Quasi-Interpolants of genuine Bernstein-Durrmeyer operators

Margareta Heilmann, Martin Wagner

University of Wuppertal
Department of Mathematics and Informatics

Jaen Conference on Approximation Theory
June 26th-July 1st, 2011

Contents

Introduction

Genuine Bernstein-Durrmeyer Operators

Definition

For $f \in C[0,1],\ n \in \mathbb{N}$, the genuine Bernstein-Durrmeyer operators are given by

$$\begin{aligned}(U_n f)(x) :=& f(0)p_{n,0}(x) + f(1)p_{n,n}(x)\\ &+ (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt,\end{aligned}$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

Properties

- U_n is positive and linear
- $U_n e_i = e_i,\ i = 0, 1$ and $(U_n e_2)(x) = x^2 + \frac{2x(1-x)}{n+1}$
- $(U_n f)(0) = f(0)$ and $(U_n f)(1) = f(1)$
- $\|U_n f\|_\infty \leq \|f\|_\infty$

Properties

- U_n is positive and linear
- $U_n e_i = e_i, \; i = 0, 1$ and $(U_n e_2)(x) = x^2 + \frac{2x(1-x)}{n+1}$
- $(U_n f)(0) = f(0)$ and $(U_n f)(1) = f(1)$
- $\|U_n f\|_\infty \leq \|f\|_\infty$

Properties

- U_n is positive and linear
- $U_n e_i = e_i,\ i = 0, 1$ and $(U_n e_2)(x) = x^2 + \frac{2x(1-x)}{n+1}$
- $(U_n f)(0) = f(0)$ and $(U_n f)(1) = f(1)$
- $\|U_n f\|_\infty \leq \|f\|_\infty$

Properties

- U_n is positive and linear
- $U_n e_i = e_i,\ i = 0, 1$ and $(U_n e_2)(x) = x^2 + \frac{2x(1-x)}{n+1}$
- $(U_n f)(0) = f(0)$ and $(U_n f)(1) = f(1)$
- $\|U_n f\|_\infty \leq \|f\|_\infty$

Properties II

Lemma

For $n \in \mathbb{N},\ m \in \mathbb{N}_0$ *it holds*

$$(U_n p_m)(x) = \lambda_{n,m} p_m(x),$$

where

$$\lambda_{n,m} := \begin{cases} \frac{(n-1)!n!}{(n-1+m)!(n-m)!} & , \text{ for } m \le n \\ 0 & , \text{ for } m > n \end{cases}.$$

and $p_0(x) := 1,\ p_1(x) := x$ *and*

$$p_m(x) := D^{m-2}\left[x^{m-1}(1-x)^{m-1}\right] \text{ for } m \ge 2,$$

respectively.

Properties II

Lemma

For $n \in \mathbb{N},\ m \in \mathbb{N}_0$ *it holds*

$$\left(U_n p_m\right)(x) = \lambda_{n,m} p_m(x),$$

where

$$\lambda_{n,m} := \begin{cases} \frac{(n-1)!n!}{(n-1+m)!(n-m)!} & , \text{ for } m \leq n \\ 0 & , \text{ for } m > n \end{cases}.$$

and $p_0(x) := 1,\ p_1(x) := x$ *and*

$$p_m(x) := D^{m-2}\left[x^{m-1}(1-x)^{m-1}\right] \text{ for } m \geq 2,$$

respectively.

Properties II

Lemma

For $n \in \mathbb{N},\ m \in \mathbb{N}_0$ *it holds*

$$(U_n p_m)(x) = \lambda_{n,m} p_m(x),$$

where

$$\lambda_{n,m} := \begin{cases} \frac{(n-1)!n!}{(n-1+m)!(n-m)!} & , \text{ for } m \leq n \\ 0 & , \text{ for } m > n \end{cases}.$$

and $p_0(x) := 1,\ p_1(x) := x$ *and*

$$p_m(x) := D^{m-2}\left[x^{m-1}(1-x)^{m-1}\right] \text{ for } m \geq 2,$$

respectively.

Properties III

Lemma

- *For arbitrary but fixed* $f \in C[0,1]$, $x \in [0,1]$ *and* $n \in \mathbb{N}$ *it holds*

$$\lim_{\alpha,\beta \to -1^+} (M_n^{(\alpha,\beta)} f)(x) = (U_n f)(x),$$

where

$$(M_n^{(\alpha,\beta)} f)(x) := \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 p_{n,k}(t) f(t) t^\alpha (1-t)^\beta dt}{\int_0^1 p_{n,k}(t) t^\alpha (1-t)^\beta dt},$$
$$\alpha, \beta > -1,$$

are the Bernstein-Durrmeyer operators with Jacobi weights. However, there is no convergence in the operator norm.

- $(U_n f)' = M_{n-1}^{(0,0)} f'$, *for* $f \in C^1[0,1]$

Properties III

Lemma

- *For arbitrary but fixed* $f \in C[0,1]$, $x \in [0,1]$ *and* $n \in \mathbb{N}$ *it holds*

$$\lim_{\alpha,\beta \to -1^+} (M_n^{(\alpha,\beta)} f)(x) = (U_n f)(x),$$

where

$$(M_n^{(\alpha,\beta)} f)(x) := \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 p_{n,k}(t) f(t) t^\alpha (1-t)^\beta dt}{\int_0^1 p_{n,k}(t) t^\alpha (1-t)^\beta dt},$$

$$\alpha, \beta > -1,$$

are the Bernstein-Durrmeyer operators with Jacobi weights. However, there is no convergence in the operator norm.

- $(U_n f)' = M_{n-1}^{(0,0)} f'$, *for* $f \in C^1[0,1]$

Quasi-Interpolants

Differential Operator $\widetilde{D}^{2l}$

Definition

$$\widetilde{D}^{2l} := D^{l-1}x^l(1-x)^l D^{l+1},\ l \in \mathbb{N} \text{ and } \widetilde{D}^0 := I$$

Lemma

For $f \in C[0,1]$ it holds

$$U_{n-1}f - U_n f = \frac{\widetilde{D}^2 U_n f}{n(n-1)},$$

and therefore

$$U_n f - f = \sum_{k=n+1}^{\infty} \frac{\widetilde{D}^2 U_k f}{k(k-1)}.$$

Differential Operator $\widetilde{D}^{2l}$

Definition

$$\widetilde{D}^{2l} := D^{l-1}x^l(1-x)^l D^{l+1},\ l \in \mathbb{N} \text{ and } \widetilde{D}^0 := I$$

Lemma

For $f \in C[0,1]$ it holds

$$U_{n-1}f - U_n f = \frac{\widetilde{D}^2 U_n f}{n(n-1)},$$

and therefore

$$U_n f - f = \sum_{k=n+1}^{\infty} \frac{\widetilde{D}^2 U_k f}{k(k-1)}.$$

Properties of $\widetilde{D}^{2l}$

Lemma

For $l \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we have

$$\left(\widetilde{D}^{2l} p_m\right)(x) = \gamma_{l,m} p_m(x),$$

where $p_m(x)$ are given by $p_0(x) := 1,\ p_1(x) := x$ and

$$p_m(x) := D^{m-2}\left[x^{m-1}(1-x)^{m-1}\right] \text{ for } m \geq 2,$$

$$\gamma_{l,m} := \begin{cases} (-1)^l \frac{(m-1+l)!}{(m-1-l)!} & \text{for } l \leq m-1 \\ 0 & \text{for } l > m-1 \end{cases}.$$

Lemma

For $f \in C^{2l}[0,1]$, $l, n \in \mathbb{N}$ it holds

$$U_n \widetilde{D}^{2l} f = \widetilde{D}^{2l} U_n f.$$

Properties of $\widetilde{D}^{2l}$

Lemma

For $l \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we have

$$\left(\widetilde{D}^{2l} p_m\right)(x) = \gamma_{l,m} p_m(x),$$

where $p_m(x)$ are given by $p_0(x) := 1,\ p_1(x) := x$ and

$$p_m(x) := D^{m-2}\left[x^{m-1}(1-x)^{m-1}\right] \text{ for } m \geq 2,$$

$$\gamma_{l,m} := \begin{cases} (-1)^l \frac{(m-1+l)!}{(m-1-l)!} & \text{for } l \leq m-1 \\ 0 & \text{for } l > m-1 \end{cases}.$$

Lemma

For $f \in C^{2l}[0,1]$, $l, n \in \mathbb{N}$ it holds

$$U_n \widetilde{D}^{2l} f = \widetilde{D}^{2l} U_n f.$$

The Quasi-Interpolants $U_n^{(r)}$

Definition

For $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r \leq n-1$ and $f \in C[0,1]$ the quasi-interpolants of the genuine Bernstein-Durrmeyer operators are defined as

$$U_n^{(r)} f := \sum_{l=0}^{r} (-1)^l \frac{(n-1-l)!}{l!(n-1)!} \widetilde{D}^{2l} U_n f.$$

Lemma

For $n \in \mathbb{N}$, $r \leq n-1$, $p \in \mathbb{P}_m,\ m \leq r+1$ we have

$$U_n^{(r)} p = p.$$

The Quasi-Interpolants $U_n^{(r)}$

Definition

For $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r \leq n-1$ and $f \in C[0,1]$ the quasi-interpolants of the genuine Bernstein-Durrmeyer operators are defined as

$$U_n^{(r)} f := \sum_{l=0}^{r} (-1)^l \frac{(n-1-l)!}{l!(n-1)!} \widetilde{D}^{2l} U_n f.$$

Lemma

For $n \in \mathbb{N}$, $r \leq n-1$, $p \in \mathbb{P}_m,\ m \leq r+1$ we have

$$U_n^{(r)} p = p.$$

Spectral Characteristics of $U_n^{(r)}$

For $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r \leq n-1$ we have

- $\left(U_n^{(r)} p_m\right)(x) = \lambda_{n,m}^{(r)} p_m(x)$,
  where
  $$\lambda_{n,m}^{(r)} = \lambda_{n,m} \sum_{l=0}^{r} (-1)^l \frac{(n-1-l)!}{l!(n-1)!} \gamma_{l,m}.$$
- $\lambda_{n,m}^{(r)} - \lambda_{n-1,m}^{(r)} = (-1)^{r+1} \frac{(n-2-r)!}{r!n!} \lambda_{n,m} \gamma_{r+1,m}$

The Bernstein inequality

Bernstein Inequality I

Theorem

For $r, n \in \mathbb{N},\ r \le n-1$ *it holds*

$$\|\widetilde{D}^{2r} U_n f\|_\infty \le C n^r \|f\|_\infty.$$

Sketch of the proof:

Bernstein Inequality I

Theorem

For $r, n \in \mathbb{N},\ r \leq n-1$ *it holds*

$$\|\widetilde{D}^{2r} U_n f\|_\infty \leq C n^r \|f\|_\infty.$$

Sketch of the proof: It is sufficient to show

$$\sum_{k=0}^{n} \left| \widetilde{D}^{2r} p_{n,k}(x) \right| \leq C n^r.$$

Using the Leibniz-Formula we need the following:

$$\sum_{k=0}^{n} \left| [x(1-x)]^{l+1} D^{r+1+l} p_{n,k}(x) \right| \leq C n^r \quad \forall\ 0 \leq l \leq r-1.$$

Bernstein Inequality I

Theorem

For $r, n \in \mathbb{N},\ r \leq n-1$ *it holds*

$$\|\widetilde{D}^{2r} U_n f\|_\infty \leq C n^r \|f\|_\infty.$$

Sketch of the proof: It is sufficient to show

$$\sum_{k=0}^{n} \left| \widetilde{D}^{2r} p_{n,k}(x) \right| \leq C n^r.$$

Using the Leibniz-Formula we need the following:

$$\sum_{k=0}^{n} \left| [x(1-x)]^{l+1} D^{r+1+l} p_{n,k}(x) \right| \leq C n^r \quad \forall\ 0 \leq l \leq r-1.$$

Bernstein Inequality II

Theorem

For $r, n \in \mathbb{N},\ r \leq n-1$ it holds

$$\|\widetilde{D}^{2r} U_n f\|_\infty \leq C n^r \|f\|_\infty.$$

Sketch of the proof: Divide $[0,1]$ into $E_n := \left[\frac{1}{n}, 1-\frac{1}{n}\right]$ and $E_n^c := [0,1] \backslash E_n$ and give the estimate separately.

Use for $x \in E_n^c$ that

$$D^{r+1+l} p_{n,k}(x) =$$

$$\frac{n!}{(n-(r+1+l))!} \sum_{i=0}^{r+1+l} (-1)^i \binom{r+1+l}{i} p_{n-(r+1+l),k-(r+1+l-i)}(x)$$

Bernstein Inequality II

Theorem

For $r, n \in \mathbb{N},\ r \leq n-1$ *it holds*

$$\|\widetilde{D}^{2r} U_n f\|_\infty \leq C n^r \|f\|_\infty.$$

Sketch of the proof: Divide $[0,1]$ into $E_n := \left[\frac{1}{n}, 1-\frac{1}{n}\right]$ and $E_n^c := [0,1] \backslash E_n$ and give the estimate separately.
Use for $x \in E_n^c$ that

$$D^{r+1+l} p_{n,k}(x) =$$

$$\frac{n!}{(n-(r+1+l))!} \sum_{i=0}^{r+1+l} (-1)^i \binom{r+1+l}{i} p_{n-(r+1+l),k-(r+1+l-i)}(x)$$

Bernstein Inequality III

Theorem

For $r, n \in \mathbb{N},\ r \leq n-1$ *it holds*

$$\|\widetilde{D}^{2r} U_n f\|_\infty \leq C n^r \|f\|_\infty.$$

Sketch of the proof: Use for $x \in E_n$ that we can represent the $(r+1+l)$-th derivative of $p_{n,k}(x)$ as sum of terms of the type

$$q_{\lambda,m}(x) \frac{(k-nx)^{r+1+l-2\lambda-m}}{[x(1-x)]^{r+1+l-\lambda}} n^\lambda p_{n,k}(x),$$

where $\lambda \geq 0,\ m \geq 0,\ r+1+l-2\lambda-m \geq 0$ and $q_{\lambda,m}(x)$ polynomials in x independent of k and n.

Further we have

$$\sum_{k=0}^{n} |k-nx|^s p_{n,k}(x) \leq C n^{\frac{s}{2}} \left[x(1-x)\right]^{\frac{s}{2}},$$

for $x \in E_n$.

Bernstein Inequality III

Theorem

For $r, n \in \mathbb{N},\ r \leq n-1$ *it holds*

$$\|\widetilde{D}^{2r} U_n f\|_\infty \leq C n^r \|f\|_\infty .$$

Sketch of the proof: Use for $x \in E_n$ that we can represent the $(r+1+l)$-th derivative of $p_{n,k}(x)$ as sum of terms of the type

$$q_{\lambda,m}(x) \frac{(k-nx)^{r+1+l-2\lambda-m}}{[x(1-x)]^{r+1+l-\lambda}} n^\lambda p_{n,k}(x),$$

where $\lambda \geq 0,\ m \geq 0,\ r+1+l-2\lambda-m \geq 0$ and $q_{\lambda,m}(x)$ polynomials in x independent of k and n.

Further we have

$$\sum_{k=0}^{n} |k-nx|^s p_{n,k}(x) \leq C n^{\frac{s}{2}} [x(1-x)]^{\frac{s}{2}} ,$$

for $x \in E_n$.

Direct Results

Norm of $U_n^{(r)}$

Theorem

Let $f \in C[0,1]$, then ist holds

$$\|U_n^{(r)} f\|_\infty \leq C_r \|f\|_\infty$$

and

$$\|f - U_n^{(r)} f\|_\infty \leq C_r \|f\|_\infty.$$

Theorem

Let $f \in C[0,1]$, then it holds

$$\lim_{n\to\infty} ||f - U_n^{(r)} f||_\infty = 0.$$

Norm of $U_n^{(r)}$

Theorem

Let $f \in C[0,1]$, then ist holds

$$\|U_n^{(r)} f\|_\infty \le C_r \|f\|_\infty$$

and

$$\|f - U_n^{(r)} f\|_\infty \le C_r \|f\|_\infty.$$

Theorem

Let $f \in C[0,1]$, then it holds

$$\lim_{n\to\infty} ||f - U_n^{(r)} f||_\infty = 0.$$

Jackson-Type Inequality

Theorem

For $f \in C^{2(r+1)}[0,1]$ it holds

$$f - U_n^{(r)} f = \frac{(-1)^{r+1}}{r!} \sum_{l=n+1}^{\infty} \frac{(l-r-2)!}{l!} U_l \left(\widetilde{D}^{2(r+1)} f \right),$$

where the convergence of the series ist absolute and uniformly in $[0,1]$.
Furthermore we get

$$\| f - U_n^{(r)} f \|_\infty \leq \frac{(n-r-1)!}{n!(r+1)!} \| \widetilde{D}^{2(r+1)} f \|_\infty .$$

Definition

$$K_{2l}(f,t)_\infty := \inf_{g\in C^{2l}} \left\{ \|f-g\|_\infty + t\|\widetilde{D}^{2l} g\|_\infty \right\} \quad l>0$$

Theorem

Let $f \in C[0,1]$, then it holds

$$\|f - U_n^{(r)} f\|_\infty \le C_r K_{2(r+1)} \left(f, \frac{(n-r-1)!}{n!(r+1)!} \right)_\infty .$$

Outlook/ Forthcoming results

Outlook/ Forthcoming results

- Asympotic Expansion
- Simultanous Approximation
- Strong Converse Results
- L_p- Approximation
- Linear Combinations of U_n

Outlook/ Forthcoming results

- Asympotic Expansion
- Simultanous Approximation
- Strong Converse Results
- L_p- Approximation
- Linear Combinations of U_n

Outlook/ Forthcoming results

- Asympotic Expansion
- Simultanous Approximation
- Strong Converse Results
- L_p- Approximation
- Linear Combinations of U_n

Outlook/ Forthcoming results

- Asympotic Expansion
- Simultanous Approximation
- Strong Converse Results
- L_p- Approximation
- Linear Combinations of U_n

Outlook/ Forthcoming results

- Asympotic Expansion
- Simultanous Approximation
- Strong Converse Results
- L_p- Approximation
- Linear Combinations of U_n

Thank you for your attention!