# Asymptotic Expansions for a Durrmeyer Variant of Baskakov and Meyer-König and Zeller Operators and Quasi-Interpolants

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# Contents

# Introduction

# Baskakov-Durrmeyer Operators

Definition

*Let* $(1+\cdot)^{-(n-1)}f(\cdot) \in L_\infty[0,\infty),\ \sigma \in [0,\infty)$, *then the Baskakov-Durrmeyer operators are given by*

$$(B_{n+1}f)(\sigma) := \sum_{k=0}^{\infty} b_{n+1,k}(\sigma) \cdot n \int_0^\infty b_{n+1,k}(\tau) f(\tau) d\tau,$$

*where*

$$b_{n+1,k}(\sigma) := \binom{n+k}{k} \sigma^k (1+\sigma)^{-(n+1+k)}.$$

# MKZD Operators and Relation to BD Operators

## Definition

*Let* $(1-\cdot)^{(n-1)} f(\cdot) \in L_\infty[0,1),\ x \in [0,1)$, *then the „natural“ Durrmeyer variant of Meyer-König and Zeller operators are given by*

$$(M_n f)(x) := \sum_{k=0}^{\infty} m_{n,k}(x) \cdot n \int_0^1 m_{n,k}(t)(1-t)^{-2} f(t)dt,$$

*where*

$$m_{n,k}(x) := \binom{n+k}{k} x^k (1-x)^{(n+1)}.$$

Let $\sigma : [0,1) \longrightarrow [0,\infty),\ \sigma(x) := \frac{x}{1-x},\ f(\cdot) = \tilde{f}(\sigma(\cdot))$. Then

$$(M_n f)(x) = (B_{n+1}\tilde{f})(\sigma(x)).$$

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# Differentialoperators

## Definition

*Let* $\sigma \in [0, \infty),\ r \in \mathbb{N}$, *then*

$$\widetilde{D}_B^{2r} := \frac{d^r}{d\sigma^r}\left(\sigma^r (1+\sigma)^r \frac{d^r}{d\sigma^r}\right)$$

*respectively for* $x \in [0, 1)$

$$\widetilde{D}_M^{2r} := U^r \frac{x^r}{(1-x)^{2r}} U^r, \ \textit{with}$$

$U := (1-x)^2 \frac{d}{dx},\ U^r = U^{r-1} \circ U.$

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# Eigenfunctions and Eigenvalues

# Eigenfunctions and Eigenvalues of BD Operators

Let $m \in \mathbb{N}_0,\ m \leq n-1$ and let

$$\widetilde{g_m}(\sigma) := \frac{d^m}{d\sigma^m}\left(\sigma^m(1+\sigma)^m\right)$$

then

$$(B_{n+1}\widetilde{g_m})(\sigma) = \lambda_{n,m}\widetilde{g_m}(\sigma)$$

where

$$\lambda_{n,m} := \frac{(n-m-1)!(n+m)!}{(n-1)!n!}.$$

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$$g_m(x) := U^m\left(\frac{x^m}{(1-x)^{2m}}\right),$$

where the differential operator $D$ is defined by

$$U := (1-x)^2\frac{d}{dx},\ U^m = U^{m-1}\circ U,$$

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## Lemma

$$\widetilde{D}_B^{2r}\widetilde{g_m}(\sigma) = \gamma_{r,m}\widetilde{g_m}(\sigma),$$

*respectively*

$$\widetilde{D}_M^{2r} g_m(x) = \gamma_{r,m} g_m(x),$$

*where*

$$\gamma_{r,m} := \begin{cases} \frac{(m+r)!}{(m-r)!} & , \text{ for } r \leq m \\ 0 & , \text{ otherwise} \end{cases}$$

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### Lemma

*For $m \leq n-1$ it holds*

$$\lambda_{n,m} = \sum_{k=0}^{m} \frac{(n-1-k)!}{k!(n-1)!}\gamma_{k,m} = 1 + \sum_{k=1}^{m} \frac{(n-1-k)!}{k!(n-1)!}\gamma_{k,m}.$$

### Corollary

*For $p \in \mathbb{P}_q, \ q \leq n-1$ it holds*

$$B_{n+1}p = p + \sum_{k=1}^{q} \frac{(n-1-k)!}{k!(n-1)!} \widetilde{D}_B^{2k} p.$$

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# Asymptotic Expansions

# Classes of Functions

Let $I$ be an Intervall, then we denote with $H^{(q)}(x)$ the set of all functions $f : I \to \mathbb{R}$ possessing the following properties:

- $f$ is $q$ times ($q \in \mathbb{N}$) differentiable at $x$
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# Theorem of Sikkema

### Theorem

*For $q \in \mathbb{N}$, let $\{L_n\},\ n \in \mathbb{N}$, be a sequence of linear positive operators, $L_n : H^{(2q)}(x) \to C(I)$. Let the operators be applicable to $(t-x)^{2q+1}$ and to $(t-x)^{2q+2}$ and let*

$$\left(L_n(t-x)^r\right)(x) = \mathcal{O}\left(n^{\left\lfloor -\frac{r+1}{2}\right\rfloor}\right)\ (n \to \infty),\ r = 0, 1, ..., 2q+2,$$

*then we have*

$$(L_n f)(x) = \sum_{\nu=0}^{2q} \frac{f^{(\nu)}(x)}{\nu!}(L_n(t-x)^\nu)(x) + o(n^{-q}) \quad (n \to \infty).$$

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**Theorem**

*Let $\sigma \in [0,\infty)$, $q \in \mathbb{N}$ and $f \in H^{(2q)}(\sigma)$, then we have for $n \to \infty$*

$$(B_{n+1}f)(\sigma) = f(\sigma) + \sum_{k=1}^{q} \frac{(n-1-k)!}{k!(n-1)!} \left(\widetilde{D}_B^{2k} f\right)(\sigma) + o\left(n^{-q}\right).$$

**Theorem**

*Let $x \in [0,1]$, $q \in \mathbb{N}$ and $f \in H^{(2q)}(x)$, then we have for $n \to \infty$*

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# Quasi-Interpolants

# Quasi-Interpolants of Baskakov and MKZD Operators

## Definition

*Let* $r \in \mathbb{N}_0,\ n \in \mathbb{N},\ (1+\sigma)^{-(n-1)} f(\sigma) \in L_\infty[0,\infty)$*, then the quasi-interpolants of the Baskakov-operators are defined as follows*

$$B_{n+1}^{(r)} f := \sum_{k=0}^{r} \frac{n!}{(n+k)!} \frac{(-1)^k}{k!} \widetilde{D}_B^{2k}\big(B_{n+1} f\big).$$

## Definition

*Let* $(1-x)^{(n-1)} f(x) \in L_\infty[0,1)$*, then the quasi-interpolants of the MKZD-operators are defined as follows*

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# Eigenfunctions and Eigenvalues of the Quasi-Interpolants

**Lemma**

*Let* $m, r \in \mathbb{N}_0,\ m \leq n-1$, *then*

$$(B_{n+1}^{(r)}\widetilde{g_m})(\sigma) = \lambda_{n,m}^{(r)}\widetilde{g_m}(\sigma)$$

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$$(M_n^{(r)}g_m)(x) = \lambda_{n,m}^{(r)}g_m(x)$$

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*For* $m, r \in \mathbb{N}_0$, $m \leq n-1$ *it holds*

$$\lambda_{n,m}^{(r)} = \begin{cases} 1 & \text{for } m \leq r \\ 1 + \sum_{k=r+1}^{m} (-1)^r \gamma_{k,m} \frac{(n-1-k)!}{k!(n-1)!} \binom{k-1}{r} & \text{for } m > r \end{cases} .$$

## Corollary

*For* $p \in \mathbb{P}_q$, $q \leq n-1$ *it holds*

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# Asymptotic Expansion Quasi-Interpolants

## Theorem

*Let $\sigma \in [0,\infty)$, $q,r \in \mathbb{N}$ and $f \in H^{(2q+2r)}(\sigma)$, then we have for $n \to \infty$*

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Thank you for your attention!