product rule yields $(\partial - (a+b) \cdot \mathrm{id}_{\mathfrak{A}})^k \mu(x,y) = \sum_{i=0}^{k-1} \binom{k-1}{i} \mu\big((\partial - a \cdot \mathrm{id}_{\mathfrak{A}})^i x, (\partial - b \cdot \mathrm{id}_{\mathfrak{A}})^{k-i} y\big) + \sum_{i=1}^{k} \binom{k-1}{i-1} \mu\big((\partial - a \cdot \mathrm{id}_{\mathfrak{A}})^i x, (\partial - b \cdot \mathrm{id}_{\mathfrak{A}})^{k-i} y\big)$. Now $\binom{k-1}{i} + \binom{k-1}{i-1} = \binom{k}{i}$, for all $i \in \{1, \ldots, k-1\}$, and $\binom{k}{0} = 1 = \binom{k}{k}$, implies the claim.

Using this we get $(\partial - (a+b) \cdot \mathrm{id}_{\mathfrak{A}})^{2n} \mu(x,y) = \sum_{i=0}^{2n} \binom{k}{i} \mu\big((\partial - a \cdot \mathrm{id}_{\mathfrak{A}})^i x, (\partial - b \cdot \mathrm{id}_{\mathfrak{A}})^{k-i} y\big) = 0$, for all $x \in \mathfrak{A}_a$ and $y \in \mathfrak{A}_b$. Noting that $\mathfrak{A}_c = T_{(X-c)^n}(\partial) = T_{(X-c)^{2n}}(\partial)$, for all $c \in K$, we infer $\mu(\mathfrak{A}_a, \mathfrak{A}_b) \leq_K \mathfrak{A}_{a+b}$, for all $a, b \in K$. This yields $\partial_s \mu(x,y) = (a+b)\mu(x,y) = \mu(ax, y) + \mu(x, by) = \mu(\partial_s x, y) + \mu(x, \partial_s y)$, for all $x \in \mathfrak{A}_a$ and $y \in \mathfrak{A}_b$. Since $\mathfrak{A} = \bigoplus_{a \in K} \mathfrak{A}_a$, this says that ∂_s is a derivation. $\sharp$

(6.4) Theorem: Cartan's Criterion. Let K be a field such that $\mathrm{char}(K) = 0$, let V be a finite-dimensional K-vector space, and let $\mathfrak{L} \subseteq \mathfrak{gl}(V)$ be a Lie K-subalgebra. Then $\mathfrak{L}$ is solvable if and only if $\mathrm{Tr}(AB) = 0$, for all $A \in [\mathfrak{L}, \mathfrak{L}]$ and $B \in \mathfrak{L}$.

Before proceeding to the proof we need a lemma:

Lemma. Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, let V be a finite-dimensional K-vector space, let $\mathcal{U} \leq_K \mathcal{W} \leq_K \mathfrak{gl}(V)$, and let $\mathcal{M} := \{A \in \mathfrak{gl}(V); [A, \mathcal{W}] \leq_K \mathcal{U}\} \leq_K \mathfrak{gl}(V)$. Moreover, let $A \in \mathcal{M}$ such that $\mathrm{Tr}(AB) = 0$, for all $B \in \mathcal{M}$. Then A is nilpotent.

Proof. Let $A = A_s + A_n \in \mathfrak{gl}(V)$ be the Jordan-Chevalley decomposition of A, where A_s is semisimple and A_n is nilpotent; hence we have to show that $A_s = 0$. By choosing a suitable K-basis of V we may assume that $A_s = \mathrm{diag}[a_1, \ldots, a_n]$ is a diagonal matrix, where $n := \dim_K(V) \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in K$ are the eigenvalues of A_s; we may assume that $n \geq 1$. Hence letting $\mathcal{E} := \langle a_1, \ldots, a_n \rangle_{\mathbb{Q}} \leq_{\mathbb{Q}} K$

be the finite-dimensional $\mathbb{Q}$-subspace of K generated by the eigenvalues of A_s, and $\mathcal{E}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{E}, \mathbb{Q})$ be its dual, we have to show that $\mathcal{E}^* = \{0\}$:

Let $\lambda \in \mathcal{E}^*$ and $B := \mathrm{diag}[\lambda(a_1), \ldots, \lambda(a_n)] \in \mathfrak{gl}(V)$. By Lagrange interpolation there is $f \in K[X]$ such that $f(a_i) = \lambda(a_i) \in \mathbb{Q} \subseteq K$, for all $i \in \{1, \ldots, n\}$, hence $B = f(A_s)$. Moreover, we have $A_s = f_s(A)$, for some $f_s \in K[X]$, implying that $B = f(f_s(A))$ commutes with A, and thus commutes with A_n. This implies that $A_n B$ is nilpotent, thus $\mathrm{Tr}(AB) = \mathrm{Tr}(A_s B) + \mathrm{Tr}(A_n B) = \mathrm{Tr}(A_s B)$.

Moreover, letting $\{E_{11}, \ldots, E_{nn}\} \subseteq \mathfrak{gl}(V)$ be the standard K-basis, we have $\mathrm{ad}_{\mathfrak{gl}(V)}(A_s)(E_{ij}) = (a_i - a_j)E_{ij}$, and similarly, using the $\mathbb{Q}$-linearity of λ, we get $\mathrm{ad}_{\mathfrak{gl}(V)}(B)(E_{ij}) = \lambda(a_i - a_j)E_{ij}$, for all $i, j \in \{1, \ldots, n\}$. By Lagrange interpolation there is $g \in K[X]$ such that $g(a_i - a_j) = \lambda(a_i - a_j) \in \mathbb{Q} \subseteq K$, for all $i, j \in \{1, \ldots, n\}$; note that for $i = j$ we get $g(0) = 0$.

Hence we have $\mathrm{ad}_{\mathfrak{gl}(V)}(B) = g\big(\mathrm{ad}_{\mathfrak{gl}(V)}(A_s)\big)$. Moreover, we have $\mathrm{ad}_{\mathfrak{gl}(V)}(A_s) = \big(\mathrm{ad}_{\mathfrak{gl}(V)}(A)\big)_s = \widetilde{f}_s\big(\mathrm{ad}_{\mathfrak{gl}(V)}(A)\big)$, for some $\widetilde{f}_s \in K[X]$ such that $\widetilde{f}_s(0) = 0$. We conclude that $\mathrm{ad}_{\mathfrak{gl}(V)}(B) = h\big(\mathrm{ad}_{\mathfrak{gl}(V)}(A)\big)$, where $h := g(\widetilde{f}_s) \in K[X]$ fulfills $h(0) = 0$. By assumption on A we have $\mathrm{ad}_{\mathfrak{gl}(V)}(A) \cdot \mathcal{W} \leq_K \mathcal{U}$, hence we infer that $\mathrm{ad}_{\mathfrak{gl}(V)}(B) \cdot \mathcal{W} \leq_K \mathcal{U}$ as well, that is $B \in \mathcal{M}$.

This implies $0 = \mathrm{Tr}(AB) = \mathrm{Tr}(A_s B) = \sum_{i=1}^n \lambda(a_i) a_i \in \mathcal{E}$. Applying λ yields $\sum_{i=1}^n \lambda(a_i)^2 = 0 \in \mathbb{Q}$, thus $\lambda(a_i) = 0$, for all $i \in \{1, \ldots, n\}$, entailing $\lambda = 0$. $\sharp$

Proof: Cartan's Criterion. Let $\overline{K}$ be an algebraic closure of K, and let $?^{\overline{K}} := ? \otimes_K \overline{K}$ denote the associated scalar extensions. Then we have $(\mathfrak{L}^{\overline{K}})^{(k)} = (\mathfrak{L}^{(k)})^{\overline{K}}$, for all $k \in \mathbb{N}_0$. Hence $\mathfrak{L}$ is solvable if and only if $\mathfrak{L}^{\overline{K}}$ is. Moreover, the trace condition is fulfilled for all $A \in [\mathfrak{L}, \mathfrak{L}]$ and $B \in \mathfrak{L}$ if and only if it is so for all $A \in [\mathfrak{L}^{\overline{K}}, \mathfrak{L}^{\overline{K}}]$ and $B \in \mathfrak{L}^{\overline{K}}$. Thus we may assume that $K = \overline{K}$ is algebraically closed.

Let $\mathfrak{L}$ be solvable. Then by Lie's Theorem we may assume that $\mathfrak{L} \subseteq \mathfrak{b}_d(K) \subseteq \mathfrak{gl}_d(K)$, where $d := \dim_K(V) \in \mathbb{N}_0$. Then we have $[\mathfrak{L}, \mathfrak{L}] \subseteq [\mathfrak{b}_d(K), \mathfrak{b}_d(K)] = \mathfrak{n}_d(K)$. Thus we get $AB \in \mathfrak{b}_d(K)\mathfrak{n}_d(K) = \mathfrak{n}_d(K)$, in particular implying $\mathrm{Tr}(AB) = 0$, for all $A \in [\mathfrak{L}, \mathfrak{L}]$ and $B \in \mathfrak{L}$.

Now let $\mathfrak{L}$ fulfill the asserted trace condition. By Lie's Theorem again, $\mathfrak{L}$ is solvable if and only if $[\mathfrak{L}, \mathfrak{L}]$ is nilpotent. In turn, by Engel's Theorem $[\mathfrak{L}, \mathfrak{L}]$ is nilpotent if and only if $\mathrm{ad}_{[\mathfrak{L},\mathfrak{L}]}([\mathfrak{L}, \mathfrak{L}])$ consists of nilpotent K-endomorphisms of $[\mathfrak{L}, \mathfrak{L}]$. Finally, the latter if fulfilled if $[\mathfrak{L}, \mathfrak{L}] \subseteq \mathfrak{gl}(V)$ consists of nilpotent K-endomorphisms of V, which we proceed to show:

We aim at applying the above lemma with $\mathcal{U} := [\mathfrak{L}, \mathfrak{L}] \leq_K \mathfrak{L} =: \mathcal{W}$, hence let $\mathcal{M} := \{C \in \mathfrak{gl}(V); [C, \mathfrak{L}] \leq_K [\mathfrak{L}, \mathfrak{L}]\}$; then $[\mathfrak{L}, \mathfrak{L}] \subseteq \mathfrak{L} \subseteq \mathcal{M}$. Letting $A, B \in \mathfrak{L}$ and $C \in \mathcal{M}$, we have $\mathrm{Tr}([A, B]C) = \mathrm{Tr}(ABC - BAC) = \mathrm{Tr}(BCA - CBA) = \mathrm{Tr}([B, C]A)$, where by definition of $\mathcal{M}$ we have $[B, C] \in [\mathfrak{L}, \mathfrak{L}]$, and thus by assumption $Tr([B, C]A) = 0$. This shows that $\mathrm{Tr}([\mathfrak{L}, \mathfrak{L}] \cdot C) = 0$, for all $C \in \mathcal{M}$, hence the above lemma says that $[\mathfrak{L}, \mathfrak{L}]$ consists of nilpotent endomorphisms. $\sharp$

Corollary. A finite-dimensional Lie K-algebra $\mathfrak{L}$ is solvable if and only if $\mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(y)) = 0$, for all $x \in [\mathfrak{L}, \mathfrak{L}]$ and $y \in \mathfrak{L}$.

Proof. The Lie K-algebra $\mathfrak{L}$ is solvable if and only if $\mathfrak{L}/Z(\mathfrak{L}) \cong \mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) \subseteq \mathfrak{gl}(\mathfrak{L})$ is so. Since $[\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}), \mathrm{ad}_{\mathfrak{L}}(\mathfrak{L})] = \mathrm{ad}_{\mathfrak{L}}([\mathfrak{L}, \mathfrak{L}])$, by Cartan's Criterion the solvability of $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L})$ is equivalent to the trace condition given. ♯

II Semisimplicity

7 Semisimple algebras

(7.1) Semisimple Lie algebras. a) Let K be a field, and let $\mathfrak{L}$ be a finite-dimensional Lie K-algebra. Then the sum $\mathrm{rad}(\mathfrak{L}) \trianglelefteq \mathfrak{L}$ of all solvable ideals of $\mathfrak{L}$ is solvable again, hence is the unique maximal solvable ideal; it is called the **(solvable) radical** of $\mathfrak{L}$. Similarly, the sum $\mathrm{nil}(\mathfrak{L}) \trianglelefteq \mathfrak{L}$ of all nilpotent ideals of $\mathfrak{L}$ is nilpotent again, hence is the unique maximal nilpotent ideal; it is called the **nil radical** of $\mathfrak{L}$. We have $\mathrm{nil}(\mathfrak{L}) \subseteq \mathrm{rad}(\mathfrak{L})$.

Lemma. We have $\mathrm{rad}(\mathfrak{L}) = \{0\}$ if and only if $\mathrm{nil}(\mathfrak{L}) = \{0\}$ if and only if $\mathfrak{L}$ does not possess any non-zero commutative ideals:

Proof. Since $\mathrm{nil}(\mathfrak{L}) \subseteq \mathrm{rad}(\mathfrak{L})$, and any commutative ideal of $\mathfrak{L}$ is nilpotent, we only have to show that the latter property implies $\mathrm{rad}(\mathfrak{L}) = \{0\}$: Assume to the contrary that $\mathrm{rad}(\mathfrak{L}) \neq \{0\}$. Since $\mathrm{rad}(\mathfrak{L}) \trianglelefteq \mathfrak{L}$ is an ideal, the derived series of $\mathrm{rad}(\mathfrak{L})$ consists of ideals of $\mathfrak{L}$, whose second-last term is commutative. ♯

If $\mathrm{rad}(\mathfrak{L}) = \{0\}$ then $\mathfrak{L}$ is called **semisimple**; more generally, if $\mathrm{rad}(\mathfrak{L}) = Z(\mathfrak{L})$ then $\mathfrak{L}$ is called **reductive**. In particular, $\mathfrak{L} = \{0\}$ is the only semisimple solvable Lie K-algebra, but any commutative Lie K-algebra is reductive.

In any case, we have $\mathrm{rad}(\mathfrak{L}/\mathrm{rad}(\mathfrak{L})) = \{0\}$, that is $\mathfrak{L}/\mathrm{rad}(\mathfrak{L})$ is semisimple: If $\mathrm{rad}(\mathfrak{L}) \subseteq \mathfrak{I} \trianglelefteq \mathfrak{L}$ is an ideal such that $\overline{\mathfrak{I}} := \mathfrak{I}/\mathrm{rad}(\mathfrak{L}) \trianglelefteq \mathfrak{L}/\mathrm{rad}(\mathfrak{L})$ is solvable, then since $\mathrm{rad}(\mathfrak{L})$ is solvable $\mathfrak{I}$ is solvable as well, hence $\mathfrak{I} \subseteq \mathrm{rad}(\mathfrak{L})$ and thus $\overline{\mathfrak{I}} = \{0\}$.

b) Let $\mathfrak{L}_1, \dots, \mathfrak{L}_n$ be finite-dimensional Lie K-algebras, where $n \in \mathbb{N}_0$. Then the direct sum $\mathfrak{L} := \bigoplus_{i=1}^n \mathfrak{L}_i$ of K-vector spaces becomes a Lie K-algebra with respect to the componentwise Lie product, being called the **direct sum** of the $\mathfrak{L}_i$; for $n = 0$ we let $\mathfrak{L} := \{0\}$. From $[\mathfrak{L}_i, \mathfrak{L}_j] = \{0\}$, for all $i \neq j \in \{1, \dots, n\}$, we conclude that any ideal of $\mathfrak{L}_i$ is an ideal of $\mathfrak{L}$, in particular $\mathfrak{L}_i \trianglelefteq \mathfrak{L}$ is an ideal.

Lemma. We have $\mathrm{rad}(\mathfrak{L}) = \bigoplus_{i=1}^n \mathrm{rad}(\mathfrak{L}_i)$ and $\mathrm{nil}(\mathfrak{L}) = \bigoplus_{i=1}^n \mathrm{nil}(\mathfrak{L}_i)$.

Proof. We have $\mathrm{rad}(\mathfrak{L}_i) \trianglelefteq \mathfrak{L}_i$, hence $\mathrm{rad}(\mathfrak{L}_i) \trianglelefteq \mathfrak{L}$ is a solvabe ideal, thus we have $\bigoplus_{i=1}^n \mathrm{rad}(\mathfrak{L}_i) \leq_K \mathrm{rad}(\mathfrak{L})$. Similarly, since $[\mathfrak{L}_i, \mathfrak{L}_j] = \{0\}$, for all $i \neq j$, we conclude that $\mathrm{nil}(\mathfrak{L}_i) \trianglelefteq \mathfrak{L}$ is a nilpotent ideal, thus $\bigoplus_{i=1}^n \mathrm{nil}(\mathfrak{L}_i) \leq_K \mathrm{nil}(\mathfrak{L})$.

As for the converse inclusions, we consider the natural projections $\pi_i\colon \mathfrak{L} \to \mathfrak{L}_i$ associated with the direct sum decomposition of $\mathfrak{L}$; then π_i is an epimorphism of Lie K-algebras. Hence $\pi_i(\mathrm{rad}(\mathfrak{L})) \trianglelefteq \mathfrak{L}_i$ is a solvable ideal, implying that $\pi_i(\mathrm{rad}(\mathfrak{L})) \leq_K \mathrm{rad}(\mathfrak{L}_i)$, for all $i \in \{1, \ldots, n\}$. Thus we conclude that $\mathrm{rad}(\mathfrak{L}) \subseteq (\bigoplus_{i=1}^n \pi_i)(\mathrm{rad}(\mathfrak{L})) \leq_K \bigoplus_{i=1}^n \mathrm{rad}(\mathfrak{L}_i)$. Similarly, $\pi_i(\mathrm{nil}(\mathfrak{L})) \trianglelefteq \mathfrak{L}_i$ is a nilpotent ideal, implying that $\pi_i(\mathrm{nil}(\mathfrak{L})) \leq_K \mathrm{nil}(\mathfrak{L}_i)$, for all $i \in \{1, \ldots, n\}$. Thus we conclude that $\mathrm{nil}(\mathfrak{L}) \subseteq (\bigoplus_{i=1}^n \pi_i)(\mathrm{nil}(\mathfrak{L})) \leq_K \bigoplus_{i=1}^n \mathrm{nil}(\mathfrak{L}_i)$. $\sharp$

Hence $\mathfrak{L}$ is semisimple if and only if the $\mathfrak{L}_i$ are so, for all $i \in \{1, \ldots, n\}$. In particular, this is the case if the $\mathfrak{L}_i$ are simple, for all $i \in \{1, \ldots, n\}$. We will show in (7.4) that, for K a field such that $\mathrm{char}(K) = 0$, this is indeed a characterisation of semisimplicity.

Example: General and special linear algebras. Let K be a field such that $\mathrm{char}(K) = 0$, and let $n \in \mathbb{N}$. Then the Lie K-algebra $\mathfrak{L} := \mathfrak{sl}_n(K)$ is semisimple, and the Lie K-algebra $\widehat{\mathfrak{L}} := \mathfrak{gl}_n(K)$ is reductive: (We will show later that $\mathfrak{L}$ is actually simple for $n \geq 2$.)

Let $\overline{K}$ be an algebraic closure of K, then we have $\mathfrak{L}^{\overline{K}} \cong \mathfrak{sl}_n(\overline{K})$ and $\widehat{\mathfrak{L}}^{\overline{K}} \cong \mathfrak{gl}_n(\overline{K})$. Since $\mathrm{rad}(\mathfrak{L})^{\overline{K}} \trianglelefteq \mathfrak{L}^{\overline{K}}$ is solvable, we have $\mathrm{rad}(\mathfrak{L})^{\overline{K}} \subseteq \mathrm{rad}(\mathfrak{L}^{\overline{K}})$, and similarly $\mathrm{rad}(\widehat{\mathfrak{L}})^{\overline{K}} \subseteq \mathrm{rad}(\widehat{\mathfrak{L}}^{\overline{K}})$, while $Z(\widehat{\mathfrak{L}}) = \mathfrak{z}_n(K) = \mathfrak{z}_n(\overline{K}) \cap \widehat{\mathfrak{L}} = Z(\widehat{\mathfrak{L}}^{\overline{K}}) \cap \widehat{\mathfrak{L}}$. Hence we may assume that $K = \overline{K}$ is algebraically closed.

For $A \in \mathrm{GL}_n(K)$ let $\mathrm{Ad}_{\widehat{\mathfrak{L}}}(A)\colon \widehat{\mathfrak{L}} \to \widehat{\mathfrak{L}}\colon M \mapsto AMA^{-1}$ be the associated inner automorphism of the associative K-algebra $\widehat{\mathfrak{L}}$. Hence $\mathrm{Ad}_{\widehat{\mathfrak{L}}}(A)$ also is an automorphism of $\widehat{\mathfrak{L}}$ as Lie K-algebras. Since $\mathrm{rad}(\widehat{\mathfrak{L}}) \trianglelefteq \widehat{\mathfrak{L}}$ is the sum of all solvable ideals of $\widehat{\mathfrak{L}}$, we conclude that it is $\mathrm{Aut}(\widehat{\mathfrak{L}})$-invariant, in particular is $\mathrm{Ad}_{\widehat{\mathfrak{L}}}(\mathrm{GL}_n(K))$-invariant. Moreover, since $\mathrm{Tr}(AMA^{-1}) = \mathrm{Tr}(M)$, for all $M \in \widehat{\mathfrak{L}}$ and all $A \in \mathrm{GL}_n(K)$, this yields the Lie K-algebra automorphism $\mathrm{Ad}_{\mathfrak{L} \subseteq \widehat{\mathfrak{L}}}(A)$ of $\mathfrak{L}$; and since $\mathrm{rad}(\mathfrak{L}) \trianglelefteq \mathfrak{L}$ is $\mathrm{Aut}(\mathfrak{L})$-invariant, it is $\mathrm{Ad}_{\mathfrak{L} \subseteq \widehat{\mathfrak{L}}}(\mathrm{GL}_n(K))$-invariant.

i) If $\mathfrak{K} \subseteq \widehat{\mathfrak{L}}$ is a solvable subalgebra, then by Lie's Theorem there is $A \in \mathrm{GL}_n(K)$ such that $A \cdot \mathfrak{K} \cdot A^{-1} \subseteq \mathfrak{b}_n(K)$. Hence $\mathfrak{K} \subseteq A^{-1} \cdot \mathfrak{b}_n(K) \cdot A$, implying that all maximal solvable subalgebras of $\widehat{\mathfrak{L}}$ are $\mathrm{Ad}_{\widehat{\mathfrak{L}}}(\mathrm{GL}_n(K))$-conjugate to $\mathfrak{b}_n(K)$. If $A \cdot \mathfrak{b}_n(K) \cdot A^{-1}$, for some $A \in \mathrm{GL}_n(K)$, is a maximal solvable subalgebra of $\mathfrak{L}$ containing $\mathrm{rad}(\widehat{\mathfrak{L}})$, then $\mathrm{rad}(\widehat{\mathfrak{L}}) = A^{-1} \cdot \mathrm{rad}(\widehat{\mathfrak{L}}) \cdot A \subseteq \mathfrak{b}_n(K)$.

Letting $\mathfrak{b}_n^-(K) := \{A = [a_{ij}]_{ij} \in \widehat{\mathfrak{L}}; a_{ij} = 0 \text{ for } i < j\}$ be the **opposite Borel subalgebra** of **lower triangular** matrices, then we also get $\mathrm{rad}(\widehat{\mathfrak{L}}) \subseteq \mathfrak{b}_n^-(K)$. Thus we have $\mathrm{rad}(\widehat{\mathfrak{L}}) \subseteq \mathfrak{b}_n(K) \cap \mathfrak{b}_n^-(K) = \mathfrak{t}_n(K)$. Letting $A = \mathrm{diag}[a_1, \ldots, a_n] \in \mathrm{rad}(\widehat{\mathfrak{L}})$, then for $i \neq j \in \{1, \ldots, n\}$ we have $[A, E_{ij}] = (a_i - a_j)E_{ij} \in \mathrm{rad}(\widehat{\mathfrak{L}}) \subseteq \mathfrak{t}_n(K)$, hence we infer that $a_i = a_j$. This implies that $\mathrm{rad}(\widehat{\mathfrak{L}}) = \mathfrak{z}_n(K) = Z(\widehat{\mathfrak{L}})$.

ii) Similarly, if $\mathfrak{K} \subseteq \mathfrak{L}$ is a solvable subalgebra, then by Lie's Theorem there is $A \in \mathrm{GL}_n(K)$ such that $A \cdot \mathfrak{K} \cdot A^{-1} \subseteq \mathfrak{b}_n(K) \cap \mathfrak{L}$. Hence $\mathfrak{K} \subseteq A^{-1} \cdot (\mathfrak{b}_n(K) \cap \mathfrak{L}) \cdot A$, implying that all maximal solvable subalgebras of $\mathfrak{L}$ are $\mathrm{Ad}_{\mathfrak{L} \subseteq \widehat{\mathfrak{L}}}(\mathrm{GL}_n(K))$-

conjugate to $\mathfrak{b}_n(K) \cap \mathfrak{L}$. If $A \cdot (\mathfrak{b}_n(K) \cap \mathfrak{L}) \cdot A^{-1}$, for some $A \in \mathrm{GL}_n(K)$, is a maximal solvable subalgebra of $\mathfrak{L}$ containing $\mathrm{rad}(\mathfrak{L})$, then $\mathrm{rad}(\mathfrak{L}) = A^{-1} \cdot \mathrm{rad}(\mathfrak{L}) \cdot A \subseteq \mathfrak{b}_n(K) \cap \mathfrak{L}$. We also get $\mathrm{rad}(\mathfrak{L}) \subseteq \mathfrak{b}_n^-(K) \cap \mathfrak{L}$. Thus we have $\mathrm{rad}(\mathfrak{L}) \subseteq \mathfrak{b}_n(K) \cap \mathfrak{b}_n^-(K) \cap \mathfrak{L} = \mathfrak{t}_n(K) \cap \mathfrak{L}$. Now letting $A = \mathrm{diag}[a_1, \ldots, a_n] \in \mathrm{rad}(\mathfrak{L})$, then for $i \neq j \in \{1, \ldots, n\}$ we have $[A, E_{ij}] = (a_i - a_j)E_{ij} \in \mathrm{rad}(\mathfrak{L}) \subseteq \mathfrak{t}_n(K)$, hence we infer that $a_i = a_j$. This implies that $\mathrm{rad}(\mathfrak{L}) \subseteq \mathfrak{z}_n(K) \cap \mathfrak{L} = \{0\}$. ♯

(7.2) The Killing form. a) Let K be a field, and let $\mathfrak{L}$ be a finite-dimensional Lie K-algebra. The K-bilinear form $\kappa = \kappa_\mathfrak{L} \colon \mathfrak{L} \times \mathfrak{L} \to K$ on $\mathfrak{L}$ defined by $\kappa(x, y) := \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y))$, for all $x, y \in \mathfrak{L}$, is called the associated **Killing form**. Since $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y)) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(y)\mathrm{ad}_\mathfrak{L}(x)) = \kappa(y, x)$ the Killing form is symmetric.

We have $\kappa([x, y], z) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}([x, y])\mathrm{ad}_\mathfrak{L}(z)) = \mathrm{Tr}([\mathrm{ad}_\mathfrak{L}(x), \mathrm{ad}_\mathfrak{L}(y)]\mathrm{ad}_\mathfrak{L}(z)) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y)\mathrm{ad}_\mathfrak{L}(z) - \mathrm{ad}_\mathfrak{L}(y)\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(z)) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y)\mathrm{ad}_\mathfrak{L}(z) - \mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(z)\mathrm{ad}_\mathfrak{L}(y)) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)[\mathrm{ad}_\mathfrak{L}(y), \mathrm{ad}_\mathfrak{L}(z)]) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}([y, z])) = \kappa(x, [y, z])$, for all $x, y, z \in \mathfrak{L}$, hence the Killing form is **associative**; note that we have seen this argument on traces in (6.4) already.

Lemma. Let $\mathfrak{I} \trianglelefteq \mathfrak{L}$ be an ideal. Then we have $\kappa_\mathfrak{I} = \kappa_\mathfrak{L}|_{\mathfrak{I} \times \mathfrak{I}}$.

Proof. If $U \leq_K V$ are finite-dimensional K-vector spaces, and $\varphi \in \mathrm{End}_K(V)$ is such that $\varphi(V) \leq_K U$, then U is φ-invariant, and considering the matrix of φ with respect to a K-basis of V obtained by extending a K-basis of U shows that $\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi|_U)$.

We apply this for $U := \mathfrak{I} \leq_K \mathfrak{L} = V$, where for $x \in \mathfrak{I}$ we have $\mathrm{ad}_\mathfrak{L}(x) \colon \mathfrak{L} \to \mathfrak{I}$. Thus we have $\kappa_\mathfrak{L}(x, y) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y)) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)|_\mathfrak{I} \cdot \mathrm{ad}_\mathfrak{L}(y)|_\mathfrak{I}) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{I}(x)\mathrm{ad}_\mathfrak{I}(y)) = \kappa_\mathfrak{I}(x, y)$, for all $x, y \in \mathfrak{I}$. ♯

b) Let $\mathrm{rad}(\kappa) := \{x \in \mathfrak{L}; \kappa(x, \mathfrak{L}) = \{0\}\} \leq_K \mathfrak{L}$ be the radical of the K-bilinear form κ. Then $\mathrm{rad}(\kappa) \trianglelefteq \mathfrak{L}$ even is an ideal: For $x \in \mathrm{rad}(\kappa)$ and $y, z \in \mathfrak{L}$ we have $\kappa([x, y], z) = \kappa(x, [y, z]) = 0$, thus $[x, y] \in \mathrm{rad}(\kappa)$.

In particular, if $\mathfrak{L}$ is nilpotent, then $\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y)$ is nilpotent, hence $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(x)\mathrm{ad}_\mathfrak{L}(y)) = 0$, for all $x, y \in \mathfrak{L}$, implying that κ is the zero form, in other words $\mathrm{rad}(\kappa) = \mathfrak{L}$. Moreover, if $\mathrm{char}(K) = 0$ then Cartan's Criterion says that $\mathfrak{L}$ is solvable if and only if $[\mathfrak{L}, \mathfrak{L}] \subseteq \mathrm{rad}(\kappa)$.

Example: The special linear algebra of degree 2**.** Let $\mathfrak{L} := \mathfrak{sl}_2(K)$. Letting $\{E, H, F\} \subseteq \mathfrak{L}$ be the standard K-basis, by (2.2) the adjoint representation $\mathrm{ad}_\mathfrak{L} \colon \mathfrak{L} \to \mathfrak{gl}_3(K)$ equals

$$\mathrm{ad}_\mathfrak{L}(E) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_\mathfrak{L}(H) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathrm{ad}_\mathfrak{L}(F) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Hence all of $\mathrm{ad}_{\mathfrak{L}}(E)^2$, $\mathrm{ad}_{\mathfrak{L}}(F)^2$, $\mathrm{ad}_{\mathfrak{L}}(E)\mathrm{ad}_{\mathfrak{L}}(H)$ and $\mathrm{ad}_{\mathfrak{L}}(H)\mathrm{ad}_{\mathfrak{L}}(F)$ are triangular matrices with zero diagonal entries, implying that $\kappa(E,E) = \kappa(F,F) = \kappa(E,H) = \kappa(H,F) = 0$, while

$$\mathrm{ad}_{\mathfrak{L}}(H)^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathrm{ad}_{\mathfrak{L}}(E)\mathrm{ad}_{\mathfrak{L}}(F) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

yields $\kappa(H,H) = 8$ and $\kappa(E,F) = 4$. This shows that the Gram matrix of the Killing form with respect to the standard K-basis is given as

$$G(\kappa) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

In particular, κ is non-degenerate if and only if $\mathrm{char}(K) \neq 2$; note that in the latter case $\kappa = 0$, and indeed $\mathfrak{L}$ is nilpotent. $\sharp$

(7.3) Theorem. Let K be a field, let $\mathfrak{L}$ be a finite-dimensional Lie K-algebra, and let κ be the associated Killing form.

a) If κ is non-degenerate then $\mathfrak{L}$ is semisimple.

b) If $\mathrm{char}(K) = 0$ then we have $\mathrm{rad}(\kappa) \subseteq \mathrm{rad}(\mathfrak{L})$. In particular, the converse of a) holds: If $\mathfrak{L}$ is semisimple then κ is non-degenerate.

Proof. a) We show that, in general, any commutative ideal $\mathfrak{I} \trianglelefteq \mathfrak{L}$ is contained in $\mathrm{rad}(\kappa)$: For $x \in \mathfrak{I}$ and $y \in \mathfrak{L}$ we have $(\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(y))^2 \colon \mathfrak{L} \to \mathfrak{L} \to \mathfrak{I} \to \mathfrak{I} \to [\mathfrak{I},\mathfrak{I}] = \{0\}$, thus $\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(y)$ is nilpotent, implying that $\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(y)) = 0$, thus $x \in \mathrm{rad}(\kappa)$, and hence $\mathfrak{I} \subseteq \mathrm{rad}(\kappa)$. Note that this does not imply that $\mathrm{rad}(\mathfrak{L}) \subseteq \mathrm{rad}(\kappa)$.

Thus, specifically, if κ is non-degenerate, that is $\mathrm{rad}(\kappa) = \{0\}$, then $\mathfrak{L}$ does not have any non-zero commutative ideal, which is equivalent to $\mathrm{rad}(\mathfrak{L}) = \{0\}$.

b) Let $\mathfrak{I} := \mathrm{rad}(\kappa) \trianglelefteq \mathfrak{L}$. For $x \in \mathfrak{I}$ and $y \in \mathfrak{L}$ we have $\mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(y)) = \kappa(x,y) = 0$; in particular for $y \in [\mathfrak{I},\mathfrak{I}]$. Since $[\mathrm{ad}_{\mathfrak{L}}(\mathfrak{I}), \mathrm{ad}_{\mathfrak{L}}(\mathfrak{I})] = \mathrm{ad}_{\mathfrak{L}}([\mathfrak{I},\mathfrak{I}])$, Cartan's Criterion implies that $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{I}) \subseteq \mathfrak{gl}(\mathfrak{L})$ is solvable. Since $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{I}) \cong \mathfrak{I}/(\mathfrak{I} \cap Z(\mathfrak{L}))$, this entails that $\mathfrak{I}$ is solvable. Hence we have $\mathfrak{I} \subseteq \mathrm{rad}(\mathfrak{L})$. $\sharp$

This elucidates the relationship of semisimplicity and non-degeneration of the Killing form. As for part b), the converse inclusion does not hold in general, and the assumption on the characteristic of the underlying field K cannot be dispensed of, inasmuch the assertion elsewise does not hold in general either. This now yields the following name-giving characterisation of semisimplicity:

(7.4) Theorem. Let K be a field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple finite-dimensional Lie K-algebra. Then $\mathfrak{L} = \bigoplus_{i=1}^{n} \mathfrak{L}_i$ is the direct sum of simple Lie K-algebras $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$, for some $n \in \mathbb{N}_0$; in other words, $\mathfrak{L}$ is a semisimple $\mathfrak{L}$-module with respect to the adjoint representation.

Moreover, we have the orthogonal direct sum $\kappa_{\mathfrak{L}} = \bigoplus_{i=1}^n \kappa_{\mathfrak{L}_i}$ of Killing forms, the subalgebras $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$ are precisely the minimal non-zero ideals of $\mathfrak{L}$; any ideal $\mathfrak{I} \trianglelefteq \mathfrak{L}$ is semisimple, and has the form $\mathfrak{I} = \bigoplus_{i \in \mathcal{I}} \mathfrak{L}_i$, for some $\mathcal{I} \subseteq \{1, \ldots, n\}$; any quotient $\mathfrak{L}/\mathfrak{I} \cong \bigoplus_{i \in \{1,\ldots,n\} \setminus \mathcal{I}} \mathfrak{L}_i$ is semisimple; and $\mathfrak{L}$ is perfect.

Proof. Since $\mathfrak{L}$ is semisimple, the Killing form $\kappa = \kappa_{\mathfrak{L}}$ of $\mathfrak{L}$ is non-degenerate. Recall that for any ideal $\mathfrak{I} \trianglelefteq \mathfrak{L}$ we have $\kappa_{\mathfrak{I}} = \kappa|_{\mathfrak{I} \times \mathfrak{I}}$.

i) Let first $\mathfrak{I} \trianglelefteq \mathfrak{L}$ be an ideal, then $\mathfrak{I}^\perp := \{x \in \mathfrak{L}; \kappa(x, \mathfrak{I}) = \{0\}\} \trianglelefteq \mathfrak{L}$ is an ideal as well: For $x \in \mathfrak{I}^\perp$ and $y \in \mathfrak{L}$ we have $\kappa([x,y]), z) = \kappa(x, [y,z]) = 0$, for all $z \in \mathfrak{I}$, hence $[x,y] \in \mathfrak{I}^\perp$. For $\mathfrak{I} \cap \mathfrak{I}^\perp \trianglelefteq \mathfrak{L}$ we conclude that $\kappa_{\mathfrak{I} \cap \mathfrak{I}^\perp} = 0$, hence Cartan's Criterion implies that $\mathfrak{I} \cap \mathfrak{I}^\perp$ is solvable, and thus $\mathfrak{I} \cap \mathfrak{I}^\perp = \{0\}$. Since $\dim_K(\mathfrak{I}) + \dim_K(\mathfrak{I}^\perp) = \dim_K(\mathfrak{L})$, this shows that $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{I}^\perp$ as K-vector spaces. Moreover, from $[\mathfrak{I}, \mathfrak{I}^\perp] \leq_K \mathfrak{I} \cap \mathfrak{I}^\perp = \{0\}$ we infer that $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{I}^\perp$ as Lie K-algebras. Thus any ideal of $\mathfrak{I}$ is an ideal of $\mathfrak{L}$, and we have the orthogonal direct sum $\kappa = \kappa_{\mathfrak{I}} \oplus \kappa_{\mathfrak{I}^\perp}$, in particular entailing that $\kappa_{\mathfrak{I}}$ in non-degenerate.

Now we proceed by induction on $\dim_K(\mathfrak{L}) \in \mathbb{N}_0$; the case $\mathfrak{L} = \{0\}$ being trivial. Let $\{0\} \neq \mathfrak{I} \trianglelefteq \mathfrak{L}$ be a minimal non-zero ideal. Then minimality implies that $\mathfrak{I}$ does not have any non-zero proper ideals, hence being non-commutative $\mathfrak{I}$ is simple. Thus if $\mathfrak{I} = \mathfrak{L}$ then we are done. If $\mathfrak{I} \triangleleft \mathfrak{L}$ then we have $\{0\} \neq \mathfrak{I}^\perp \triangleleft \mathfrak{L}$, thus since $\mathfrak{I}^\perp$ is semisimple again we are done by induction.

ii) We have $[\mathfrak{L}, \mathfrak{L}] = \bigoplus_{i=1}^n [\mathfrak{L}_i, \mathfrak{L}_i] = \bigoplus_{i=1}^n \mathfrak{L}_i = \mathfrak{L}$, that is $\mathfrak{L}$ is perfect.

Finally, if $\mathfrak{I} \trianglelefteq \mathfrak{L}$ is an ideal, then the above argument shows that $\mathfrak{I}$ is semisimple again and a direct sum of minimal non-zero ideals of $\mathfrak{L}$. Hence it remains to be shown that any minimal non-zero ideal $\{0\} \neq \mathfrak{I} \triangleleft \mathfrak{L}$ is amongst $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$: Indeed, $[\mathfrak{I}, \mathfrak{L}] \trianglelefteq \mathfrak{L}$ and $[\mathfrak{I}, \mathfrak{L}_i] \trianglelefteq \mathfrak{L}$ are ideals, for all $i \in \{1, \ldots, n\}$, where since $Z(\mathfrak{L}) = \{0\}$ we have $[\mathfrak{I}, \mathfrak{L}] \neq \{0\}$, thus $\{0\} \neq \mathfrak{I} = [\mathfrak{I}, \mathfrak{L}] = \bigoplus_{i=1}^n [\mathfrak{I}, \mathfrak{L}_i]$ by minimality shows that $\mathfrak{I} = \mathfrak{L}_i$, for a unique $i \in \{1, \ldots, n\}$. ♯

(7.5) Theorem: Zassenhaus. Let K be a field. If $\mathfrak{L}$ is a finite-dimensional Lie K-algebra with non-degenerate Killing form, then we have $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) = \mathrm{Der}_K(\mathfrak{L})$.

In particular, if $\mathrm{char}(K) = 0$ and $\mathfrak{L}$ is a semisimple Lie K-algebra, then any derivation of $\mathfrak{L}$ is inner.

Proof. Let $\partial \in \mathrm{Der}_K(\mathfrak{L})$. Then we have the K-linear map $\mathfrak{L} \to K \colon x \mapsto \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x) \cdot \partial)$. Since the Killing form $\kappa = \kappa_{\mathfrak{L}}$ of $\mathfrak{L}$ is non-degenerate, there is a unique $d \in \mathfrak{L}$ such that $\kappa(x, d) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x) \cdot \partial)$, for all $x \in \mathfrak{L}$. We show that $\partial = \mathrm{ad}_{\mathfrak{L}}(d) \in \mathrm{Der}_K(\mathfrak{L})$:

To this end let $\delta := \partial - \mathrm{ad}_{\mathfrak{L}}(d)$. Then we have $\mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x) \cdot \delta) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x) \cdot \partial) - \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x)\mathrm{ad}_{\mathfrak{L}}(d)) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x) \cdot \partial) - \kappa(x, d) = 0$, for all $x \in \mathfrak{L}$. Recalling that $[\delta, \mathrm{ad}_{\mathfrak{L}}(x)] = \mathrm{ad}_{\mathfrak{L}}(\delta x) \in \mathrm{Der}_K(\mathfrak{L})$, we get $\kappa(\delta x, y) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(\delta x)\mathrm{ad}_{\mathfrak{L}}(y)) =$

$\mathrm{Tr}([\delta, \mathrm{ad}_{\mathfrak{L}}(x)] \cdot \mathrm{ad}_{\mathfrak{L}}(y)) = \mathrm{Tr}(\delta \cdot [\mathrm{ad}_{\mathfrak{L}}(x), \mathrm{ad}_{\mathfrak{L}}(y)]) = 0$, for all $x, y \in \mathfrak{L}$. Thus from κ being non-degenerate, we infer that $\delta x = 0$, hence $\delta = 0$. ♯

(7.6) Abstract Jordan-Chevalley decomposition. a) Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple Lie K-algebra. The above considerations allow us to define Jordan-Chevalley decompositions in $\mathfrak{L}$, without presupposing an embedding into a linear Lie algebra:

Let $x \in \mathfrak{L}$. Since $\mathrm{Der}_K(\mathfrak{L}) \leq_K \mathrm{End}_K(\mathfrak{L})$ has Jordan-Chevalley decompositions, see (6.3), we have $\mathrm{ad}_{\mathfrak{L}}(x) = \mathrm{ad}_{\mathfrak{L}}(x)_s + \mathrm{ad}_{\mathfrak{L}}(x)_n \in \mathrm{Der}_K(\mathfrak{L})$, where $\mathrm{ad}_{\mathfrak{L}}(x)_s \in \mathrm{End}_K(\mathfrak{L})$ is semisimple, $\mathrm{ad}_{\mathfrak{L}}(x)_n \in \mathrm{End}_K(\mathfrak{L})$ is nilpotent, as well as $[\mathrm{ad}_{\mathfrak{L}}(x)_s, \mathrm{ad}_{\mathfrak{L}}(x)_n] = 0$. Now we have $\mathfrak{L} \cong \mathfrak{L}/Z(\mathfrak{L}) \cong \mathrm{ad}_{\mathfrak{L}}(\mathfrak{L}) = \mathrm{Der}_K(\mathfrak{L})$, hence there are unique elements $x_s, x_n \in \mathfrak{L}$ such that $\mathrm{ad}_{\mathfrak{L}}(x_s) = \mathrm{ad}_{\mathfrak{L}}(x)_s$ and $\mathrm{ad}_{\mathfrak{L}}(x_n) = \mathrm{ad}_{\mathfrak{L}}(x)_n$; hence $x_s \in \mathfrak{L}$ is **ad-semisimple** and $x_n \in \mathfrak{L}$ is ad-nilpotent.

The elements $x_s \in \mathfrak{L}$ and $x_n \in \mathfrak{L}$ are called the **semisimple** and **nilpotent parts** of $x \in \mathfrak{L}$, respectively. We have $\mathrm{ad}_{\mathfrak{L}}(x_s + x_n) = \mathrm{ad}_{\mathfrak{L}}(x_s) + \mathrm{ad}_{\mathfrak{L}}(x_n) = \mathrm{ad}_{\mathfrak{L}}(x)_s + \mathrm{ad}_{\mathfrak{L}}(x)_n = \mathrm{ad}_{\mathfrak{L}}(x)$, implying $x = x_s + x_n$; and we have $\mathrm{ad}_{\mathfrak{L}}([x_s, x_n]) = [\mathrm{ad}_{\mathfrak{L}}(x_s), \mathrm{ad}_{\mathfrak{L}}(x_n)] = [\mathrm{ad}_{\mathfrak{L}}(x)_s, \mathrm{ad}_{\mathfrak{L}}(x)_n] = 0$, implying $[x_s, x_n] = 0$.

Proposition. Let $\mathfrak{I} \trianglelefteq \mathfrak{L}$, and let $\mathfrak{L} \to \mathfrak{L}/\mathfrak{I} \colon x \mapsto \overline{x}$ be the natural map. If $x = x_s + x_n \in \mathfrak{L}$ is the abstract Jordan-Chevalley decomposition of $x \in \mathfrak{L}$, then $\overline{x} = \overline{x_s} + \overline{x_n} \in \mathfrak{L}$ is the abstract Jordan-Chevalley decomposition of $\overline{x} \in \mathfrak{L}$.

Proof. Since $\overline{\mathfrak{L}} := \mathfrak{L}/\mathfrak{I}$ is semisimple again, it has abstract Jordan-Chevalley decompositions. Moreover, we have $\mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{x}) \colon \overline{\mathfrak{L}} \to \overline{\mathfrak{L}} \colon \overline{y} \mapsto \overline{[x, y]} = \overline{\mathrm{ad}_{\mathfrak{L}}(x)(y)}$, that is $\mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{x})$ is the map induced by $\mathrm{ad}_{\mathfrak{L}}(x)$ by naturally passing to the quotient.

Hence, since $\mathrm{ad}_{\mathfrak{L}}(x_s) \in \mathfrak{gl}(\mathfrak{L})$ is semisimple, that is has a multiplicity-free minimum polynomial, we infer that $\mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{x_s}) \in \mathfrak{gl}(\overline{\mathfrak{L}})$ is semisimple as well. Similarly, since $\mathrm{ad}_{\mathfrak{L}}(x_n) \in \mathfrak{gl}(\mathfrak{L})$ is nilpotent, that is has a power which is the zero map, we infer that $\mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{x_n}) \in \mathfrak{gl}(\overline{\mathfrak{L}})$ is nilpotent as well. Finally, we have $[\overline{x_s}, \overline{x_n}] = \overline{[x_s, x_n]} = 0$, hence we infer that $[\mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{x_s}), \mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{x_n})] = \mathrm{ad}_{\overline{\mathfrak{L}}}([\overline{x_s}, \overline{x_n}]) = \mathrm{ad}_{\overline{\mathfrak{L}}}(\overline{[x_s, x_n]}) = 0$. Thus, by uniqueness, we conclude that $\overline{x} = \overline{x_s} + \overline{x_n}$ is the abstract Jordan-Chevalley decomposition of $\overline{x} \in \overline{\mathfrak{L}}$. ♯

In particular, if $\varphi \colon \mathcal{L} \to \mathfrak{gl}(V)$ is a representation, where V is a finite-dimensional K-vector space, then $\varphi(x) = \varphi(x_s) + \varphi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\varphi(x) \in \varphi(\mathfrak{L})$, where $\varphi(\mathfrak{L}) \subseteq \mathfrak{gl}(V)$ is a linear Lie algebra. This leads to the following question:

b) If $\mathfrak{L} \subseteq \mathfrak{gl}(V)$ is a linear Lie algebra, there also might be Jordan-Chevalley decompositions in $\mathfrak{L}$ being inherited from $\mathfrak{gl}(V)$, which a priorily need not coincide with abstract Jordan-Chevalley decompositions. We will show in (8.3) that these always do coincide. In the following case this is actually immediate:

Let $\mathfrak{L} := \mathfrak{sl}_n(K) \subseteq \mathfrak{gl}_n(K) =: \widehat{\mathfrak{L}}$, where $n \in \mathbb{N}$. For $A \in \mathfrak{L}$ let $A = A_s + A_n \in \widehat{\mathfrak{L}}$ be the Jordan-Chevalley decomposition, where $A_s \in \widehat{\mathfrak{L}}$ is semisimple and $A_n \in \widehat{\mathfrak{L}}$ is nilpotent. Then we have $\mathrm{Tr}(A_n) = 0$, thus $A_n \in \mathfrak{L}$, and hence $A_s = A - A_n \in \mathfrak{L}$ as well. This shows that $\mathfrak{L}$ has Jordan-Chevalley decompositions.

Now $\mathfrak{L}$ is semisimple, thus has abstract Jordan-Chevalley decompositions. By (6.3), $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A_s)$ is semisimple, hence $\mathrm{ad}_{\mathfrak{L}}(A_s)$ is so as well; similarly, $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A_n)$ is nilpotent, hence $\mathrm{ad}_{\mathfrak{L}}(A_n)$ is so as well; finally we have $[\mathrm{ad}_{\mathfrak{L}}(A_s), \mathrm{ad}_{\mathfrak{L}}(A_n)] = \mathrm{ad}_{\mathfrak{L}}([A_s, A_n]) = 0$. Thus, by uniqueness, we conclude that $A = A_s + A_n$ coincides with the abstract Jordan-Chevalley decomposition of $A \in \mathfrak{L}$.

8 Semisimple modules

(8.1) Casimir elements. Let K be a field, and let $\mathfrak{L}$ be a finite-dimensional Lie K-algebra.

a) Let $\beta\colon \mathfrak{L} \times \mathfrak{L} \to K$ be a non-degenerate symmetric associative K-bilinear form on $\mathfrak{L}$; for example, the Killing form κ of $\mathfrak{L}$ has these properties. Letting $\mathcal{B} := \{x_1, \ldots, x_n\} \subseteq \mathfrak{L}$ be a K-basis, where $n := \dim_K(\mathfrak{L}) \in \mathbb{N}_0$, there is a unique **dual** K-basis $\mathcal{B}^* := \{x_1^*, \ldots, x_n^*\} \subseteq \mathfrak{L}$ with respect to β, defined by $\beta(x_i, x_j^*) = \delta_{ij}$, for all $i, j \in \{1, \ldots, n\}$.

The action $\mathrm{ad}_{\mathfrak{L}}(x) \in \mathfrak{gl}(\mathfrak{L})$ of $x \in \mathfrak{L}$ in the adjoint representation is given in terms of structure constants. More precisely, with respect to the K-basis $\mathcal{B}$ we have $[x, x_j] = \sum_{i=1}^n a_{ij}(x)x_i$, where $a_{ij}(x) \in K$, hence we let $\mathrm{ad}_{\mathcal{B}}(x) := [a_{ij}(x)]_{ij} \in \mathfrak{gl}_n(K)$. Likewise, with respect to the K-basis $\mathcal{B}^*$ we have $[x, x_j^*] = \sum_{i=1}^n a_{ij}^*(x)x_i^*$, where $a_{ij}^*(x) \in K$, hence we let $\mathrm{ad}_{\mathcal{B}^*}(x) := [a_{ij}^*(x)]_{ij} \in \mathfrak{gl}_n(K)$.

Using associativity we have $a_{ij}(x) = \beta(\sum_{k=1}^n a_{kj}(x)x_k, x_i^*) = \beta([x, x_j], x_i^*) = -\beta([x_j, x], x_i^*) = -\beta(x_j, [x, x_i^*]) = -\beta(x_j, \sum_{k=1}^n a_{ki}^*(x)x_k^*) = -a_{ji}^*(x)$, for all $i, j \in \{1, \ldots, n\}$, that is the representing matrices fulfill $\mathrm{ad}_{\mathcal{B}^*}(x) = -\mathrm{ad}_{\mathcal{B}}(x)^{\mathrm{tr}}$.

b) Let V be a finite-dimensional K-vector space, and let $\varphi\colon \mathfrak{L} \to \mathfrak{gl}(V)$ be a representation. Then let $C_\varphi(\beta, \mathcal{B}) := \sum_{i=1}^n \varphi(x_i)\varphi(x_i^*) \in \mathfrak{gl}(V)$ be the **Schur element** of φ associated with β and $\mathcal{B}$. Note that $C_\varphi(\beta, \mathcal{B})$ is contained in the unital associative K-subalgebra of $\mathfrak{gl}(V)$ generated by $\varphi(\mathfrak{L})$.

We proceed to determine $[\varphi(x), C_\varphi(\beta, \mathcal{B})] = \sum_{i=1}^n [\varphi(x), \varphi(x_i)\varphi(x_i^*)] \in \mathfrak{gl}(V)$: Using the fact that the adjoint map is a derivation, we obtain $[\varphi(x), C_\varphi(\beta, \mathcal{B})] = \sum_{i=1}^n \big([\varphi(x), \varphi(x_i)]\varphi(x_i^*) + \varphi(x_i)[\varphi(x), \varphi(x_i^*)]\big)$. Since φ is a representation, we infer $[\varphi(x), C_\varphi(\beta, \mathcal{B})] = \sum_{i=1}^n \big(\varphi([x, x_i])\varphi(x_i^*) + \varphi(x_i)\varphi([x, x_i^*])\big)$, where using structure constants yields $[\varphi(x), C_\varphi(\beta, \mathcal{B})] = \sum_{i=1}^n \sum_{j=1}^n \big(a_{ji}(x)\varphi(x_j)\varphi(x_i^*) + a_{ji}^*(x)\varphi(x_i)\varphi(x_j^*)\big) = \sum_{i=1}^n \sum_{j=1}^n \big(a_{ij}(x)\varphi(x_i)\varphi(x_j^*) + a_{ji}^*(x)\varphi(x_i)\varphi(x_j^*)\big) = 0$, for all $x \in \mathfrak{L}$. In other words, $C_\varphi(\beta, \mathcal{B})$ commutes with $\varphi(\mathfrak{L})$, that is $C_\varphi(\beta, \mathcal{B}) \in \mathrm{End}_{\mathfrak{L}}(V)$ is an endomorphism of V as an $\mathfrak{L}$-module.

c) Moreover, φ gives rise to a K-bilinear form on $\mathfrak{L}$ defined by $\beta_\varphi(x, y) := \mathrm{Tr}(\varphi(x)\varphi(y))$, for all $x, y \in \mathfrak{L}$.

Since $\beta_\varphi(x, y) = \mathrm{Tr}(\varphi(x)\varphi(y)) = \mathrm{Tr}(\varphi(y)\varphi(x)) = \beta_\varphi(y, x)$ and $\beta_\varphi([x, y], z) = \mathrm{Tr}(\varphi([x, y])\varphi(z)) = \mathrm{Tr}([\varphi(x), \varphi(y)]\varphi(z)) = \mathrm{Tr}(\varphi(x)\varphi(y)\varphi(z) - \varphi(y)\varphi(x)\varphi(z)) =$

$\mathrm{Tr}(\varphi(x)\varphi(y)\varphi(z)-\varphi(x)\varphi(z)\varphi(y)) = \mathrm{Tr}(\varphi(x)[\varphi(y),\varphi(z)]) = \mathrm{Tr}(\varphi(x)\varphi([y,z])) = \beta_\varphi(x,[y,z])$, for all $x,y,z \in \mathfrak{L}$, the form β_φ is symmetric and associative. In particular, associativity implies that $\mathfrak{I} := \mathrm{rad}(\beta_\varphi) \trianglelefteq \mathfrak{L}$ is an ideal: For $x \in \mathfrak{I}$ and $y,z \in \mathfrak{L}$ we have $\beta_\varphi([x,y],z) = \beta_\varphi(x,[y,z]) = 0$, thus $[x,y] \in \mathfrak{I}$.

d) Now let $\mathrm{char}(K) = 0$ and $\mathfrak{L}$ be semisimple, and moreover let φ be **faithful**, that is $\ker(\varphi) = \{0\}$. Then for $x \in \mathfrak{I}$ and $y \in \mathfrak{L}$ we have $\mathrm{Tr}(\varphi(x),\varphi(y)) = \beta_\varphi(x,y) = 0$; in particular this holds for $y \in [\mathfrak{I},\mathfrak{I}]$. Since $[\varphi(\mathfrak{I}),\varphi(\mathfrak{I})] = \varphi([\mathfrak{I},\mathfrak{I}])$, Cartan's Criterion implies that $\mathfrak{I} \cong \varphi(\mathfrak{I}) \subseteq \mathfrak{gl}(V)$ is solvable, hence $\mathfrak{I} = \{0\}$, that is β_φ is non-degenerate.

Note that we have seen these arguments in (7.2) and (7.3) already: Indeed, since $\ker(\mathrm{ad}_\mathfrak{L}) = Z(\mathfrak{L}) = \{0\}$ the adjoint representation $\mathrm{ad}_\mathfrak{L}$ is faithful, and we just have $\beta_{\mathrm{ad}_\mathfrak{L}} = \kappa$, the Killing form of $\mathfrak{L}$.

Then the Schur element $C_\varphi(\mathcal{B}) := C_\varphi(\beta_\varphi,\mathcal{B}) \in \mathfrak{gl}(V)$ is called the **Casimir element** of φ with respect to $\mathcal{B}$. Then we have $C_\varphi(\mathcal{B}) \in \mathrm{End}_\mathfrak{L}(V)$, and its trace equals $\mathrm{Tr}(C_\varphi(\mathcal{B})) = \sum_{i=1}^n \mathrm{Tr}(\varphi(x_i)\varphi(x_i^*)) = \sum_{i=1}^n \beta_\varphi(x_i,x_i^*) = n = \dim_K(\mathfrak{L})$.

If φ additionally is irreducible, then the Casimir element $C_\varphi(\mathcal{B}) \in \mathfrak{gl}_d(K)$, where $d := \dim_K(V) \in \mathbb{N}$, by Schur's Lemma and $\mathrm{Tr}(C_\varphi(\mathcal{B})) \neq 0$, is an invertible matrix. In particular, if K is algebraically closed then it is a scalar matrix, where from $\mathrm{Tr}(C_\varphi(\mathcal{B})) = n$ we get $C_\varphi := C_\varphi(\mathcal{B}) = \frac{n}{d} \cdot E_d = \frac{\dim_K(\mathfrak{L})}{\dim_K(V)} \cdot E_d$, which in this case is independent of the K-basis $\mathcal{B} \subseteq \mathfrak{L}$ chosen.

Example: The special linear algebra of degree 2. Let $\mathrm{char}(K) = 0$ and $\mathfrak{L} := \mathfrak{sl}_2(K)$, which has the standard K-basis $\mathcal{S} := \{E,H,F\} \subseteq \mathfrak{L}$, see (2.2).

i) We consider the tautological representation $\mathrm{id}_\mathfrak{L}\colon \mathfrak{L} \to \mathfrak{gl}_2(K)$, which is irreducible: Assume to the contrary that there is a proper non-zero $\mathfrak{L}$-submodule $U \leq K^{2\times 1}$, which hence is 1-dimensional, then U is contained in an eigenspace with respect to H, thus $U = \langle [1,0]^{\mathrm{tr}} \rangle_K$ or $U = \langle [0,1]^{\mathrm{tr}} \rangle_K$, where the former is not F-invariant and the latter is not E-invariant, a contradiction.

Let $\beta := \beta_{\mathrm{id}_\mathfrak{L}}$ be the K-bilinear form on $\mathfrak{L}$ associated with $\mathrm{id}_\mathfrak{L}$. All of E^2, F^2, EH and HF are triangular matrices with zero diagonal entries, implying that $\beta(E,E) = \beta(F,F) = \beta(E,H) = \beta(H,F) = 0$, while $H^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $EF = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ yields $\beta(H,H) = 2$ and $\beta(E,F) = 1$. Hence the Gram matrix of β with respect to $\mathcal{S}$ is given as

$$G(\beta) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence the dual K-basis of $\mathfrak{L}$ associated with $\mathcal{S}$ is $\mathcal{S}^* := \{F, \frac{1}{2}H, E\} \subseteq \mathfrak{L}$. Thus we get the Casimir element $C_{\mathrm{id}_{\mathfrak{L}}} = C_{\mathrm{id}_{\mathfrak{L}}}(\mathcal{S}) = EF + \frac{1}{2}H^2 + FE$, that is

$$C_{\mathrm{id}_{\mathfrak{L}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} = \frac{3}{2} \cdot E_2 \in \mathfrak{gl}_2(K);$$

for K algebraically closed this was to be expected from $\mathrm{Tr}(C_{\mathrm{id}_{\mathfrak{L}}}) = 3$.

ii) We consider the adjoint representation $\mathrm{ad}_{\mathfrak{L}}$, which since $\mathfrak{L}$ is simple is irreducible. The associated K-bilinear form is the Killing form $\beta_{\mathrm{ad}_{\mathfrak{L}}} = \kappa$, whose Gram matrix with respect to $\mathcal{S}$ is given as, see (7.2),

$$G(\kappa) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

Hence the dual K-basis associated with $\mathcal{S}$, with respect to the Killing form κ, is $\mathcal{S}^* := \{\frac{1}{4}F, \frac{1}{8}H, \frac{1}{4}E\} \subseteq \mathfrak{L}$. Thus we get the Casimir element $C_{\mathrm{ad}_{\mathfrak{L}}} = C_{\mathrm{ad}_{\mathfrak{L}}}(\mathcal{S}) = \frac{1}{4}\mathrm{ad}_{\mathfrak{L}}(E)\mathrm{ad}_{\mathfrak{L}}(F) + \frac{1}{8}\mathrm{ad}_{\mathfrak{L}}(H)\mathrm{ad}_{\mathfrak{L}}(H) + \frac{1}{4}\mathrm{ad}_{\mathfrak{L}}(F)\mathrm{ad}_{\mathfrak{L}}(E) \in \mathfrak{gl}_3(K)$, thus

$$C_{\mathrm{ad}_{\mathfrak{L}}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = E_3 \in \mathfrak{gl}_3(K);$$

for K algebraically closed this was to be expected from $\mathrm{Tr}(C_{\mathrm{ad}_{\mathfrak{L}}}) = 3$.

(8.2) Theorem: Weyl. Let K be a field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple Lie K-algebra. Then any finite-dimensional $\mathfrak{L}$-module is semisimple.

Proof. Let $V \neq \{0\}$ be a finite-dimensional $\mathfrak{L}$-module.

i) Let first $U <_{\mathfrak{L}} V$ be an $\mathfrak{L}$-submodule such that $\dim_K(V/U) = 1$. Then the induced representation $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}] \to [\mathfrak{gl}(V/U), \mathfrak{gl}(V/U)] \leq_K \mathfrak{sl}(V/U) = \{0\}$ is trivial, hence we may just write $V/U \cong K$. In order to show that U has a complement in V, we proceed by induction on $\dim_K(U) \in \mathbb{N}_0$; the case of $U = \{0\}$ being trivial, we may assume that $U \neq \{0\}$.

If U is reducible, then let $\{0\} \neq U' <_{\mathfrak{L}} U$ be a non-zero proper $\mathfrak{L}$-submodule. Thus we have $U/U' <_{\mathfrak{L}} V/U'$ such that $(V/U')/(U/U') \cong V/U \cong K$. Since $\dim_K(U/U') < \dim_K(U)$, by induction there is $U' <_{\mathfrak{L}} X <_{\mathfrak{L}} V$ such that $X/U' <_{\mathfrak{L}} V/U'$ is a complement of U/U'; that is we have $V/U' = U/U' \oplus X/U'$, where $X/U' \cong K$. Since $\dim_K(U') < \dim_K(U)$, by induction again there is a complement $W <_{\mathfrak{L}} X$ of U'; hence we have $X = U' \oplus W$, where $W \cong K$. Thus we have $V = U + X = U + U' + W = U + W$, where $\dim_K(U) + \dim_K(W) = \dim_K(U/U') + \dim_K(U') + \dim_K(W) = \dim_K(U/U') + \dim_K(X) = \dim_K(U/U') + \dim_K(X/U') + \dim_K(U') = \dim_K(V/U') + \dim_K(U') = \dim_K(V)$ shows that $U \cap W = \{0\}$, implying that $V = U \oplus W$.

Hence we may now assume assume that U is simple. Let $C_V \in \mathfrak{gl}(V)$ be a Casimir element of V, with respect to some K-basis of $\mathfrak{L}$; note that since any quotient of $\mathfrak{L}$ is semisimple again we may assume that V is faithful. Since C_V is contained in the unital associative K-algebra generated by the image of $\mathfrak{L}$ in $\mathfrak{gl}(V)$, we conclude that C_V acts both on U and $V/U \cong K$. With respect to the latter, C_V acts by the zero map. Moreover, since by (8.1) we have $\mathrm{Tr}_V(C_V) = \dim_K(\mathfrak{L}) \neq 0$, we infer that C_V cannot possibly act on U by the zero map as well. Hence Schur's Lemma implies that C_V acts invertibly on U. Thus we conclude that $\dim_K(\ker(C_V)) = 1$, and $V = U \oplus \ker(C_V)$ as K-vector spaces. Since $C_V \in \mathrm{End}_{\mathfrak{L}}(V)$ we have $C_V(xv) = x(C_V v) = 0$, for all $x \in \mathfrak{L}$ and $v \in \ker(C_V)$, that is $\ker(C_V) \leq_{\mathfrak{L}} V$ is a complement of U in V.

ii) We are now prepared to tackle the general case: In order to show that V is semisimple, let $\{0\} \neq U \leq_{\mathfrak{L}} V$ be an $\mathfrak{L}$-submodule, for which we show that it has a complement in V: To this end we consider the $\mathfrak{L}$-module $\mathrm{Hom}_K(V, U)$. Let $\mathcal{V} := \{\varphi \in \mathrm{Hom}_K(V, U); \varphi|_U = \lambda_\varphi \cdot \mathrm{id}_U \text{ for some } \lambda_\varphi \in K\} \leq_K \mathrm{Hom}_K(V, U)$. Then, since $U \neq \{0\}$, we have a surjective K-linear map $\mathcal{V} \to K \colon \varphi \mapsto \lambda_\varphi$, having kernel $\mathcal{U} := \{\varphi \in \mathcal{V}; \lambda_\varphi = 0\} \leq_K \mathcal{V}$ such that $\dim_K(\mathcal{V}/\mathcal{U}) = 1$. Moreover we have $(x\varphi)(u) = x(\varphi(u)) - \varphi(xu) = \lambda_\varphi \cdot xu - \lambda_\varphi \cdot xu = 0$, for all $\varphi \in \mathcal{V}$ and $x \in \mathfrak{L}$ and $u \in U$, thus $(x\varphi)|_U = 0$, in other words $x\varphi \in \mathcal{U}$. This shows that $\mathcal{U} \leq_{\mathfrak{L}} \mathcal{V} \leq_{\mathfrak{L}} \mathrm{Hom}_K(V, U)$ such that $\mathfrak{L} \cdot \mathcal{V} \leq_K \mathcal{U}$, hence $\mathcal{V}/\mathcal{U}$ is a trivial $\mathfrak{L}$-module.

Thus applying (i) there is $\varphi \in \mathcal{V}$ such that $\lambda_\varphi \neq 0$ and $\mathcal{V} = \mathcal{U} \oplus \langle \varphi \rangle_K$. Since $\varphi|_U = \lambda_\varphi \cdot \mathrm{id}_U$ we conclude that $\varphi \colon V \to U$ is surjective and $U \cap \ker(\varphi) = \{0\}$, entailing that $V = U \oplus \ker(\varphi)$ as K-vector spaces. Finally, from $x\varphi = 0$, for all $x \in \mathfrak{L}$, we get $0 = (x\varphi)(v) = x(\varphi(v)) - \varphi(xv)$, for all $v \in V$, that is $\varphi(xv) = x(\varphi(v))$, or equivalently that φ is an $\mathfrak{L}$-homomorphism, thus $\ker(\varphi) \leq_{\mathfrak{L}} V$. ♯

(8.3) Theorem. Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, let V be a finite-dimensional K-vector space, and let $\mathfrak{L} \subseteq \mathfrak{gl}(V)$ be a semisimple linear Lie K-algebra. Then $\mathfrak{L}$ has Jordan-Chevalley decompositions, that is for any $A \in \mathfrak{L}$ the semisimple and nilpotent parts $A_s \in \mathfrak{gl}(V)$ and $A_n \in \mathfrak{gl}(V)$, respectively, are elements of $\mathfrak{L}$ as well. Moreover, Jordan-Chevalley decompositions in $\mathfrak{L}$ and abstract Jordan-Chevalley decompositions in $\mathfrak{L}$ coincide.

Proof. Letting $\widehat{\mathfrak{L}} := \mathfrak{gl}(V)$, by (6.3) we have $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)_s = \mathrm{ad}_{\widehat{\mathfrak{L}}}(A_s) \in \mathfrak{gl}(\widehat{\mathfrak{L}})$ and $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)_n = \mathrm{ad}_{\widehat{\mathfrak{L}}}(A_n) \in \mathfrak{gl}(\widehat{\mathfrak{L}})$. Since $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)_s$ and $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)_n$ are polynomials in $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)$, from $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A) \cdot \mathfrak{L} \leq_K \mathfrak{L}$ we get $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A_s) \cdot \mathfrak{L} \leq_K \mathfrak{L}$ and $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A_n) \cdot \mathfrak{L} \leq_K \mathfrak{L}$ as well. In other words, we have both $A_s, A_n \in N_{\widehat{\mathfrak{L}}}(\mathfrak{L})$.

But we have $\mathfrak{L} + Z(\widehat{\mathfrak{L}}) \trianglelefteq N_{\widehat{\mathfrak{L}}}(\mathfrak{L})$, where $Z(\widehat{\mathfrak{L}}) = K \cdot \mathrm{id}_V$ and $Z(\widehat{\mathfrak{L}}) \cap \mathfrak{L} = \{0\}$, so that this is not sufficient to show straightaway that A_s and A_n belong to $\mathfrak{L}$. Thus we look for a smaller Lie K-subalgebra of $\widehat{\mathfrak{L}}$ containing $\mathfrak{L}$:

For any $\mathfrak{L}$-submodule $U \leq_{\mathfrak{L}} V$, we consider the Lie K-subalgebra $\mathfrak{L}_U := \{M \in \widehat{\mathfrak{L}}; M \cdot U \leq_K U, \mathrm{Tr}_U(M) = 0\} \subseteq \widehat{\mathfrak{L}}$; in particular we have $\mathfrak{L}_V = \mathfrak{sl}(V)$, where $\mathfrak{sl}(V) \cap Z(\widehat{\mathfrak{L}}) = \{0\}$. Let $\mathfrak{K} := N_{\widehat{\mathfrak{L}}}(\mathfrak{L}) \cap \bigcap_{U \leq_{\mathfrak{L}} V} \mathfrak{L}_U \subseteq \widehat{\mathfrak{L}}$, which is a Lie K-subalgebra such that $\mathfrak{K} \cap Z(\widehat{\mathfrak{L}}) = \{0\}$. Since for any $U \leq_{\mathfrak{L}} V$ we have $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}] \to [\mathfrak{gl}(U), \mathfrak{gl}(U)] \leq_K \mathfrak{sl}(U)$, we conclude that $\mathfrak{L} \leq_K \mathfrak{K}$ and hence $\mathfrak{L} \trianglelefteq \mathfrak{K}$.

Moreover, since $A_s, A_n \in \widehat{\mathfrak{L}}$ are polynomials in A, from $A \cdot U \leq_K U$ we infer that $A_s \cdot U \leq_K U$ and $A_n \cdot U \leq_K U$ as well, and then from $\mathrm{Tr}_U(A) = 0$ and A_n being nilpotent we infer that $\mathrm{Tr}_U(A_n) = 0 = \mathrm{Tr}_U(A_s)$ as well. Thus we conclude that both $A_s, A_n \in \mathfrak{K}$. We proceed to show that $\mathfrak{L} = \mathfrak{K}$:

The Lie subalgebra $\mathfrak{K} \subseteq \widehat{\mathfrak{L}}$ is an $\mathfrak{L}$-submodule, with respect to the adjoint representation of $\widehat{\mathfrak{L}}$ restricted to $\mathfrak{L}$. Hence by Weyl's Theorem there is an $\mathfrak{L}$-submodule $\mathcal{E} \leq_{\mathfrak{L}} \mathfrak{K}$ such that $\mathfrak{K} = \mathfrak{L} \oplus \mathcal{E}$. We have to show that $\mathcal{E} = \{0\}$: Since $[\mathfrak{L}, \mathfrak{K}] \leq_K [\mathfrak{L}, N_{\widehat{\mathfrak{L}}}(\mathfrak{L})] \leq_K \mathfrak{L}$ we have $[\mathfrak{L}, \mathcal{E}] = \{0\}$, in other words $\mathcal{E}$ is a trivial $\mathfrak{L}$-module. Hence for any $M \in \mathcal{E}$ we have $[\mathfrak{L}, M] = \{0\}$, that is $M \in \mathrm{End}_{\mathfrak{L}}(V)$.

Letting $U \leq_{\mathfrak{L}} V$ be a simple $\mathfrak{L}$-submodule, by Schur's Lemma we have $M|_U = \lambda \cdot \mathrm{id}_U$, for some $\lambda \in K$, where from $\mathrm{Tr}_U(M) = 0$ we infer that $\lambda = 0$, hence M acts on U by the zero map. Again by Weyl's Theorem, V is the direct sum of simple $\mathfrak{L}$-submodules, entailing $M = 0 \in \widehat{\mathfrak{L}}$. Hence we conclude that $\mathcal{E} = \{0\}$.

Finally, $\mathrm{ad}_{\mathfrak{L}}(A_s) = \mathrm{ad}_{\widehat{\mathfrak{L}}}(A_s)|_{\mathfrak{L}}$ is semisimple, $\mathrm{ad}_{\mathfrak{L}}(A_n) = \mathrm{ad}_{\widehat{\mathfrak{L}}}(A_n)|_{\mathfrak{L}}$ is nilpotent, and $[\mathrm{ad}_{\mathfrak{L}}(A_s), \mathrm{ad}_{\mathfrak{L}}(A_n)] = \mathrm{ad}_{\mathfrak{L}}([A_s, A_n]) = 0$. Hence by the uniqueness of abstract Jordan-Chevalley decompositions we conclude that $A = A_s + A_n \in \mathfrak{L}$ is the abstract Jordan-Chevalley decomposition. $\sharp$

In particular, as was promised in (7.6), if $\widetilde{\mathcal{L}}$ is a semisimple Lie K-algebra and $\varphi \colon \widetilde{\mathcal{L}} \to \mathfrak{gl}(V)$ is a representation, then for $x \in \widetilde{\mathcal{L}}$ with abstract Jordan-Chevalley decomposition $x = x_s + x_n$, where $x_s \in \widetilde{\mathcal{L}}$ is semisimple and $x_n \in \widetilde{\mathcal{L}}$ is nilpotent, $\varphi(x) = \varphi(x_s) + \varphi(x_n) \in \varphi(\widetilde{\mathcal{L}}) \subseteq \mathfrak{gl}(V)$ is the (abstract) Jordan-Chevalley decomposition of $\varphi(x)$, where $\varphi(x_s)$ is semisimple and $\varphi(x_n)$ is nilpotent.

9 Modules for $\mathfrak{sl}_2$

(9.1) Weights. Let K be a field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L} := \mathfrak{sl}_2(K)$ be the special linear algebra of degree 2; recall that $\mathfrak{L}$ is simple. Let $\{E, H, F\} \subseteq \mathfrak{L}$ be the standard K-basis, that is $E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then we have $[E, F] = H$ and $[H, E] = 2E$ and $[H, F] = -2F$.

Let V be a finite-dimensional $\mathfrak{L}$-module. Let $V_\lambda := T_\lambda(H) = \{v \in V; Hv = \lambda v\} \leq_K V$ be the eigenspace of the action of H on V with respect to $\lambda \in K$. If $V_\lambda \neq \{0\}$, that is $\lambda \in K$ is an eigenvalue of the action of H, then λ is called a **weight** of H on V, any vector $0 \neq v \in V_\lambda$ is called a **weight vector**, and V_λ is called the associated **weight space**.

Weight spaces are related to each other as follows: Letting $v \in V_\lambda$, for some $\lambda \in K$, we have $HEv = [H, E] \cdot v + EHv = 2Ev + \lambda Ev = (\lambda + 2)Ev$ and $HFv = [H, F] \cdot v + FHv = -2Fv + \lambda Fv = (\lambda - 2)Fv$, implying that $Ev \in V_{\lambda+2}$ and $Fv \in V_{\lambda-2}$. The action of E and F on V is also called the **upward** and **downward ladder operator**, respectively.

A weight vector $v \in V$ such that $Ev = 0$ is called **maximal**; similarly, if $Fv = 0$

then it is called **minimal**. Moreover, a weight $\lambda \in K$ such that $V_{\lambda+2} = \{0\}$ is called **maximal**; similarly, if $V_{\lambda-2} = \{0\}$ then it is called **minimal**. Any weight vector with respect to a maximal weight is maximal, and any with respect to a minimal weight is minimal. As soon as there is a weight at all, then since there are only finitely many of them, there also are maximal and minimal weights.

(9.2) Weight strings. We keep the setting of (9.1).

In order to show that weights always exist, let $\overline{K}$ be an algebraic closure of K. Then we have $\overline{\mathfrak{L}} := \mathfrak{sl}_2(\overline{K}) \cong \mathfrak{L} \otimes_K \overline{K}$ as Lie $\overline{K}$-algebras, where the standard K-basis $\{E, H, F\} \subseteq \mathfrak{L}$ can be identified with the standard $\overline{K}$-basis of $\overline{\mathfrak{L}}$.

We consider the $\overline{\mathfrak{L}}$-module $\overline{V} := V \otimes_K \overline{K}$. Since $H \in \overline{\mathfrak{L}} \subseteq \mathfrak{gl}_2(\overline{K})$ is semisimple, hence is abstractly semisimple, we conclude that H acts semisimply on $\overline{V}$, thus we have $\overline{V} = \bigoplus_{\lambda \in \overline{K}} \overline{V}_\lambda$ as $\overline{K}$-vector spaces. Similarly, since $E, F \in \overline{\mathfrak{L}} \subseteq \mathfrak{gl}_2(\overline{K})$ are nilpotent, hence are abstractly nilpotent, we conclude that E and F act nilpotently on $\overline{V}$, and thus on V as well; alternatively, this also follows from the fact that there are only finitely many weights of H on $\overline{V}$. Now let $V \neq \{0\}$.

Let $v_0 \in \overline{V}_\lambda$ be a maximal vector, where $\lambda \in \overline{K}$, and for $i \in \mathbb{N}$ let $v_i := \frac{1}{i!} F^i v_0 \in \overline{V}_{\lambda-2i} \leq_{\overline{K}} \overline{V}$. Then by definition we have $Fv_i = (i+1)v_{i+1}$, for all $i \in \mathbb{N}_0$. Moreover, we have $Ev_i = (\lambda - i + 1)v_{i-1}$, for all $i \in \mathbb{N}_0$, where we additionally let $v_{-1} := 0 \in \overline{V}$: Proceeding by induction on $i \in \mathbb{N}_0$, the case $i = 0$ is clear by the definition of maximality; hence letting $i \geq 1$ we have $iEv_i = EFv_{i-1} = [E, F] \cdot v_{i-1} + FEv_{i-1} = Hv_{i-1} + (\lambda - i + 2)Fv_{i-2} = (\lambda - 2i + 2)v_{i-1} + (\lambda - i + 2)(i-1)v_{i-1} = i(\lambda - i + 1)v_{i-1}$; finally the result follows from dividing by i.

Since F acts nilpotently, let $m \in \mathbb{N}_0$ be such that v_m is minimal, that is $v_m \neq 0$ but $v_{m+1} = 0$. In particular, for $i = m + 1$ we get $0 = Ev_{m+1} = (\lambda - m)v_m$, which since $v_m \neq 0$ implies that $\lambda = m \in \mathbb{N}_0$. Thus in particular any maximal weight is a non-negative integer. Moreover, if $v \in \overline{V}_\lambda$ is any weight vector, where $\lambda \in \overline{K}$, then since E acts nilpotently there is $l \in \mathbb{N}_0$ such that $E^l v \in \overline{V}_{\lambda+2l}$ is maximal, implying that $\lambda + 2l \in \mathbb{N}_0$, thus $\lambda \in \mathbb{Z}$ is an integer.

Hence we conclude that all the eigenvalues of the action of $H \in \overline{\mathfrak{L}}$ on $\overline{V}$ are in $\mathbb{Z} \subseteq K$. Thus we actually have $V = \bigoplus_{\lambda \in K} V_\lambda$ as K-vector spaces. Hence, picking a maximal vector $v_0 \in V_\lambda$, where $\lambda \in \mathbb{N}_0 \subseteq K$, all of the above discussion holds verbally for V instead of $\overline{V}$.

Now let $U := \langle v_0, \ldots, v_m \rangle_K \leq_K V$. Since $v_0, \ldots, v_m$ are non-zero and belong to distinct weight spaces, we conclude that $\{v_0, \ldots, v_m\}$ is K-linearly independent, and thus $\dim_K(U) = m + 1$. Moreover, the above formulae show that $U \leq_{\mathfrak{L}} V$ actually is an $\mathfrak{L}$-submodule. With respect to the K-basis given, H acts by the diagonal matrix $H^{(m)} := \operatorname{diag}[\lambda, \lambda - 2, \ldots, \lambda - 2m] = \operatorname{diag}[m, m-2, \ldots, -m] \in \mathfrak{gl}_{m+1}(K)$, in particular saying that each weight occurring has a 1-dimensional weight space associated with it, while E and F act by the following strictly upper and lower triangular matrices in $\mathfrak{gl}_{m+1}(K)$, respectively:

$$E^{(m)} := \begin{bmatrix} 0 & m & & & & \\ & 0 & m-1 & & & \\ & & 0 & m-2 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}, \quad F^{(m)} := \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & m & 0 \end{bmatrix}.$$

(9.3) Theorem: Simple $\mathfrak{sl}_2$-modules. We keep the setting of (9.2).

Let V be simple. Then V is uniquely determined by $m := \dim_K(V) - 1 \in \mathbb{N}_0$. In this case, V has precisely $m+1$ weights $\{m, m-2, \ldots, -m\}$; in particular, $m \in \mathbb{N}_0$ is the unique maximal weight, being called the associated **highest weight**. All weight spaces are 1-dimensional; in particular, up to scalar multiples there is a unique maximal weight vector, being called a **highest weight vector**.

Conversely, for any $m \in \mathbb{N}_0$ there actually exists a simple $\mathfrak{L}$-module $V^{(m)}$ such that $\dim_K(V^{(m)}) = m+1$.

Proof. Let V be simple. Picking a maximal vector $v_0 \in V_m \leq_K V$, where $m \in \mathbb{N}_0$ is the associated weight, and letting $U := \langle v_0, \ldots, v_m \rangle_K$ as above, from V being simple we infer that $U = V$. Hence we have $\dim_K(V) = m+1$, showing that m is uniquely determined. Consequently, V is uniquely determined as well, and has the asserted properties.

It remains to be shown that for any $m \in \mathbb{N}_0$ the K-vector space $K^{(m+1)\times 1}$ becomes an $\mathfrak{L}$-module, where E, H and F act by the matrices $E^{(m)}, H^{(m)}, F^{(m)} \in \mathfrak{gl}_{m+1}(K)$, respectively, given above. To do so, we have to verify the commutators $[E^{(m)}, F^{(m)}] = H^{(m)}$ and $[H^{(m)}, E^{(m)}] = 2E^{(m)}$ and $[H^{(m)}, F^{(m)}] = 2F^{(m)}$:

We have $[H^{(m)}, F^{(m)}]_{i+1,i} = (m-2i)\cdot i - i\cdot\big(m-2(i-1)\big) = -2i = (-2F^{(m)})_{i+1,i}$, for all $i \in \{1, \ldots, m\}$, the other entries of $[H^{(m)}, F^{(m)}]$ are zero anyway. Similarly, for the non-zero entries of $[H^{(m)}, E^{(m)}]$ we get $[H^{(m)}, E^{(m)}]_{i,i+1} = \big(m-2(i-1)\big)\big(m-(i-1)\big) - \big(m-(i-1)\big)(m-2i) = 2m-2(i-1) = (2E^{(m)})_{i,i+1}$, for all $i \in \{1, \ldots, m\}$. Finally, the off-diagonal entries of $[E^{(m)}, F^{(m)}]$ are zero, and we have $[E^{(m)}, F^{(m)}]_{ii} = \big(m-(i-1)\big)\cdot i - (i-1)\big(m-(i-2)\big) = m-2(i-1) = (H^{(m)})_{ii}$, for all $i \in \{1, \ldots, m+1\}$. $\sharp$

Example. For $m = 0$ we have $\dim_K(V^{(0)}) = 1$ and $H^{(0)} = E^{(0)} = F^{(0)} = \big[0\big]$, thus we get the trivial representation. For $m = 1$ we have $\dim_K(V^{(1)}) = 2$, where $H^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $E^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $F^{(1)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, thus we recover the tautological representation; this shows again that the latter is irreducible.

For $m = 2$ we have $\dim_K(V^{(2)}) = 3$, where

$$H^{(2)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ and } E^{(2)} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } F^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

This representation is equivalent to the adjoint representation $\mathrm{ad}_{\mathfrak{L}}$: Since $\mathfrak{L}$ is simple, $\mathrm{ad}_{\mathfrak{L}}$ is irreducible, hence it suffices to observe that $\dim_K(\mathfrak{L}) = 3$. More precisely, $\mathrm{ad}_{\mathfrak{L}}(H)$ has the weight spaces $\mathfrak{L}_2 = \langle E\rangle_K$ and $\mathfrak{L}_0 = \langle H\rangle_K$ and $\mathfrak{L}_{-2} = \langle F\rangle_K$. Choosing the highest weight vector $v_0 := E$, we get $v_1 := \mathrm{ad}_{\mathfrak{L}}(F)(E) = [F, E] = -H$ and $v_2 := \frac{1}{2}\mathrm{ad}_{\mathfrak{L}}(F)^2(E) = \frac{1}{2}[F, [F, E]] = -F$. Using the matrices of $\mathrm{ad}_{\mathfrak{L}}$ with respect to the standard K-basis $\{E, H, F\} \subseteq \mathfrak{L}$, with respect to the K-basis $\{E, -H, -F\} \subseteq \mathfrak{L}$ we indeed get

$$\mathrm{diag}[1,-1,-1] \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \cdot \mathrm{diag}[1,-1,-1] = H^{(2)},$$

$$\mathrm{diag}[1,-1,-1] \cdot \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \mathrm{diag}[1,-1,-1] = E^{(2)},$$

$$\mathrm{diag}[1,-1,-1] \cdot \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \cdot \mathrm{diag}[1,-1,-1] = F^{(2)}.$$

(9.4) Theorem: Arbitrary $\mathfrak{sl}_2$-modules. We keep the setting of (9.3).

Let V be arbitrary, and let $V = \bigoplus_{i=1}^k V^{(m_i)}$ be a decomposition of V as a direct sum of simple $\mathfrak{L}$-submodules, where $k \in \mathbb{N}_0$ and $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$; recall that by Weyl's Theorem V is semisimple.

Then the number k and the highest weights $m_1, \ldots, m_k$ occurring are uniquely determined by V; more precisely, we have $k = \dim_K(V_0) + \dim_K(V_1)$. Moreover, we have $\dim_K(V_\lambda) = \dim_K(V_{-\lambda})$, for all $\lambda \in \mathbb{Z}$.

Proof. If $\varphi\colon \mathfrak{L} \to \mathfrak{gl}(V)$ is the representation associated with V, the **Jordan-Hölder Theorem**, applied to the unital associative K-subalgebra of $\mathfrak{gl}(V)$ generated by $\varphi(\mathfrak{L})$, implies the uniqueness of the multiset of isomorphism types of simple modules occurring in a composition series of V as a $\varphi(\mathfrak{L})$-module. But having the description of the simple $\mathfrak{L}$-modules at hand, we may argue much more precisely using weights:

We proceed by induction on $\dim_K(V) \in \mathbb{N}_0$; the case of $V = \{0\}$ being clear, we let $V \neq \{0\}$. For any $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{N}_0$, we have $V_\lambda^{(m)} = \{0\}$ if $\lambda > m$, and $V_m^{(m)} \neq \{0\}$. Hence we infer that $V_\lambda = \bigoplus_{i=1}^k V_\lambda^{(m_i)} = \{0\}$ if $\lambda > m_1$, and $V_{m_1} \neq \{0\}$. In other words, we have $m_1 = \max\{\lambda \in \mathbb{N}_0; V_\lambda \neq \{0\}\}$, showing that m_1 is uniquely determined by V. Now we have $V/V^{(m_1)} \cong \bigoplus_{i=2}^k V^{(m_i)}$, and since $V^{(m_1)} \neq \{0\}$ we are done by induction.

Moreover, for m even we have $\dim_K(V_0^{(m)}) = 1$ and $\dim_K(V_1^{(m)}) = 0$, and for m odd we have $\dim_K(V_0^{(m)}) = 0$ and $\dim_K(V_1^{(m)}) = 1$, showing that $\dim_K(V_0^{(m)}) + \dim_K(V_1^{(m)}) = 1$ in all cases, implying the assertion on k. Finally, we have $\dim_K(V_\lambda^{(m)}) = \dim_K(V_{-\lambda}^{(m)}) \in \{0,1\}$, for all $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{N}_0$. ♯

The above leads to the following combinatorial decomposition algorithm: We have $V = V' \oplus V''$, where $V' := \bigoplus_{i=1}^r V^{(2m_i-2)}$ and $V'' := \bigoplus_{i=1}^s V^{(2n_i-1)}$ for some $r, s \in \mathbb{N}_0$, and where we may assume that $m_1 \geq m_2 \geq \cdots \geq m_r \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_s \geq 1$. Now, factorising the characteristic polynomial of the action of H on V yields the K-dimensions $d_\lambda := \dim_K(V_\lambda) \in \mathbb{N}_0$ of the weight spaces, for all $\lambda \in \mathbb{N}_0$; note that we can safely ignore negative weights here.

Thus we get $d' := [d'_j; j \in \mathbb{N}] := [d_{2\lambda-2} \in \mathbb{N}_0; \lambda \in \mathbb{N}]$ and $d'' := [d''_j; j \in \mathbb{N}] := [d_{2\lambda-1} \in \mathbb{N}_0; \lambda \in \mathbb{N}]$. The sequences $d'_1 \geq d'_2 \geq \cdots$ and $d''_1 \geq d''_2 \geq \cdots$ are non-increasing, hence can be considered as partitions of $\sum_{j\geq 1} d'_j$ and $\sum_{j \geq 1} d''_j$, respectively. Thus we have $m_i := |\{j \geq 1; d'_j \geq i\}|$, for all $i \in \{1, \ldots, r\}$, and $n_i := |\{j \geq 1; d''_j \geq i\}|$, for all $i \in \{1, \ldots, s\}$, in other words, the associated conjugate partitions are given as $(d')' = [m_i; i \in \mathbb{N}]$ and $(d'')' = [n_i; i \in \mathbb{N}]$. ♯

10 Cartan decomposition

(10.1) Toral subalgebras. Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple Lie K-algebra. A Lie K-subalgebra of $\mathfrak{L}$ is called **toral** if it consists entirely of semisimple elements, in the sense of abstract Jordan-Chevalley decompositions.

Proposition. a) There is a non-zero toral Lie K-subalgebra of $\mathfrak{L}$.
b) Any toral Lie K-subalgebra of $\mathfrak{L}$ is commutative.

Proof. a) It suffices show that $\mathfrak{L}$ possesses a non-zero semisimple element; indeed, if $x \in \mathfrak{L}$ is semisimple, then $\langle x \rangle_K$ is a toral Lie K-subalgebra:

By the abstract Jordan-Chevalley decomposition, the existence of a non-zero semisimple element is equivalent to the existence of a non-nilpotent element. To see this, assume to the contrary that all elements of $\mathfrak{L}$ are nilpotent, that is ad-nilpotent. Then by Engel's Theorem $\mathfrak{L}$ is nilpotent, a contradiction.

b) Let $\mathfrak{T} \subseteq \mathfrak{L}$ be a toral Lie K-subalgebra. We have to show that $\mathrm{ad}_{\mathfrak{T}}(x) = 0$, for all $x \in \mathfrak{T}$: Since x is semisimple, $\mathrm{ad}_{\mathfrak{L}}(x)$ is semisimple, hence $\mathrm{ad}_{\mathfrak{T}}(x)$ is semisimple as well. Thus we show that $\mathrm{ad}_{\mathfrak{T}}(x)$ has no non-zero eigenvalue:

Assume to the contrary that $0 \neq y \in T_\lambda(x) := T_\lambda(\mathrm{ad}_{\mathfrak{T}}(x)) \leq_K \mathfrak{T}$ is an eigenvector of $\mathrm{ad}_{\mathfrak{T}}(x)$ with respect to some eigenvalue $0 \neq \lambda \in K$, that is we have $[x, y] = \lambda y \neq 0 \in \mathfrak{T}$. In particular, we have $[y, y] = 0$, thus $y \in T_0(y)$. Now y

being semisimple, $\mathrm{ad}_{\mathfrak{T}}(y)$ is semisimple as well, and we have $x = \sum_{i=1}^n x_i \in \mathfrak{T}$, where $n := \dim_K(\mathfrak{T}) \in \mathbb{N}_0$ and $x_i \in T_{\lambda_i}(y)$, for some $\lambda_i \in K$. Then $0 \neq -\lambda y = [y, x] = \sum_{i,\, \lambda_i \neq 0} \lambda_i x_i \in T_0(y) \cap \bigoplus_{0 \neq \lambda \in K} T_\lambda(y) = \{0\}$ is a contradiction. ♯

(10.2) Cartan decomposition. a) Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple Lie K-algebra. Moreover, let $\{0\} \neq \mathfrak{H} \subseteq \mathfrak{L}$ be a maximal toral Lie K-subalgebra, that is $\mathfrak{H}$ is a toral Lie K-subalgebra, and for any toral Lie K-subalgebra $\mathfrak{H} \subseteq \mathfrak{T} \subseteq \mathfrak{L}$ we already have $\mathfrak{T} \subseteq \mathfrak{H}$. The chosen subalgebra $\mathfrak{H} \subseteq \mathfrak{L}$ is kept fixed in the sequel.

Since $\mathfrak{H}$ is commutative, we infer that $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{H}) \leq_K \mathrm{End}_K(\mathfrak{L})$ consists of pairwise commuting semisimple K-endomorphisms of $\mathfrak{L}$. Hence $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{H})$ is simultaneously diagonalisable, that is there is a K-basis of $\mathfrak{L}$ consisting of eigenvectors for all elements of $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{H})$. Hence considering the K-linear forms induced by $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{H})$ on its simultaneous eigenspaces, we are led to the following:

Let $\mathfrak{L}_\alpha := \bigcap_{h \in \mathfrak{H}} T_{\alpha(h)}(h) = \{x \in \mathfrak{L}; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{H}\} \leq_K \mathfrak{L}$, for all $\alpha \in \mathfrak{H}^* := \mathrm{Hom}_K(\mathfrak{H}, K)$. In particular, for $\alpha = 0 \in \mathfrak{H}^*$ we get $\mathfrak{L}_0 = \{x \in \mathfrak{L}; [h, x] = 0 \text{ for all } h \in \mathfrak{H}\} = C_{\mathfrak{L}}(\mathfrak{H})$, the centraliser of $\mathfrak{H}$ in $\mathfrak{L}$, which is a K-Lie subalgebra of $\mathfrak{L}$. Since $\mathfrak{H}$ is commutative, we have $\{0\} \neq \mathfrak{H} \subseteq C_{\mathfrak{L}}(\mathfrak{H})$, and since $\mathfrak{H} \not\subseteq Z(\mathfrak{L}) = \{0\}$ we infer that $C_{\mathfrak{L}}(\mathfrak{H}) \neq \mathfrak{L}$.

If $\mathfrak{L}_\alpha \neq \{0\}$, for some $0 \neq \alpha \in \mathfrak{H}^*$, then α is called a **root** of $\mathfrak{L}$, and $\mathfrak{L}_\alpha \leq_K \mathfrak{L}$ is called the associated **root space**. Let $\Phi \subseteq \mathfrak{H}^* \setminus \{0\}$ be the set of roots of $\mathfrak{L}$; since $\mathfrak{L}_0 = C_{\mathfrak{L}}(\mathfrak{H}) \neq \mathfrak{L}$ we have $\Phi \neq \emptyset$. Actually, in view of the following lemma we deduce that Φ is finite, and that we have the **Cartan decomposition** or **root space decomposition** $\mathfrak{L} = C_{\mathfrak{L}}(\mathfrak{H}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha$ as $\mathfrak{H}$-modules.

Lemma. Let $\alpha_1, \ldots, \alpha_n \in \mathfrak{H}^*$ be pairwise distinct, for some $n \in \mathbb{N}$, and let $0 \neq x_i \in \mathfrak{L}_{\alpha_i}$, for all $i \in \{1, \ldots, n\}$. Then $\{x_1, \ldots, x_n\}$ is K-linearly independent.

Proof. Assume to the contrary that $\{x_1, \ldots, x_n\}$ is K-linearly independent, where we may assume $n \geq 2$ to be chosen minimal, and such that we have $\sum_{i=1}^n \lambda_i x_i = 0$ for some $\lambda_1, \ldots, \lambda_n \in K$ such that $\lambda_{n-1} \neq 0 \neq \lambda_n$. Let $h \in \mathfrak{H}$ such that $\alpha_n(h) \neq \alpha_{n-1}(h)$; interchanging α_n and α_{n-1} if necessary we may assume that $\alpha_n(h) = 1 \neq \alpha_{n-1}(h)$. Then we have $0 = \mathrm{ad}_{\mathfrak{L}}(h)(\sum_{i=1}^n \lambda_i x_i) = \sum_{i=1}^n \lambda_i \mathrm{ad}_{\mathfrak{L}}(h)(x_i) = \sum_{i=1}^n \lambda_i \alpha_i(h) x_i$, thus subtracting $\sum_{i=1}^n \lambda_i x_i = 0$ yields $\sum_{i=1}^n \lambda_i(\alpha_i(h) - 1)x_i = 0$, where $\alpha_n(h) = 1$ and $\alpha_{n-1}(h) \neq 1$ imply that this is a shorter non-trivial K-linear combination yielding zero, a contradiction. ♯

b) We set out to examine how the root spaces interfere with the Killing form of $\mathfrak{L}$. Here is the first result into that direction, which will be needed in the following theorem; subsequently we will elucidate this much further:

Proposition. Let κ be the Killing form of $\mathfrak{L}$. If $\alpha, \beta \in \Phi \,\dot\cup\, \{0\}$ such that $\alpha \neq -\beta$, then we have $\kappa(\mathfrak{L}_\alpha, \mathfrak{L}_\beta) = 0$, that is $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$ are orthogonal to each other. In particular, the restriction of κ to $C_{\mathfrak{L}}(\mathfrak{H})$ is non-degenerate.

Proof. The associativity of κ yields $\alpha(h)\kappa(x,y) = \kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y]) = -\beta(h)\kappa(x,y)$, for all $x \in \mathfrak{L}_\alpha$ and $y \in \mathfrak{L}_\beta$. Thus letting $h \in \mathfrak{H}$ such that $(\alpha+\beta)(h) \neq 0$, we infer that $\kappa(x,y) = 0$.

Now let $z \in \mathrm{rad}(\kappa|_{C_\mathfrak{L}(\mathfrak{H})\times C_\mathfrak{L}(\mathfrak{H})})$. Since $C_\mathfrak{L}(\mathfrak{H}) = \mathfrak{L}_0 \leq_K \mathfrak{L}_\alpha^\perp$, for all $\alpha \in \Phi$, we infer that $z \in \mathrm{rad}(\kappa)$. Since κ is non-degenerate this entails $z = 0$. $\sharp$

Example: The special linear algebra of degree 2. Let $\mathfrak{L} := \mathfrak{sl}_2(K)$. We show that any non-zero toral Lie K-subalgebra $\mathfrak{H} \subseteq \mathfrak{L}$ satisfies $C_\mathfrak{L}(\mathfrak{H}) = \mathfrak{H}$, and is $\mathrm{GL}_2(K)$-conjugate to $\langle H\rangle_K \subseteq \mathfrak{L}$:

If $0 \neq A \in \mathfrak{L}$ is semisimple, then it has eigenvalues $\pm\lambda$, for some $0 \neq \lambda \in K$. Hence A is $\mathrm{GL}_2(K)$-conjugate to λH. Thus we may assume that $H \in \mathfrak{H}$. Next, any element of $M \in \mathfrak{gl}_2(K)$ centralisíng H leaves the eigenspaces of H invariant: For any $v \in T_\lambda(H) \leq_K K^{2\times 1}$, where $\lambda \in \{\pm 1\}$, we have $HMv = MHv = \lambda Mv$, saying that $Mv \in T_\lambda(H)$ as well. Thus, since $T_1(H) = \langle e_1\rangle_K$ and $T_{-1}(H) = \langle e_2\rangle_K$, we have $C_{\mathfrak{gl}_2(K)}(H) = \mathfrak{t}_2(K)$, and hence $C_\mathfrak{L}(H) = \mathfrak{t}_2(K) \cap \mathfrak{L} = \langle H\rangle_K$. Thus, $\mathfrak{H}$ being commutative, we have $\mathfrak{H} \subseteq C_\mathfrak{L}(H) = \langle H\rangle_K \subseteq \mathfrak{H}$. $\sharp$

Hence we may assume that $\mathfrak{H} = C_\mathfrak{L}(H) = \langle H\rangle_K$. Let $H^* \in \mathfrak{H}^*$ be the element dual to $H \in \mathfrak{H}$, that is $H^*(H) = 1$. Then we have $\mathfrak{L}_{2H^*} = \{A \in \mathfrak{L}; [H,A] = 2A\} = \langle E\rangle_K$ and $\mathfrak{L}_{-2H^*} = \{A \in \mathfrak{L}; [H,A] = -2A\} = \langle F\rangle_K$, yielding the Cartan decomposition $\mathfrak{L} = \langle H\rangle_K \oplus \langle E\rangle_K \oplus \langle F\rangle_K$. Hence the root spaces of $\mathfrak{L}$ are precisely the weight spaces of H on the adjoint module, and we have $\Phi \,\dot\cup\, \{0\} = \{\lambda H^*; \lambda \in \{2,0,-2\}\} \subseteq \mathfrak{H}^*$, where $\{2,0,-2\}$ are the weights occurring. Finally, recall that the Gram matrix of the Killing form of $\mathfrak{L}$ with respect to the standard K-basis $\{E,H,F\} \subseteq \mathfrak{L}$ is given as follows, see (7.2), reflecting the above orthogonality properties:

$$G(\kappa) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

(10.3) Theorem. Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple Lie K-algebra. Then any maximal toral Lie K-subalgebra $\mathfrak{H} \subseteq \mathfrak{L}$ is **self-centralising**, that is we have $C_\mathfrak{L}(\mathfrak{H}) = \mathfrak{H}$. In particular, the restriction of κ to $\mathfrak{H}$ is non-degenerate.

Proof. i) Let $\mathfrak{K} := C_\mathfrak{L}(\mathfrak{H}) = \{x \in \mathfrak{L}; \mathrm{ad}_\mathfrak{L}(x)(\mathfrak{H}) = \{0\}\}$. Recall that $\mathrm{ad}_\mathfrak{L}(x_s) = \mathrm{ad}_\mathfrak{L}(x)_s$ and $\mathrm{ad}_\mathfrak{L}(x_n) = \mathrm{ad}_\mathfrak{L}(x)_n$ are polynomials without constant coefficient in $\mathrm{ad}_\mathfrak{L}(x)$, for all $x \in \mathfrak{L}$. Hence for $x \in \mathfrak{K}$ we have $\mathrm{ad}_\mathfrak{L}(x_s)(\mathfrak{H}) = \{0\} = \mathrm{ad}_\mathfrak{L}(x_n)(\mathfrak{H})$ as well, saying that $x_s, x_n \in \mathfrak{K}$, too. In other words, $\mathfrak{K}$ contains the semisimple and nilpotent parts of its elements.

Now let $x \in \mathfrak{K}$ be semisimple. Then $[x,\mathfrak{H}] = \{0\}$ implies that $\mathfrak{H} + \langle x\rangle_K \leq_K \mathfrak{K}$ is a Lie K-subalgebra. Since the sum of two commuting semisimple elements is semisimple again we conclude that $\mathfrak{H} + \langle x\rangle_K$ is toral, whence maximality of

$\mathfrak{H}$ entails that $\mathfrak{H} + \langle x \rangle_K = \mathfrak{H}$, that is $x \in \mathfrak{H}$. In other words, the semisimple elements of $\mathfrak{K}$ are contained in $\mathfrak{H}$.

Letting $x = x_s + x_n \in \mathfrak{K}$ be arbirary again, we have $\mathrm{ad}_{\mathfrak{K}}(x) = \mathrm{ad}_{\mathfrak{K}}(x_s) + \mathrm{ad}_{\mathfrak{K}}(x_n)$. Since $x_s \in \mathfrak{H}$ we have $[x_s, \mathfrak{K}] = \{0\}$, that is $\mathrm{ad}_{\mathfrak{K}}(x_s) = 0$. Thus $\mathrm{ad}_{\mathfrak{K}}(x) = \mathrm{ad}_{\mathfrak{K}}(x_n) = \mathrm{ad}_{\mathfrak{L}}(x_n)|_{\mathfrak{K}} = \mathrm{ad}_{\mathfrak{L}}(x)_n|_{\mathfrak{K}}$ is nilpotent, hence Engel's Theorem implies that $\mathfrak{K}$ is nilpotent.

ii) Next, we record the following fact: If $x \in \mathfrak{L}$ is nilpotent and $y \in C_{\mathfrak{L}}(x)$, then we have $[\mathrm{ad}_{\mathfrak{L}}(x), \mathrm{ad}_{\mathfrak{L}}(y)] = \mathrm{ad}_{\mathfrak{L}}([x, y]) = 0$, that is $\mathrm{ad}_{\mathfrak{L}}(x)$ and $\mathrm{ad}_{\mathfrak{L}}(y)$ commute, hence $\mathrm{ad}_{\mathfrak{L}}(x)$ being nilpotent implies that $\mathrm{ad}_{\mathfrak{L}}(x) \cdot \mathrm{ad}_{\mathfrak{L}}(y)$ is nilpotent as well, thus $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(x) \cdot \mathrm{ad}_{\mathfrak{L}}(y)) = 0$.

Now we consider the center $Z(\mathfrak{K})$ of $\mathfrak{K} = C_{\mathfrak{L}}(\mathfrak{H})$, where thus $\mathfrak{H} \subseteq Z(\mathfrak{K})$. Hence for $x \in Z(\mathfrak{K}) \subseteq \mathfrak{K}$ we have $x_s, x_n \in \mathfrak{K}$ as well, where moreover $x_s \in \mathfrak{H} \subseteq Z(\mathfrak{K})$, thus we have $x_n \in Z(\mathfrak{K})$ as well. In other words, $Z(\mathfrak{K})$ contains the semisimple and nilpotent parts of its elements.

We show that $Z(\mathfrak{K})$ is toral; this by the maximality of $\mathfrak{H}$ implies that $Z(\mathfrak{K}) = \mathfrak{H}$: Assume to the contrary that there is $x \in Z(\mathfrak{K})$ which is not semisimple, then we may assume that $0 \neq x \in Z(\mathfrak{K})$ is nilpotent. Hence we have $\kappa(x, y) = 0$, for all $y \in \mathfrak{K}$, that is $0 \neq x \in \mathrm{rad}(\kappa|_{\mathfrak{K} \times \mathfrak{K}}) = \{0\}$, a contradiction.

iii) We show that $\kappa|_{\mathfrak{H} \times \mathfrak{H}}$ is non-degenerate: Let $x \in \mathrm{rad}(\kappa|_{\mathfrak{H} \times \mathfrak{H}})$, that is $\kappa(x, \mathfrak{H}) = \{0\}$, and let $y \in \mathfrak{K}$, where we may assume that y is semisimple or nilpotent. If y is semisimple, then we have $y \in \mathfrak{H}$, by assumption implying that $\kappa(x, y) = 0$. If y is nilpotent, then since x and y commute we get $\kappa(x, y) = 0$ as well. Hence we infer that $\kappa(x, \mathfrak{K}) = \{0\}$, that is $x \in \mathrm{rad}(\kappa|_{\mathfrak{K} \times \mathfrak{K}}) = \{0\}$.

In particular, this entails that $\mathfrak{H} \cap [\mathfrak{K}, \mathfrak{K}] = \{0\}$: For $x \in \mathfrak{H}$ and $y, z \in \mathfrak{K}$, by the associativity of κ we have $\kappa(x, [y, z]) = \kappa([x, y], z) = 0$, that is $\kappa(\mathfrak{H}, [\mathfrak{K}, \mathfrak{K}]) = \{0\}$, thus $\mathfrak{H} \cap [\mathfrak{K}, \mathfrak{K}] \leq_K \mathrm{rad}(\kappa|_{\mathfrak{H} \times \mathfrak{H}}) = \{0\}$.

Hence it now suffices to show that $\mathfrak{K}$ is commutative; then we have $\mathfrak{H} = Z(\mathfrak{K}) = \mathfrak{K}$: Assume to the contrary that $[\mathfrak{K}, \mathfrak{K}] \neq \{0\}$. Since $\mathfrak{K}$ is nilpotent, and $[\mathfrak{K}, \mathfrak{K}] \trianglelefteq \mathfrak{K}$ is $\mathrm{ad}_{\mathfrak{K}}$-invariant, by (5.3) there is $0 \neq z \in [\mathfrak{K}, \mathfrak{K}]$ such that $[\mathfrak{K}, z] = 0$, that is $z \in Z(\mathfrak{K}) = \mathfrak{H}$. Thus we infer that $0 \neq z \in \mathfrak{H} \cap [\mathfrak{K}, \mathfrak{K}] = \{0\}$, a contradiction. $\sharp$

Corollary. The maximal toral subalgebra $\mathfrak{H}$ is a **Cartan subalgebra** of $\mathfrak{L}$, that is a self-normalising nilpotent Lie K-subalgebra; in particular, $\mathfrak{H}$ is a maximal nilpotent Lie K-subalgebra.

Proof. Recall first that any proper subalgebra of a nilpotent Lie algebra is strictly contained in its normaliser; hence the last statement follows from the first. Thus we only have to show that $N_{\mathfrak{L}}(\mathfrak{H}) \subseteq \mathfrak{H}$: Hence let $x \in N_{\mathfrak{L}}(\mathfrak{H})$.

By the Cartan decomposition we may write $x = x_0 + \sum_{\alpha \in \Phi} x_\alpha$, where $x_0 \in \mathfrak{H}$ and $x_\alpha \in \mathfrak{L}_\alpha$, and where $\Phi \subseteq \mathfrak{H}^*$ are the associated roots. Hence we have $[h, x] = [h, x_0] + \sum_{\alpha \in \Phi} [h, x_\alpha] = \sum_{\alpha \in \Phi} \alpha(h) x_\alpha \in \mathfrak{H}$, for all $h \in \mathfrak{H}$. This implies

$\alpha(h)x_\alpha = 0$, for all $\alpha \in \Phi$ and $h \in \mathfrak{H}$. Now, for any $\alpha \in \Phi$ there is $h \in \mathfrak{H}$ such that $\alpha(h) \neq 0$, implying that $x_\alpha = 0$. Hence we conclude that $x = x_0 \in \mathfrak{H}$. $\sharp$

11 Roots

(11.1) Roots. Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L}$ be a semisimple Lie K-algebra, with Killing form κ, and Cartan decomposition $\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha$ as $\mathfrak{H}$-modules, where $\mathfrak{H} \subseteq \mathfrak{L}$ is a maximal toral Lie K-algebra and $\Phi \subseteq \mathfrak{H}^*$ are the associated roots. We proceed to elucidate the structure of the set of roots. We first collect a few immediate properties:

Proposition. a) We have $\langle \Phi \rangle_K = \mathfrak{H}^*$.

b) For $\alpha \in \Phi$ and any $0 \neq x \in \mathfrak{L}_\alpha$ we have $\kappa(x, \mathfrak{L}_{-\alpha}) \neq \{0\}$. In particular, we have $\kappa(\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}) \neq \{0\}$, and thus $\Phi = -\Phi$.

c) For $\alpha, \beta \in \Phi \,\dot\cup\, \{0\}$ we have $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \leq_K \mathfrak{L}_{\alpha+\beta}$; recall that $\mathfrak{L}_0 = \mathfrak{H}$. In particular, for $\alpha \in \Phi$ the elements of $\mathfrak{L}_\alpha$ are nilpotent.

Proof. a) Assume to the contrary that $\langle \Phi \rangle_K \neq \mathfrak{H}^*$. Then we get $\bigcap_{\alpha\in\Phi} \ker(\alpha) = \bigcap_{\alpha\in\langle\Phi\rangle_K} \ker(\alpha) \neq \{0\}$. Hence let $0 \neq h \in \mathfrak{H}$ such that $\alpha(h) = 0$, for all $\alpha \in \Phi$. Then we have $[h, \mathfrak{L}_\alpha] = \{0\}$, for all $\alpha \in \Phi$. Since $[h, \mathfrak{H}] = \{0\}$ as well, we conclude that $h \in Z(\mathfrak{L}) = \{0\}$, a contradiction.

b) Assume to the contrary that $\kappa(x, \mathfrak{L}_{-\alpha}) = \{0\}$; then we have $\kappa(x, \mathfrak{L}_\beta) = \{0\}$, for all $\beta \in \Phi \,\dot\cup\, \{0\}$, that is $0 \neq x \in \mathrm{rad}(\kappa)$, a contradiction. In particular, this shows that $\mathfrak{L}_{-\alpha} \neq \{0\}$, that is $-\alpha \in \Phi$.

c) For $h \in \mathfrak{H}$ we have $[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h) \cdot [x, y]$, for all $\in \mathfrak{L}_\alpha$ and $y \in \mathfrak{L}_\beta$, thus $[x, y] \in \mathfrak{L}_{\alpha+\beta}$.

If $x \in \mathfrak{L}_\alpha$, then we get $\mathrm{ad}_{\mathfrak{L}}(x)^k(\mathfrak{L}_\beta) \leq_K \mathfrak{L}_{k\alpha+\beta}$, for all $\beta \in \Phi \,\dot\cup\, \{0\}$ and $k \in \mathbb{N}_0$. If $\alpha \neq 0$, since $\Phi \,\dot\cup\, \{0\} \subseteq \mathfrak{H}^*$ is finite, there is $k = k_\beta \in \mathbb{N}_0$ such that $k\alpha + \beta \notin \Phi \,\dot\cup\, \{0\}$, hence $\mathrm{ad}_{\mathfrak{L}}(x)^k = 0$ for $k := \max\{k_\beta \in \mathbb{N}_0; \beta \in \Phi \,\dot\cup\, \{0\}\}$. $\sharp$

Now, since the restriction of κ to $\mathfrak{H}$ is non-degenerate, we have the isomorphism $\mathfrak{H} \to \mathfrak{H}^* \colon h \mapsto \kappa_h$ of K-vector spaces, where $\kappa_h \colon \mathfrak{H} \to K \colon x \mapsto \kappa(h, x)$, allowing to identify $\mathfrak{H}^*$ with $\mathfrak{H}$. Conversely, for $\alpha \in \mathfrak{H}^*$ let $t_\alpha \in \mathfrak{H}$ such that $\kappa_{t_\alpha} = \alpha$, that is $t_\alpha \in \mathfrak{H}$ is the unique element such that $\alpha(h) = \kappa(t_\alpha, h) \in K$, for all $h \in \mathfrak{H}$.

Theorem. a) We have $\langle t_\alpha; \alpha \in \Phi \rangle = \mathfrak{H}$.

b) For $\alpha \in \Phi$ and any $x \in \mathfrak{L}_\alpha$ and $y \in \mathfrak{L}_{-\alpha}$ we have $[x, y] = \kappa(x, y) \cdot t_\alpha \in \mathfrak{H}$; in particular we have $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}] = \langle t_\alpha \rangle_K \neq \{0\}$.

c) For $\alpha \in \Phi$ we have $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

Hence we may let $h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)} \cdot t_\alpha \in \mathfrak{H}$, being called the associated **coroot**; note that we have $\alpha(h_\alpha) = 2$ and $h_{-\alpha} = \frac{2}{\kappa(-t_\alpha, -t_\alpha)} \cdot (-t_\alpha) = -\frac{2}{\kappa(t_\alpha, t_\alpha)} \cdot t_\alpha = -h_\alpha$.

d) For $\alpha \in \Phi$ let $0 \neq e_\alpha \in \mathfrak{L}_\alpha$. Then letting $f_\alpha \in \mathfrak{L}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ gives rise to the Lie K-subalgebra $\mathfrak{K}_\alpha := \langle e_\alpha, h_\alpha, f_\alpha \rangle_K \subseteq \mathfrak{L}$, such that $\mathfrak{sl}_2(K) \to \mathfrak{K}_\alpha \colon E \mapsto e_\alpha, H \mapsto h_\alpha, F \mapsto f_\alpha$ is an isomorphism of Lie K-algebras.

Proof. a) By the above identification we have $\mathfrak{H}^* = \langle \Phi \rangle_K \cong \langle t_\alpha; \alpha \in \Phi \rangle_K \leq \mathfrak{H}$ as K-vector spaces, thus $\langle t_\alpha; \alpha \in \Phi \rangle_K = \mathfrak{H}$.

b) By the associativity of κ we have $\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(h, \kappa(x, y)t_\alpha)$, for all $h \in \mathfrak{H}$. Thus the non-degeneracy of the restriction of κ to $\mathfrak{H}$ implies that $[x, y] = \kappa(x, y)t_\alpha$.

c) Assume to the contrary that $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = 0$. Then let $x \in \mathfrak{L}_\alpha$ and $y \in \mathfrak{L}_{-\alpha}$ such that $\kappa(x, y) \neq 0$; we may assume that $\kappa(x, y) = 1$. Then we have $[x, y] = t_\alpha$, and by assumption we have $[t_\alpha, x] = \alpha(t_\alpha)x = \kappa(t_\alpha, t_\alpha)x = 0$ and $[t_\alpha, y] = -\alpha(t_\alpha)y = -\kappa(t_\alpha, t_\alpha)y = 0$. Thus $\mathfrak{K} := \langle x, y, t_\alpha \rangle_K \subseteq \mathfrak{L}$ is a 3-dimensional nilpotent Lie K-subalgebra, such that $[\mathfrak{K}, \mathfrak{K}] = Z(\mathfrak{K}) = \langle t_\alpha \rangle_K$.

Hence $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{K}) \subseteq \mathfrak{gl}(\mathfrak{L})$ is nilpotent and thus solvable. Thus by Lie's Theorem $\mathrm{ad}_{\mathfrak{L}}(\mathfrak{K})$ stabilises a flag in $\mathfrak{L}$, thus is contained in a Borel subalgebra of $\mathfrak{gl}(\mathfrak{L})$. The derived subalgebra of the latter consists of nilpotent matrices, hence since $t_\alpha \in [\mathfrak{K}, \mathfrak{K}]$ we infer that $\mathrm{ad}_{\mathfrak{L}}(t_\alpha)$ is nilpotent. Since $t_\alpha \in \mathfrak{H} \subseteq \mathfrak{L}$ is semisimple, we infer that $\mathrm{ad}_{\mathfrak{L}}(t_\alpha)$ is semisimple as well. This entails that $\mathrm{ad}_{\mathfrak{L}}(t_\alpha) = 0$, hence $0 \neq t_\alpha \in Z(\mathfrak{L}) = \{0\}$, a contradiction.

d) Since $\kappa(e_\alpha, \mathfrak{L}_{-\alpha}) \neq \{0\}$ there is $0 \neq y \in \mathfrak{L}_{-\alpha}$ such that $\kappa(e_\alpha, y) \neq 0$. Hence we may choose $f_\alpha \neq 0$ such that $[e_\alpha, f_\alpha] = h_\alpha$; recall that $h_\alpha \neq 0$. This shows that $\dim_K(\mathfrak{K}_\alpha) = 3$. Moreover, we have $[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = 2e_\alpha$ and $[h_\alpha, f_\alpha] = -\alpha(h_\alpha)f_\alpha = -2f_\alpha$. Thus the K-basis $\{e_\alpha, h_\alpha, f_\alpha\} \subseteq \mathfrak{K}_\alpha$ satisfies the commutator rules of the standard K-basis $\{E, H, F\} \subseteq \mathfrak{sl}_2(K)$, hence $\mathfrak{K} \subseteq \mathfrak{L}$ is a Lie K-subalgebra, and the map given is an isomorphism of Lie K-algebras. $\sharp$

(11.2) Theorem. We keep the setting of (11.1), and let $\alpha \in \Phi$. Then we have

a) $\langle \alpha \rangle_K \cap \Phi = \{\pm\alpha\}$ and **b)** $\dim_K(\mathfrak{L}_\alpha) = 1$.

Proof. Let $\mathfrak{K}_\alpha := \langle e_\alpha, h_\alpha, f_\alpha \rangle_K$ be any Lie K-subalgebra as above, and let $V := \mathfrak{H} \oplus \bigoplus_{0 \neq c \in K} \mathfrak{L}_{c\alpha} \leq_K \mathfrak{L}$; recall that $\mathfrak{H} = \mathfrak{L}_0$. Since $[h_\alpha, \mathfrak{L}_{c\alpha}] \leq_K \mathfrak{L}_{c\alpha}$ and $[e_\alpha, \mathfrak{L}_{c\alpha}] \leq_K \mathfrak{L}_{(c+1)\alpha}$ and $[f_\alpha, \mathfrak{L}_{c\alpha}] \leq_K \mathfrak{L}_{(c-1)\alpha}$ we conclude that V is a $\mathfrak{K}_\alpha$-submodule of $\mathfrak{L}$, with respect to the adjoint representation. Since $[h_\alpha, \mathfrak{H}] = \{0\}$ and $\alpha(h_\alpha) = 2$, we conclude that the weight space decomposition of V as a $\mathfrak{K}_\alpha$-module is as given above. Moreover, the non-zero weights occurring are of the form $c\alpha(h_\alpha) = 2c \in \mathbb{Z}$, entailing that we have $\mathfrak{L}_{c\alpha} \neq \{0\}$ only if $0 \neq c \in \frac{1}{2}\mathbb{Z}$.

We have $\mathfrak{H} = \ker(\alpha) \oplus \langle h_\alpha \rangle_K$. For $h \in \ker(\alpha)$ we have $[e_\alpha, h] = -\alpha(h)e_\alpha = 0$ and $[f_\alpha, h] = \alpha(h)f_\alpha = 0$. Hence since $[h_\alpha, h] = 0$ anyway we conclude that $\ker(\alpha) \leq_K V$ is a trivial $\mathfrak{K}_\alpha$-submodule of V. Moreover, $\mathfrak{K}_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle_K$ is a (simple) $\mathfrak{K}_\alpha$-submodule of V as well, with associated weights 0 and ± 2. Hence we get the $\mathfrak{K}_\alpha$-submodule $U := \ker(\alpha) \oplus \mathfrak{K}_\alpha$ of V, whose weight space decomposition is given as $U = \mathfrak{H} \oplus \langle e_\alpha \rangle_K \oplus \langle f_\alpha \rangle_K$.

Since V is a semisimple $\mathfrak{K}_\alpha$-module, U can be considered as summand in a direct sum decomposition of V. Since U encompasses the full weight space $\mathfrak{H}$ of $\mathfrak{L}$ associated with the weight 0, we conclude that U encompasses all the weight spaces of V associated with an even weight. Thus the only even non-zero weights of V are ± 2, each having a 1-dimensional weight space. Hence we have $\mathfrak{L}_{c\alpha} \neq \{0\}$, for some $0 \neq c \in \mathbb{Z}$, if and only if $c \in \{\pm 1\}$, in which case we have $\mathfrak{L}_\alpha = \langle e_\alpha \rangle_K$ and $\mathfrak{L}_{-\alpha} = \langle f_\alpha \rangle_K$, showing (b). In particular, we have $\mathfrak{L}_{2\alpha} = \{0\}$.

Now, assuming that $\mathfrak{L}_{\frac{1}{2}\alpha} \neq \{0\}$, repeating the above argument with $\frac{1}{2}\alpha$ instead of α yields $\mathfrak{L}_\alpha = \{0\}$, a contradiction. Hence we have $\mathfrak{L}_{\frac{1}{2}\alpha} = \{0\}$, saying that 1 is not a weight of V, in turn entailing that V does not have any odd weights, in other words $V = \mathfrak{H} \oplus \mathfrak{L}_\alpha \oplus \mathfrak{L}_{-\alpha} = \mathfrak{H} \oplus \langle e_\alpha \rangle_K \oplus \langle f_\alpha \rangle_K = U$, showing (a). ♯

Corollary. For any $0 \neq e_\alpha \in \mathfrak{L}_\alpha$ there is a unique $f_\alpha \in \mathfrak{L}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$. Hence we have $\mathfrak{K}_\alpha = \langle h_\alpha \rangle_K \oplus \mathfrak{L}_\alpha \oplus \mathfrak{L}_{-\alpha}$ as K-vector spaces, being the Lie K-subalgebra of $\mathfrak{L}$ generated by $\{\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}\}$. Moreover, $\mathfrak{L}$ is as a Lie K-algebra generated by $\{\mathfrak{L}_\alpha; \alpha \in \Phi\}$.

Proof. The Lie K-algebra $\mathfrak{K}_\alpha$ is generated by $\{e_\alpha, f_\alpha\}$. Since $\langle h_\alpha; \alpha \in \Phi \rangle_K = \langle t_\alpha; \alpha \in \Phi \rangle_K = \mathfrak{H}$ we have $\mathfrak{L} = \sum_{\alpha \in \Phi} \mathfrak{K}_\alpha$ as K-vector spaces. ♯

(11.3) Theorem. We keep the setting of (11.2), and let $\beta \neq \pm\alpha \in \Phi$.

a) Let $r, s \in \mathbb{N}_0$ be chosen maximal such that $\beta - r\alpha \in \Phi$ and $\beta + s\alpha \in \Phi$, respectively. Then we have $\beta + i\alpha \in \Phi$, for all $i \in \{-r, \ldots, s\}$, being called the **α-string through β**. Moreover, we have $\beta(h_\alpha) = r - s \in \mathbb{Z}$, being called the associated **Cartan integer**; in particular, we have $\beta - \beta(h_\alpha)\alpha \in \Phi$.

b) We have $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] = \mathfrak{L}_{\alpha+\beta}$; recall that $\mathfrak{L}_{\alpha+\beta} \neq \{0\}$ if and only if $\alpha + \beta \in \Phi$.

Proof. Let $V := \bigoplus_{i \in \mathbb{Z}} \mathfrak{L}_{\beta+i\alpha} \leq_K \mathfrak{L}$. Since we have $[h_\alpha, \mathfrak{L}_{\beta+i\alpha}] \leq_K \mathfrak{L}_{\beta+i\alpha}$ and $[e_\alpha, \mathfrak{L}_{\beta+i\alpha}] \leq_K \mathfrak{L}_{\beta+(i+1)\alpha}$ and $[f_\alpha, \mathfrak{L}_{\beta+i\alpha}] \leq_K \mathfrak{L}_{\beta+(i-1)\alpha}$, we conclude that V is a $\mathfrak{K}_\alpha$-submodule of $\mathfrak{L}$, with respect to the adjoint representation. Since we have $(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i$, we conclude that the weight space decomposition of V as a $\mathfrak{K}_\alpha$-module is as given above. Since we have $\dim_K(\mathfrak{L}_{\beta+i\alpha}) \leq 1$, we get $\dim_K(V_0) + \dim_K(V_1) \leq 1$. Thus V is a simple $\mathfrak{K}_\alpha$-module. In particular, we have $e_\alpha \cdot V_{\beta(h_\alpha)} = V_{\beta(h_\alpha)+2}$, translating into $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] = \mathfrak{L}_{\beta+\alpha}$, showing (b).

The weights of V form an arithmetic progression with steps of width 2. Thus the associated roots form the string $\beta - r\alpha, \ldots, \beta, \ldots, \beta + s\alpha$, where $r, s \in \mathbb{N}_0$ are as given above. Comparing the minimal and the maximal weights of V we get $\beta(h_\alpha) - 2r = (\beta - r\alpha)(h_\alpha) = -(\beta + s\alpha)(h_\alpha) = -\beta(h_\alpha) - 2s$, thus we finally deduce that $\beta(h_\alpha) = r - s \in \mathbb{Z}$, completing the proof of (a). ♯

Note that the α-string through α is $\{-\alpha, 0, \alpha\}$, where slightly more generally we allow for 0 occurring; hence we have $r = 2$ and $s = 0$, where indeed $\alpha(h_\alpha) = 2 = r - s$ and $\alpha - \alpha(h_\alpha)\alpha = -\alpha$. Similarly, the α-string through $-\alpha$ is $\{-\alpha, 0, \alpha\}$, hence $r = 0$ and $s = 2$, where $-\alpha(h_\alpha) = -2 = r - s$ and $-\alpha + \alpha(h_\alpha)\alpha = \alpha$.

(11.4) Root systems. We keep the setting of (11.3).

a) Recall the mutually inverse identifications $\mathfrak{H} \to \mathfrak{H}^*\colon h \mapsto \kappa_h$ and $\mathfrak{H}^* \to \mathfrak{H}\colon \alpha \mapsto t_\alpha$, where $t_\alpha \in \mathfrak{H}$ is the unique element such that $\alpha(h) = \kappa_{t_\alpha}(h) = \kappa(t_\alpha, h) \in K$, for all $h \in \mathfrak{H}$. We get a symmetric non-degenerate K-bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{H}^*$ by $\langle \alpha, \beta \rangle := \kappa(t_\alpha, t_\beta) = \alpha(t_\beta) = \beta(t_\alpha) \in K$, for all $\alpha, \beta \in \mathfrak{H}^*$.

In particular, for $\alpha \in \Phi$ the coroot $h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)} \cdot t_\alpha \in \mathfrak{H}$ can be identified with $\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \cdot \alpha \in \mathfrak{H}^*$; by a slight abuse the latter is also called the associated **coroot**, and we let $\Phi^\vee := \{\alpha^\vee; \alpha \in \Phi\} \subseteq \mathfrak{H}^*$. Hence for $\alpha, \beta \in \Phi$ we get $\langle \beta, \alpha^\vee \rangle = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \beta(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}) = \beta(h_\alpha) \in \mathbb{Z}$, the associated Cartan integer. In particular, we have $\beta - \beta(h_\alpha)\alpha = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = \beta - \langle \beta, \alpha^\vee \rangle \alpha \in \Phi$.

Let $\Delta := \{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi \subseteq \mathfrak{H}^*$ be a K-basis of $\mathfrak{H}^*$ consisting of roots, where $l := \dim_K(\mathfrak{H}) \in \mathbb{N}$ is called the **rank** of $\mathfrak{L}$ with respect to $\mathfrak{H}$; then $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\} \subseteq \mathfrak{H}$ is a K-basis as well. Let $C_\Delta := [\langle \alpha_i, \alpha_j^\vee \rangle]_{ij} \in \mathbb{Z}^{l \times l}$ be the associated **Cartan matrix**. Since $\Delta \subseteq \mathfrak{H}^*$ and $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\} \subseteq \mathfrak{H}$ are K-bases, we infer that C_Δ is invertible over K, and thus over $\mathbb{Q}$.

Proposition. Let $\mathcal{E}_0 := \langle \Delta \rangle_\mathbb{Q} \subseteq \mathfrak{H}^*$ be the $\mathbb{Q}$-subspace of $\mathfrak{H}^*$ with $\mathbb{Q}$-basis Δ. Then we have $\Phi \subseteq \mathcal{E}_0$ and $\Phi^\vee \subseteq \mathcal{E}_0$, and $\langle \cdot, \cdot \rangle$ restricts to a positive definite symmetric $\mathbb{Q}$-bilinear form on $\mathcal{E}_0$.

Proof. For $\beta \in \Phi$ let $\beta = \sum_{i=1}^l c_i \alpha_i$, for some $c_i \in K$. Then we have $\langle \beta, \alpha_j^\vee \rangle = \sum_{i=1}^l c_i \langle \alpha_i, \alpha_j^\vee \rangle$, for all $j \in \{1, \ldots, l\}$, which in terms of matrices yields the system of linear equations $[\langle \beta, \alpha_1^\vee \rangle, \ldots, \langle \beta, \alpha_l^\vee \rangle] = [c_1, \ldots, c_l] \cdot C$ for the unknowns $[c_1, \ldots, c_l] \in K^l$. Since $[\langle \beta, \alpha_1^\vee \rangle, \ldots, \langle \beta, \alpha_l^\vee \rangle] \in \mathbb{Z}^l$ and $C \in \mathrm{GL}_l(\mathbb{Q})$ we infer that $[c_1, \ldots, c_l] \in \mathbb{Q}^l$, thus $\beta \in \mathcal{E}_0$, showing that $\Phi \subseteq \mathcal{E}_0$.

Next, for any $\beta, \gamma \in \mathfrak{H}^*$, by the Cartan decomposition $\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha$ as $\mathfrak{H}$-modules, we get $\langle \beta, \gamma \rangle = \kappa(t_\beta, t_\gamma) = \mathrm{Tr}(\mathrm{ad}_\mathfrak{L}(t_\beta) \cdot \mathrm{ad}_\mathfrak{L}(t_\gamma)) = \sum_{\alpha \in \Phi} \alpha(t_\beta)\alpha(t_\gamma) = \sum_{\alpha \in \Phi} \kappa(t_\alpha, t_\beta)\kappa(t_\alpha, t_\gamma) = \sum_{\alpha \in \Phi} \langle \alpha, \beta \rangle \langle \alpha, \gamma \rangle$.

Now let $\beta \in \Phi$. Then we have $\langle \beta, \beta \rangle = \kappa(t_\beta, t_\beta) \neq 0$, that is β is not isotropic. Moreover, we have $\langle \beta, \beta \rangle = \sum_{\alpha \in \Phi} \langle \alpha, \beta \rangle^2$, thus dividing by $\langle \beta, \beta \rangle^2$ yields $\frac{1}{\langle \beta, \beta \rangle} = \sum_{\alpha \in \Phi} \big(\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\big)^2 = \frac{1}{4} \cdot \sum_{\alpha \in \Phi} \langle \alpha, \beta^\vee \rangle^2 \in \mathbb{Q}$, entailing that $\langle \beta, \beta \rangle \in \mathbb{Q}$. Hence we have $\beta^\vee = \frac{2}{\langle \beta, \beta \rangle}\beta \in \mathcal{E}_0$, showing that $\Phi^* \subseteq \mathcal{E}_0$.

We have $\langle \alpha, \beta \rangle = \frac{1}{2}\langle \alpha, \beta^\vee \rangle \langle \beta, \beta \rangle \in \mathbb{Q}$, for all $\alpha, \beta \in \Phi$. This shows that $\langle \cdot, \cdot \rangle$ restricts to a $\mathbb{Q}$-bilinear form on $\mathcal{E}_0$. From $\langle \beta, \beta \rangle = \sum_{\alpha \in \Phi} \langle \alpha, \beta \rangle^2$, for all $\beta \in \mathcal{E}_0$, being a sum of squares in $\mathbb{Q}$, we infer that $\langle \beta, \beta \rangle \geq 0$. If $\langle \beta, \beta \rangle = 0$ then $\langle \alpha, \beta \rangle = 0$, for all $\alpha \in \Phi$, which since Φ contains a K-basis of $\mathfrak{H}^*$ entails that $\beta \in \mathrm{rad}(\langle \cdot, \cdot \rangle) = \{0\}$. Thus $\langle \cdot, \cdot \rangle$ is positive definite on $\mathcal{E}_0$. ♯

b) Now let $\mathcal{E} := \mathcal{E}_0 \otimes_\mathbb{Q} \mathbb{R}$ be the scalar extension of $\mathcal{E}_0$ associated with the field extension $\mathbb{Q} \subseteq \mathbb{R}$. Hence identifying $\mathcal{E}_0$ with $\mathcal{E}_0 \otimes 1 \subseteq \mathcal{E}$, we get the $\mathbb{R}$-basis

$\Delta \subseteq \mathcal{E}$. Moreover, $\langle \cdot, \cdot \rangle$ gives rise to a positive definite symmetric $\mathbb{R}$-bilinear form on $\mathcal{E}$, that is a scalar product on $\mathcal{E}$, so that $\mathcal{E}$ becomes a Euclidean space.

The finite subset $\Phi \subseteq \mathcal{E} \setminus \{0\}$ is a **(reduced) root system**, that is **i)** $\langle \Phi \rangle_{\mathbb{R}} = \mathcal{E}$, **ii)** $\langle \alpha \rangle_{\mathbb{R}} \cap \Phi = \{\pm\alpha\}$, **iii)** $\frac{2\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$, and **iv)** $\beta - \frac{2\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle}\alpha \in \Phi$, for all $\alpha, \beta \in \Phi$.

Root systems are intimately connected to the following notion:

(11.5) The Weyl group. a) Let $\mathcal{E} \neq \{0\}$ be an Euclidean space with scalar product $\langle \cdot, \cdot \rangle$. For $0 \neq \alpha \in \mathcal{E}$ let $\alpha^\vee := \frac{2}{\langle \alpha,\alpha \rangle}\alpha \in \mathcal{E}$; note that we have $\langle \alpha, \alpha \rangle > 0$.

For $0 \neq \alpha \in \mathcal{E}$ let $s_\alpha \in \mathrm{End}_{\mathbb{R}}(\mathcal{E})$ be defined by $s_\alpha \colon \beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha$, for all $\beta \in \mathcal{E}$. Then using $\langle \alpha, \alpha^\vee \rangle = 2$ we get $s_\alpha(\alpha) = \alpha - \langle \alpha, \alpha^\vee \rangle \alpha = -\alpha$. This yields $s_\alpha s_\alpha(\beta) = s_\alpha(\beta) - \langle \beta, \alpha^\vee \rangle s_\alpha(\alpha) = \beta - \langle \beta, \alpha^\vee \rangle \alpha + \langle \beta, \alpha^\vee \rangle \alpha = \beta$, for all $\beta \in \mathcal{E}$, saying that $(s_\alpha)^2 = \mathrm{id}_{\mathcal{E}}$, in particular $s_\alpha \in \mathrm{GL}(\mathcal{E})$.

Moreover, we have $\langle s_\alpha(\beta), s_\alpha(\gamma) \rangle = \langle \beta - \langle \beta, \alpha^\vee \rangle \alpha, \gamma - \langle \gamma, \alpha^\vee \rangle \alpha \rangle = \langle \beta, \gamma \rangle - \langle \beta, \alpha^\vee \rangle \langle \alpha, \gamma \rangle - \langle \gamma, \alpha^\vee \rangle \langle \beta, \alpha \rangle + \langle \beta, \alpha^\vee \rangle \langle \gamma, \alpha^\vee \rangle \langle \alpha, \alpha \rangle = \langle \beta, \gamma \rangle - \frac{2}{\langle \alpha,\alpha \rangle}(\langle \beta, \alpha \rangle \langle \alpha, \gamma \rangle + \langle \gamma, \alpha \rangle \langle \beta, \alpha \rangle - 2\langle \beta, \alpha \rangle \langle \gamma, \alpha \rangle) = \langle \beta, \gamma \rangle$, for all $\beta, \gamma \in \mathcal{E}$, saying that s_α is an isometry, that is $s_\alpha \in O(\mathcal{E})$ is an orthogonal map.

Indeed, s_α fixes $\langle \alpha \rangle_{\mathbb{R}}^\perp \leq \mathcal{E}$ elementwise: We have $\langle \beta, \alpha^\vee \rangle = \frac{2}{\langle \alpha,\alpha \rangle}\langle \beta, \alpha \rangle = 0$, for all $\beta \in \langle \alpha \rangle_{\mathbb{R}}^\perp$, and hence $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta$. Thus from $\mathcal{E} = \langle \alpha \rangle_{\mathbb{R}} \oplus \langle \alpha \rangle_{\mathbb{R}}^\perp$ we infer that $s_\alpha \in O(\mathcal{E})$ is the reflection in the hyperplane orthogonal to α. In parricular, we have $\det(s_\alpha) = -1$, that is $s_\alpha \in O(\mathcal{E}) \setminus \mathrm{SO}(\mathcal{E})$.

b) Let $\Phi \subseteq \mathcal{E}$ be a root system. Then let $W := \langle s_\alpha \in O(\mathcal{E}); \alpha \in \Phi \rangle$ be the associated **Weyl group**.

For all $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha \in \Phi$ again. Thus W permutes Φ, hence we get a permutation representation $\rho \colon W \to \mathcal{S}_\Phi$ into the symmetric group on Φ. Since $\langle \Phi \rangle_{\mathbb{R}} = \mathcal{E}$ we conclude that ρ is faithful, that is the only element of W fixing Φ elementwise is the identity map. Thus ρ is an embedding, so that W can be viewed as a subgroup of the finite group $\mathcal{S}_\Phi$; in particular W is finite.

Here we end our developments. To summarize, given a semisimple Lie K-algebra $\mathfrak{L}$ over an algebraically closed field K such that $\mathrm{char}(K) = 0$, we have managed to exhibit a root system and a Weyl group associated with $\mathfrak{L}$. But this is merely the beginning of a longer story: The root system coming up does not depend on the choice of a maximal toral subalgebra. Moreover, semisimple Lie algebras are isomorphic if and only if they have isomorphic root systems. Moreover, root systems are of a combinatorial nature, and are rigid enough to allow for a complete classification, where finally for any root systems conversely there is a semisimple Lie algebra attached to it.

But this we do not prove here. Instead, we are content with giving an immediate result to indicate the flavour of the combinatorics involved, and to present the special linear algebras as an explicitly worked example.

(11.6) Lemma. In the setting of (11.3), let $\alpha, \beta \in \Phi$. Then the α-string through β contains at most four roots. In particular we have $|\langle \beta, \alpha^\vee \rangle| \leq 3$.

Proof. Considering the α-string $\{\beta - r\alpha, \ldots, \beta, \ldots, \beta + s\alpha\} \subseteq \Phi$ through β, where $r, s \in \mathbb{N}_0$, the first statement is equivalent to saying that $r + s \leq 3$, which using $\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) = r - s$ entails the second statement. Now:

Since the α-string through $\pm\alpha$ is given as $\{-\alpha, 0, \alpha\}$ we may assume that $\beta \neq \pm\alpha$. Now assume to the contrary that the α-string through β contains at least five roots, where we may additionally assume that $\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha \in \Phi$. Then $(\beta + 2\alpha) + \beta = 2(\alpha + \beta) \notin \Phi$ and $(\beta + 2\alpha) - \beta = 2\alpha \notin \Phi$ shows that the β-string through $\beta + 2\alpha$ consists of $\beta + 2\alpha$ alone. This implies $\langle \beta + 2\alpha, \beta^\vee \rangle = 0$ and hence $\langle \beta + 2\alpha, \beta \rangle = 0$. Similarly, $(\beta - 2\alpha) + \beta = 2(\beta - \alpha) \notin \Phi$ and $(\beta - 2\alpha) - \beta = -2\alpha \notin \Phi$ shows that the β-string through $\beta - 2\alpha$ consists of $\beta - 2\alpha$ alone, implying $\langle \beta - 2\alpha, \beta^\vee \rangle = 0$ and hence $\langle \beta - 2\alpha, \beta \rangle = 0$. Adding yields $\langle \beta, \beta \rangle = 0$, that is β is isotropic, a contradiction. $\sharp$

(11.7) Example: Special linear algebras. Let K be an algebraically closed field such that $\mathrm{char}(K) = 0$, and let $\mathfrak{L} := \mathfrak{sl}_n(K)$, where $n \in \mathbb{N}$, having standard K-basis

$$\big\{E_{ii} - E_{i+1,i+1}; i \in \{1, \ldots, n-1\}\big\} \,\dot\cup\, \big\{E_{ij}; i \neq j \in \{1, \ldots, n\}\big\} \subseteq \mathfrak{L}.$$

Recall that $[E_{kk}, E_{ij}] = (\delta_{k,i} - \delta_{k,j})E_{ij} \in \mathfrak{gl}_n(K)$, for all $i, j, k \in \{1, \ldots, n\}$.

i) Let $\mathfrak{H} := \mathfrak{t}_n(K) \cap \mathfrak{L} = \langle H_1, \ldots, H_{n-1} \rangle_K \subseteq \mathfrak{L}$, where we let $H_k := E_{kk} - E_{k+1,k+1} \in \mathfrak{L}$, for $k \in \{1, \ldots, n-1\}$. Since $\mathfrak{H}$ consists of semisimple matrices, we conclude that $\mathfrak{H}$ is toral. From the above commutator rules we infer that $\{E_{ij}; i \neq j \in \{1, \ldots, n\}\} \subseteq \mathfrak{L}$ consists of simultaneous eigenvectors of $\mathfrak{H}$.

We show that $\mathfrak{H}$ is a maximal toral Lie K-subalgebra of $\mathfrak{L}$, or equivalently that $\mathfrak{H}$ is is self-centralising: Let $x \in C_{\mathfrak{L}}(\mathfrak{H})$, where since $\mathfrak{H} \subseteq C_{\mathfrak{L}}(\mathfrak{H})$ we may assume that $x = \sum_{i \neq j} a_{ij} E_{ij} \in \mathfrak{L}$, for some $a_{ij} \in K$, and we have to show that $x = 0$. Now we have $0 = [E_{kk} - E_{ll}, x] = \sum_{i \neq j} a_{ij} [E_{kk} - E_{ll}, E_{ij}] = \sum_{i \neq j} a_{ij} (\delta_{k,i} - \delta_{k,j} - \delta_{l,i} + \delta_{l,j}) E_{ij}$, for all $k \neq l \in \{1, \ldots, n\}$. In particular, for $k = i$ and $l = j$ we get $\delta_{k,i} - \delta_{k,j} - \delta_{l,i} + \delta_{l,j} = 2$, implying that $a_{kl} = 0$.

ii) From $[H_k, E_{ij}] = [E_{kk} - E_{k+1,k+1}, E_{ij}] = (\delta_{k,i} - \delta_{k+1,i} - \delta_{k,j} + \delta_{k+1,j})E_{ij}$, for all $i \neq j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n-1\}$, the associated root $\alpha_{ij} \in \mathfrak{H}^*$ is given as follows: We have $\alpha_{ji} = -\alpha_{ij}$, hence assuming that $i < j$ we get

$$\alpha_{ij}(H_k) = \begin{cases} 2, & \text{if } \{i, j\} = \{k, k+1\}, \\ -1, & \text{if } i = k+1 \text{ or } j = k, \\ 1, & \text{if } j - 1 \neq i = k \text{ or } i + 1 \neq j = k, \\ 0, & \text{if } \{i, j\} \cap \{k, k+1\} = \emptyset. \end{cases}$$

In particular, the latter roots are pairwise distinct. Hence the Cartan decomposition $\mathfrak{L}$ with respect to $\mathfrak{H}$ is given as $\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{i \neq j} \langle E_{ij} \rangle_K$, and we have

$\Phi = \{\alpha_{ij} \in \mathfrak{H}^*; i \neq j \in \{1, \ldots, n\}\}$. Alternatively, since we know that all root spaces are 1-dimensional, the roots given must be pairwise distinct.

iii) From $[E_{rs}, E_{st}] = E_{rt}$, for all $r < s < t \in \{1, \ldots, n\}$, we get $\alpha_{rt} = \alpha_{rs} + \alpha_{st}$. In particular, we infer that $\alpha_{ij} = \sum_{r=i}^{j-1} \alpha_{r,r+1}$, for all $i < j \in \{1, \ldots, n\}$. Thus letting $\alpha_i := \alpha_{i,i+1}$, for all $i \in \{1, \ldots, n-1\}$, and $\Delta := \{\alpha_1, \ldots, \alpha_{n-1}\} \subseteq \Phi$ we infer that $\langle \Delta \rangle_K = \langle \Phi \rangle_K = \mathfrak{H}^*$, so that $\Delta \subseteq \mathfrak{H}^*$ is a K-basis consisting of roots.

For $i < j \in \{1, \ldots, n\}$, we observe that any $\alpha_{ij} \in \Phi$ is a non-negative linear combination of Δ, and hence that any $\alpha_{ji} = -\alpha_{ij} \in \Phi$ is a non-positive linear combination of Δ. Thus these are called **positive** and **negative** roots, respectively, and Φ is the disjoint union of positive and negative roots.

The linear combinations in question have integral coefficients, the latter are even in $\{0, \pm 1\}$. Hence we conclude that for positive roots $\alpha \neq \beta$, α-string through β consists of β alone, or of $\{\alpha, \alpha+\beta\}$ or of $\{\alpha - \beta, \alpha\}$. In other words, using the earlier notation, we have $r = s = 0$, or $r = 0$ and $s = 1$, or $r = 1$ and $s = 0$. Thus the associated Cartan integer is $\langle \beta, \alpha^\vee \rangle \in \{0, -1, 1\}$, respectively. (For both phenomena, integrality and positivity, we have not seen explanations.)

iv) We determine the Killing form $\kappa = \kappa_{\mathfrak{L}}$ of $\mathfrak{L}$. To this end let $\widehat{\mathfrak{L}} := \mathfrak{gl}_n(K)$. Then we have $\widehat{\mathfrak{L}} = Z(\widehat{\mathfrak{L}}) \oplus [\widehat{\mathfrak{L}}, \widehat{\mathfrak{L}}] = \langle E_n \rangle_K \oplus \mathfrak{L}$ as $\widehat{\mathfrak{L}}$-modules. The adjoint action of $A \in \widehat{\mathfrak{L}}$ being given as $X \mapsto [A, X] = AX - XA$, for all $X \in \widehat{\mathfrak{L}}$, with respect to the standard K-basis of $\widehat{\mathfrak{L}}$ we get $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A) = A \otimes E_n - E_n \otimes A^{\mathrm{tr}}$, where $\otimes$ denotes the Kronecker product of matrices.

Hence we get $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A) \cdot \mathrm{ad}_{\widehat{\mathfrak{L}}}(B) = (A \otimes E_n - E_n \otimes A^{\mathrm{tr}})(B \otimes E_n - E_n \otimes B^{\mathrm{tr}}) = AB \otimes E_n - A \otimes B^{\mathrm{tr}} - B \otimes A^{\mathrm{tr}} + E_n \otimes A^{\mathrm{tr}} B^{\mathrm{tr}}$, for all $A, B \in \widehat{\mathfrak{L}}$, which yields $\kappa_{\widehat{\mathfrak{L}}}(A, B) = \mathrm{Tr}(\mathrm{ad}_{\widehat{\mathfrak{L}}}(A) \cdot \mathrm{ad}_{\widehat{\mathfrak{L}}}(B)) = \mathrm{Tr}(AB \otimes E_n - A \otimes B^{\mathrm{tr}} - B \otimes A^{\mathrm{tr}} + E_n \otimes A^{\mathrm{tr}} B^{\mathrm{tr}}) = \mathrm{Tr}(AB)\mathrm{Tr}(E_n) - \mathrm{Tr}(A)\mathrm{Tr}(B^{\mathrm{tr}}) - \mathrm{Tr}(B)\mathrm{Tr}(A^{\mathrm{tr}}) + \mathrm{Tr}(E_n)\mathrm{Tr}(A^{\mathrm{tr}} B^{\mathrm{tr}}) = 2n\mathrm{Tr}(AB) - 2\mathrm{Tr}(A)\mathrm{Tr}(B)$. Since we have $\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)|_{Z(\widehat{\mathfrak{L}})} = 0$ anyway, we conclude that $\mathrm{Tr}(\mathrm{ad}_{\widehat{\mathfrak{L}}}(A)|_{\mathfrak{L}} \cdot \mathrm{ad}_{\widehat{\mathfrak{L}}}(B)|_{\mathfrak{L}}) = \mathrm{Tr}(\mathrm{ad}_{\widehat{\mathfrak{L}}}(A) \cdot \mathrm{ad}_{\widehat{\mathfrak{L}}}(B))$. Hence we get $\kappa(A, B) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{L}}(A) \cdot \mathrm{ad}_{\mathfrak{L}}(B)) = 2n\mathrm{Tr}(AB)$, for all $A, B \in \mathfrak{L}$.

Hence the Gram matrix of the restriction $\kappa_{\mathfrak{H} \times \mathfrak{H}}$ of κ to $\mathfrak{H}$ is given as, with respect to the K-basis $\{H_1, , \ldots, H_{n-1}\} \subseteq \mathfrak{H}$,

$$G(\kappa_{\mathfrak{H} \times \mathfrak{H}}) = 2n \cdot \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{Z}^{(n-1) \times (n-1)}.$$

v) We determine the elements $t_{ij} := t_{\alpha_{ij}} \in \mathfrak{H}$, for all $i \neq j \in \{1, \ldots, n\}$. Since the identification map $\mathfrak{H}^* \to \mathfrak{H} \colon \alpha \mapsto t_\alpha$ is K-linear, we have $-t_{ji} = t_{ij} = \sum_{r=i}^{j-1} t_r$, for all $i < j \in \{1, \ldots, n\}$, where $t_i := t_{i,i+1} \in \mathfrak{H}$. Thus it suffices to determine the associated elements for the roots in $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$. For those we have $[\alpha_i(H_1), \ldots, \alpha_i(H_{n-1})] = [0, \ldots, 0, -1, 2, -1, 0, \ldots, 0]$, the entry

2 occurring in position i, for all $i \in \{1, \dots, n-1\}$. Hence by inspection we get $t_i = \frac{1}{2n} H_i = \frac{1}{2n}(E_{ii} - E_{i+1,i+1}) \in \mathfrak{H}$. This in turn entails $t_{ij} = \sum_{r=i}^{j-1} t_r = \frac{1}{2n} \sum_{r=i}^{j-1} (E_{rr} - E_{r+1,r+1}) = \frac{1}{2n}(E_{ii} - E_{jj}) \in \mathfrak{H}$, for all $i \neq j \in \{1, \dots, n\}$

We determine the coroots $h_{ij} := h_{\alpha_{ij}} \in \mathfrak{H}$, for all $i < j \in \{1, \dots, n\}$. We have $\kappa(t_{ij}, t_{ij}) = \frac{1}{4n^2} \kappa(E_{ii} - E_{jj}, E_{ii} - E_{jj}) = \frac{1}{4n^2}(\kappa_{\widehat{\mathfrak{L}}}(E_{ii}, E_{ii}) - 2\kappa_{\widehat{\mathfrak{L}}}(E_{ii}, E_{jj}) + \kappa_{\widehat{\mathfrak{L}}}(E_{jj}, E_{jj})) = \frac{1}{4n^2}(2 \cdot (2n-2) + 2 \cdot 2) = \frac{1}{n}$. Thus the associated coroot is given as $h_{ij} = \frac{2}{\kappa(t_{ij}, t_{ij})} t_{ij} = 2n \cdot t_{ij} = E_{ii} - E_{jj} \in \mathfrak{H}$; in particular we have $h_i := h_{i,i+1} = H_i \in \mathfrak{H}$. Thus from $[E_{ij}, E_{ji}] = E_{ii} - E_{jj} = H_{ij}$ we infer that $\mathfrak{K}_{ij} := \mathfrak{K}_{\alpha_{ij}} \subseteq \mathfrak{L}$ has standard K-basis $\{E_{ij}, H_{ij}, E_{ji}\} \subseteq \mathfrak{K}_{ij}$.

vi) Pulling back with the identification map we get the associated coroot $\alpha_{ij}^\vee := 2n \cdot \alpha_{ij} \in \mathfrak{H}^*$, for all $i < j \in \{1, \dots, n\}$. For the K-bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{H}^*$ pulled back from $\kappa_{\mathfrak{H} \times \mathfrak{H}}$ we have

$$\langle \alpha_i, \alpha_j^\vee \rangle = \alpha_i(h_j) = \kappa(t_i, h_j) = \frac{1}{2n} \kappa(H_i, H_j) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i-j| = 1, \\ 0, & \text{if } |i-j| > 1, \end{cases}$$

for all $i, j \in \{1, \dots, n-1\}$. Hence we obtain the associated Cartan matrix

$$C_\Delta = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{Z}^{(n-1) \times (n-1)}.$$

(We observe that C_Δ is symmetric, for which we have not seen an explanation.)

We have $\langle \alpha_{ij}, \alpha_{ij} \rangle = \frac{1}{2n} \langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = \frac{1}{n}$, for all $i < j \in \{1, \dots, n\}$, hence all roots have the same length $\sqrt{\langle \alpha_{ij}, \alpha_{ij} \rangle} = \frac{1}{\sqrt{n}}$; similarly, we have $\langle \alpha_{ij}^\vee, \alpha_{ij}^\vee \rangle = 2n \cdot \langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = 4n$, hence all roots have the same length $\sqrt{\langle \alpha_{ij}^\vee, \alpha_{ij}^\vee \rangle} = 2\sqrt{n}$. Moreover, we have $\frac{\langle \alpha_i, \alpha_j^\vee \rangle}{\sqrt{\langle \alpha_i, \alpha_i \rangle} \cdot \sqrt{\langle \alpha_j^\vee, \alpha_j^\vee \rangle}} = \frac{1}{2} \langle \alpha_i, \alpha_j^\vee \rangle$, for all $i, j \in \{1, \dots, n-1\}$; hence α_i and α_j are perpendicular if and only if $|i-j| > 1$, while for $|i-j| = 1$ the angle between α_i and α_j equals $\frac{2\pi}{3}$.

III References

12 References

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