# 1 Definition and basic properties

All linear spaces will be over the scalar field $\mathbb{K} = \mathbb{C}$ oder $\mathbb{R}$.

**Definition.**

A Fréchet space is a metrizable, complete locally convex vector space.

We recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological vector space is a Cauchy sequence iff for every neighborhood of zero $U$ there is $n_0$ so that for $n, m \geq n_0$ we have $x_n - x_m \in U$. Of course a metrizable topological vector space is complete iff every Cauchy sequence is convergent.

If $E$ is a vector space and $A \subset E$ absolutely convex then the *Minkowski functional* $\| \ \|_A$ is defined as

$$\|x\|_A = \inf\{\, t > 0 \mid x \in tA \,\}.$$

$\| \ \|_A$ is a extended real valued seminorm and $\|x\|_A < \infty$ if and only if $x \in \operatorname{span}\{A\} = \bigcup_{t>0} tA$.

$$\ker \| \ \|_A = \{\, x \mid \|x\|_A = 0 \,\} = \bigcap_{t>0} tA$$

is the largest linear space contained in $A$.

We have

$$\{\, x \mid \|x\|_A < 1 \,\} \subset A \subset \{\, x \mid \|x\|_A \leq 1 \,\}.$$

If $p$ is a seminorm, $A = \{\, x \mid p(x) \leq 1 \,\}$ then $\| \ \|_A = p$.

We set

$$E_A = (\operatorname{span}(A)/\ker \| \ \|_A, \| \ \|_A)^\wedge,$$

where "$\wedge$" denotes the completion. Notice that $\|x + y\|_A = \|x\|_A$ for $y \in \ker \| \ \|_A$. So $\| \ \|_A$ defines a norm on $\operatorname{span}(A)/\ker \| \ \|_A$.

If $E$ is a topological vector space and $A$ has an interior point, then $\| \ \|_A$ is continuous. In this case 0 is an interior point of $A$ and

$$\overset{\circ}{A} = \{\, x \mid \|x\|_A < 1 \,\}, \quad \bar{A} = \{\, x \mid \|x\|_A \leq 1 \,\}.$$

A space is called locally convex if it has a basis of absolutely convex neighborhoods of zero. Since for such a neighborhood of zero $U$, which may be assumed open, $\| \ \|_U$ is continuous and $x \in U$ iff $\|x\|_U < 1$, we have for a generalized sequence $(x_\tau)_{\tau \in T}$ that $x_\tau \longrightarrow x$ iff $p(x_\tau - x) \longrightarrow 0$ for every continuous seminorm.

**Definition.** A set $\mathscr{P}$ of continuous seminorms on the locally convex space $E$ is called fundamental system iff for every continuous seminorm $q$ there is $p \in \mathscr{P}$ and $C > 0$ so that $q \leq C \cdot p$.

Of course, we have for every generalized sequence $(x_\tau)_{\tau \in T}$ that $x_\tau \longrightarrow x$ if and only if $p(x_\tau - x) \longrightarrow 0$ for all $p \in \mathscr{P}$.

If a fundamental system of seminorms $\mathscr{P}$ is countable then we may assume that $\mathscr{P} = \{ \| \ \|_k \mid x \in \mathbb{N} \}$ where $\| \ \|_1 \leq \| \ \|_2 \leq \ldots$. This can be achieved by setting $\|x\|_k = \max\limits_{j=1,\ldots,k} p_j(x)$ where $\{ p_j \mid j \in \mathbb{N} \}$ is a given countable fundamental system of seminorms.

**Lemma 1.1** *For a locally convex space $E$ the following are equivalent:*

*(1) $E$ is metrizable.*

*(2) $E$ has a countable basis of neighborhoods of zero.*

*(3) $E$ has a countable fundamental system of seminorms.*

*(4) The topology of $E$ can be given by a translation invariant metric.*

*Proof.* (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are obvious.

To show (2) $\Rightarrow$ (3) we assume that $U_1 \supset U_2 \supset \ldots$ is a basis of absolutely convex neighborhoods of zero. We set $\| \ \|_k := \| \ \|_{U_k}$. Then all $\| \ \|_k$ are continuous seminorms, then we may choose $k$ so that $U_k \subset \{ x \mid p(x) \leq 1 \}$. This implies $p(x) \leq \|x\|_k$ for all $x$.

Finally, for (3) $\Rightarrow$ (4) we put

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|x-y\|_k}{1+\|x-y\|_k}.$$

It is an elementary exercise to show that $d(\cdot,\cdot)$ is a translation invariant metric which gives the topology of $E$. □

We notice that for a translation invariant metric $d$ on a topological vector space $E$ is complete if and only if $E$ is complete with respect to $d$.

For a not translation invariant metric this needs not to be the case as the example of $\mathbb{R}$ with the metric $d(x,y) = \|\arctan x - \arctan y\|$ shows.

As in the case of Banach space we show:

**Theorem 1.2** *If $E$ is a Fréchet space and $F \subset E$ a closed subspace then $F$ and $E/F$ are Fréchet spaces.*

A subset $M$ of a linear space $E$ is called *absorbant* if $\bigcup_{t>0} tM = E$. A topological vector space $E$ is called *barrelled* iff every closed, absolutely convex, absorbant set ("*barrel*") is a neighborhood of zero.

**Theorem 1.3** *Every Fréchet space is barrelled.*

*Proof.* Obviously $E = \bigcup_{n\in\mathbb{N}} nM$. By Baire's theorem there is $n_0$ so that $n_0 M$ and therefore also $M$ has an interior point. So 0 is an interior point of $M$, which had to be proved. □

**Lemma 1.4** *If $E$ and $F$ are locally convex, $F$ a Fréchet space and $A\colon E \longrightarrow F$ linear and surjective, then $A$ is* nearly open *, i.e. for every neighborhood of zero $U \subset E$ the set $\overline{AU}$ is a neighborhood of zero in $F$.*

*Proof.* We may assume that $U$ is absolutely convex, hence $\overline{AU}$ is a barrel in $F$. Theorem 1.3 yields the result. □

As in the case of Banach spaces we prove the following lemma.

**Lemma 1.5 (Schauder)** *If $E$ and $F$ are metrizable spaces, $E$ complete, and $A\colon E \longrightarrow F$ linear, continuous and nearly open, then $A$ is surjective and open.*

These two lemmas yield:

**Theorem 1.6 (Open Mapping Theorem)** *If $E$ and $F$ are Fréchet spaces, $A\colon E \longrightarrow F$ linear, continuous and surjective, then $A$ is open.*

**Corollary 1.7 (Banach's Isomorphy Theorem)** *If $E$ and $F$ are Fréchet spaces, $A\colon E \longrightarrow F$ linear, continuous and bijective, then $A^{-1}$ is continuous.*

**Theorem 1.8 (Closed Graph Theorem)** *If $E$ and $F$ are Fréchet spaces, $A\colon E \longrightarrow F$ linear and $Graph\, A = \{\, (x, Ax) \mid x \in E \,\}$ closed in $E \times F$, then $A$ is continuous.*

*Proof.* We consider the maps $\pi_1\colon (x,y) \mapsto x$, $\pi_2\colon (x,y) \mapsto y$ from $Graph\, A$ to $E$, resp. $F$, we notice that $\pi_1$ is continuous linear and bijective, hence by Corollary 1.7 $\pi_1^{-1}$ is continuous and therefore also $A = \pi_2 \circ \pi_1^{-1}$. □

If $E$ is a Fréchet space, $\| \ \|_1 \leq \| \ \|_2 \leq \ldots$ a fundamental system of seminorms, then the Banach spaces $E_k := (E/\| \ \|_k, \| \ \|_k)^\wedge$ are called "*local Banach spaces*".

Since $\ker \| \ \|_{k+1} \subset \ker \| \ \|_k$ we have natural quotient maps $E/\ker \| \ \|_{k+1} \longrightarrow E/\ker \| \ \|_k$, which extend to continuous linear maps $\imath_{k+1}^k\colon E_{k+1} \longrightarrow E_k$, ("*linking maps*"). Of course we may define in a similar way $\imath_m^n\colon E_m \longrightarrow E_n$ for $m \geq n$ and obtain $\imath_m^n = \imath_{n+1}^n \circ \ldots \circ \imath_m^{m-1}$. By $\imath^k\colon E \longrightarrow E_k$ we denote the quotient map. We have for $m > n$ that $\imath_m^n \circ \imath^m = \imath^n$. We define continuous linear maps

$$\begin{aligned} \imath\colon E \longrightarrow \Pi_k E_k \text{ by} \quad & \imath x & &= (\imath^k x)_{k\in\mathbb{N}} \\ \sigma\colon \Pi_k E_k \longrightarrow \Pi_k E_k \text{ by} \quad & \sigma((x_k)_{k\in\mathbb{N}}) & &= (x_k - \imath_{k+1}^k x_{k+1})_{k\in\mathbb{N}}. \end{aligned}$$

**Theorem 1.9** *By*

$$0 \longrightarrow E \xrightarrow{\ \imath\ } \Pi_k E_k \xrightarrow{\ \sigma\ } \Pi_k E_k \longrightarrow 0$$

*we obtain a short exact sequence ("*canonical resolution*").*

*Proof.* $\imath$ is clearly injective since $\imath^n = 0$ for all $k$ implies $\|x\|_k = 0$ for all $k$, hence $x = 0$.

$\operatorname{im}\imath \subset \ker\sigma$ because $\imath^k x - \imath^k_{k+1}\imath^{k+1}x = \imath^k x - \imath^k x = 0$. To prove $\ker\sigma \subset \operatorname{im}\sigma$ we notice that

$$\ker\sigma = \{\, (x_k)_{k\in\mathbb{N}} \mid x_k = \imath^k_{k+1}x_{k+1} \text{ for all } k \,\}.$$

This implies $x_n = \imath^n_m x_m$ for all $n \le m$. Let $x \in \ker\sigma$. For every $x_k$ we choose $\xi_k \in E$ so that $\|x_k - \imath^k\xi_k\|_k \le 2^{-k}$.

For $k \le n < m$ we obtain

$$\begin{aligned}\|\xi_n - \xi_m\|_k &= \|\imath^k\xi_n - \imath^k_n x_n + \imath^k_m x_m - \imath^k\xi_m\|_k \\ &\le \|\imath^n\xi_n - x_n\|_n + \|x_m - \imath^m\xi_m\|_m \\ &\le 2^{-k+1}\end{aligned}$$

Therefore $(\xi_n)_n$ is a Cauchy sequence, hence convergent. We put $x = \lim_{k\to\infty}\xi_k$. We have

$$\begin{aligned}\|x_k - \imath^k x\|_k &= \lim_{n\to\infty}\|x_k - \imath^k\xi_n\|_k \\ &= \lim_{n\to\infty}\|\imath^k_n x_n - \imath^n_n\imath^n\xi_n\|_k \\ &\le \lim_{n\to\infty}\|x_n - \imath^n\xi_n\|_n = 0.\end{aligned}$$

So $x_k = \imath^k x$ for all $k$.

To prove surjectivity of $\sigma$ we consider an element $(0,\dots,0,x_k,0\dots)$, $x_k \in E_k$. There exists $\xi_k \in E$ such that $\|x_k - \imath^k\xi_k\|_k < 2^{-k}$. We set $y_k = x_k - \imath^k\xi_k \in E_k$, then $\|y_k\|_k < 2^{-k}$.

We define elements

$$\begin{aligned}a_k &= (0,\dots,0,-\imath^{k+1}\xi_k,-\imath^{k+2}\xi_k,\dots) \\ b_k &= (\imath^1_k y_k,\dots,\imath^{k-1}_k y_k, y_k, 0,\dots).\end{aligned}$$

Then we have

$$\begin{aligned}\sigma(a_k) &= (0,\dots,0\imath^k\xi_k,0\dots) \\ \sigma(b_k) &= (0,\dots,0,y_k,0,\dots).\end{aligned}$$

Now we put $u_k = a_k + b_k =: (\eta^k_1,\eta^k_2,\dots)$. We obtain $\sigma(u_k) = (0,\dots,0,x_k,0\dots)$ with $\|\eta^k_j\|_j < 2^{-j}$ for $j = 1,\dots,k$. From there it is easily seen that $u := \sum_k u_k$ converges in $\prod_k E_k$. We have $\sigma(u) = \sum_k \sigma(u_k) = x$. □

## 2 The dual space of a Fréchet space

For any locally convex space $E$ we need the following notation for polar sets:

For $M \subset E$ we put

$$M^\circ = \{\, y \in E' \mid |y(x)| \leq 1 \text{ for all } x \in M \,\}$$

and for $N \subset E'$ we put

$$N^\circ = \{\, x \in E \mid |y(x)| \leq 1 \text{ for all } y \in N \,\}.$$

If $E'$ is equipped with a locally convex topology ( e.g. the strong topology, see below ) then the second notation might be ambiguous. If there is any danger of confusion we will specify whether we mean $N^\circ \subset E$ or $N^\circ \subset E''$.

We recall

**Theorem 2.1 (Bipolar Theorem)** *If $E$ is locally convex and $M \subset E$ absolutely convex then $M^{\circ\circ} = \overline{M}$.*

If $p$ is a continuous seminorm on the locally convex space $E$ and $U = \{\, x \in E \mid p(x) \leq 1 \,\}$ then we have

$$U^\circ = \{\, y \in E' \mid \sup_{x \in U} |y(x)| \leq 1 \,\}.$$

We put

$$p^*(y) = \sup_{x \in U} |y(x)|$$

and obtain

$$U^\circ = \{\, y \mid p^*(y) \leq 1 \,\}.$$

Therefore $p^*$ is the Minkowski functional of $U^\circ$.

If $E$ is a Fréchet space and

$$\| \|_1 \leq \| \|_2 \leq \dots$$

a fundamental system of seminorms, then

$$\| \|_1^* \geq \| \|_2^* \geq \dots$$

is a decreasing sequence of extended real valued norms. For the unit balls we have

$$B_k := \{\, y \mid \|y\|_k^* \leq 1 \,\} = U_k^\circ.$$

We set

$$E_k^* = \operatorname{span} U_k^\circ = \bigcup_{t>0} tU_k^\circ.$$

Then $E_k^*$ is the space of linear form in $E$ which are continuous with respect to $\| \ \|_k$. $(E_k^*, \| \ \|_k^*)$ is a normed space. The following remark shows that it is even a Banach space.

**Remark.** $\imath^{k'} : E_k' \longrightarrow E'$ is an isometric isomorphism from $E_k'$ onto $E_k^*$.

Therefore we have an increasing sequence $E_1^* \subset E_2^* \subset \ldots$ of Banach spaces, so that $E' = \bigcup_k E_k^*$.

**Definition.** $E'$ is made into a locally convex space by the seminorms

$$p_B(y) = \sup_{x \in B} |y(x)|$$

where $B$ runs through the bonded subsets of $E$.

We recall that a subset of a locally convex space is *bounded* if it is bounded with respect to all continuous seminorms on $E$ or, equivalently, with respect to all seminorms of a fundamental system.

Since for every $y \in E$ the function $x \mapsto |y(x)|$ is a seminorm, the $p_B$ are finite.

**Theorem 2.2** *Let $E$ be metrizable locally convex and $F$ locally convex. If $A\colon E \to F$ is linear and $A$ maps bounded sets into bounded sets then $A$ is continuous.*

*Proof.* If $x_n \longrightarrow 0$ then $\|x_n\|_k \longrightarrow 0$ for all $k$, where $\| \ \|_1 \leq \| \ \|_2 \leq \ldots$ is a fundamental system of seminorms. There is a sequence $(\lambda_n)_n$ in $\mathbb{K}$, so that $|\lambda_n| \longrightarrow +\infty$ but

$$\|\lambda_n x_n\|_k = |\lambda_n| \|x_n\|_k \longrightarrow 0$$

for all $k$. In particular $B = \{\, \lambda_n x_n \mid n \in \mathbb{N} \,\}$ is bounded. Therefore $A(B)$ is bounded. If $p$ is a continuous seminorm on $F$ then

$$p(Ax_n) = \frac{1}{|\lambda_n|} p(A(\lambda_n x_n)) \leq \frac{1}{|\lambda_n|} \sup_{x \in B} p(Ax) \underset{n \to \infty}{\longrightarrow} 0.$$

So $Ax_n \longrightarrow 0$ and $A$ is continuous. □

This holds, of course, in particular for $F = \mathbb{K}$. That means, a linear form on $E$ is continuous iff it is bounded on bounded subsets of $E$.

**Theorem 2.3** *If $E$ is metrizable, then $E'$ is complete.*

*Proof.* If $(y_\tau)_{\tau\in T}$ in a Cauchy net in $E'$, then $(y_\tau(x))_{\tau\in T}$ is a Cauchy net for every $x \in E$. Hence for every $x$ there is $y(x)$ so that $y_\tau(x) \longrightarrow y(x)$. Clearly $y$ is a linear form, which is bounded on bounded subsets of $E'$, therefore continuous, and $p_B(y_\tau - y) \longrightarrow 0$. □

As a consequence we see that for a metrizable locally convex space $E'$ is metrizable if and only if $E$ is normed. This is because Baire's theorem implies that for some $k$ the set $kU_k^\circ$ has an interior point. So $U_m^\circ \subset C_m U_k^\circ$ for all $m$ with suitable $C_m$ and the Bipolar Theorem implies that $E$ is normed.

**Definition.** If $E$ is locally convex then:

(1) the weak topology $\sigma(E, E')$ on $E$ is given by the fundamental system of seminorms
$$p_e(x) = \sup_{y\in e} |y(x)|$$
where $e$ runs through the finite subsets of $E'$

(2) the weak$^*$ topology $\sigma(E', E)$ on $E'$ is given by the fundamental system of seminorms
$$p_e(y) = \sup_{x\in e} |y(x)|$$
where $e$ runs through the finite subsets of $E$.

We recall that $(E, \sigma(E, E'))' = E'$ and $(E', \sigma(E', E))' = E$. And we have the following theorem.

**Theorem 2.4 (Alaoglu-Bourbaki)** *If $E$ is locally convex and $U \subset E$ a neighborhood of zero then $U^\circ$ is $\sigma(E', E)$-compact.*

For the *proof* see MV 23.5.

**Theorem 2.5** *If $E$ is a Fréchet space then for $M \subset E'$ the following are equivalent:*

*(1) $M$ is weak* bounded*

*(2) $M$ is weak* relatively compact*

*(3) $M$ is bounded*

*(4) There is $k \in \mathbb{N}$, $C > 0$ so that $M \subset CU_k^\circ$.*

*Proof.* (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are obvious. (4) $\Longrightarrow$ (2) follows from the Alaoglu - Bourbaki theorem 2.4.

For (4) $\Longrightarrow$ (3) it is enough to show that $U_k^\circ$ is bounded. Let $B \subset E$ be bounded and $y \in U_k^\circ$ then

$$p_B(y) \leq \|y\|_k^* \sup_{x \in B} \|x\|_k \leq \sup_{x \in B} \|x\|_k.$$

So it remains to show that (1) $\Longrightarrow$ (4). Now $M^\circ \subset E$ is obviously absolutely convex and closed. For $x \in E$ we have $\sup_{y \in M} |y(x)| = C_x < +\infty$, hence $x \in C_x M^\circ$. Therefore $M^\circ$ is absorbant. By Theorem 1.3 $U := M^\circ$ is a neighborhood of zero in $E$. Hence there are $k \in \mathbb{N}$ and $C > 0$ so that $\frac{1}{C} U_k \subset U$. This implies

$$M \subset M^{\circ\circ} = U^\circ \subset CU_k^\circ.$$

□

**Corollary 2.6** *If $E$ is a Fréchet space then $E'$ is a complete locally convex space which has a countable fundamental system of bounded sets.*

A set $\mathscr{B}$ of bounded sets in a locally convex space $E$ is called a *fundamental system of bounded sets* if for every bounded set $B \subset E$ there is $B_0 \in \mathscr{B}$ and $C > 0$ so that $B \subset CB_0$.

A subset $M \subset E$, $E$ locally convex, is called *bornivorous* if for every bounded set $B \subset E$ there is $t > 0$ so that $B \subset tM$.

**Definition.** $E$ is called bornological if every absolutely convex bornivorous set is a neighborhood of zero.

**Remark.** For a locally convex space the following are equivalent

(1) $E$ is bornological

(2) for every locally convex space $G$ and every linear map $A\colon E \longrightarrow G$ which maps bounded sets into bounded sets, $A$ is continuous

(3) for every Banach space $G$ and every linear map $A\colon E \longrightarrow G$ which maps bounded sets into bounded sets, $A$ is continuous.

*Proof.* (1) $\Longrightarrow$ (2) If $U$ is an absolutely convex neighborhood of zero in $G$ and $A$ maps bounded sets into bounded sets, then $A^{-1}U$ is locally convex and bornivorous. Hence it is a neighborhood of zero.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) If $M \subset E$ is absolutely convex and bornivorous, in particular absorbant, the space $E_M = (E/\ker \| \ \|_M, \| \ \|_M)^\wedge$ is a Banach space. The quotient map $\varphi\colon E \longrightarrow E_M$ maps bounded sets into bounded sets so, by assumption, it is continuous. Therefore $M \subset \{\, x \mid \|qx\|_M < 1 \,\}$ is a neighborhood of zero. □

**Example.** By Theorem 2.2 every metrizable locally convex space is bornological. The dual $E' = \bigcup_k E_k^*$ of a Fréchet space $E$ is bornological if the following holds:

A linear map $A\colon E' \longrightarrow G$, $G$ locally convex is continuous iff the restriction $A|_{E_k^*}$ is continuous $E_k^* \longrightarrow G$ for all $k$.

**Theorem 2.7** *Let $E$ be metrizable locally convex. Then the following are equivalent:*

*(1) $E'$ is bornological*

*(2) $E'$ is barrelled.*

*Proof.* (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1) Let $M \subset E'$ be absolutely convex and bornivorous. We put $B_n = U_n^\circ$ for all $n \in \mathbb{N}$.

For every $k$ there is $\varepsilon_k > 0$ so that $2\varepsilon_k B_k \subset M$. Therefore

$$C_n := \Gamma \bigcup_{k=1}^{n} \varepsilon_k B_k \subset \frac{1}{2} M$$

for all $n \in \mathbb{N}$. Here $\Gamma$ denotes the absolutely convex hull.

$C_n$ is weak$^*$-compact. This is because $C_n$ is the image of the weak$^*$-continuous map

$$\{\, z \in \mathbb{K}^n \mid |z|_{f_1} \leq 1 \,\} \times B_1 \times \ldots \times B_n \longrightarrow E'$$

given by

$$(z, x_1, \ldots, x_n) \mapsto \sum_{k=1}^{n} z_k \varepsilon_k x_k.$$

Therefore $C_n$ is weak$^*$-closed, hence closed in $E'$. We set

$$D := \Gamma \bigcup_{k=1}^{\infty} \varepsilon_k B_k = \bigcup_n C_n \subset \frac{1}{2} M.$$

We want to show that $\overline{D} \subset 2D$, because if we have shown this, then $\overline{D} \subset M$. Clearly $\overline{D}$ is a barrel (i. e. absolutely convex, closed and absorbant) so, by assumption, a neighborhood of zero, and we are done.

So let $x_0 \in E \setminus 2D$. Since $C_n$ is closed there is an absolutely convex, weak$^*$-closed neighborhood of zero in $E'$ (i. e. $V_n = B^\circ$, $B \subset E$ bounded), so that

$$(x_0 + V_n) \cap 2C_n = \emptyset.$$

Therefore

$$(x_0 + V_n + C_n) \cap C_n = \emptyset.$$

Since $V_n$ and $C_n$ are weak$^*$-compact, $V_n + C_n$ is weak$^*$-compact (as range of the map $(x, y) \mapsto x + y$ from $V_n \times C_n \longrightarrow E'$).

Therefore

$$W = \bigcap_n (V_n + C_n)$$

is weak*-closed, hence closed.

$W$ is a barrel. It is clearly absolutely convex, closed by the previous and absorbant by the following:

For $x \in E$ we have $x \in \lambda B_k$ for certain $\lambda$ and $k$. Therefore $x \in \frac{\lambda}{\varepsilon_k} C_n$ for all $n \geq k$. There is $\mu > 0$ so that $x \in \mu V_n$ for $n = 1, \ldots, k-1$, so $x \in \max(\frac{\lambda}{\varepsilon_k}, \mu) W$.

Since $E'$ is barrelled, by assumption $W$ is a neighborhood of zero. We have $(x_0 + W) \cap C_n = \emptyset$ for all $n$, therefore $(x_0 + W) \cap D = \emptyset$ which implies $x \notin \overline{D}$. □

**Definition.** A Fréchet space is called distinguished if the equivalent conditions of Theorem 2.5 are satisfied.

We refer to MV, page 270 and recall that we have a canonical imbedding $\mathrm{J}\colon E \hookrightarrow E'$ given by $(\mathrm{J}(x))[y] = y(x)$ for $x \in E$ and $y \in E'$. By Theorem 2.5 a fundamental system of seminorms in $E''$ is given by

$$\|z\|_k = \sup_{y \in U_k^\circ} |z(y)|, \quad k \in \mathbb{N}.$$

Therefore $\|\mathrm{J}x\|_k = \|x\|_k$ and J imbeds $E$ topologically into $E''$.

**Definition.** The Fréchet space $E$ is called reflexive if $\mathrm{J}E = E''$.

If there is no danger of confusion we will omit J in future and consider $E$ in a natural way as a subspace of $E''$.

**Theorem 2.8** *If the Fréchet space $E$ is reflexive then it is distinguished.*

*Proof.* We show that $E'$ is barrelled. Let $U$ be a barrel in $E'$. Then, due to the bipolar theorem 2.1, $U^{\circ\circ} = U$. Here the first polar $U^\circ$ has to be taken in $E$. However, due to reflexively, we may take it in $E''$. Since $U \cap E_k^*$ is a barrel in the Banach space $E_k^*$ it is a neighborhood of zero there, so there is $\varepsilon_k > 0$ with $\varepsilon_k U_k^\circ \subset U$ and therefore $U^\circ \subset \frac{1}{\varepsilon_k} U_k$. This holds for all $k$, so $U^\circ$ is bounded in $E$ and therefore $U$ a neighborhood of zero in $E'$. □

A criterion for reflexivity is:

**Theorem 2.9** *The Fréchet space $E$ is reflexive iff every bounded set in $E$ is weak relatively compact.*

*Proof.* Let $E$ be reflexive and $B \subset E$ bounded. Then $B \subset B^{\circ\circ}$ and $B^\circ$ is a neighborhood of zero in $E'$. So due to Theorem 2.4 $(B^\circ)^\circ \subset E''$ is $\delta(E'', E')$-compact. Due to reflexively we may take $B^{\circ\circ} \subset E$ and it is $\sigma(E, E') = \sigma(E'', E')$-compact.

To prove the reverse take $z \in E''$, then there is a neighborhood of zero $U \subset E'$ so that $z \in U^\circ$. We may assume $U = B^\circ$ where $B \subset E$ is bounded, absolutely convex and closed, hence, due to $B^{\circ\circ} = B$, weakly closed. So $B$ is $\sigma(E, E')$-compact.

Now, considering polars (except $B^\circ$) with respect to the duality $E'$, $E''$ we have

$$U^\circ = (B^\circ)^\circ = \overline{\mathrm{J}B}^{\sigma(E'', E')} = \mathrm{J}B.$$

This is because J is a topological imbedding, also with respect to the weak $\to$ weak$^*$-topologies hence $\mathrm{J}B$ is $\sigma(E'', E')$-compact, and therefore closed. □

**Corollary 2.10** *If $E$ is a reflexive Fréchet space and $F \subset E$ a closed subspace, then $F$ is reflexive.*

*Proof.* Due to Theorem 2.1, which implies $F^{\circ\circ} = F$, the space $F$ is weakly closed. If $B \subset F$ is bounded, then $\overline{B}^{\sigma(E, E')} \subset F$. Because of Theorem 2.9 it is $\sigma(E, E')$ and therefore, due to the Hahn - Banach theorem, also $\sigma(F, F')$-compact. □

**Example.** Let $E_k$, $k \in \mathbb{N}$ be Banach spaces and $E = \prod_k E_k$. It is easily verified, that $E' = \bigoplus_k E_k'$, equipped with the seminorms $p(x) = \sum_{k=1}^\infty c_k \|x_k\|_k^*$ for $x = \oplus_{k=1}^\infty x_k$, where $(c_k)_k$ runs through all positive sequences.

Again, it is easily verified that $E'' = \prod_k E_k''$. Therefore $E$ is reflexive iff $E_k$ is reflexive for all $k$.

**Lemma 2.11** *If $E$ has a fundamental system of seminorms so that all local Banach spaces $E_k$ are reflexive, then $E$ is reflexive.*

*Proof.* $E$ can be considered as a closed subspace of $\prod_k E_k$ and, due to the previous, $\prod_k E_k$ is reflexive. So Corollary 2.10 gives the result. □

**Definition.** $E$ is called Fréchet-Hilbert space if there exists a fundamental system of seminorms, so that for all $k$ and $x \in E$ we have $\|x\|_k^2 = < x, x >_k$ where $< \cdot, \cdot >_k$ is a semiscalar product on $E$.

**Remark.** In this case all local Banach spaces $E_k$ are Hilbert spaces.

**Theorem 2.12** *A Fréchet-Hilbert space is reflexive hence distinguished.*

**Lemma 2.13** *If $E$ is a Fréchet-Hilbert space, $F \subset E$ a closed subspace, then $F$ and $E/F$ are Fréchet-Hilbert spaces.*

*Proof.* This is easily verified, since subspaces and quotient spaces of Hilbert spaces are Hilbert spaces. □

**Corollary 2.14** *If $E$ is a Fréchet-Hilbert space, $F \subset E$ a closed subspace, then $F$ and $E/F$ are reflexive, hence distinguished.*