# Lectures on Fréchet spaces

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# 1 Definition and basic properties

All linear spaces will be over the scalar field  $\mathbb{K} = \mathbb{C}$  oder  $\mathbb{R}$ .

**Definition:** A Fréchet space is a metrizable, complete locally convex vector space.

We recall that a sequence  $(x_n)_{n \in \mathbb{N}}$  in a topological vector space is a Cauchy sequence if for every neighborhood of zero U there is  $n_0$  so that for  $n, m \ge n_0$ we have  $x_n - x_m \in U$ . Of course a metrizable topological vector space is complete if every Cauchy sequence is convergent.

If E is a vector space and  $A \subset E$  absolutely convex then the *Minkowski* functional  $|| ||_A$  is defined as

$$||x||_A = \inf\{t > 0 \mid x \in tA\}.$$

 $\| \|_A$  is a extended real valued seminorm and  $\|x\|_A < \infty$  if and only if  $x \in \text{span}\{A\} = \bigcup_{t>0} tA$ .

$$\ker \| \|_A = \{ x \mid \|x\|_A = 0 \} = \bigcap_{t>0} tA$$

is the largest linear space contained in A.

We have

$$\{x \mid ||x||_A < 1\} \subset A \subset \{x \mid ||x||_A \le 1\}.$$

If p is a seminorm,  $A = \{x \mid p(x) \le 1\}$  then  $|| ||_A = p$ .

We set

$$E_A = (\operatorname{span}(A)/\ker \parallel \parallel_A, \parallel \parallel_A)^{\wedge},$$

where "^" denotes the completion. Notice that  $||x + y||_A = ||x||_A$  for  $y \in \ker || ||_A$ . So  $|| ||_A$  defines a norm on  $\operatorname{span}(A)/\ker || ||_A$ .

If E is a topological vector space and A has an interior point, then  $|| ||_A$  is continuous. In this case 0 is an interior point of A and

$$\tilde{A} = \{ x \mid ||x||_A < 1 \}, \quad \bar{A} = \{ x \mid ||x||_A \le 1 \}.$$

A space is called *locally convex* if it has a basis of absolutely convex neighborhoods of zero. Since for such a neighborhood of zero U, which may be

assumed open,  $\| \|_U$  is continuous and  $x \in U$  if, and only if,  $\|x\|_U < 1$ , we have for a generalized sequence  $(x_\tau)_{\tau \in T}$  that  $x_\tau \longrightarrow x$  if, and only if,  $p(x_\tau - x) \longrightarrow 0$  for every continuous seminorm.

**Definition:** A set  $\mathscr{P}$  of continuous seminorms on the locally convex space E is called fundamental system if for every continuous seminorm q there is  $p \in \mathscr{P}$  and C > 0 so that  $q \leq C \cdot p$ .

Of course, we have for every generalized sequence  $(x_{\tau})_{\tau \in T}$  that  $x_{\tau} \longrightarrow x$  if, and only if,  $p(x_{\tau} - x) \longrightarrow 0$  for all  $p \in \mathscr{P}$ .

If a fundamental system of seminorms  $\mathscr{P}$  is countable then we may assume that  $\mathscr{P} = \{ \| \|_k \mid x \in \mathbb{N} \}$  where  $\| \|_1 \leq \| \|_2 \leq \ldots$  This can be achieved by setting  $\|x\|_k = \max_{j=1,\ldots,k} p_j(x)$  where  $\{ p_j \mid j \in \mathbb{N} \}$  is a given countable fundamental system of seminorms.

**1.1 Lemma:** For a locally convex space E the following are equivalent:

- (1) E is metrizable.
- (2) E has a countable basis of neighborhoods of zero.
- (3) E has a countable fundamental system of seminorms.
- (4) The topology of E can be given by a translation invariant metric.

*Proof:*  $(1) \Rightarrow (2)$  and  $(4) \Rightarrow (1)$  are obvious.

To show (2)  $\Rightarrow$  (3) we assume that  $U_1 \supset U_2 \supset \ldots$  is a basis of absolutely convex neighborhoods of zero. We set  $|| ||_k := || ||_{U_k}$ . Then all  $|| ||_k$  are continuous seminorms. If p is a continuous seminorm, then we may choose k such that  $U_k \subset \{x \mid p(x) \leq 1\}$ . This implies  $p(x) \leq ||x||_k$  for all x.

Finally, for  $(3) \Rightarrow (4)$  we put

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|x-y\|_k}{1+\|x-y\|_k}.$$

It is an elementary exercise to show that  $d(\cdot, \cdot)$  is a translation invariant metric which gives the topology of E.

We notice that for a translation invariant metric d on a topological vector space E is complete if and only if E is complete with respect to d.

For a not translation invariant metric this needs not to be the case as the example of  $\mathbb{R}$  with the metric  $d(x, y) = \|\arctan x - \arctan y\|$  shows.

As in the case of Banach spaces one can show:

**1.2 Theorem:** If E is a Fréchet space and  $F \subset E$  a closed subspace then F and E/F are Fréchet spaces.

A subset M of a linear space E is called *absorbant* if  $\bigcup_{t>0} tM = E$ . A topological vector space E is called *barrelled* if every closed, absolutely convex, absorbant set ("*barrel*") is a neighborhood of zero.

1.3 Theorem: Every Fréchet space is barrelled.

Proof: Obviously  $E = \bigcup_{n \in \mathbb{N}} nM$ . By Baire's theorem there is  $n_0$  so that  $n_0M$  and therefore also M has an interior point. Then also 0 is an interior point of M, which had to be proved.

**1.4 Lemma:** If E and F are locally convex, F a Fréchet space and  $A: E \longrightarrow F$  linear and surjective, then A is nearly open , i.e. for every neighborhood of zero  $U \subset E$  the set  $\overline{AU}$  is a neighborhood of zero in F.

*Proof:* We may assume that U is absolutely convex, hence  $\overline{AU}$  is a barrel in F. Theorem 1.3 yields the result.  $\Box$ 

As in the case of Banach spaces one can prove the following lemma.

**1.5 Lemma (Schauder):** If E and F are metrizable spaces, E complete, and  $A: E \longrightarrow F$  linear, continuous and nearly open, then A is surjective and open.

These two lemmas yield:

**1.6 Theorem (Open Mapping Theorem):** If E and F are Fréchet spaces,  $A: E \longrightarrow F$  linear, continuous and surjective, then A is open.

**1.7 Corollary (Banach's Isomorphy Theorem):** If E and F are Fréchet spaces,  $A: E \longrightarrow F$  linear, continuous and bijective, then  $A^{-1}$  is continuous.

**1.8 Theorem (Closed Graph Theorem):** If E and F are Fréchet spaces,  $A: E \longrightarrow F$  linear and Graph  $A := \{(x, Ax) \mid x \in E\}$  closed in  $E \times F$ , then A is continuous.

Proof: We consider the maps  $\pi_1: (x, y) \mapsto x, \pi_2: (x, y) \mapsto y$  from Graph A to E, resp. F, we notice that  $\pi_1$  is continuous linear and bijective, hence by Corollary 1.7  $\pi_1^{-1}$  is continuous and therefore also  $A = \pi_2 \circ \pi_1^{-1}$ .

If E is a Fréchet space,  $\| \|_1 \leq \| \|_2 \leq \ldots$  a fundamental system of seminorms, then the Banach spaces  $E_k := (E/\| \|_k, \| \|_k)^{\wedge}$  are called "local Banach spaces".

Since ker  $|| ||_{k+1} \subset \text{ker } || ||_k$  we have natural quotient maps  $E/\text{ker } || ||_{k+1} \longrightarrow E/\text{ker } || ||_k$ , which extend to continuous linear maps  $i_{k+1}^k \colon E_{k+1} \longrightarrow E_k$ , ("linking maps"). Of course we may define in a similar way  $i_m^n \colon E_m \longrightarrow E_n$  for  $m \ge n$  and obtain  $i_m^n = i_{n+1}^n \circ \ldots \circ i_m^{m-1}$ . By  $i^k \colon E \longrightarrow E_k$  we denote the quotient map. We have for m > n that  $i_m^n \circ i^m = i^n$ . We define continuous linear maps

$$i: E \longrightarrow \Pi_k E_k \text{ by } ix = (i^k x)_{k \in \mathbb{N}}$$
  
$$\sigma: \Pi_k E_k \longrightarrow \Pi_k E_k \text{ by } \sigma((x_k)_{k \in \mathbb{N}}) = (x_k - i^k_{k+1} x_{k+1})_{k \in \mathbb{N}}.$$

1.9 Theorem: By

$$0 \longrightarrow E \xrightarrow{i} \Pi_k E_k \xrightarrow{\sigma} \Pi_k E_k \longrightarrow 0$$

we obtain a short exact sequence ("canonical resolution").

Proof: *i* is clearly injective since  $i^n = 0$  for all k implies  $||x||_k = 0$  for all k, hence x = 0.

im  $i \subset \ker \sigma$  because  $i^k x - i^k_{k+1} i^{k+1} x = i^k x - i^k x = 0$ . To prove  $\ker \sigma \subset \operatorname{im} i$  we notice that

$$\ker \sigma = \{ (x_k)_{k \in \mathbb{N}} \mid x_k = i_{k+1}^k x_{k+1} \text{ for all } k \}.$$

This implies  $x_n = \imath_m^n x_m$  for all  $n \leq m$ . Let  $x \in \ker \sigma$ . For every  $x_k$  we choose  $\xi_k \in E$  so that  $||x_k - \imath^k \xi_k||_k \leq 2^{-k}$ .

For  $k \leq n < m$  we obtain

$$\begin{aligned} \|\xi_n - \xi_m\|_k &= \|i^k \xi_n - i^k_n x_n + i^k_m x_m - i^k \xi_m\|_k \\ &\leq \|i^n \xi_n - x_n\|_n + \|x_m - i^m \xi_m\|_m \\ &\leq 2^{-k+1}. \end{aligned}$$

Therefore  $(\xi_n)_n$  is a Cauchy sequence, hence convergent. We put  $x = \lim_{k \to \infty} \xi_k$ . We have

$$\begin{aligned} \|x_k - i^k x\|_k &= \lim_{n \to \infty} \|x_k - i^k \xi_n\|_k \\ &= \lim_{n \to \infty} \|i_n^k x_n - i_n^k i^n \xi_n\|_k \\ &\leq \lim_{n \to \infty} \|x_n - i^n \xi_n\|_n = 0. \end{aligned}$$

So  $x_k = i^k x$  for all k.

To prove surjectivity of  $\sigma$  we consider an element  $(0, \ldots, 0, x_k, 0 \ldots), x_k \in E_k$ . There exists  $\xi_k \in E$  such that  $||x_k - i^k \xi_k||_k < 2^{-k}$ . We set  $y_k = x_k - i^k \xi_k \in E_k$ , then  $||y_k||_k < 2^{-k}$ .

We define elements

$$a_k = (0, \dots, 0, -i^{k+1}\xi_k, -i^{k+2}\xi_k, \dots)$$
  

$$b_k = (i_k^1 y_k, \dots, i_k^{k-1} y_k, y_k, 0, \dots).$$

Then we have

$$\sigma(a_k) = (0, \dots, 0, i^k \xi_k, 0 \dots)$$
  
$$\sigma(b_k) = (0, \dots, 0, y_k, 0, \dots).$$

Now we put  $u_k = a_k + b_k =: (\eta_1^k, \eta_2^k, \dots)$ . We obtain  $\sigma(u_k) = (0, \dots, 0, x_k, 0 \dots)$ and  $\|\eta_j^k\|_j = \|i_k^j y_k\|_j \le \|y_k\|_k < 2^{-k}$  for  $j \le k$ . From there it is easily seen that  $u := \sum_k u_k$  converges in  $\prod_k E_k$ . We have  $\sigma(u) = \sum_k \sigma(u_k) = x$ .  $\Box$ 

# 2 The dual space of a Fréchet space

For any locally convex space E we use the following notation of polar sets: For  $M \subset E$  we put

$$M^{\circ} = \{ y \in E' \mid |y(x)| \le 1 \text{ for all } x \in M \}$$

and for  $N \subset E'$  we put

$$N^{\circ} = \{ x \in E \mid |y(x)| \le 1 \text{ for all } y \in N \}.$$

If E' is equipped with a locally convex topology (e.g. the strong topology, see below) then the second notation might be ambiguous. If there is any danger of confusion we will specify whether we mean  $N^{\circ} \subset E$  or  $N^{\circ} \subset E''$ .

We recall

**2.1 Theorem (Bipolar Theorem):** If *E* is locally convex and  $M \subset E$  absolutely convex then  $M^{\circ \circ} = \overline{M}$ .

If p is a continuous seminorm on the locally convex space E and  $U = \{ x \in E \mid p(x) \le 1 \}$  then we have

$$U^{\circ} = \{ y \in E' \mid \sup_{x \in U} |y(x)| \le 1 \}.$$

We put

$$p^*(y) = \sup_{x \in U} |y(x)|$$

and obtain

$$U^{\circ} = \{ y \mid p^*(y) \le 1 \}.$$

Therefore  $p^*$  is the Minkowski functional of  $U^{\circ}$ .

If E is a Fréchet space and

$$\| \|_1 \leq \| \|_2 \leq \dots$$

a fundamental system of seminorms, then

$$\| \|_1^* \ge \| \|_2^* \ge \dots$$

is a decreasing sequence of extended real valued norms. For the unit balls we have

$$B_k := \{ y \mid \|y\|_k^* \le 1 \} = U_k^{\circ}.$$

We set

$$E_k^* = \operatorname{span} U_k^\circ = \bigcup_{t>0} t U_k^\circ.$$

Then  $E_k^*$  is the space of linear form in E which are continuous with respect to  $\| \|_k$ .  $(E_k^*, \| \|_k^*)$  is a normed space. The following remark shows that it is even a Banach space.

**Remark:**  $i^{k'}: E'_k \longrightarrow E'$  is an isometric isomorphism from  $E'_k$  onto  $E^*_k$ .

Therefore we have an increasing sequence  $E_1^* \subset E_2^* \subset \ldots$  of Banach spaces, such that  $E' = \bigcup_k E_k^*$ .

**Definition:** E' is made into a locally convex space by the seminorms

$$p_B(y) = \sup_{x \in B} |y(x)|$$

where B runs through the bounded subsets of E.

We recall that a subset of a locally convex space is *bounded* if it is bounded with respect to all continuous seminorms on E or, equivalently, with respect to all seminorms of a fundamental system.

Since for  $y \in E'$  the function  $x \mapsto |y(x)|$  is a seminorm,  $p_B(y) = \sup_{x \in B} |y(x)|$  is finite for every  $y \in E$  and every bounded set  $B \subset E$ .

**2.2 Theorem:** Let *E* be metrizable locally convex and *F* locally convex. If  $A: E \to F$  is linear and *A* maps bounded sets into bounded sets then *A* is continuous.

Proof: If  $x_n \to 0$  then  $||x_n||_k \to 0$  for all k, where  $|||_1 \leq |||_2 \leq \ldots$  is a fundamental system of seminorms. There is a sequence  $(\lambda_n)_n$  in  $\mathbb{K}$ , such that  $|\lambda_n| \to +\infty$  and

$$\|\lambda_n x_n\|_k = |\lambda_n| \|x_n\|_k \longrightarrow 0$$

for all k. In particular  $B = \{\lambda_n x_n \mid n \in \mathbb{N}\}$  is bounded. Therefore A(B) is bounded. If p is a continuous seminorm on F then

$$p(Ax_n) = \frac{1}{|\lambda_n|} p(A(\lambda_n x_n)) \le \frac{1}{|\lambda_n|} \sup_{x \in B} p(Ax) \xrightarrow[n \to \infty]{} 0.$$

So  $Ax_n \longrightarrow 0$  and A is continuous.

This holds, of course, in particular for  $F = \mathbb{K}$ . That means, a linear form on E is continuous if, and only if, it is bounded on bounded subsets of E.

**2.3 Theorem:** If E is metrizable, then E' is complete.

Proof: If  $(y_{\tau})_{\tau \in T}$  is a Cauchy net in E', then  $(y_{\tau}(x))_{\tau \in T}$  is a Cauchy net for every  $x \in E$ . Hence for every x there is y(x) so that  $y_{\tau}(x) \longrightarrow y(x)$ . Clearly y is a linear form, which is bounded on bounded subsets of E', therefore continuous, and  $p_B(y_{\tau} - y) \longrightarrow 0$ . As a consequence we see that for a metrizable locally convex space the space E' is metrizable if and only if E is normed. This is because Baire's theorem implies that for some k the set  $kU_k^{\circ}$  has an interior point, hence is a neighborhood of zero. So  $U_m^{\circ} \subset C_m U_k^{\circ}$  for all m with suitable  $C_m$  and the Bipolar Theorem implies that E is normed.

**Definition:** If E is locally convex then:

(1) the weak topology  $\sigma(E, E')$  on E is given by the fundamental system of seminorms

$$p_e(x) = \sup_{y \in e} |y(x)|$$

where e runs through the finite subsets of E'

(2) the weak\* topology  $\sigma(E', E)$  on E' is given by the fundamental system of seminorms

$$p_e(y) = \sup_{x \in e} |y(x)|$$

where e runs through the finite subsets of E.

We recall that  $(E, \sigma(E, E'))' = E'$  and  $(E', \sigma(E', E))' = E$ . And we have the following theorem.

**2.4 Theorem (Alaoglu-Bourbaki):** If E is locally convex and  $U \subset E$  a neighborhood of zero then  $U^{\circ}$  is  $\sigma(E', E)$ -compact.

For the proof see [1, 23.5].

**2.5 Theorem:** If E is a Fréchet space then for  $M \subset E'$  the following are equivalent:

- (1) M is weak<sup>\*</sup> bounded
- (2) M is weak<sup>\*</sup> relatively compact
- (3) M is bounded
- (4) There is  $k \in \mathbb{N}$ , C > 0 such that  $M \subset CU_k^{\circ}$ .

*Proof:*  $(2) \Longrightarrow (1)$  and  $(3) \Longrightarrow (1)$  are obvious.  $(4) \Longrightarrow (2)$  follows from the Alaoglu - Bourbaki Theorem 2.4.

For (4)  $\implies$  (3) it is enough to show that  $U_k^{\circ}$  is bounded. Let  $B \subset E$  be bounded and  $y \in U_k^{\circ}$  then

$$p_B(y) \leq \|y\|_k^* \sup_{x \in B} \|x\|_k \leq \sup_{x \in B} \|x\|_k.$$

So it remains to show that  $(1) \Longrightarrow (4)$ . Now  $M^{\circ} \subset E$  is obviously absolutely convex and closed. For  $x \in E$  we have  $\sup_{y \in M} |y(x)| = C_x < +\infty$ , hence  $x \in C_x M^{\circ}$ . Therefore  $M^{\circ}$  is absorbant. By Theorem 1.3  $U := M^{\circ}$  is a neighborhood of zero in E. Hence there are  $k \in \mathbb{N}$  and C > 0 so that  $\frac{1}{C}U_k \subset U$ . This implies

$$M \subset M^{\circ \circ} = U^{\circ} \subset CU_k^{\circ}.$$

**2.6 Corollary:** If E is a Fréchet space then E' is a complete locally convex space which has a countable fundamental system of bounded sets.

A set  $\mathscr{B}$  of bounded sets in a locally convex space E is called a *fundamental* system of bounded sets if for every bounded set  $B \subset E$  there is  $B_0 \in \mathscr{B}$  and C > 0 so that  $B \subset CB_0$ .

A subset  $M \subset E$ , E locally convex, is called *bornivorous* if for every bounded set  $B \subset E$  there is t > 0 so that  $B \subset tM$ .

**Definition:** E is called bornological if every absolutely convex bornivorous set is a neighborhood of zero.

Remark: For a locally convex space the following are equivalent

- (1) E is bornological
- (2) for every locally convex space G and every linear map  $A: E \longrightarrow G$  which maps bounded sets into bounded sets, A is continuous
- (3) for every Banach space G and every linear map  $A: E \longrightarrow G$  which maps bounded sets into bounded sets, A is continuous.

Proof: (1)  $\implies$  (2) If U is an absolutely convex neighborhood of zero in G and A maps bounded sets into bounded sets, then  $A^{-1}U$  is locally convex and bornivorous. Hence it is a neighborhood of zero.

 $(2) \Longrightarrow (3)$  is obvious.

(3)  $\Longrightarrow$  (1) If  $M \subset E$  is absolutely convex and bornivorous, in particular absorbant, the space  $E_M = (E/\ker || ||_M, || ||_M)^{\wedge}$  is a Banach space. The quotient map  $\varphi \colon E \longrightarrow E_M$  maps bounded sets into bounded sets so, by assumption, it is continuous. Therefore  $M \supset \{x \mid ||\varphi x||_M < 1\}$  is a neighborhood of zero.  $\Box$ 

**Example:** By Theorem 2.2 every metrizable locally convex space is bornological. The dual  $E' = \bigcup_k E_k^*$  of a Fréchet space E is bornological if the following holds:

A linear map  $A: E' \longrightarrow G$ , G locally convex is continuous if, and only if, the restriction  $A|_{E_k^*}$  is continuous  $E_k^* \longrightarrow G$  for all k.

**2.7 Theorem:** Let *E* be metrizable locally convex. Then the following are equivalent:

- (1) E' is bornological
- (2) E' is barrelled.

Proof:  $(1) \Longrightarrow (2)$  is obvious.

(2)  $\implies$  (1) Let  $M \subset E'$  be absolutely convex and bornivorous. We put  $B_n = U_n^{\circ}$  for all  $n \in \mathbb{N}$ .

For every k there is  $\varepsilon_k > 0$  such that  $2\varepsilon_k B_k \subset M$ . Therefore

$$C_n := \Gamma \bigcup_{k=1}^n \varepsilon_k B_k \subset \frac{1}{2}M$$

for all  $n \in \mathbb{N}$ . Here  $\Gamma$  denotes the absolutely convex hull.

 $C_n$  is weak\*-compact. This is because  $C_n$  is the image of the weak\*-continuous map

$$\{z \in \mathbb{K}^n \mid |z|_{\ell_1} \le 1\} \times B_1 \times \ldots \times B_n \longrightarrow E'$$

given by

$$(z, x_1, \ldots, x_n) \mapsto \sum_{k=1}^n z_k \varepsilon_k x_k.$$

Therefore  $C_n$  is weak\*-closed, hence closed in E'. We set

$$D := \Gamma \bigcup_{k=1}^{\infty} \varepsilon_k B_k = \bigcup_n C_n \subset \frac{1}{2}M.$$

We want to show that  $\overline{D} \subset 2D$ , because if we have shown this, then  $\overline{D} \subset M$ . Clearly  $\overline{D}$  is a barrel (i. e. absolutely convex, closed and absorbant) so, by assumption, a neighborhood of zero, and we are done.

So let  $x_0 \in E \setminus 2D$ . Since  $C_n$  is closed there is an absolutely convex, weak<sup>\*</sup>closed neighborhood of zero  $V_n$  in E' (that is  $V_n = B^\circ$ ,  $B \subset E$  bounded), such that

$$(x_0 + V_n) \cap 2C_n = \emptyset.$$

Therefore

$$(x_0 + V_n + C_n) \cap C_n = \emptyset.$$

Since  $V_n$  is weak\*-closed and  $C_n$  is weak\*-compact, the set  $V_n + C_n$  is weak\*-closed.

Therefore

$$W = \bigcap_{n} (V_n + C_n)$$

is weak\*-closed, hence closed.

W is a barrel. It is clearly absolutely convex, closed by the previous and absorbant by the following:

For  $x \in E'$  we have  $x \in \lambda B_k$  for certain  $\lambda$  and k. Therefore  $x \in \frac{\lambda}{\varepsilon_k} C_n$  for all  $n \geq k$ . There is  $\mu > 0$  such that  $x \in \mu V_n$  for  $n = 1, \ldots, k - 1$ , so  $x \in \max(\frac{\lambda}{\varepsilon_k}, \mu)W$ .

Since E', by assumption, is barrelled W is a neighborhood of zero. We have  $(x_0 + W) \cap C_n = \emptyset$  for all n, therefore  $(x_0 + W) \cap D = \emptyset$  which implies  $x \notin \overline{D}$ .

**Definition:** A Fréchet space is called distinguished if the equivalent conditions of Theorem 2.5 are satisfied.

We refer to [1, page 270] and recall that we have a canonical imbedding J:  $E \hookrightarrow E'$  given by (J(x))[y] = y(x) for  $x \in E$  and  $y \in E'$ . By Theorem 2.5 a fundamental system of seminorms in E'' is given by

$$||z||_k = \sup_{y \in U_k^\circ} |z(y)|, \ k \in \mathbb{N}.$$

Therefore  $||Jx||_k = ||x||_k$  and J imbeds E topologically into E''.

**Definition:** The Fréchet space E is called reflexive if JE = E''.

If there is no danger of confusion we will omit J in future and consider E in a natural way as a subspace of E''.

**2.8 Theorem:** If the Fréchet space *E* is reflexive then it is distinguished.

Proof: We show that E' is barrelled. Let U be a barrel in E'. Then, due to the Bipolar Theorem 2.1,  $U^{\circ\circ} = U$ . Here the first polar  $U^{\circ}$  has to be taken in E''. However, due to reflexivity, we may take it in E. Since  $U \cap E_k^*$  is a barrel in the Banach space  $E_k^*$  it is a neighborhood of zero there, so there is  $\varepsilon_k > 0$  with  $\varepsilon_k U_k^{\circ} \subset U$  and therefore  $U^{\circ} \subset \frac{1}{\varepsilon_k} U_k$ . This holds for all k, so  $U^{\circ}$  is bounded in E and therefore U a neighborhood of zero in E'.

A criterion for reflexivity is:

**2.9 Theorem:** The Fréchet space E is reflexive if, and only if, every bounded set in E is relatively weakly compact.

Proof: Let E be reflexive and  $B \subset E$  bounded. Then  $B \subset B^{\circ\circ}$  and  $B^{\circ}$  is a neighborhood of zero in E'. So, due to Theorem 2.4,  $(B^{\circ})^{\circ} \subset E''$  is  $\sigma(E'', E')$ -compact. Due to reflexivity, we may take  $B^{\circ\circ} \subset E$  and it is  $\sigma(E, E') = \sigma(E'', E')$ -compact.

To prove the reverse take  $z \in E''$ , then there is a neighborhood of zero  $U \subset E'$  so that  $z \in U^{\circ}$ . We may assume  $U = B^{\circ}$  where  $B \subset E$  is bounded, absolutely convex and closed, hence, due to  $B^{\circ\circ} = B$ , weakly closed. So B is  $\sigma(E, E')$ -compact.

Now, considering polars (except  $B^{\circ}$ ) with respect to the duality E', E'' we have

$$U^{\circ} = (B^{\circ})^{\circ} = \overline{\mathrm{J}B}^{\sigma(E^{\prime\prime},E^{\prime})} = \mathrm{J}B.$$

This is because J is a topological imbedding, also with respect to the topologies  $\sigma(E, E')$  and  $\sigma(E'', E')$ , hence JB is  $\sigma(E'', E')$ -compact, and therefore closed.

**2.10 Corollary:** If E is a reflexive Fréchet space and  $F \subset E$  a closed subspace, then F is reflexive.

Proof: Due to Theorem 2.1, which implies  $F^{\circ\circ} = F$ , the space F is weakly closed. If  $B \subset F$  is bounded, then  $M := \overline{B}^{\sigma(E, E')} \subset F$ . Because of Theorem 2.9 M is  $\sigma(E, E')$ - compact and therefore, due to the Hahn - Banach theorem, also  $\sigma(F, F')$ -compact.

**Example:** Let  $E_k, k \in \mathbb{N}$ , be Banach spaces and  $E = \prod_k E_k$ . It is easily verified, that  $E' = \bigoplus_k E'_k$ , equipped with the seminorms  $p(x) = \sum_{k=1}^{\infty} c_k ||x_k||_k^*$  for  $x = \bigoplus_{k=1}^{\infty} x_k$ , where  $(c_k)_k$  runs through all positive sequences.

Again, it is easily verified that  $E'' = \prod_k E''_k$ . Therefore E is reflexive if, and only if,  $E_k$  is reflexive for all k.

**2.11 Lemma:** If the Fréchet space E has a fundamental system of seminorms such that all local Banach spaces  $E_k$  are reflexive, then E is reflexive.

*Proof:* E can be considered as a closed subspace of  $\prod_k E_k$  and, due to the previous,  $\prod_k E_k$  is reflexive. So Corollary 2.10 gives the result.  $\Box$ 

**Definition:** The Fréchet space E is called *Fréchet-Hilbert space* if there exists a fundamental system of seminorms, such that for all k and  $x \in E$  we have  $||x||_k^2 = \langle x, x \rangle_k$  where  $\langle \cdot, \cdot \rangle_k$  is a semiscalar product on E.

**Remark:** In this case all local Banach spaces  $E_k$  are Hilbert spaces.

**2.12 Theorem:** A Fréchet-Hilbert space is reflexive hence distinguished.

**2.13 Lemma:** If E is a Fréchet-Hilbert space,  $F \subset E$  a closed subspace, then F and E/F are Fréchet-Hilbert spaces.

*Proof:* This is easily verified, since subspaces and quotient spaces of Hilbert spaces are Hilbert spaces.  $\Box$ 

**2.14 Corollary:** If E is a Fréchet-Hilbert space,  $F \subset E$  a closed subspace, then F and E/F are reflexive, hence distinguished.

#### **3** Schwartz spaces and nuclear spaces

Let X be a linear space over  $\mathbb{K}$  and V, U absolutely convex. We use the following notation:  $V \prec U$  if there is t > 0 so that  $V \subset tU$ .

If  $V \prec U$  then V is called U-precompact if for every  $\varepsilon > 0$  there are finitely many elements  $x_1, \ldots, x_m$  such that  $V \subset \bigcup_{i=1}^m (x_i + \varepsilon U)$ .

If we denote by  $E_V$ ,  $E_U$  the local Banach spaces with respect to  $|| ||_V$ ,  $|| ||_U$ , respectively, that is

$$E_V = (E/\ker \| \|_V, \| \|_V)^{\wedge}, \quad E_U = (E/\ker \| \|_U, \| \|_U)^{\wedge},$$

and denote by  $i_V^U: E_V \longrightarrow E_U$  the canonical map, then V is U-precompact if, and only i,  $i_V^U$  is compact.

**Definition:** A locally convex space E is called *Schwartz space* if for every absolutely convex neighborhood of zero U there is an absolutely convex neighborhood of zero  $V \prec U$ , which is U-precompact.

By use of the previous remarks we can state:

**3.1 Proposition:** The locally convex space E is a Schwartz space if, and only if, for every absolutely convex neighborhood of zero U there is an absolutely convex neighborhood of zero  $V \subset U$ , so that the canonical map  $i_V^U: E_V \longrightarrow E_U$  is compact.

**3.2 Lemma:** In a complete Schwartz space *E* all bounded sets are relatively compact.

Proof: Let  $\mathscr{U}$  be the set of all absolutely convex neighborhoods of zero in E. For  $U \in \mathscr{U}$  let  $E_U$  be the local Banach space,  $i^U : E \longrightarrow E_U$  the quotient map. Then

$$i: E \longrightarrow \prod_{U \in \mathscr{U}} E_U, \ x \mapsto (i^U x)_{U \in \mathscr{U}}$$

is a topological imbedding. Since E is complete, iE is closed in  $\prod_{U \in \mathscr{U}} E_U$ .

Let  $B \subset E$  be bounded. For  $U \in \mathscr{U}$  we find  $V \in \mathscr{U}$ , so that  $\imath_V^U \colon E_V \longrightarrow E_U$  is compact. Hence  $\imath^U B = \imath_V^U(\imath^V B)$  is relatively compact in  $E_U$ . By Tychonoff's theorem  $\imath B$  is relatively compact in  $\prod_{U \in \mathscr{U}} E_U$  and therefore in  $\imath E$ .  $\Box$ 

This yields immediately:

**3.3 Theorem:** Every Fréchet-Schwartz space is reflexive, hence distinguished.

*Proof:* The bounded sets are relatively compact, hence relatively weakly compact, so by 2.9 the space is reflexive and by 2.8 it is distinguished.  $\Box$ 

We will now try to understand better the structure of compact sets in a Fréchet space. In the following  $\Gamma$  denotes the absolutely convex hull of a set.

**3.4 Lemma:** If E is locally convex and complete and  $(x_k)_{k \in \mathbb{N}}$  a weak null sequence in E, then

$$\overline{\Gamma\{x_k \mid k \in \mathbb{N}\}} = \left\{ \sum_{k=1}^{\infty} \xi_k x_k \mid \sum_{k=1}^{\infty} |\xi_k| \le 1 \right\}$$

and this set is weakly compact.

*Proof:* We define a continuous linear map  $\varphi \colon \ell_1 \longrightarrow E$  by

$$\varphi(\xi) := \sum_{j} \xi_j x_j, \quad \xi = (\xi_j)_{j \in \mathbb{N}} \in \ell_1.$$

Existence follows from the completeness and continuity from the boundedness of  $\{x_k \mid k \in \mathbb{N}\}$  and the triangle inequality.

For  $y \in E'$  and  $\xi \in \ell_1$  we have

$$y(\varphi(\xi)) = \sum_{j} \xi_j y(x_j)$$

and  $(y(x_j))_{j\in\mathbb{N}}$  is, by assumption, a null sequence. This shows that

$$\varphi \colon (\ell_1, \, \sigma(\ell_1, \, c_0)) \longrightarrow (E, \, \sigma(E, \, E'))$$

is continuous. Since, by Theorem 2.4, the closed unit ball  $U_{\ell_1}$  of  $\ell_1$  is  $\sigma(\ell_1, c_0)$ -compact, the set

$$\varphi(U_{\ell_1}) = \left\{ \sum_k \xi_k x_k \mid \sum_k |\xi_k| \le 1 \right\}$$

is weakly compact, hence weakly closed, hence closed in E. Since  $\Gamma\{x_k \mid k \in \mathbb{N}\} \subset \varphi(U_{\ell_1})$  this implies

$$\overline{\Gamma\{x_k \mid k \in \mathbb{N}\}} \subset \left\{ \sum_{k=1}^{\infty} \xi_k x_k \mid \sum_{k=1}^{\infty} |\xi_k| \le 1 \right\}.$$

The reverse inclusion is obvious.

**3.5 Corollary:** If E is locally convex and complete and  $(x_k)_{k \in \mathbb{N}}$  is a null sequence then

$$\overline{\Gamma\{x_k \mid k \in \mathbb{N}\}} = \left\{ \sum_{k=1}^{\infty} \xi_k x_k \mid \sum_{k=1}^{\infty} |\xi_k| \le 1 \right\}$$

is compact.

*Proof:* We assume first that E is a Banach space. We consider the map  $\varphi \colon \ell_1 \longrightarrow E$  defined by

$$\varphi(\xi) = \sum_{k=1}^{\infty} \xi_k x_k, \quad \xi = (\xi_k)_k \in \ell_1$$

and for  $n \in \mathbb{N}$ 

$$\varphi_n(\xi) = \sum_{k=1}^n \xi_k x_k, \quad \xi = (\xi_k)_k \in \ell_1$$

Obviously

$$\|\varphi - \varphi_n\| \le \sup_{k>n} \|x_k\| \xrightarrow[n \to \infty]{} 0.$$

Therefore  $\varphi$  is compact, hence  $\varphi(\{\xi \mid ||\xi||_{\ell_1} \leq 1\})$  relatively compact and, because it is closed by Lemma 3.4, compact.

In the general case we conclude from the previous that, in the notation of the proof of Lemma 3.2,  $i \circ \varphi(U_{\ell_1}) \subset \prod_{U \in \mathscr{U}} E_U$  is compact. Since, due to the completeness of E, the space i(E) is closed and i is an isomorphism onto its range, we obtain the result.  $\Box$ 

**3.6 Lemma:** If E is a Fréchet space and  $K \subset E$  compact, then there is a null sequence  $(x_k)_{k \in \mathbb{N}}$  so that

$$K \subset \left\{ \sum_{k=1}^{\infty} \xi_k x_k \mid \sum_{k=1}^{\infty} |\xi_k| \le 1 \right\}.$$

Proof: Let  $U_1 \supset U_2 \supset \ldots$  be a basis of closed absolutely convex neighborhoods of zero. We set up an inductive procedure and put  $K_1 = K$ . If the compact set  $K_k$  is determined, we find  $x_1^{(k)}, \ldots, x_{m_k}^{(k)} \in K_k$ , such that

$$K_k \subset \bigcup_{j=1}^{m_k} \left( x_j^{(k)} + \frac{1}{4^{k+1}} U_k \right)$$

and put

$$K_{k+1} = \bigcup_{j=1}^{m_k} \left( K_k - x_j^{(k)} \right) \cap \frac{1}{4^{k+1}} U_k.$$

Since  $x_j^{(k)} \in 4^{-k}U_{k-1}$  for  $k > 1, j = 1 \dots m_k$ , the sequence

$$(2x_1^{(1)}, \dots, 2x_{m_1}^{(1)}, 4x_1^{(2)}, \dots, 4x_{m_2}^{(2)}, \dots, 2^k x_1^{(k)}, \dots, 2^k x_{m_k}^{(k)}, \dots)$$

is a null sequence.

For  $x \in K$  we find  $x_{j_1}^{(1)}$  so that

$$x = x_{j_1}^{(1)} + u_1, \quad u_1 \in \frac{1}{4^2} U_1.$$

Therefore

$$u_1 = x - x_{j_1}^{(1)} \in \left(K - x_{j_1}^{(1)}\right) \cap \frac{1}{4^2} U_1 \subset K_2.$$

Proceeding inductively we find a sequence  $j_k, j_k \in \{1, \ldots, m_k\}$ , such that

$$x = \sum_{k=1}^{\infty} x_{j_k}^{(k)} = \sum_{k=1}^{\infty} \frac{1}{2^k} (2^k x_{j_k}^{(k)}).$$

In particular  $x \in \{\sum_{k=1}^{\infty} \xi_k x_k \mid \sum_{k=1}^{\infty} |\xi_k| \le 1\}.$ 

As an immediate consequence of the previous we obtain

**3.7 Lemma:** If E is a Fréchet space,  $K \subset E$  compact, then there is an absolutely convex compact set  $L \subset E$ , so that,  $K \subset L$  and K is compact in the Banach space  $E_L$ .

*Proof:* We choose a null sequence  $(x_k)_{k \in \mathbb{N}}$  such that

$$K \subset \left\{ \sum_{k=1}^{\infty} \xi_k x_k \mid \sum_k |\xi_k| \le 1 \right\} =: K_1$$

and a sequence  $1 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ ,  $\lambda_k \to \infty$  such that  $\lambda_k x_k \to 0$  (cf. the proof of 2.2). We put

$$L := \left\{ \sum_{k=1}^{\infty} \xi_k \lambda_k x_k \mid \sum_k |\xi_k| \le 1 \right\}.$$

Then, due to Corollary 3.5, L is absolutely convex and compact in E, clearly  $K_1 \subset L$ . Since  $||x_k||_L \leq \frac{1}{\lambda_k}$  we have  $x_k \to 0$  in  $E_L$ .

Therefore  $K_1$  is compact in  $E_L$ , hence K relatively compact in  $E_L$ . Since K is closed in E and therefore in  $E_L$ , K is compact in  $E_L$ .

**3.8 Theorem:** If E is a Fréchet-Schwartz space then E' is a Schwartz space.

Proof: Let  $U = \overset{\circ}{K}$  be a neighborhood of zero in E'. K may be assumed absolutely convex and, due to 3.2, compact. We choose L according to Lemma 3.7 and put  $V = \overset{\circ}{L}$ . Since  $E_K \hookrightarrow E_L$  is compact also the canonical map  $E'_V \longrightarrow E'_U$  is compact.  $\Box$ 

The above conclusion comes from the following consequence of the Arzelá-Ascoli Theorem.

**3.9 Theorem (Schauder):** Let X, Y be Banach spaces and  $A: X \longrightarrow Y$  continuous and linear. Then A is compact if and only if A' is compact.

**Definition:** A locally convex space E is *nuclear* if, and only if, for every absolutely convex neighborhood of zero U there is a neighborhood of zero V and a finite positive Radon measure on the weak\*-compact set  $V^{\circ}$ , such that

$$||x||_U \le \int_{V^\circ} |y(x)| d\mu(y)$$

for all  $x \in E$ .

**Example:** Let  $\Omega \subset \mathbb{R}^n$  be open,  $\omega_1 \subset \omega_2 \subset \ldots$  an exhaustion by open sets. Then  $\mathscr{C}^{\infty}(\Omega)$  is a Fréchet space with the seminorms

$$||f||_k = \sup_{\substack{x \in \omega_k \\ |\alpha| \le k}} |f^{(\alpha)}(x)|.$$

To show that  $\mathscr{C}^{\infty}(\Omega)$  is nuclear we notice that for every  $y \in \mathscr{D}(\mathbb{R}^n)$  we have

$$\sup_{x \in \mathbb{R}^n} |g(x)| \le \int_{\mathbb{R}^n} |g^{(\vec{1})}(x)| dx$$

where  $\vec{1} = (1, ..., 1)$ .

For any k we choose  $\varphi \in \mathscr{D}(\omega_{k+1}), \varphi \equiv 1$  on  $\omega_k$  and obtain

$$\begin{split} \|f\|_{k} &\leq \sup_{|\alpha| \leq k} \int_{\omega_{k+1}} |(\varphi f)^{(\alpha+\vec{1})}(x)| dx \\ &\leq \sum_{|\alpha| \leq k} \int_{\omega_{k+1}} |(\varphi f)^{(\alpha+\vec{1})}(x)| dx \\ &\leq \sum_{|\beta| \leq k+n} \int_{\omega_{k+n}} |f^{(\beta)}(x)| h_{\beta}(x) dx \end{split}$$

with suitable nonnegative  $h_{\beta} \in \mathscr{C}(\mathbb{R}^n)$  with compact support in  $\omega_{n+1}$ . We set

$$M_{\beta} = \{ \, \delta_x^{(\beta)} \mid x \in \overline{\omega}_{k+n} \, \}$$

and

$$M = \bigcup_{|\beta| \le k+n} M_{\beta} \subset U_{k+n}^{\circ}$$

and define a measure on  $U_{k+n}^{\circ}$  by

$$\mu \varphi = \sum_{|\beta| \le k+n} \int_{\omega_{k+n}} \varphi(\delta_x^{(\beta)}) h_\beta(x) dx$$

for  $\varphi \in \mathscr{C}(U_{k+n}^{\circ})$ . Since the summands define measures on the disjoint compact sets  $M_{\beta}$  we may rewrite the above estimate for  $f \in C^{\infty}(\Omega)$  as

$$||f||_k \le \int_{U_{k+n}^{\circ}} |y(f)| d\mu(y).$$

**3.10 Lemma:** If E is nuclear then it admits a fundamental system of Hilbert seminorms.

*Proof:* Let U be an absolutely convex neighborhood of zero. Then we find V and  $\mu$  according to the definition of nuclearity, V may be assumed absolutely convex. We obtain:

$$\begin{aligned} \|x\|_U &\leq \int_{V^\circ} |y(x)| d\mu(x) \\ &\leq \mu(V^\circ)^{\frac{1}{2}} \left( \int_{V^\circ} |y(x)|^2 d\mu(y) \right)^{\frac{1}{2}} \\ &\leq \mu(V^\circ) \|x\|_V. \end{aligned}$$

Therefore the seminorms

$$p_V(x) := \left( \int_{V^\circ} |y(x)|^2 d\mu(y) \right)^{\frac{1}{2}}$$

which come from the semi-scalar product

$$\langle x_1, x_2 \rangle = \int_{V^\circ} y(x_1) \overline{y(x_2)} d\mu(y)$$

are a fundamental system of Hilbert seminorms.

3.11 Corollary: Every nuclear Fréchet space is a Fréchet-Hilbert space.

**3.12 Lemma:** If X, Y are Hilbert spaces,  $U = \{x \in X \mid ||x|| \le 1\}$ , A:  $X \longrightarrow Y$  linear, such that

$$\|Ax\| \leq \int_U |\langle x,y\rangle| d\mu(y)$$

for a finite positive measure y, then A is a Hilbert-Schmidt operator.

*Proof:* Let  $(e_i)_{i \in I}$  be an orthonormal basis in X and  $M \subset I$  finite. Then

$$\sum_{i \in M} ||Ae_i||^2 \leq \sum_{i \in M} \left( \int_U |\langle e_i, y \rangle| d\mu(y) \right)^2$$
$$\leq \mu(U) \sum_{i \in M} \int_U |\langle e_i, y \rangle|^2 d\mu(y)$$
$$\leq \mu(U)^2.$$

Therefore

$$\sum_{i \in I} \|Ae_i\|^2 < \infty$$

and A is Hilbert-Schmidt.

**3.13 Corollary:** Every nuclear space is a Schwartz space.

**3.14 Theorem:** If E is nuclear and  $F \subset E$  a closed subspace, then F and E/F are nuclear.

Proof: Let U be an absolutely convex neighborhood of zero in E. We choose V and  $\mu$  according to the definition of nuclearity. To show the nuclearity of F we consider the quotient map  $q: E' \longrightarrow E'/F^{\circ} = F'$  and set  $W^{\circ} = q V^{\circ}$ . We notice that it corresponds to  $(V \cap F)^{\circ}$  in F'. By

$$\int_{W^\circ} f(y) d\nu(y) := \int_{V^\circ} f(q(y)) d\mu(y)$$

for  $f \in \mathscr{C}(W^{\circ})$  we get a positive finite measure  $\nu$  on  $W^{\circ}$ , such that for  $x \in F$ 

$$\|x\|_U \le \int_{V^\circ} |y(x)| d\mu(y) = \int_{W^\circ} |\eta(x)| d\nu(\eta).$$

To show the nuclearity of E/F we choose V as given by a Hilbert seminorm. Then  $E_V$  is a Hilbert space. We notice that the inequality in the definition of nuclearity extends to

$$\|\iota_V^U x\|_U \le \int_{\widehat{V}} |\langle x, y \rangle| d\mu(y)$$

for all  $x \in E_V$  where  $\widehat{V}$  is the unit ball of  $E_V$  and  $\langle \cdot, \cdot \rangle$  the scalar product of  $E_V$ . Let  $\pi$  be the orthogonal projection onto  $(i^V F)^{\perp} \subset E_V$ . For  $x \in E$  we set  $\widehat{x} = x + F$ . By  $\| \|_U$  we denote the quotient seminorm on E/F of  $\| \|_U$ . We obtain

$$\|\hat{x}\|_{U}^{\widetilde{}} \leq \|\imath_{V}^{U}\pi\imath^{V}x\|_{U} \leq \int_{\widehat{V}} |\langle \pi\imath^{V}x,y\rangle| d\mu(y) = \int_{\widehat{V}} |\langle\imath^{V}x,\pi y\rangle| d\mu(y).$$

The first inequality holds because  $i^V x - \pi i^V x \in \overline{i^V F}^{\parallel \parallel_V}$ , and therefore  $i^U x - i^U_V \pi i^V x \in \overline{i^U F}^{\parallel \parallel_U}$ . Clearly the last integral depends only on  $\hat{x}$ .

For  $\varphi \in \mathscr{C}(\widehat{V} \cap (\imath^V F)^{\perp})$  we set

$$\nu(\varphi) := \int_{\widehat{V}} \varphi(\pi y) d\mu(y).$$

Then  $\nu$  is a positive finite measure on  $\widehat{V} \cap (i^V F)^{\perp}$ . The elements in this set are the Riesz representations under  $\langle \cdot, \cdot \rangle$  of the elements in  $V^{\circ} \cap F^{\circ} = Q(V)^{\circ}$ ,  $Q: E \longrightarrow E/F$  the quotient map. We have

$$\|\hat{x}\|_U^{\sim} \leq \int_{Q(V)^{\circ}} |y(\hat{x})| d\nu(y).$$

**Example:** Since  $\mathscr{C}^{\infty}(\Omega)$  is nuclear for any open  $\Omega \subset \mathbb{R}^n$ , so is every closed subspace. Therefore the space  $h(\Omega)$  of harmonic functions with the compact open topology (which coincides with the topology inherited from  $\mathscr{C}^{\infty}(\Omega)$ ) is nuclear.

**Definition:** Let X, Y be Banach spaces,  $A: X \longrightarrow Y$  a linear map. Then A is called nuclear if there are sequences  $\xi_j \in X'$ ,  $\eta_j \in Y$  such that

- (1)  $\sum_{j} \|\xi_{j}\|^{*} \|\eta_{j}\| < +\infty$
- (2)  $Ax = \sum_{j} \xi_j(x) \eta_j$  for all  $x \in E$ .

Clearly a nuclear map is continuous and, as a norm limit of finite dimensional maps, compact. We recall (see e.g. [1, Lemma 16.21]):

**3.15 Theorem:** If X, Y, Z are Hilbert spaces,  $A: X \longrightarrow Y$ ,  $B: Y \longrightarrow Z$ Hilbert-Schmidt operators then BA is nuclear.

We obtain the following characterization of nuclear spaces.

- **3.16 Theorem:** For a locally convex space E the following are equivalent:
  - (1) E is nuclear.
  - (2) E has a fundamental system  $\mathscr{U}$  of neighborhoods of zero given by Hilbert seminorms and for every  $U \in \mathscr{U}$  there is  $V \in \mathscr{U}, V \subset U$ , such that the canonical map  $E_V \longrightarrow E_U$  is Hilbert-Schmidt.
  - (3) For every absolutely convex neighborhood of zero U there is an absolutely convex neighborhood of zero  $V \subset U$ , such that the canonical map  $E_V \longrightarrow E_U$  is nuclear.

Proof:  $(1) \Longrightarrow (2)$  follows from 3.10 and 3.12.

(2)  $\implies$  (3) It is obviously sufficient to show (3) for  $U \in \mathscr{U}$ . We choose  $V \in \mathscr{U}$ , so that  $E_V \longrightarrow E_U$  is Hilbert-Schmidt and then  $W \in \mathscr{U}$  so that  $E_W \longrightarrow E_V$  is Hilbert-Schmidt. Due to 3.15  $E_W \longrightarrow E_U$  is nuclear.

(3)  $\Longrightarrow$  (1) Let  $i_V^U : E_V \longrightarrow E_U$  be nuclear, i.e.

$$\imath_V^U x = \sum_{j=1}^\infty \xi_j(x) \eta_j$$

with  $\|\xi_j\|_V^* \leq 1$  for all j and  $\sum_j \|\eta_j\|_U < +\infty$ . We define a measure on  $V^\circ$  by setting for  $\varphi \in \mathscr{C}(V^\circ)$ 

$$\mu(\varphi) = \sum_{j=1}^{\infty} \varphi(\xi_j) \|\eta_j\|_U.$$

We obtain

$$||x||_U \le \sum_{j=1}^{\infty} |\xi_j(x)| ||y_j||_U = \int_{V^\circ} |y(x)| d\mu(y).$$

Therefore (3) implies that E is nuclear.

For a further description of nuclear Fréchet spaces we need the following lemma.

**3.17 Lemma:** A locally convex space E is nuclear if, and only if, for every neighborhood of zero U there exists a sequence  $(y_j)_{j \in \mathbb{N}}$  in E' with following properties:

- (1)  $U^{\circ} \subset \overline{\Gamma\{y_j \mid j \in \mathbb{N}\}}$
- (2) For every k there is a neighborhood of zero  $U_k \subset E$ , such that the set  $\{j^k y_j \mid j \in \mathbb{N}\}$  is contained in  $U_k^{\circ}$ .

*Proof:* We first assume that E is nuclear and we may assume that U is an absolutely convex, closed Hilbert disk. We find inductively a sequence of absolutely convex, closed neighborhoods of zero such that

- (a)  $U_0 = U$
- (b)  $U_k$  is a Hilbert disk
- (c)  $\iota_{U_{k+1}}^{U_k} \colon U_{u_{k+1}} \longrightarrow U_{u_k}$  is nuclear.

We set  $E_k := E_{U_k}$ ,  $i_{k+1}^k := i_{U_{k+1}}^{U_k}$  and  $j_k^{k+1} := (i_{k+1}^k)' : E_k^* \longrightarrow E_{k+1}^*$ . Then also maps  $j_k^{k+1}$  are nuclear, i.e. in the Schatten 1-class  $S_1(E_k^*, E_{k+1}^*)$  (see [1, Lemma 16.7, (2.)]). So for the singular numbers  $s_n(j_k^{k+1})$  we have

$$\sum_{n} s_n(j_k^{k+1}) < +\infty.$$

Since the sequence of singular numbers is decreasing we have

$$\lim_{n \to \infty} n s_n(j_k^{k+1}) = 0$$

for all k. From [1, Lemma 16.6, (7.)] we conclude that for  $j_k := \imath_k^{0'} = j_{k-1}^k \circ \ldots \circ j_0^1 \colon E_0^* \longrightarrow E_k^*$  we obtain

$$\lim_{n \to \infty} n^k s_n(j_k) = 0.$$

Let

$$j_k(x) = \sum_{n=0}^{\infty} s_n(j_k) \langle x, e_n^k \rangle_0 f_n^k$$

be the Schmidt-representation of  $j_k$ ,  $(e_n^k)_{n\in\mathbb{N}}$  and  $(f_n^k)_{n\in\mathbb{N}}$  orthonormal systems in  $E_0^*$  and  $E_k^*$  respectively.  $(e_n^k)_n$  is complete since  $j_k$  is injective. We arrange the double indexed sequence  $(e_n^k)_{n\in\mathbb{N}}$  to a sequence  $(\tilde{e}_n)_{n\in\mathbb{N}}$  in the following way

$$(\tilde{e}_0, \tilde{e}_1, \ldots) = (e_0^0, e_0^1, e_1^0, e_0^2, e_1^1, e_2^0, e_0^3, e_1^2, e_2^1, e_3^0, e_0^4, \ldots)$$

We apply the Gram-Schmidt orthogonalization procedure to the sequence  $(\tilde{e}_n)_{n \in \mathbb{N}}$ , throwing away linearly dependent vectors in the order they appear, and obtain an orthonormal basis  $(e_n)_{n \in \mathbb{N}_0}$  of  $E_0^*$ .

Obviously we have  $e_m \perp e_n^k$  for

$$m > 1 + 2 + \ldots + (k + n) = \frac{1}{2}(k + n)(k + n + 1).$$
 (1)

We fix k. Then  $\langle e_m, e_n^k \rangle_0 = 0$  for  $n < \sqrt{2m} - k - 1$ , therefore for  $n < [\sqrt{m}]$  if m is large enough.

So we obtain for m large enough

$$\begin{aligned} \|e_{m}\|_{k}^{*2} &= \sum_{n \ge [\sqrt{m}]} s_{n}^{2}(j_{k}) |\langle e_{m}, e_{n}^{k} \rangle_{0}|^{2} \\ &\leq s_{[\sqrt{m}]}^{2}(j_{k}) \sum_{n} |\langle e_{m}, e_{n}^{k} \rangle_{0}|^{2} \\ &= s_{[\sqrt{m}]}^{2}(j_{k}). \end{aligned}$$

Consequently

$$\limsup_{m} \|m^{k} e_{m}\|_{2k}^{*} \leq \lim_{m} [\sqrt{m}]^{2k} s_{[\sqrt{m}]}(j_{2k}) = 0.$$

This means that for suitable  $C_k > 0$ 

$$\{m^k e_m \mid m \in \mathbb{N}_0\} \subset C_k U_{2k}^\circ.$$

For  $y \in U^{\circ}$  we have

$$y = \sum_{m=0}^{\infty} \langle y, e_m \rangle_0 \, e_m$$

where the series converges in  $E_0^*$ , hence in E'.

We put  $y_m = \lambda(m+1)e_m$ ,  $\lambda^2 = \sum_{m=0}^{\infty} \frac{1}{(m+1)^2}$  and  $\xi_m = \frac{1}{\lambda(m+1)} \langle y, e_m \rangle_0$ . Then for  $y \in U^\circ$  we have

$$y = \sum_{m=0}^{\infty} \xi_m y_m$$

with

$$\sum_{m=0}^{\infty} |\xi_m| = \sum_{m=0}^{\infty} \frac{1}{\lambda(m+1)} |\langle y, e_m \rangle_0| \le 1.$$

and

$$m^k y_m \in D_k U_{2k+2}^{\circ}$$

for all m and k with suitable  $D_k$ .

For the reverse direction we choose  $(y_j)_{j\in\mathbb{N}}$  according to the assumption, and obtain

$$\|x\|_U = \sup_{y \in U^\circ} |y(x)| \le \sup_{j \in \mathbb{N}} |y_j(x)|$$
$$\le \sum_{j \in \mathbb{N}} |y_j(x)| = \sum_{j \in \mathbb{N}} j^{-2} |(j^2 y_j(x))|$$

We put  $V = U_2$ , then  $j^2 y_j \in V^\circ$  for all j. We define a measure on  $V^\circ$  by setting

$$\mu(\varphi) = \sum_{j \in \mathbb{N}} j^{-2} \varphi(j^2 y_j)$$

for  $\varphi \in \mathscr{C}(V^{\circ})$ , and obtain

$$\|x\|_U \le \int_{V^\circ} |y(x)| d\mu(y).$$

**Remark:** The previous argument shows that the following is sufficient for the nuclearity of E: for every U there are V, a sequence  $(y_j)_{j \in \mathbb{N}}$  in E' and a positive sequence  $(\lambda_j)_{j \in \mathbb{N}}$  with  $\sum_j \frac{1}{\lambda_j} < \infty$ , such that  $U^\circ \subset \overline{\Gamma\{y_j \mid j \in \mathbb{N}\}}$ and  $\lambda_j y_j \in V^\circ$  for all j.

**Definition:** A sequence  $(y_j)_{j \in \mathbb{N}}$  in E' is called rapidly decreasing if for every k there is a neighborhood of zero U so that  $\{j^k y_j \mid j \in \mathbb{N}\} \subset U^\circ$ .

**Remark:** From Theorem 2.5 we easily derive that a sequence  $(y_j)_{j\in\mathbb{N}}$  in the dual space E' of a Fréchet space E is rapidly decreasing if, and only if,  $j^k y_j \longrightarrow 0$  for all k.

If E is locally convex then we may define a locally convex topology  $t_N$  as given by the seminorms  $p_y(x) = \sup_j |y_j(x)|$  where y runs through the rapidly decreasing sequences in E'. It is easily seen to be nuclear and it is the finest nuclear locally convex topology on E coarser than the given topology. It is called the *associated nuclear topology* and Lemma 3.17 says that E is nuclear if and only if its topology coincides with the associated nuclear topology.

We use Lemma 3.17 to show that every nuclear Fréchet space is isomorphic to a subspace of  $s^{\mathbb{N}}$  or likewise of  $\mathscr{C}^{\infty}(\mathbb{R})$ . So these spaces are universal spaces for the nuclear Fréchet space, which had been conjectured by Grothendieck.

#### **Definition:** We set

$$s = \{ \xi = (\xi_j)_{j \in \mathbb{N}} \mid |\xi|_k^2 = \sum_j j^{2k} |\xi_j|^2 < \infty \text{ for all } k \}$$
$$= \{ \xi = (\xi_j)_{j \in \mathbb{N}} \mid \|\xi\|_k = \sup_j j^k |\xi_j| < \infty \text{ for all } k \}.$$

The norm systems  $(| |_k)_{k \in \mathbb{N}_0}$  and  $(|| ||_k)_{k \in \mathbb{N}_0}$  are easily seen to be equivalent. s equipped with the topology given by these systems is a nuclear space. This is because the local Banach space for  $| |_k$  is

$$s_k = \{ \xi = (\xi_j)_{j \in \mathbb{N}} \mid |\xi|_k^2 = \sum_j j^{2k} |\xi_j|^2 < \infty \}$$

and the canonical map  $s_{k+1} \longrightarrow s_k$  is the identical imbedding which is Hilbert-Schmidt, since

$$i_{k+1}^{k}x = x = \sum_{n=0}^{\infty} \frac{1}{n+1} \langle x, (n+1)^{-k-1} e_{n+1} \rangle_{k+1} \left( (n+1)^{-k} e_{n+1} \right)$$

is the Schmidt representation where  $e_n := (0, \ldots, 0, 1, 0, \ldots)$  and  $\langle , \rangle_{k+1}$  is the scalar product for  $||_{k+1}$ .

We obtain the following theorem, which solves a conjecture of Grothendieck:

**3.18 Theorem (T. and Y. Komura):** A Fréchet space E is nuclear if, and only if, it is isomorphic to a subspace of  $s^{\mathbb{N}}$ .

*Proof:* Since s is nuclear, so is  $s^{\mathbb{N}}$  and therefore every subspace. This proves one implication.

For the reverse implication we choose a fundamental system of seminorms  $\| \|_1 \leq \| \|_2 \leq \ldots$  and for  $U_k = \{ x \in E \mid \|x\|_k \leq 1 \}$  a rapidly decreasing sequence  $y^{(k)}$  in E', so that

$$U_k^{\circ} \subset \overline{\Gamma\{y_j^{(k)} \mid j \in \mathbb{N}\}}, \quad \{j^{\nu} y_j^{(k)} \mid j \in \mathbb{N}\} \subset C_{k,\nu} U_{\mu(k,\nu)}^{\circ}.$$

We define a map  $\varphi_k \colon E \longrightarrow s$  by

$$\varphi_k(x) = (y_j^{(k)}(x))_{j \in \mathbb{N}}.$$

We have to explain that this is a continuous map into s.

$$\begin{aligned} |\varphi_k(x)|_m &= (\sum_j j^{2m} |y_j^{(k)}(x)|^2)^{\frac{1}{2}} \\ &\leq (\sum_j j^{-2})^{\frac{1}{2}} \sup_j j^{m+1} |y_j^{(k)}(x)| \\ &\leq (\sum_j j^{-2})^{\frac{1}{2}} C_{k,m+1} ||x||_{\mu(k,m+1)} \end{aligned}$$

Moreover we have

$$||x||_k \le \sup_j |y_j^{(k)}(x)| \le |\varphi_j(x)|_0$$

From these estimates we conclude that

$$\varphi(x) := (\varphi_k(x))_{k \in \mathbb{N}}$$

defines a topological imbedding into  $s^{\mathbb{N}}$ .

We shall prove that  $s^{\mathbb{N}} \cong \mathscr{C}^{\infty}(\Omega)$  for every open  $\Omega \subset \mathbb{R}^n$ . For that we need the following lemmas.

**3.19 Lemma:** If X is complemented subspace of  $s^{\mathbb{N}}$  then  $X \times s^{\mathbb{N}} \cong s^{\mathbb{N}}$ .

*Proof:* Let  $X \times Y \cong s^{\mathbb{N}}$  then we have

$$s^{\mathbb{N}} \cong (s^{\mathbb{N}})^{\mathbb{N}} \cong (X \times Y)^{\mathbb{N}} \cong X \times X^{\mathbb{N}} \times Y^{\mathbb{N}} \cong X \times (X \times Y)^{\mathbb{N}} \cong X \times s^{\mathbb{N}}.$$

The isomorphims are self explaining.

**3.20 Lemma:** If *E* is isomorphic to a complemented subspace of  $s^{\mathbb{N}}$  and  $s^{\mathbb{N}}$  is isomorphic to a complemented subspace of *E*, then  $E \cong s^{\mathbb{N}}$ .

Proof: Let X be a complement for  $s^{\mathbb{N}}$  in E. Then  $E \cong X \times s^{\mathbb{N}}$ . Let Y be a complement of E in  $s^{\mathbb{N}}$ , then  $Y \times E \cong s^{\mathbb{N}}$ . Therefore  $s^{\mathbb{N}} \cong Y \times E \cong X \times (Y \times s^{\mathbb{N}})$ . So X is isomorphic to a complemented subspace of  $s^{\mathbb{N}}$  and therefore  $E \cong X \times s^{\mathbb{N}} \cong s^{\mathbb{N}}$  by Lemma 3.19.

For  $r = (r_1, \ldots, r_n), r_j > 0$  for all j, we set

$$\mathscr{C}_r^{\infty} = \{ f \in \mathscr{C}^{\infty}(\mathbb{R}^n) \mid f(x+r) = f(x) \text{ for all } x \}.$$

This is a closed subspace of  $\mathscr{C}^{\infty}(\mathbb{R}^n)$  hence a nuclear Fréchet space. For r > 0 we put  $\mathscr{C}^{\infty}_{(r)} = \mathscr{C}^{\infty}_{(r,\dots,r)}$ .

# **3.21 Theorem:** $\mathscr{C}_r^{\infty} \cong s$ .

Proof: Obviously we may assume  $r = (2\pi, \ldots, 2\pi)$ . The topology induced by  $\mathscr{C}^{\infty}(\mathbb{R}^n)$  can be expressed by the seminorms

$$||f||_k = \sup\{ |f^{(\alpha)}(x)| \mid |\alpha| \le k, \ 0 \le x_j \le 2\pi, \ j = 1, \dots, n \}$$

or, equivalently, by the seminorms

$$|f|_k^2 = \frac{1}{(2\pi)^n} \sum_{|\alpha| \le k} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |f^{(\alpha)}(x)|^2 dx_1 \dots dx_n,$$

which, in terms of the Fourier expansion

*(* )

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} c_{\nu} e^{i\nu x}, \quad \nu x = \nu_1 x_1 + \ldots + \nu_n x_n,$$

can be expressed as

$$|f|_k^2 = \sum_{|\alpha| \le k} \sum_{\nu \in \mathbb{Z}^n} |c_\nu|^2 \nu^{2\alpha}$$
$$= \sum_{\nu \in \mathbb{Z}^n} |c_\nu|^2 (\sum_{|\alpha| \le k} \nu^{2\alpha}).$$

Here  $\nu^{\alpha} = \nu_1^{\alpha_1} \dots \nu_n^{\alpha_n}$ .

Since

$$(1 + \nu_1^2 + \ldots + \nu_n^2)^k = \sum_{|\alpha| \le k} \frac{k!}{\alpha_1! \ldots \alpha_n! (k - |\alpha|)!} \nu^{2\alpha}$$

we obtain for  $\nu \neq 0$ 

$$\sum_{|\alpha| \le k} \nu^{2\alpha} \le (1 + \nu_1^2 + \ldots + \nu_n^2)^k \le (n+1)^k \sup_{|\alpha| \le k} \nu^{2\alpha} \le (n+1)^k \sum_{|\alpha| \le k} \nu^{2\alpha}.$$

Moreover, we have with  $|\nu| = |\nu_1| + \cdots + |\nu_n|$ 

$$(1+|\nu|)^k \le (1+\nu_1^2+\ldots+\nu_n^2)^k \le (1+|\nu|)^{2k}.$$

Therefore norm system  $|f|_k, k \in \mathbb{N}_0$  is equivalent to the norm system

$$\begin{aligned} |f|_{k}^{\sim^{2}} &:= \sum_{\nu \in \mathbb{Z}^{n}} |c_{\nu}|^{2} (1+|\nu|)^{2k} \\ &= \sum_{m=0}^{\infty} \left( \sum_{|\nu|=m} |c_{\nu}|^{2} \right) (1+m)^{2k}, \quad k \in \mathbb{N}_{0} \end{aligned}$$

If  $M_m = \{ \nu \in \mathbb{Z} \mid |\nu| \le m \}$  then we have for all  $m \in \mathbb{N}_0$ 

$$\left(2\left[\frac{m}{n}\right]+1\right)^n \le \#M_m \le (2m+1)^n.$$

We count the  $\nu \in \mathbb{Z}$  with increasing  $|\nu|$ . Let  $\nu \in \mathbb{Z}^n$  carry the number  $\ell(\nu)$ . We obtain for  $|\nu| = m \ge 1$ 

$$\left(2\left[\frac{m-1}{n}\right]+1\right)^n \le \#M_{m-1} < \ell(\nu) \le \#M_m \le (2m+1)^n.$$

Therefore we have

$$\frac{1}{n^n}(m+1)^n \le \ell(\nu) \le 2^n(m+1)^n.$$

where the left inequality holds for  $m \ge n+1$ , the right one for all m. We set  $b_{\ell} = c_{\ell(\nu)}$  for  $\ell = \ell(\nu)$ . Then

$$|f|_{k}^{\sim^{2}} \geq \frac{1}{4^{k}} \sum_{m=0}^{\infty} \left( \sum_{|\nu|=m} |c_{\nu}|^{2} \ell(\nu)^{2\frac{k}{n}} \right)$$
$$= \frac{1}{4^{k}} \sum_{\ell=1}^{\infty} |b_{\ell}|^{2} \ell^{2\frac{k}{n}}$$

and therefore

$$\begin{aligned} |f|_{k}^{\sim^{2}} &\leq n^{2k} \sum_{m=0}^{\infty} \left( \sum_{|\nu|=m} |c_{\nu}|^{2} \ell(\nu)^{\frac{2k}{n}} \right) \\ &\leq n^{2k} \sum_{\ell=1}^{\infty} |b_{\ell}|^{2} \ell^{\frac{2k}{n}}. \end{aligned}$$

Therefore

$$n^{-k}|f|_{k}^{\sim} \leq \left(\sum_{\ell=1}^{\infty} |b_{\ell}|^{2} \ell^{2k}\right)^{\frac{1}{2}} \leq 4^{nk}|f|_{nk}^{\sim}$$

which proves that  $f \mapsto (b_\ell)_{\ell \in \mathbb{N}}$  defines an isomorphism of  $\mathscr{C}_r^{\infty}$  into s, which is clearly surjective since the image contains the finite sequences.  $\Box$ 

**3.22 Theorem:** If  $\Omega \in \mathbb{R}^n$  is open then  $\mathscr{C}^{\infty}(\Omega) \cong s^{\mathbb{N}}$ .

Proof: We use Lemma 3.20 and show first that  $s^{\mathbb{N}}$  is isomorphic to a complemented subspace of  $\mathscr{C}^{\infty}(\Omega)$ . We choose a function  $\varphi \in \mathscr{D}(\mathbb{R})$ , with  $\varphi \geq 0$ and  $\operatorname{supp} \varphi \subset [0,3]$ , such that  $\sum_{k \in \mathbb{Z}} \varphi^2(x-k) \equiv 1$ . This can be achieved by setting  $\varphi(x) = (\int_1^2 \chi(x-t)dt)^{\frac{1}{2}}$  where  $\chi \in \mathscr{D}(\mathbb{R})$ ,  $\operatorname{supp} \chi \subset [-1,+1]$ ,  $\int_{-\infty}^{+\infty} \chi(t)dt = 1$ . For r > 0 and  $x = (x_2, \ldots, x_n) \in \mathbb{R}^n$  we put  $\varphi_r(x) = \varphi(\frac{x_1}{r}) \ldots \varphi(\frac{x_n}{r})$ . Then we obtain

$$\sum_{k \in \mathbb{Z}^n} \varphi_r^2(x - rk) \equiv 1.$$

We choose a discrete sequence  $x_1, x_2, \ldots \in \Omega$  and functions  $\psi_j(x) := \varphi_{r_j}(x - x_j)$  such that  $\operatorname{supp} \psi_j \subset \Omega$  for all j and  $\operatorname{supp} \psi_j \cap \operatorname{supp} \psi_k = \emptyset$  for  $j \neq k$ .

We define a continuous linear map

$$\Phi\colon \prod_{j} \mathscr{C}^{\infty}_{(r_{j})} \longrightarrow \mathscr{C}^{\infty}(\Omega)$$

by

$$\Phi((f_j)_{j\in\mathbb{N}}) = \sum_j \psi_j f_j$$

and we define a continuous linear map

$$\Psi\colon \mathscr{C}^{\infty}(\Omega) \longrightarrow \prod_{j} \mathscr{C}^{\infty}_{(r_{j})}$$

by

$$\Psi(f) = \left(\sum_{k \in \mathbb{Z}^n} \psi_j(x - r_j k) f(x - r_j k)\right)_{j \in \mathbb{N}}$$

For  $(f_j)_{j\in\mathbb{N}}\in\prod_j \mathscr{C}^{\infty}_{(r_j)}$  we obtain for the *i*-th component  $g_i$  of  $\Psi(\Phi(f_j)_{j\in\mathbb{N}})$ )

$$g_i(x) = \sum_{k \in \mathbb{Z}^n} \psi_i(x - r_i k) \sum_j \psi_j(x - r_i k) f_j(x - r_i k)$$
$$= \sum_{k \in \mathbb{Z}^n} \psi_i^2(x - r_i k) f_i(x - r_i k)$$
$$= f_i(x).$$

Therefore  $\Psi \circ \Phi = \text{id.}$  Notice that by Theorem 3.21  $\mathscr{C}^{\infty}_{(r_j)} \cong s$  for all j.

Now we show that  $\mathscr{C}^{\infty}(\Omega)$  is isomorphic to a complemented subspace of  $s^{\mathbb{N}}$ . We choose functions  $\varphi_j \in \mathscr{D}(\Omega)$  such that  $\sum_j \varphi_j^2 \equiv 1$  and each  $x \in \Omega$  has a neighborhood which meets only finitely many of the supp  $\varphi_j$ . We choose  $r_j > 0$  such that supp  $\varphi_j$  is contained in the interior of a period cube. We define a map

$$\Phi\colon \mathscr{C}^{\infty}(\Omega) \longrightarrow \prod_{j} \mathscr{C}^{\infty}_{(r_{j})}$$

by

$$\Phi(f) = \left(\sum_{k \in \mathbb{Z}^n} \varphi_j(x - r_j k) f(x - r_j k)\right)_{j \in \mathbb{N}}$$

Notice that the summands above have disjoint supports. And we define a continuous linear map

$$\Psi\colon \prod_{j} \mathscr{C}^{\infty}_{(r_{j})} \longrightarrow \mathscr{C}^{\infty}(\Omega)$$

by

$$\Psi((f_j)_{j\in\mathbb{N}}) = \sum_j \varphi_j f_j.$$

For  $f \in \mathscr{C}^{\infty}(\Omega)$  we obtain

$$\Psi(\Phi f)(x) = \sum_{j} \varphi_{j}(x) \sum_{k \in \mathbb{Z}^{n}} \varphi_{j}(x - r_{j}k) f(x - r_{j}k)$$
$$= \sum_{j} \sum_{k \in \mathbb{Z}^{n}} \varphi_{j}(x) \varphi_{j}(x - r_{j}k) f(x - r_{j}k)$$
$$= \sum_{j} \varphi_{j}^{2}(x) f(x)$$
$$= f(x).$$

Therefore  $\Psi \circ \Phi = \text{id and } \mathscr{C}^{\infty}(\Omega)$  is isomorphic to a complemented subspace of  $\prod_{j} \mathscr{C}^{\infty}_{(r_{j})} \cong s^{\mathbb{N}}$ .  $\Box$ 

For both sides of the proof we used the following fact:

**3.23 Lemma:** Let X, Y be topological vector spaces,  $\Phi: X \longrightarrow Y, \Psi: Y \longrightarrow X$  continuous linear and  $\Psi \circ \Phi = id_X$ . Then X is isomorphic to a complemented subspace of Y.

Proof:  $\Phi: X \longrightarrow \text{im } \Phi \text{ is an isomorphism since } \Psi|_{\text{im } \Phi} \text{ is its continuous linear inverse map. } \Phi \circ \Psi \text{ is a projection in } Y, \text{ since } \Phi \circ \Psi \circ \Phi \circ \Psi = \Phi \circ \text{id } \circ \Psi = \Phi \circ \Psi.$ Because  $\Psi$  is surjective we have  $\text{im}(\Phi \circ \Psi) = \text{im } \Phi.$ 

Finally we have proved

**3.24 Theorem:** *E* is a nuclear Fréchet space if, and only if, it is isomorphic to a closed subspace of  $\mathscr{C}^{\infty}(\mathbb{R})$  (resp.  $\mathscr{C}^{\infty}(\Omega)$ ,  $\Omega \subset \mathbb{R}^n$  open).

### 4 Diametral Dimension

Let E be a linear space over  $\mathbb K.$  We consider absolutely convex subsets  $U,\,V\subset E.$ 

We recall the following notation:

 $V\prec U$  if there is t>0 such that  $V\subset tU.$  For  $V\prec U$  and any linear subspace  $F\subset E$  we set

$$\delta(V, U; F) = \inf \{ \delta > 0 \mid V \subset \delta U + F \}.$$

**Definition:** For  $V \prec U$  and  $n \in \mathbb{N}_0$  the number

 $\delta_n(V, U) = \inf \left\{ \, \delta(V, U; F) \mid \dim F \le n \, \right\}$ 

is called the n-th Kolmogoroff diameter of V with respect to U.

**4.1 Lemma:** We have for any  $V \prec U$ :

- (1)  $\delta_n(V, U) \ge \delta_{n+1}(V, U)$  for all n.
- (2)  $\delta_n(V, U) \to 0$  if and only if V is precompact with respect to U.

Proof: (1) is obvious. To show (2) we first assume that V is precompact with respect to U. Then we have for every  $\varepsilon > 0$  elements  $x_1, \ldots, x_m$ such that  $V \subset \bigcup_{j=1}^m (x_j + \varepsilon U)$ . Putting  $F = \text{span} \{x_1, \ldots, x_m\}$  we obtain  $\delta(V, U; F) \leq \varepsilon$  and therefore  $\delta_n(V, U) \leq \varepsilon$  for  $n \geq m$ .

To prove the reverse implication we assume that  $\delta_n(V, U) < \varepsilon$ . Then there exists a subspace  $F \subset E$ , dim  $F \leq n$ , such that

$$V \subset \varepsilon U + F.$$

We may assume  $F \subset \operatorname{span} U$ . Namely if  $v = \varepsilon u + f$  we have  $f = \varepsilon u - v \in \operatorname{span} U$ , so if necessary we can replace F by  $F \cap \operatorname{span} U$ .

We may assume that  $|| ||_U$  is a norm on F. For otherwise we set  $F_0 = F \cap \ker || ||_U$  and  $F = F_0 \oplus F_1$ . Then  $V \subset \varepsilon U + F_1$ , and we can replace F by  $F_1$ .

Since  $V \prec U$  there is t > 0 such that  $V \subset tU$  hence, arguing as above, we obtain

$$V \subset \varepsilon U + F \cap (t + \varepsilon)U$$
$$= \varepsilon U + M.$$

Since  $M = F \cap (t + \varepsilon)U$  is bounded in the finite dimensional space  $(F, || ||_U)$ we may choose  $x_1, \ldots, x_m \in F$  such that  $M \subset \bigcup_{j=1}^m (x_j + \varepsilon U)$  and therefore

$$V \subset \bigcup_{j=1}^{m} (x_j + 2\varepsilon U).$$

**4.2 Corollary:** A locally convex space E is a Schwartz space if and only if for every neighborhood of zero U there is a neighborhood of zero  $V \prec U$  such that  $\delta_n(V, U) \rightarrow 0$ .

For the special case of Hilbert discs the Kolmogoroff diameters coincide with the singular numbers of the canonical imbedding. This is contained in the following lemma.

**4.3 Lemma:** Let X, Y be Hilbert spaces,  $U_X, U_Y$  their unit balls,  $T: X \longrightarrow Y$  continuous and linear. Then

- (1) T is compact if, and only if,  $\delta_n(TU_X, U_Y) \to 0$ .
- (2) If T is compact and

$$Tx = \sum_{n=0}^{\infty} s_n \langle x, \varphi_n \rangle e_n$$

its Schmidt representation then  $s_n = \delta_n(TU_X, U_Y)$ .

*Proof:* (1) is a consequence of Lemma 4.1 so it remains to show (2).

We set  $F = \text{span} \{ e_0, \dots, e_{m-1} \}$ . Then dim F = m and for  $x \in U_X$  we have

$$Tx \in \sum_{n=m}^{\infty} s_n \langle x, \varphi_n \rangle e_n + F$$

and

$$\|\sum_{n=m}^{\infty} s_n \langle x, \varphi_n \rangle e_n \|^2 = \sum_{n=m}^{\infty} s_n^2 |\langle x, \varphi_n \rangle|^2 \le s_m^2 \sum_{n=m}^{\infty} |\langle x, \varphi_n \rangle|^2 \le s_m^2.$$

Therefore  $\delta_m(TU_X, U_Y) \leq s_m$ .

The reverse inequality is trivial if  $s_m = 0$ , so we assume that  $s_m > 0$  and

$$TU_X \subset \delta U_Y + F, \quad \dim F \le m.$$
 (2)

Let  $h_1, \ldots, h_{\mu}, \mu \leq m$ , be a basis of F. There is a nontrivial solution  $y \in \text{span} \{ e_0, \ldots, e_m \}$  of the equations  $\langle y, h_j \rangle = 0, j = 1, \ldots, \mu$ . Hence  $y \in F^{\perp}$ .

We may choose  $\xi_0, \ldots, \xi_m$  such that:

$$y = \sum_{n=0}^{m} s_n \xi_n e_n = T(\sum_{n=0}^{m} \xi_n e_n) = Tx.$$
 (3)

Since  $x \neq 0$  we may set x' = tx and y' = ty where t = 1/||x|| and we still have

$$y' = Tx'$$
 and  $y' \perp F$ .

So we may assume (3) with ||x|| = 1. For  $f \in F$  we obtain

$$||Tx - f||^2 = ||y - f||^2 = ||y||^2 + ||f||^2$$
  

$$\geq \sum_{n=0}^m s_n^2 |\xi_n|^2 \ge s_m^2 \sum_{n=0}^m |\xi_n|^2$$
  

$$= s_m^2.$$

Therefore we have  $\delta \geq s_m$ . Since  $\delta$  and F, fulfilling (2), were arbitrarily chosen we have shown that

$$\delta_m(TU_X, U_Y) \ge s_m.$$

This proves assertion (2).

An easy consequence of the definition is the following:

**4.4 Lemma:** For any V, U we have:

- (1) If  $V_o \subset V \prec U \subset U_o$  then  $\delta_n(V_o, U_o) \leq \delta_n(V, U)$  for all n.
- (2) If  $W \prec V \prec U$  then  $\delta_{n+m}(W, U) \leq \delta_n(W, V) \, \delta_m(V, U)$  for all n, m.

Proof: (1) is obvious. To show (2) we assume that

$$W \subset \delta V + F, \qquad \dim F \le n$$
$$V \subset \delta' U + G, \qquad \dim G \le m.$$

Then we have

 $W \subseteq \delta \delta' U + F + G.$ 

Since dim  $(F + G) \le n + m$  we obtain the assertion.

We want now to express nuclearity in terms of the Kolmogoroff diameters. One implication of the intended characterization we have almost done.

**4.5 Proposition:** If E is nuclear then for every neighborhood of zero U and  $k \in \mathbb{N}$  there is a neighborhood of zero  $V \subset U$ , such that

$$\lim_{n \to \infty} n^k \delta_n(V, U) = 0.$$

*Proof:* We prove it by induction over k.

For k = 1 we choose a hilbertian zero neighborhood  $U_1 \subset U$  and then a hilbertian zero neighborhood  $V \subset U_1$  such that  $j_V^{U_1}$  is nuclear. Then, due to Lemma 4.3 and Lemma 4.4, we have  $s_n(j_V^{U_1}) = \delta_n(V, U_1) \geq \delta_n(V, U)$  and therefore

$$\sum_{n} \delta_n(V, U) < +\infty.$$

Since the sequence of the  $\delta_n(V, U)$  is decreasing, this implies

$$\lim_{n} n \,\delta_n(V, \, U) = 0$$

Assume that we found  $V_1 \subset U$ , such that  $\lim_n n^k \delta_n(V_1, U) = 0$ , then we find  $V \subset V_1$ , such that  $\lim_n n \delta_n(V, V_1) = 0$  and therefore, due to Lemma 4.4,

$$\lim_{n} n^{k+1} \delta_{2n}(V, U) = 0$$

which implies

$$\lim_{n} n^{k+1} \delta_{2n+1}(V, U) = 0.$$

This implies the assertion.

To prove the converse of Proposition 4.5 we recall some facts about nuclear operators between Banach spaces.

Let X, Y Banach spaces then we denote by  $\mathcal{N}(X, Y)$  the linear space of nuclear operators. With the nuclear norm

$$\nu(A) := \inf \{ \sum_{k} \|\xi_{k}\|^{*} \|\eta_{k}\| \mid \xi_{k} \in X', \eta_{k} \in Y \text{ for all } k, \\ Ax = \sum_{k} \xi_{k}(x) y_{k} \text{ for all } x \}$$

 $\mathcal{N}(X, Y)$  is a Banach space (see [1]).

Every operator  $A \colon X \longrightarrow Y$  with finite dimensional range is nuclear and

$$||A|| \le \nu(A) \le n ||A||$$

where  $n = \dim R(A)$ .

While the left estimate follows from the triangular inequality, the right one follows from Auerbach's lemma [1, 10.5]. Let  $e_1, \ldots, e_n$  be an Auerbach basis of  $Y_0 = R(A)$ , and  $f_1, \ldots, f_n \in Y'$  such that

$$f_j(e_k) = \delta_{j,k}, \qquad ||e_1|| = \ldots = ||e_n|| = ||f_1||^* = \ldots = ||f_n||^* = 1.$$

Then

$$Ax = \sum_{k=1}^{n} f_k(Ax)e_k$$

and

$$\nu(A) \le \sum_{k=1}^{n} \|f_k \circ A\| \|e_k\| \le n \|A\|.$$
(4)

We obtain:

**4.6 Lemma:** Let X, Y be Banach spaces,  $U_X$ ,  $U_Y$  their unit balls, A:  $X \rightarrow Y$  continuous and linear with

$$\sum_{n} n^2 \delta_n(AU_X, U_Y) < +\infty.$$

Then A is nuclear.

*Proof:* We choose a decreasing sequence  $\varepsilon_n > \delta_n(AU_X, U_Y)$  such that  $\sum_n n^2 \varepsilon_n < +\infty$ .

For every n there is a subspace  $F_n \subset Y$ , dim  $F_n = m_n \leq n$ , such that

$$AU_X \subset \varepsilon_n U_Y + F_n. \tag{5}$$

Let  $e_1^n, \ldots, e_{m_n}^n, f_1^n, \ldots, f_{m_n}^n$  be an Auerbach basis of  $F_n$ . Then

$$P_n x := \sum_{k=1}^{m_n} f_k^n(x) e_k^n$$

is a projection onto  $F_n$  with  $||P_n|| \leq n$ ,

If we apply  $\operatorname{id} -P_n$  to the inclusion (5) and notice that  $\|\operatorname{id} -P_n\| \le n+1$ we see that

$$(\mathrm{id} - P_n)AU_X \subset (n+1)\varepsilon_n U_Y$$

or

$$||A - P_n A|| \le (n+1)\varepsilon_n.$$
(6)

We put  $P_0 = 0$  and  $U_n = P_{n+1}A - P_nA$ . Then dim  $R(U_n) \le 2n + 1$  and, because of (6),

$$||U_n|| = ||(A - P_n A) - (A - P_{n+1} A)|| \le (2n+1)\varepsilon_n.$$

Therefore, due to (4),

$$\nu(U_n) \le (2n+1)^2 \varepsilon_n.$$

and therefore

$$\sum_k \nu(U_n) < +\infty.$$

Hence there is  $U \in \mathcal{N}(X, Y)$  such that

$$U = \sum_{n} U_{n}$$

in  $\mathcal{N}(X, Y)$  and therefore also in L(X, Y). Since, because of (6),

$$\sum_{n=0}^{m-1} U_n = P_m A \underset{m \to \infty}{\longrightarrow} A$$

in L(X, Y) we have  $A = U \in \mathcal{N}(X, Y)$ .

**4.7 Theorem:** A locally convex space E is nuclear if, and only if, for every neighborhood of zero U and every  $k \in \mathbb{N}$  there is a neighborhood of zero V, such that

$$\lim_{n \to \infty} n^k \delta_n(V, U) = 0.$$

*Proof:* One implication is Proposition 4.5, to prove the other one we choose V for k = 4 and obtain

$$\sum_{n} n^2 \delta_n(V, U) < +\infty.$$

We apply Lemma 4.6 to  $i_V^U : E_V \longrightarrow E_U$  taking into account that obviously  $\delta_n(V, U) = \delta_n(i_V^U \widehat{V}, \widehat{U})$  where  $\widehat{V}, \widehat{U}$  are the closed unit balls in  $E_V$  and  $E_U$ , respectively.  $\Box$ 

Since in the proof of Proposition 4.5 we used from nuclearity only that for every U there is V with  $\sum_n \delta_n(V, U) < +\infty$  we have also proved the following theorem.

**4.8 Theorem:** A locally convex space E is nuclear if, and only if, for every neighborhood of zero U there is a neighborhood of zero  $V \subset U$ , such that

$$\sum_{n} \delta_n(V, U) < +\infty.$$

The description of Schwartz spaces and nuclear spaces in 4.2 and 4.5 by means of Kolmogoroff diameters motivate the following definition.

**Definition:** For a locally convex space E the set

$$\Delta(E) = \{ t = (t_0, t_1, \ldots) \in \mathbb{K}_0^{\mathbb{N}} \mid \forall U \exists V : \lim_n t_n \delta_n(V, U) = 0 \}$$

is called the *diametral dimension* of E.

**Remark:** In 4.2 and 4.5 there was shown:

- (1) E is a Schwartz space if, and only if,  $(1, 1, 1, ...) \in \Delta(E)$ .
- (2) E is nuclear if, and only if,  $(n^k)_n \in \Delta(E)$  for all k.

**4.9 Theorem:** The diametral dimension is a linear topological invariant, that is, if  $E \cong F$  then  $\Delta(E) = \Delta(F)$ .

*Proof:* This follows from the obvious fact that for a linear bijective T

$$\delta_n(TV, TU) = \delta_n(V, U).$$

**Remark:** If  $\mathscr{U}$  is a fundamental system of absolutely convex neighborhoods of zero then

$$\Delta(E) = \{ t = (t_0, t_1, \ldots) \mid \forall U \in \mathscr{U} \exists V \in \mathscr{U} : t_n \delta_n(V, U) \to 0 \}.$$

Here  $\mathscr{U}$  is called fundamental system of neighborhoods of zero if for every neighborhood of zero W there is  $U \in \mathscr{U}$  and  $\varepsilon > 0$  such that  $\varepsilon U \subset W$ , that is, if the  $|| \parallel_U$  are a fundamental system of seminorms.

If  $V \prec U$  are absolutely convex subsets of E and  $T: E \longrightarrow G$  a linear map then clearly  $TV \prec TU$  and  $\delta_n(TV, TU) \leq \delta_n(V, U)$ . This yields: **4.10 Theorem:** If  $F \subset E$  is a closed subspace then  $\Delta(E/F) \supset \Delta(E)$ .

Proof: Let  $(t_0, t_1, \ldots) \in \Delta(E)$  and  $\hat{U} = q U$  an absolutely convex neighborhood of zero in E/F, then we find a neighborhood V of zero in  $E, V \prec U$ , such that  $t_n \delta_n(V, U) \to 0$  and therefore  $t_n \delta_n(qV, qU) \to 0$ .

To handle the case of subspaces we introduce another notion of relative diameters.

Let  $V \prec U$  be absolutely convex subsets of E, and  $G \subset E$  a linear subspace. We set

 $\gamma(V, U; G) := \inf\{\gamma > 0 \mid V \cap G \subset \gamma U\}.$ 

**Definition:** For  $V \prec U$  and  $n \in \mathbb{N}_0$  the number

 $\gamma_n(V, U) = \inf\{\gamma(V, U; G) \mid \operatorname{codim} G \le n\}$ 

is called the n-th Gelfand diameter of V with respect to U.

A comparison of the Kolmogoroff and Gelfand diameters is given in the following lemma.

**4.11 Lemma:** Let  $V \prec U$  be absolutely convex closed neighborhoods of zero in E.

(a) We have

$$\delta_n(V, U) \le (n+1)\gamma_n(V, U)$$
  
$$\gamma_n(V, U) \le (n+1)\delta_n(V, U).$$

for all  $n \in \mathbb{N}_0$ .

(b) If V and U are Hilbert discs then

$$\delta_n(V, U) = \gamma_n(V, U)$$

for all  $n \in \mathbb{N}_0$ .

Proof: Let  $V \subset \delta U + F$ , dim  $F \leq n$ . As in the proof of Lemma 4.1 we may assume that  $\| \|_U$  is a norm on F. Let P be a projection from E onto F with

$$||P||_U = \sup_{x \in U} ||Px||_U < +\infty.$$

We set  $G = \ker P$ , then  $\operatorname{codim} G = \dim F \leq n$ . For  $x \in V \cap G$  we obtain

$$x = (\mathrm{id} - P)x \in \delta \| \mathrm{id} - P\|_U U,$$

hence

$$V \cap G \subset \delta \| \operatorname{id} - P \|_U U.$$

This proves that

$$\gamma_n(V, U) \le \|\operatorname{id} - P\|_U \,\delta_n(V, U)\|$$

In the general case we may use Auerbach's lemma to get P with  $||P||_U \le n$ hence  $||\operatorname{id} - P||_U \le 1 + n$ . If U is a Hilbert disc we may choose for P the orthogonal projection, so  $||\operatorname{id} - P||_U \le 1$ .

In a similar way we prove the reverse estimate. Let  $V \cap G \subset \gamma U$ ,  $\operatorname{codim} G \leq n$ . Without restriction of generality we may assume G closed with respect to  $\| \|_V$ . Let P be a  $\| \|_V$ - continuous projection with ker P = G. We set  $F := \operatorname{im} P$ . Then we have for  $x \in V$ 

$$x = (\mathrm{id} - P)x + Px \in \|\mathrm{id} - P\|_V (V \cap G) + F \subset \|\mathrm{id} - P\|_V \gamma U + F$$

hence

$$V \subset \gamma \| \operatorname{id} - P \|_V U + F$$

This proves that

$$\delta_n(V, U) \le \|\operatorname{id} - P\|_V \gamma_n(V, U).$$

Arguing as previously we complete the proof.

**Definition:** For a locally convex space E we put

$$\Gamma(E) := \{ t = (t_0, t_1, \ldots) \in \mathbb{K}_0^{\mathbb{N}} \mid \forall U \exists V \lim_n t_n \gamma_n(V, U) = 0 \}.$$

From Lemma 4.11 we obtain:

**4.12 Proposition:** If *E* has a fundamental system of Hilbert seminorms, then  $\Delta(E) = \Gamma(E)$ .

If  $V \prec U$  are absolutely convex subsets of E and  $F \subset E$  a linear subspace then clearly  $V \cap F \prec U \cap F$  and  $\gamma_n(V \cap F, U \cap F) \leq \gamma_n(V, U)$ . This yields quite easily:

**4.13 Lemma:** If  $F \subset E$  is a linear subspace then  $\Gamma(F) \supset \Gamma(E)$ .

Proof: Let  $(t_0, t_1, ...) \in \Gamma(E)$  and U an absolute convex neighborhood of zero in F. We may find an absolutely convex neighborhood of zero  $U_0 \subset E$ , such that  $U_0 \cap F \subset U$  and  $V_0 \prec U_0$ , such that  $\lim_n t_n \gamma_n(V_0, U_0) = 0$ . Putting  $V = V_0 \cap F$  we have  $V \prec U$  and  $\lim_n t_n \gamma_n(V, U) = 0$ .  $\Box$ 

**4.14 Theorem:** If *E* has a fundamental system of Hilbert seminorms and  $F \subset E$  is a subspace then  $\Delta(F) \supset \Delta(E)$ .

*Proof:* This follows from proposition 4.12 and lemma 4.13.  $\Box$ 

- **Examples:** (1) If dim  $E = m < +\infty$  and  $V \prec U$  in E then  $\delta_n(V, U) = 0$  for  $n \ge m$ . In particular  $\Delta(E) = \omega$ .
  - (2) If  $E = \omega$  and U a neighborhood of zero then there is  $m \in \mathbb{N}$  so that for all  $\delta > 0$

$$E = \delta U + \operatorname{span}\{e_1, \dots, e_m\}$$

where  $e_j$  are the canonical unit vectors. This means that for every  $V \prec U$  we have  $\delta_m(V,U) = 0$ . Therefore  $\Delta(\omega) = \omega$ .

- (3) If E is infinitely dimensional and a non Schwartzian locally convex space then there is U so that no  $V \prec U$  is U-precompact. That means  $\inf_n \delta_n(V,U) > 0$ . Therefore  $\Delta(E) = c_0$ .
- (4) If E = s and

$$U_k = \{\xi \mid |\xi|_k^2 = \sum_{j=1}^{\infty} |\xi_j|^2 j^{2k} < +\infty\}$$

then the local Banach space is  $s_k$  (see ...?...). Let  $i_m^k : s_m \hookrightarrow s_k$  be the canonical imbedding. Then

$$a_m^k \xi = \xi = \sum_{j=0}^{\infty} \frac{1}{(j+1)^{m-k}} \langle \xi, \frac{1}{(j+1)^m} e_{j+1} \rangle_m \frac{1}{(j+1)^k} e_{j+1}$$

is the Schmidt representation. Therefore

$$\delta_n(U_m, U_k) = s_n(i_m^k) = \frac{1}{(n+1)^{m-k}}.$$

From this we derive easily that

 $\Delta(s) = \{t = (t_0, t_1, \ldots) \mid \exists m \in \mathbb{N}, C > 0 \text{ such that } |t_n| \le Cn^m \text{ for all } n\}.$ 

Theorem 4.7 we may then read as follows:

**4.15 Theorem:** *E* is nuclear if, and only if,  $\Delta(E) \supset \Delta(s)$ .

Now we can continue the series of examples by:

- (5)  $\Delta(s^{\mathbb{N}}) = \Delta(s)$ . Since  $s^{\mathbb{N}}$  is nuclear we have  $\Delta(s^{\mathbb{N}}) \supset \Delta(s)$  (see 4.15). Since s is a quotient space of  $s^{\mathbb{N}}$  we have  $\Delta(s) \supset \Delta(s^{\mathbb{N}})$  (see 4.10).
- (6) If  $\Omega \subset \mathbb{R}^n$  is open then  $\Delta(\mathscr{C}^{\infty}(\Omega)) = s$ . This is because  $\mathscr{C}^{\infty}(\Omega) \cong s^{\mathbb{N}}$  (see 3.22).

At this point it seems that the diametral dimensional is not very significant for the isomorphy type of a nuclear Fréchet space. We will try to get a better understanding in the next section.

# 5 Power series spaces and their related nuclearities

We start with a short introduction to the theory of Köthe sequence spaces. We concentrate on the Fréchet Hilbert version of these.

Let  $A = (a_{j,k})_{j \in \mathbb{N}_0, k \in \mathbb{N}_0}$  be a matrix such that

- (1)  $0 \le a_{j,k} \le a_{j,k+1}$  for all j, k
- (2)  $\forall j \exists k: a_{j,k} > 0.$

A is called a Köthe-matrix.

We put

$$\lambda(A) := \{ x = (x_0, x_1, \ldots) \mid |x|_k^2 := \sum_{j=0}^{\infty} |x_j|^2 a_{j,k}^2 < +\infty \text{ for all } k \}.$$

It is easily seen that  $\lambda(A)$  is a Fréchet-Hilbert space with the semiscalar products

$$\langle x, y \rangle_k = \sum_j x_j \, \bar{y}_j \, a_{j,k}^2.$$

To determine the local Banach spaces we put

$$\mathscr{I}_k = \{j \mid a_{j,k} > 0\}$$

Then

$$\lambda_k(A) := \{ x = (x_j)_{j \in \mathscr{I}_k} \mid |x|_k^2 = \sum_{j \in \mathscr{I}_k} |x_j|^2 a_{j,k}^2 < +\infty \}$$

is the local Banach space with respect to  $| |_k$ . For m > k we have  $\mathscr{I}_m \supset \mathscr{I}_k$ and the canonical map  $i_m^k \colon \lambda_m(A) \longrightarrow \lambda_k(A)$  is given by  $i_m^k \colon (x_j)_{j \in \mathscr{I}_m} \mapsto (x_j)_{j \in \mathscr{I}_k}$ . Similar as in the case of s we may write  $i_m^k$  as

$$i_m^k x = \sum_{j \in \mathscr{I}_k} \frac{a_{j,k}}{a_{j,m}} \langle x, \frac{1}{a_{j,m}} e_j \rangle_m \left( \frac{1}{a_{j,k}} e_j \right).$$

Here  $e_j$  is the canonical *j*-th basis vector,  $\left(\frac{1}{a_{j,m}}e_j\right)_{j\in\mathscr{I}_k}$ , and  $\left(\frac{1}{a_{j,k}}e_j\right)_{j\in\mathscr{I}_k}$  are orthonormal sequences in  $\lambda_m(A)$  and  $\lambda_k(A)$ , respectively.

The map  $i_m^k$  is compact if and only if  $(\frac{a_{j,k}}{a_{j,m}})_{j \in \mathscr{I}_m}$  is a null sequence and for  $U_k = \{x \mid |x|_k \leq 1\}$  we obtain

$$\delta_n(U_m, U_k) = s_n(\imath_m^k) = b_n$$

where

$$b_n := \frac{a_{\pi(n),k}}{a_{\pi(n),m}}, \quad n \in \mathbb{N}_0$$

is a decreasing rearrangement,  $\pi : \mathbb{N}_0 \to \mathscr{I}_k$  bijective (if  $\mathscr{I}_k$  is infinite, the other case is trivial).

From the previous considerations, Corollary 4.2 and Theorem 4.8 we derive

**5.1 Theorem:**  $\lambda(A)$  is a Schwartz space if, and only if, for every k there is m > k, such that

$$\lim_{j \in \mathscr{I}_k} \frac{a_{j,k}}{a_{j,m}} = 0$$

**5.2 Theorem:**  $\lambda(A)$  is nuclear if, and only if, for every k there is m > k, such that

$$\sum_{j \in \mathscr{I}_k} \frac{a_{j,k}}{a_{j,m}} < +\infty$$

**Definition:**  $\lambda(A)$  is called regular if  $\left(\frac{a_{j,k}}{a_{j,k+1}}\right)_j$  is decreasing for all k.

In this case either  $\lambda(A) = \omega$  or there is  $k_0$  such that  $a_{j,k_0} > 0$  for all j. Hence for a regular Köthe space we will assume that  $a_{j,0} > 0$  for all j.

**5.3 Lemma:** If  $\lambda(A)$  is regular, then it is either a Schwartz space or a Banach space.

Proof: If  $\lambda(A)$  is not a Schwartz space then there exists k such that for every m > k we have  $\inf_j a_{j,k}/a_{j,m} =: C_m > 0$ . This implies  $a_{j,m} \leq C_m^{-1}a_{j,k}$  for all j and m. Therefore we obtain  $\lambda(A) = \lambda_k(A)$ .

If  $\lambda(A)$  is regular then for m > k

$$\delta_n(U_m, U_k) = \gamma_n(U_m, U_k) = s_n(i_m^k) = \frac{a_{n,k}}{a_{n,m}}$$

If  $\lambda$  is a sequence space then we set

$$M(\lambda) := \{ t = (t_0, t_1, \ldots) \mid (t_j x_j)_j \in \lambda \text{ for all } x \in \lambda \}.$$

 $M(\lambda)$  is called the space of multipliers on  $\lambda$ . Clearly  $M(\lambda)$  is an algebra with 1 with respect to pointwise addition and multiplication.

**5.4 Theorem:** If  $\lambda(A)$  is a regular Schwartz space then  $\Delta(\lambda(A)) = M(\lambda(A))$ .

Proof: If  $t = (t_0, t_1, \ldots) \in \Delta(\lambda(A))$ , then for every k there is m such that

$$t_j \frac{a_{j,k}}{a_{j,m}} \to 0.$$

In particular this sequence is bounded and for suitable C > 0 we have

$$|t_j|a_{j,k} \le Ca_{j,m}.$$

For  $x \in \lambda(A)$  this gives

$$\sum_{j=0}^{\infty} |t_j x_j|^2 a_{j,k}^2 \le C^2 \sum_{j=0}^{\infty} |x_j|^2 a_{j,m}^2 < \infty.$$

Therefore  $(t_j x_j)_j \in \lambda(A)$  and  $t \in M(\lambda(A))$ .

To prove the reverse implication we assume that  $t \in M(\lambda(A))$ . We define a linear map  $M_t \colon \lambda(A) \longrightarrow \lambda(A)$  by

$$M_t \colon x \mapsto (t_j x_j)_j.$$

The graph of  $M_t$  is closed because, assuming  $x^{(n)} \to 0$  and  $M_t x^{(n)} \to y$ , we obtain  $x_j^{(n)} \to 0$  for all  $j, t_j x_j^{(n)} \to y_j$  for all j, which implies  $y_j = 0$  for all j.

Due to the Closed Graph Theorem 1.8  $M_t$  is continuous. Hence for every k there is  $\mu \in \mathbb{N}$  and C > 0 such that  $||Mx||_k \leq C ||x||_{\mu}$ .

For  $x = e_j$  this gives  $|e_j|a_{j,k} \leq Ca_{j,\mu}$  for all j. Since  $\lambda(A)$  is a Schwartz space we can find  $m > \mu$ , such that  $\frac{a_{j,\mu}}{a_{j,m}} \to 0$ . Therefore for k we find m, such that

$$|t_j|\delta_j(U_m, U_k) = |t_j|\frac{a_{j,k}}{a_{j,m}} = |t_j|\frac{a_{j,k}}{a_{j,\mu}}\frac{a_{j,\mu}}{a_{j,m}} \longrightarrow 0.$$

This means that  $t \in \Delta(\lambda(A))$  which completes the proof.

**5.5 Corollary:** If the regular Schwartz space  $\lambda(A)$  is an algebra with 1 with respect to the coordinatewise multiplication then  $\Delta(\lambda(A)) = \lambda(A)$ .

Proof: Since  $\lambda(A)$  is an algebra we have  $\lambda(A) \subset M(\lambda(A)) = \Delta(\lambda(A))$ . Since  $1 \in \lambda(A)$  we get for  $t \in \Delta(\lambda(A)) = M(\lambda(A))$  that  $t = (t_j 1)_j \in \lambda(A)$ . Therefore  $\Delta(\lambda(A)) \subset \lambda(A)$ .

It is easily verified that for  $y \in \lambda'(A)$ 

$$||y||_k^{*^2} = \sum_{j=0}^{\infty} |y_j|^2 a_{j,k}^{-2}$$

where  $a_{j,k}^{-2} = +\infty$  for  $a_{j,k} = 0$ , and  $y_j = y(e_j)$ . With this identification

$$\lambda_k^*(A) = \{ y = (y_0, y_1, \dots) \mid ||y||_k^* < +\infty \}$$

and the imbedding  $\lambda_{k+1}^*(A) \hookrightarrow \lambda_k^*(A)$  corresponds to the identical imbedding of sequences. So

$$\lambda'(A) = \{ y = (y_0, y_1, \ldots) \mid \exists k \text{ such that } \|y\|_k^* < +\infty \}$$

In particular  $\lambda'(A)$  is identified with a sequence space. For  $x = (x_0, x_1, \ldots) \in \lambda(A)$  and  $y = (y_0, y_1, \ldots) \in \lambda'(A)$  we have

$$y(x) = \sum_{j=0}^{\infty} y_j x_j.$$

From this we derive immediately the

**Remark:**  $M(\lambda(A)) = M(\lambda'(A)).$ 

and then the

**5.6 Theorem:** If  $\lambda(A)$  is a regular space then  $\Delta(\lambda(A)) = M(\lambda'(A))$ .

And exactly as previously we obtain:

**5.7 Corollary:** If  $\lambda(A)$  is a regular Schwartz space and  $\lambda'(A)$  is an algebra with 1 with respect to the coordinatewise addition and multiplication of sequences then  $\Delta(\lambda(A)) = \lambda'(A)$ .

**Example:** We consider  $s = \lambda(A)$  with  $a_{j,k} = (j+1)^k$ . Clearly for  $\mathbf{1} = (1,1,1,\ldots)$  we have  $\|\mathbf{1}\|_1^* < +\infty$  hence  $\mathbf{1} \in s'$ . Moreover the Cauchy-Schwartz inequalitity implies for  $y, z \in \lambda(A)'$  and  $yz = (y_j z_j)_j$ 

$$||yz||_{k+m}^{*^2} \le ||y||_k^* ||z||_m^*.$$

Therefore s' is an algebra with **1** with respect to coordinatewise multiplication and we obtain

$$\Delta(s) = s'.$$

This kind of behaviour extends to a much wider class of Fréchet spaces.

Let  $\alpha$ :  $0 \le \alpha_0 \le \alpha_1 \le \ldots$  be a numerical sequence,  $\lim_k \alpha_k = +\infty$ . We call it exponent sequence. Let  $r \in \mathbb{R} \cup \{\infty\}$ . We set

$$\Lambda_r(\alpha) = \{ x = (x_0, x_1, \ldots) \mid |x|_t^2 = \sum_{j=0}^{\infty} |x_j|^2 e^{2t\alpha_j} < +\infty \text{ for all } t < r \}.$$

**Definition:**  $\Lambda_r(\alpha)$  is called power series space of infinite type if  $r = +\infty$ , of finite type if  $r < +\infty$ .

Equipped with the fundamental system of seminorms  $(| |_t)_{t < r}$  the space  $\Lambda_r(a)$  is a Fréchet space, since for every sequence  $t_k \nearrow r$  the norms  $|| ||_k := | |_{t_k}$  are a fundamental system of seminorms. In fact,  $\Lambda_r(\alpha) = \lambda(A)$  with  $a_{j,k} = e^{t_k \alpha_j}$ .

Let  $r < +\infty$ . The map  $x \mapsto (x_j e^{r\alpha_j})_j$  defines a linear topological isomorphism from  $\Lambda_r(\alpha)$  onto  $\Lambda_0(\alpha)$ . So we have

**Remark:** Let  $\alpha$  be fixed. Then all spaces  $\Lambda_r(\alpha)$ ,  $r < +\infty$  are isomorphic.

Therefore we can, without restriction of generality, confine ourselves to  $r \in \{0, \infty\}$ .

**Example:**  $s = \Lambda_{\infty}(\log(j+1)).$ 

**5.8 Theorem:** (1)  $\Lambda_r(\alpha)$  is a regular Fréchet-Schwartz space.

- (2)  $\Lambda_{\infty}(\alpha)$  is nuclear if, and only if,  $\limsup_{n \to \infty} \frac{\log n}{\alpha_n} < +\infty$ .
- (3)  $\Lambda_0(\alpha)$  is nuclear if, and only if,  $\lim_n \frac{\log n}{\alpha_n} = 0$ .

Proof: (1) is obvious.

(2) follows from the following equivalences, the first of which is a consequence of Theorem 5.2  $\,$ 

$$\begin{split} \Lambda_{\infty}(\alpha) \text{ is nuclear } &\iff \exists k \colon \sum_{n} e^{-k\alpha_{n}} < +\infty \\ &\iff \exists k, C > 0 \,\forall \, n \ge 1 \colon e^{-k\alpha_{n}} \le \frac{C}{n} \\ &\iff \exists k, c \in \mathbb{R} \,\forall \, n \ge 1 \colon \log n \le c + k\alpha_{n} \\ &\iff \exists k, n_{0} \,\forall \, n \ge n_{0} \colon \log n \le k\alpha_{n}. \end{split}$$

(3) In a similar way we have

$$\begin{split} \Lambda_0(\alpha) \text{ nuclear} &\iff \forall \varepsilon > 0 \colon \sum_n e^{-\varepsilon a_n} < +\infty \\ &\iff \forall \varepsilon > 0 \exists n_\varepsilon \forall n \ge n_\varepsilon \colon \log n \le \varepsilon \alpha_n. \end{split}$$

The calculation of  $\Delta(s) = s'$  generalizes now to:

- **5.9 Theorem:** (1) If  $\Lambda_{\infty}(\alpha)$  is nuclear, then  $\Lambda'_{\infty}(\alpha)$  is an algebra with 1 with respect to coordinatewise multiplication, hence  $\Delta(\Lambda_{\infty}(\alpha)) = \Lambda'_{\infty}(\alpha)$ .
  - (2) If  $\Lambda_0(\alpha)$  is nuclear, then  $\Lambda_0(\alpha)$  is an algebra with 1 with respect to coordinatewise multiplication, hence  $\Delta(\Lambda_0(\alpha)) = \Lambda_0(\alpha)$ .

Proof: (1) The nuclearity condition in Theorem 5.8, (2) guarantees that  $\mathbf{1} = (1, 1, \ldots) \in \Lambda'_{\infty}(\alpha)$ . This space is an algebra with respect to coordinatewise multiplication, since for  $y, z \in \Lambda_{\infty}(\alpha)'$  and  $yz := (y_j z_j)_j$  we have, on account of the Cauchy-Schwarz inequality,

$$|yz|_{t+s}^* \le |y|_t^* |z|_s^*.$$

(2) Again the nuclearity condition in Theorem 5.8, (3) guarantees that  $\mathbf{1} = (1, 1, \ldots) \in \Lambda_0(\alpha)$ . An analogous application of the Cauchy-Schwarz inequality gives for  $x, \xi \in \lambda_0(\alpha)$ 

$$|x\xi|_t^2 \le |x|_{t/2} |\xi|_{t/2}.$$

An immediate consequence is, that the nuclear power series spaces  $\Lambda_r(\alpha)$ and  $\Lambda_{\rho}(\beta)$ ,  $r, \rho \in \{0, +\infty\}$  are isomorphic if, and only if, they are equal. This, however, is true also without the assumption of nuclearity. We will show this now.

#### **5.10 Lemma:** If $\lambda(A)$ is a regular Schwartz space then

$$\Delta(\lambda(A)) = \{ t = (t_0, t_1, \ldots) \mid \forall k \exists m : \sup_j |t_j| \frac{a_{j,k}}{a_{j,m}} < +\infty \}.$$

This is shown like in the proof of Theorem 5.4.

#### 5.11 Corollary: We have

- (1)  $\Delta(\Lambda_{\infty}(\alpha)) = \{\xi = (\xi_0, \xi_1, \ldots) \mid \sup_j |\xi_j| e^{-k\alpha_j} < +\infty \text{ for some } k\},\$
- (2)  $\Delta(\Lambda_0(\alpha)) = \{\xi = (\xi_0, \xi_1, \ldots) \mid \sup_j |\xi_j| e^{t\alpha_j} < +\infty \text{ for all } t < 0\}.$

**5.12 Theorem:** If  $r, \rho \in \{0, \infty\}$  and  $\alpha, \beta$  are exponent sequences the following are equivalent:

- (1)  $\Lambda_r(\alpha) \cong \Lambda_\rho(\beta)$
- (2)  $\Delta(\Lambda_r(\alpha)) = \Delta(\Lambda_\rho(\beta))$
- (3)  $r = \rho$  and there exist C > 0 and  $j_0$  so that

$$\frac{1}{C}\alpha_j \le \beta_j \le C\alpha_j$$

for all  $j \geq j_0$ .

- (4)  $\Lambda_r(\alpha) = \Lambda_{\rho}(\beta)$  as sets.
- (5)  $\Lambda_r(\alpha) = \Lambda_{\rho}(\beta)$  as topological vector spaces.

*Proof:*  $(1) \Rightarrow (2)$  follows from Theorem 4.9.

 $(3) \Rightarrow (1)$  is obvious.

 $(2) \Rightarrow (3)$ . Assume  $\Delta(\Lambda_{\infty}(\alpha)) = C$ . Then  $(e^{\alpha_j})_j \in \Delta(\Lambda_0(\beta))$ , hence for every  $\varepsilon > 0$  we have

$$\sup_{j} e^{\alpha_j - \varepsilon \beta_j} = C_{\varepsilon} < +\infty$$

and therefore with  $c_{\varepsilon} = \log C_{\varepsilon}$ 

$$\alpha_j \le \varepsilon \beta_j + c_\varepsilon.$$

So we obtain

$$\limsup_{j} \frac{\alpha_j}{\beta_j} \le \varepsilon$$

for all  $\varepsilon > 0$ , that is

$$\lim_{j} \frac{\alpha_j}{\beta_j} = 0$$

On the other hand for any nonnegative sequence  $t_j \to 0$  we have  $\xi := (e^{t_j \beta_j})_j \in \Delta(\Lambda_0(\beta)) = \Delta(\Lambda_\infty(\alpha))$ . Therefore there id k such that

$$\sup_{j} e^{t_j \beta_j - k\alpha_j} =: C < \infty,$$

hence

$$t_j \beta_j \le k \alpha_j + c$$

with  $c = \log C$ . So we obtain

$$\limsup_{j} t_j \frac{\beta_j}{\alpha_j} < \infty$$

for all nonnegative sequences  $t_j \to 0$ . This implies

$$\limsup_j \frac{\beta_j}{\alpha_j} < \infty$$

which yields a contradiction.

Therefore we have proved  $r = \rho$ .

Assume first that  $r = \rho = +\infty$  and  $\Delta(\Lambda_{\infty}(\alpha)) = \Delta(\Lambda_{\infty}(\beta))$ . Then  $(e^{\alpha_j})_j \in \Delta(\Lambda_{\infty}(\alpha)) = \Delta(\Lambda_{\infty}(\beta))$ . Therefore we have k so that

$$\sup_{j} e^{\alpha_j - k\beta_j} = C < +\infty$$

hence with  $c = \log C$ 

$$\alpha_i \le k\beta_i + c$$

for all j, hence

$$\alpha_j \le (k+1)\beta_j$$

for all  $j \geq j_1$ .

Analogously we get l and  $j_2$ , so that

$$\beta_i \leq (l+1)\alpha_i$$

for all  $j \ge j_2$ , which completes the proof for  $r = \rho = +\infty$ .

For  $r = \rho = 0$  we have  $\Delta(\Lambda_0(\alpha)) = \Delta(\Lambda_0(\beta))$ , both being Fréchet spaces. Since the graph of the identical map is closed, as both topologies imply coordinatewise convergence, their topologies coincide. So we can find C > 0and  $\varepsilon > 0$ , such that

$$\sup_{j} |x_j| e^{-\alpha_j} \le C \sup_{j} |x_j| e^{-\varepsilon\beta_j}$$

for all  $x \in \Delta(\lambda_0(\alpha)) = \Delta(\lambda_0(\beta))$ .

This implies with  $c = \log C$ 

$$-\alpha_j \le -\varepsilon\beta_j + c$$

and therefore

$$\beta_j \le \frac{1}{\varepsilon} \alpha_j + \frac{C}{\varepsilon}.$$

Analogously we get

$$\alpha_j \le \frac{1}{\varepsilon'}\beta_j + \frac{C'}{\varepsilon'}$$

with suitable  $\varepsilon' > 0$ , C', and the proof is completed as in the infinite type case.

 $(3) \Longrightarrow (5)$  is immediate,  $(5) \Longrightarrow (1)$  and  $(5) \Longrightarrow (4)$  trivial.

 $(4) \Longrightarrow (5)$  follows from the Closed Graph Theorem like previously, since the graph of the identical map is closed, as both topologies imply coordinatewise convergence.

**Remark:** For  $r, \rho \in \mathbb{R} \cup \{+\infty\}$  assertion (1), (2), (3), of Theorem 5.12 are still equivalent. This does, of course, not hold for (4).

A consequence of Theorem 5.12 is:

**5.13 Theorem:** Let  $\alpha$ , r, be given. The following are equivalent:

- (1)  $\Lambda_r(\alpha) \cong \mathbb{K} \times \Lambda_r(\alpha)$
- (2)  $\Delta(\Lambda_r(\alpha)) = \Delta(\mathbb{K} \times \Lambda_r(\alpha))$
- (3)  $\limsup_n \frac{a_{n+1}}{a_n} < +\infty$

Proof: We apply Theorem 5.12 and the Remark after it to  $\Lambda_r(\alpha)$  and  $\Lambda_r(\beta)$  with  $\beta = (0, \alpha_0, \alpha_1, \alpha_2, \ldots)$ .

A little bit more of a proof is required by the following:

**5.14 Theorem:** Let  $\alpha$ , r be given. The following are equivalent:

- (1)  $\Lambda_r(\alpha) \times \Lambda_r(\alpha) \cong \Lambda_r(\alpha)$
- (2)  $\Delta(\Lambda_r(\alpha) \times \Lambda_r(\alpha)) = \Delta(\Lambda_r(\alpha))$
- (3)  $\Delta \left( \Lambda_r(\alpha)^{\mathbb{N}} \right) = \Delta(\Lambda_r(\alpha))$
- (4)  $\limsup_{n \to \infty} \frac{\alpha_{2n}}{\alpha_n} < +\infty.$

Proof: We apply Theorem 5.12 and the Remark after it to  $\Lambda_r(\alpha)$  and  $\Lambda_r(\beta)$  with  $\beta = (\alpha_0, \alpha_0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots)$ . Then  $\Lambda_r(\beta) \cong \Lambda_r(\alpha) \times \Lambda_r(\alpha)$  and we obtain the equivalence of (1), (2), (4).

 $(3) \Longrightarrow (2)$  follows from Theorem 4.10 by

$$\Delta(\Lambda_r(\alpha)) \supset \Delta(\Lambda_r(\alpha) \times \Lambda_r(\alpha)) \supset \Delta\left(\Lambda_r(\alpha)^{\mathbb{N}}\right)$$

since  $\Lambda_r(\alpha)$  is a quotient of  $\Lambda_r(\alpha) \times \Lambda_r(\alpha)$  and  $\Lambda_r(\alpha) \times \Lambda_r(\alpha)$  a quotient of  $\Lambda_r(\alpha)^{\mathbb{N}}$ .

So it remains to show that (4) implies (3). Since again  $\Delta(\Lambda_r(\alpha) \supset \Delta(\Lambda_r(\alpha)^{\mathbb{N}}))$  follows from Theorem 4.10 we have to show that (4) implies

$$\Delta(\Lambda_r(\alpha)) \subset \Delta\left(\Lambda_r(\alpha)^{\mathbb{N}}\right).$$

Let  $t \in \Delta(\Lambda_r(\alpha))$  then we have

$$\lim_{n} t_n e^{\tau \alpha_n} = 0$$

for some  $\tau < 0$  if  $r = +\infty$ , or for every  $\tau < 0$  if r = 0, respectively.

We want to show that  $t \in \Delta(\Lambda_r(\alpha)^{\mathbb{N}})$ , so let  $U \subset \Lambda_r(\alpha)^{\mathbb{N}}$  be a neighborhood of zero. We may assume that for some m and s < r

$$U = \{ x = (x_1, x_2, \ldots) \mid |x|_U = \sum_{k=0}^m \sum_{j=0}^\infty |x_{k,j}|^2 e^{2s\alpha_j} \le 1 \}.$$

Here  $x_k \in \Lambda_r(\alpha)$  for every k and  $x_k = (x_{k,0}, x_{k,1}, \ldots)$ .

We choose

$$V = \left\{ x = (x_1, x_2, \ldots) \mid |x|_V = \sum_{k=1}^m \sum_{j=0}^\infty |x_{k,j}|^2 e^{2\sigma\alpha_j} \le 1 \right\}$$

where  $s < \sigma < r$ .

It can easily be seen that the sequence of Kolmogoroff diameters  $\delta_n(V, U)$  or, equivalently, of singular numbers  $s_n(i_V^U)$  where  $i_V^U \colon E_V \longrightarrow E_U$  is the canonical map, has the form

$$(e^{(s-\sigma)\alpha_0},\ldots,e^{(s-\sigma)\alpha_0},e^{(s-\sigma)\alpha_1},\ldots,e^{(s-\sigma)\alpha_1},\ldots),$$

where the term  $e^{(s-\sigma)\alpha_n}$  is repeated *m* times. Therefore for  $\nu \in \mathbb{N}_0, 0 \leq \mu < m$ :

$$\delta_{\nu m+\mu}(V,U) = e^{(s-\sigma)\alpha_{\nu}}.$$

By assumption we have

$$\limsup_{n} \frac{\alpha_{2n}}{\alpha_n} =: d < +\infty$$

hence

$$\limsup_{\nu} \frac{\alpha_{2^{m+1}\nu}}{\alpha_{\nu}} \le d^{m+1}.$$

Therefore we have for large  $\nu$  and  $0 \leq \mu < m$ 

$$\alpha_{m\nu+\mu} \le \alpha_{(m+1)\nu} \le \alpha_{2^{m+1}\nu} \le d^{m+1}\alpha_{\nu},$$

and therefore with suitable c>0 for all  $\nu$ 

$$\alpha_{m\nu+\mu} \le d^{m+1}(\alpha_{\nu}+c) \,.$$

This implies

$$\delta_{\nu m+\mu}(V,U) \le e^{(\sigma-s)c} \cdot e^{(s-\sigma)d^{-m-1}\alpha_{\nu m+\mu}}.$$

Therefore

$$\delta_n(V,U) \le e^{(\sigma-s)c} \cdot e^{(s-\sigma)d^{-m-1}\alpha_n}$$

for all  $n \in \mathbb{N}_0$ .

We obtain

$$|t_n|\delta_n(V,U) \le e^{(\sigma-s)c} \cdot |t_n|e^{(s-\sigma)d^{-m-1}\alpha_n}.$$

For  $r = +\infty$  we choose  $\sigma$  so large that  $(s - \sigma)d^{-m-1} \leq \tau$ , and for  $r < +\infty$  we may choose any  $\sigma$  with  $s < \sigma < r$ . In both cases we have

$$\lim_{n} t_n \delta_n(V, U) = 0.$$

**Definition:** (1) An exponent sequence  $\alpha = (\alpha_0, \alpha_1, ...)$  is called *shift-stable* if

$$\limsup_{n} \frac{\alpha_{n+1}}{\alpha_n} < +\infty.$$

(2) It is called *stable* if

$$\limsup_n \frac{\alpha_{2n}}{\alpha_n} < +\infty.$$

The names are self-explaining in view of Theorem 5.13 and 5.14. Using the description of Theorem 4.7 we generalize now the concept of nuclearity in a twofold way.

**Definition:** A Fréchet-Hilbert space E is called

(1)  $(\alpha, \infty)$ -nuclear if the following holds:

$$\forall U, k \exists V \colon e^{k\alpha_n} \delta_n(V, U) \longrightarrow 0$$

(2)  $(\alpha, 0)$ -nuclear if the following holds:

$$\forall U \exists V, \varepsilon > 0 \colon e^{\varepsilon \alpha_n} \delta_n(V, U) \longrightarrow 0.$$

Here V and U denote absolutely convex neighborhoods of zero. As E is assumed to be a Fréchet-Hilbert space the definition does not depend on whether we use Kolmogoroff or Gelfand diameters, or singular numbers of connecting maps.

**Remark:** If  $\alpha$  is so that  $\Lambda_{\infty}(\alpha)$ , resp.  $\Lambda_0(\alpha)$ , is nuclear, then, due to Theorems 5.8 and 4.7, conditions (1), resp. (2), in the definition imply that E is nuclear. In particular it is automatically a Fréchet-Hilbert space.

**Example:** By Theorem 4.7 *E* is nuclear if, and only if, it is  $(\alpha, \infty)$ -nuclear for  $\alpha_n = \log(n+1)$ .

**5.15 Theorem:** A Fréchet-Hilbert space E is  $(\alpha, r)$ -nuclear if, and only if,  $\Delta(E) \supset \Delta(\Lambda_r(\alpha))$ .

*Proof:* This is immediate from Corollary 5.11.

By means of Theorems 4.10 and 4.14 this implies:

**5.16 Theorem:** If the Fréchet-Hilbert space E is  $(\alpha, r)$ -nuclear and  $F \subset E$  a closed subspace then F and E/F are  $(\alpha, r)$ -nuclear.

Since, due to Corollary 5.11,  $\Delta(\Lambda_{\infty}(\alpha)) \stackrel{\supset}{\Rightarrow} \Delta(\Lambda_{0}(\alpha))$  we obtain

**5.17 Proposition:** (1)  $\Lambda_{\infty}(\alpha)$  is  $(\alpha, r)$ -nuclear,  $r = 0, +\infty$ .

- (2)  $\Lambda_0(\alpha)$  is  $(\alpha, 0)$ -nuclear, however not  $(\alpha, \infty)$ -nuclear.
- (3) Every  $(\alpha, \infty)$ -nuclear space is also  $(\alpha, 0)$ -nuclear.

More generally:

**5.18 Proposition:** For any  $\alpha$  and  $\beta$  we have

- (1)  $\Lambda_{\infty}(\alpha)$  is  $(\beta, r)$ -nuclear if, and only if,  $\limsup_{n \to \infty} \frac{\alpha_n}{\beta_n} < +\infty$ .
- (2)  $\Lambda_0(\alpha)$  is  $(\beta, 0)$ -nuclear if, and only if,  $\limsup_n \frac{\alpha_n}{\beta_n} < +\infty$ .
- (3)  $\Lambda_0(\alpha)$  is  $(\beta, \infty)$ -nuclear if, and only if,  $\lim_n \frac{\alpha_n}{\beta_n} = 0$ .

Proof: Sufficiency of the condition is easily checked. For necessity we have, in case of  $(\beta, \infty)$ -nuclearity, to draw the consequences of  $(e^{-\beta_n})_n \in \Delta(\Lambda_r(\alpha))$ and in case of  $(\beta, 0)$ -nuclearity of the continuity of the imbedding  $\Delta(\Lambda_0(\alpha))$  $\hookrightarrow \Delta(\Lambda_r(\alpha))$ , the first one being a Fréchet space (cf. the proof of Theorem 5.12).

**5.19 Lemma:**  $\Lambda_r(\alpha)^{\mathbb{N}}$  is  $(\alpha, r)$ -nuclear if, and only if,  $\alpha$  is stable.

Proof: As  $\Delta(\Lambda_r(\alpha)^{\mathbb{N}}) \subset \Delta(\Lambda_r(\alpha))$  (see 4.10), Theorem 5.15 implies that  $(\alpha, r)$ -nuclearity is equivalent to  $\Delta(\Lambda_r(\alpha)^{\mathbb{N}}) = \Delta(\Lambda_r(\alpha))$ . Due to Theorem 5.14 this is equivalent to the stability of  $\alpha$ .

So we have:

**5.20 Corollary:** If  $\alpha$  is stable then every subspace of  $\Lambda_r(\alpha)^{\mathbb{N}}$  is  $(\alpha, r)$ -nuclear.

This is one direction of a generalization of Theorem 3.18. The full generalization can be given under the additional assumption of nuclearity.

**5.21 Theorem:** If  $\alpha$  is stable and  $\Lambda_r(\alpha)$  nuclear then the following are equivalent

- (1) E is  $(\alpha, r)$ -nuclear,
- (2) E is isomorphic to a subspace of  $\Lambda_r(\alpha)^{\mathbb{N}}$ .

*Proof:* (2) $\Rightarrow$ (1) follows from the previous Corollary and the fact that the diametral dimension and therefore  $(\alpha, r)$ -nuclearity is a linear topological invariant.

 $(1) \Rightarrow (2)$  is shown for  $r = 0, +\infty$  separately. First we assume  $r = +\infty$ . We proceed as in the proof of T. and Y. Komura's Theorem 3.18.

We fix K and denote by

$$j_k \colon E_K^* \hookrightarrow E_{K+k}^*$$

the canonical imbedding. Let  $(A(k))_{k\in\mathbb{N}}$  be a sequence which will be determined later. We may assume

$$s_n(j_k)e^{A(k)\alpha_n} \longrightarrow 0$$

for every k. The Schmidt representation is

$$j_k(x) = \sum_{n=0}^{\infty} s_n(j_k) \langle x, e_n^k \rangle_K f_n^k.$$
(7)

We put  $s_{n,k} := s_n(j_k)$  and set for  $k \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ 

$$l(n,k) := 2^{k-1}(1+2n) - 1.$$

We apply the Gram-Schmidt orthogonalization method with respect to  $\langle \cdot, \cdot \rangle_K$  to the vectors  $e_n^k$ ,  $l(k,n) = 0, 1, 2, \ldots$  and obtain an orthonormal sequence  $e_0, e_1, \ldots$ 

By construction we have  $e_m \perp e_n^k$  for m > l(n,k) or, equivalently,  $m \ge 2^{k-1}(1+2n)$  which is equivalent to  $n \le \frac{1}{2}(2^{-k+1}m-1)$ . We set

$$n(m) = \left[\frac{1}{2}(2^{-k+1}m - 1)\right]$$

and obtain by use of (7)

$$\|e_m\|_{K+k}^{*^2} = \sum_{n>n(m)} s_{n,k}^2 |\langle e_m, e_n^k \rangle_K|^2 \le s_{n(m)+1}^2$$
  
$$\le C_k^2 e^{-2A(k)\alpha_{n(m)+1}}$$

with suitable  $C_k$ . We have

$$n(m) + 1 = \left[\frac{1}{2}(2^{-k+1}m + 1)\right] \ge [2^{-k}m].$$

Putting  $s := \sup_{n \ge n_0} \frac{\alpha_{2n}}{\alpha_n}$ , where  $\alpha_n > 0$  for  $n \ge n_0$ , we have for large m

$$\alpha_m \le \alpha_{2^{k+1}[2^{-k}m]} \le s^{k+1}\alpha_{[2^{-k}m]} \le s^{k+1}\alpha_{n(m)+1}.$$

Therefore we obtain

$$\alpha_m \le s^{k+1} \alpha_{n(m)+1} + d_k$$

for all m. We choose  $A(k) = ks^{k+1}$  and have with possibly increased  $C_k$ 

$$\|e_m\|_{K+k}^* \le C_k e^{-2k\alpha_m}$$

For  $x \in E$  this gives

$$\sum_{m} |e_m(x)|^2 = \sum_{m} |\langle \tilde{x}, e_m \rangle_K|^2 = \|\tilde{x}\|_K^2 = \|x\|_K^2$$

Here  $\tilde{x} \in E_K^*$  is chosen by the Riesz representation theorem to fulfill  $y(x) = \langle y, \tilde{x} \rangle_K$  for  $y \in E_K^*$ .

Moreover

$$\sum_{m} |e_{m}(x)|^{2} e^{2k\alpha_{m}} \leq ||x||_{K+L}^{2} \sum_{m} ||e_{m}||_{K+L}^{*^{2}} e^{2k\alpha_{m}}$$
$$\leq ||x||_{K+L}^{2} C_{L} \sum_{m} e^{2(k-L)\alpha_{m}}$$

where the series in the last term converges for L large enough.

Therefore  $\varphi_K \colon x \mapsto (e_m(x))_{m \in \mathbb{N}_0}$  defines a continuous linear map

$$\varphi_K \colon E \longrightarrow \Lambda_\infty(\alpha)$$

such that

$$|\varphi_K x|_0 = ||x||_K.$$

Consequently  $\varphi \colon x \mapsto (\varphi_K(x))_{K \in \mathbb{N}}$  defines a continuous linear map,

$$\varphi \colon E \longrightarrow \Lambda_{\infty}(\alpha)^{\mathbb{N}}$$

such that

$$\max_{K=1,\dots,L} |\varphi_K(x)|_0 = ||x||_L.$$

Hence  $\varphi$  is a topological linear imbedding.

Next we show (1) $\Longrightarrow$ (2) for r = 0. We may assume again that all  $|| ||_k$  are Hilbertian seminorms, that  $i_{k+1}^k$  is compact and

$$s_n(\iota_{k+1}^k)e^{\varepsilon_k\alpha_n}\longrightarrow 0$$

for all k with suitable  $\varepsilon_k > 0$ .

The Schmidt representation gives with  $s_n = s_n(i_{k+1}^k)$ 

$$u_{k+1}^k(x) = \sum_{n=0}^{\infty} s_n \langle x, e_n \rangle_{k+1} f_k.$$

Here  $(e_n)_{n \in \mathbb{N}_0}$  and  $(f_n)_{n \in \mathbb{N}_0}$  are orthonormal systems in the local Banach spaces  $E_{k+1}$  and  $E_k$ , respectively. The first one needs not to be an orthonormal basis, the second one is an orthonormal basis.

We put for  $x \in E$ , omitting the maps  $i^k$  and  $i^{k+1}$ ,

$$\varphi_k(x) = (\langle x, f_n \rangle_k)_{k \in \mathbb{N}_0}$$

and obtain, using  $\langle x, f_n \rangle_k = s_n \langle x, e_n \rangle_{k+1}$ ,

$$\begin{split} \sum_{n} e^{2\varepsilon_k \alpha_n} |\langle x, f_n \rangle_k|^2 &= \sum_{n} e^{2\varepsilon_k \alpha_n} s_n^2 |\langle x, e_n \rangle_{k+1}|^2 \\ &\leq C_k^2 \sum_{n} |\langle x, e_n \rangle_{k+1}|^2 \\ &\leq C_k^2 \|x\|_{k+1}^2, \end{split}$$

with suitable  $C_k > 0$ . Therefore  $\varphi_k$  defines a continuous (even compact) linear map  $E \longrightarrow \Lambda_{\varepsilon_k}(\alpha)$ , such that

$$|\varphi_k(x)|_0 = \left(\sum_n |\langle x, f_n \rangle|_k^2\right)^{\frac{1}{2}} = ||x||_k.$$
 (8)

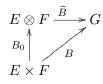
By setting  $\varphi(x) := (\varphi_k(x))_{k \in \mathbb{N}}$  we define a continuous linear map

$$\varphi \colon E \longrightarrow \prod_{k} \Lambda_{\varepsilon_{k}}(x) \cong \Lambda_{0}(\alpha)^{\mathbb{N}},$$

which is, due to (8), an isomorphic imbedding.

## 6 Tensor products and nuclearity

We recall the definition of the algebraic tensor product. Let E and F be linear spaces. A linear space  $E \otimes F$  together with a bilinear map  $B_0: E \times F \longrightarrow E \otimes F$  is called tensor product of E and F if the following holds: for every bilinear map  $B: E \times F \longrightarrow G$ , G linear space, there is exactly one linear map  $\tilde{B}: E \otimes F \longrightarrow G$  with  $\tilde{B} \circ B_0 = B$ .



If  $(E \otimes F)_1$  with  $B_1: E \times F \longrightarrow (E \otimes F)_1$  is a second tensor product then we have

$$E \otimes F \xrightarrow{\widetilde{B}_1} (E \otimes F)_1$$

$$B_0 \xrightarrow{\widetilde{B}_0} B_1$$

$$E \times F.$$

Since  $\widetilde{B}_0 \circ \widetilde{B}_1 \circ B_0 = \widetilde{B}_0 \circ B_1 = B_0 = \operatorname{id} \circ B_0$  we have  $\widetilde{B}_0 \circ \widetilde{B}_1 = \operatorname{id}$  and likewise  $\widetilde{B}_1 \circ \widetilde{B}_0 = \operatorname{id}$ . So  $E \otimes F$  and  $(E \otimes F)_1$  are isomorphic, with an isomorphism compatible with  $B_0$  and  $B_1$ . In this sense the tensor product is uniquely determined up to isomorphism.

We set  $x \otimes y := B_0(x, y)$ . Since  $\tilde{B}$  is uniquely determined for any G and B we have clearly span $\{x \otimes y \mid x \in E, y \in F\} = E \otimes F$ . So taking into account the bilinearity of  $(x, y) \mapsto x \otimes y$  we see that every  $u \in E \otimes F$  can be written as a finite sum  $u = \sum_j x_j \otimes y_j$ .

This yields also a proof of the existence of a tensor product. We set

$$H = \{(\xi_{x,y})_{x \in E, y \in F} \in \mathbb{K} \mid \text{ only finitely many } \xi_{x,y} \neq 0\}$$

$$H_{0} = \operatorname{span}\{\xi_{\lambda x_{1}+\mu x_{2},y} - \lambda \xi_{x_{1},y} - \mu \xi_{x_{2},y}, \xi_{x,\lambda y_{1}+\mu y_{2}} - \lambda \xi_{x,y_{1}} - \mu \xi_{x,y_{2}} \\ | x \in E, y \in F, \lambda, \mu \in \mathbb{K}\}$$

and set  $E \otimes F := H/H_0$  and  $B_0(x, y) = \hat{e}_{x,y}$ , where  $e_{x,y}$  is the natural unit basis vector in H and  $\hat{e}_{x,y}$  its residue class.

We turn now to locally convex spaces.

**Definition:** The  $\pi$ -tensor product of two locally convex spaces E and F is their tensor product equipped with the uniquely defined locally convex topology on  $E \otimes F$  such that  $B_0$  is continuous and for any continuous linear map  $B: E \times F \longrightarrow G$ , G locally convex, the map  $\widetilde{B}$  is continuous. It is denoted by  $E \otimes_{\pi} F$ .

The uniqueness of the topology follows from the argument proving the "uniqueness" of  $E \otimes F$ . To see the existence we put for any continuous seminorms  $p_1$  on E,  $p_2$  on F and  $u \in E \otimes F$ 

$$p_1 \otimes_{\pi} p_2(u) = \inf \bigg\{ \sum_j p_1(x_j) p_2(y_j) \mid u = \sum_j x_j \otimes y_j \bigg\}.$$

It is easy to see that  $p_1 \otimes_{\pi} p_2$  is a seminorm on  $E \otimes F$ . We further note that for locally convex spaces E, F, G a bilinear map  $B: E \times F \longrightarrow G$  is continuous if and only if for any seminorm q on G there are seminorms  $p_1$ on E and  $p_2$  on F such that

$$q(B(x,y)) \le p_1(x)p_2(y)$$

for all  $x \in E, y \in F$ .

Hence for any representation  $u = \sum_j x_j \otimes y_j$  we have

$$q(\widetilde{B}(u)) \leq \sum_{j} q(\widetilde{B}(x_{j} \otimes y_{j}))$$
$$= \sum_{j} q(B(x_{j}, y_{j}))$$
$$\leq \sum_{j} p_{1}(x_{j})q_{2}(y_{j})$$

and therefore  $q(\widetilde{B}(u)) \leq p_1(x) \otimes_{\pi} p_2(u)$ .

Obviously  $p_1 \otimes_{\pi} p_2(x \otimes y) \leq p_1(x)p_2(y)$  hence  $B_0$  is continuous. This proves the existence of the  $\pi$ -tensor product topology.

Moreover we have:

**6.1 Lemma:** For  $x \in E$ ,  $y \in F$  we have  $p_1 \otimes_{\pi} p_2(x \otimes y) = p_1(x)p_2(y)$ .

Proof:  $p_1 \otimes_{\pi} p_2(x \otimes y) \leq p_1(x)p_2(y)$  is clear. To prove the opposite inequality we choose linear forms  $\varphi \in E'$ ,  $\psi \in F'$  such that  $|\varphi(\xi)| \leq p_1(\xi)$  for all  $\xi \in E$ ,  $|\psi(\eta)| \leq p_2(\eta)$  for all  $\eta \in F$  and  $\varphi(x) = p_1(x)$ ,  $\psi(y) = p_2(y)$ . For the bilinear form  $B(\xi, \eta) = \varphi(\xi)\psi(\eta)$  we have  $|B(\xi, \eta)| \leq p_1(\xi)p_2(\eta)$  hence, due to the previous calculation  $|B(u)| \leq p_1 \otimes_{\pi} p_2(u)$  for all  $u \in E \otimes_{\pi} F$ . This gives

$$p_1(x)p_2(y) = |B(x \otimes y)| \le p_1 \otimes_{\pi} p_2(x \otimes y).$$

From the previous considerations we obtain:

**6.2 Theorem:** If  $\mathscr{P}_1$  (resp.  $\mathscr{P}_2$ ) is a fundamental system of seminorms on E (resp. F) then  $\{p_1 \otimes_{\pi} p_2 \mid p_1 \in \mathscr{P}_1, p_2 \in \mathscr{P}_2\}$  is a fundamental system of seminorms on  $E \otimes_{\pi} F$ .

As an immediate consequence of this theorem we have the

**6.3 Corollary:** If *E* and *F* are normed then  $E \otimes_{\pi} F$  is normed, if *E* and *F* are metrizable locally convex, then also  $E \otimes_{\pi} F$ .

**Definition:** The completion  $E \hat{\otimes}_{\pi} F$  is called the complete  $\pi$ -tensor product of E and F.

**Remark:** If *E* and *F* are complete then  $E \hat{\otimes}_{\pi} F$  fulfills the defining properties of a tensor product in the class of complete locally convex spaces.

We denote by  $p_1 \hat{\otimes}_{\pi} p_2$  the continuous extension of  $p_1 \otimes_{\pi} p_2$  onto  $E \hat{\otimes}_{\pi} F$ . Then clearly the seminorms  $p_1 \hat{\otimes} p_2$ ,  $p_1 \in \mathscr{P}_1$ ,  $p_2 \in \mathscr{P}_2$ , where  $\mathscr{P}_1, \mathscr{P}_2$  are fundamental systems of seminorms on E and F, respectively, are a fundamental system of seminorms for  $E \hat{\otimes}_{\pi} F$ . This proves:

**6.4 Theorem:** If E and F are Banach spaces then  $E \hat{\otimes}_{\pi} F$  is a Banach space, if E and F are Fréchet spaces then  $E \hat{\otimes}_{\pi} F$  is a Fréchet space.

If *E* and *F* are Fréchet spaces with fundamental system of seminorms  $|| ||_1 \leq || ||_2 \leq \ldots$  then we set  $|| ||_k := || ||_k \hat{\otimes}_{\pi} || ||_k$  on the Fréchet space  $E \hat{\otimes}_{\pi} F$ , and  $|| ||_1 \leq || ||_2 \leq \ldots$  is a fundamental systems of seminorms there.

**6.5 Theorem:** Let *E* and *F* be Fréchet spaces. Then every  $u \in E \hat{\otimes}_{\pi} F$  has an expansion  $u = \sum_{j=1}^{\infty} x_j \otimes y_j$  such that  $\sum_{j=1}^{\infty} ||x_j||_k ||y_j||_k < 1$  for all *k*. We have  $||u||_k = \inf\{\sum_{j=1}^{\infty} ||x_j||_k ||y_j||_k\}$  where the infimum runs through all such representations.

Proof: We fix  $\varepsilon > 0$ . Then it is easy to see that u can be written as  $u = \sum_{k=1}^{\infty} u_k$  where  $u_k \in E \otimes F$  and  $||u_k||_k < 2^{-k-1}\varepsilon$  for  $k = 2, 3, \ldots$ 

We choose for every k a representation

$$u_k = \sum_{j=1}^{m_k} x_j^k \otimes y_j^k$$

such that

$$\sum_{j=1}^{m_k} ||x_j^k||_k |y_j^k||_k \le 2^{-k-1}\varepsilon + ||u_k||_k.$$

Then we have

$$u = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} x_j^k \otimes y_j^k$$

and for any  $m \in \mathbb{N}$ 

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} ||x_j^k||_m ||y_j^k||_m \le \sum_{k=1}^{m} \sum_{j=1}^{m_k} ||x_j^k||_m ||y_j^k||_m + \sum_{k=m+1}^{\infty} 2^{-k} \varepsilon < +\infty.$$

For m = 1 we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} ||x_j^k||_1 ||y_j^k||_1 \le ||u_1||_1 + \frac{3}{4}\varepsilon$$

and by choice of the  $u_k$ 

$$||u_1||_1 \le ||u||_1 + \frac{1}{2}\varepsilon.$$

This proves

$$||u||_1 \ge \inf \{\sum_{j=1}^{\infty} ||x_j||_1 ||y_j||_1 | \ldots \}.$$

The reverse inequality follows from the triangular inequality and Lemma 6.1. Replacing 1 with k in the whole argument proves the general assertion.  $\Box$ 

**Remark:** In the previous proof we may assume only that E and F are locally convex and metrizable and we get the same representation for the elements of  $E\hat{\otimes}_{\pi}F = \hat{E}\hat{\otimes}_{\pi}\hat{F}$  which implies that in the case of Fréchet spaces the representations can be chosen from any given dense subspaces.

We study now tensor products of continuous linear maps. For locally convex spaces  $F_1$  and  $F_2$  and  $A_1 \in L(E_1, F_1)$ ,  $A_2 \in L(E_2, F_2)$  we consider the diagram

which defines the map  $A_1 \otimes A_2$ . Clearly  $A_1 \otimes A_2 \in L(E_1 \otimes E_2, F_1 \otimes F_2)$  and it extends to a map  $A_1 \otimes A_2 \in L(E_1 \otimes_{\pi} E_2, F_1 \otimes_{\pi} F_2)$ . We have  $A_1 \otimes A_2(x \otimes y) = A_1 x \otimes A_2 y$ .

If  $E_1, E_2$  are locally convex and  $A_1: E_1 \longrightarrow F_1, A_2: E_2 \longrightarrow F_2$  are surjective and open then clearly

$$A_1 \otimes A_2 \colon E_1 \otimes_{\pi} E_2 \longrightarrow F_1 \otimes_{\pi} F_2$$

is surjective and open. In general it does not follow that the extension to the completion is again surjective. However, in the case of Fréchet spaces the situation is nice: **6.6 Theorem:** If  $A_1$  and  $A_2$  are surjective,  $F_1$  and  $F_2$  Fréchet spaces, then  $A_1 \hat{\otimes} A_2 \colon E_1 \hat{\otimes}_{\pi} E_2 \longrightarrow F_1 \hat{\otimes}_{\pi} F_2$  is surjective.

Proof: Let  $|| ||_1 \leq || ||_2 \leq ...$  be fundamental systems of seminorms in the respective spaces, moreover  $|| ||_k$  in  $F_j$  the quotient seminorm of  $|| ||_k$  in  $E_j$  under  $A_j$ . Let

$$u = \sum_{j=1}^{\infty} x_j \otimes y_j, \quad \sum_{j=1}^{\infty} ||x_j||_k ||y_j||_k < +\infty \text{ for all } k$$

be a representation of  $u \in F_1 \hat{\otimes}_{\pi} F_2$ . By assumption we may find  $\xi_j \in E_1$ ,  $\eta_j \in E_2$  such that  $A_1 \xi_j = x_j$ ,  $A_2 \eta_j = y_j$  and

$$||\xi_j||_n ||\eta_j||_n \le ||x_j||_n ||y_j||_n + 2^{-j}$$

for all j.

Therefore  $v := \sum_{j=1}^{\infty} \xi_j \otimes \eta_j \in E_1 \hat{\otimes}_{\pi} E_2$  and  $A \otimes B(v) = u$ .

For the kernels we can not do much more at present, than to describe the algebraic situation.

**6.7 Lemma:**  $\ker(A_1 \otimes A_2) = \ker A_1 \otimes E_2 + E_1 \otimes \ker A_2$  for all linear maps  $A_1 : E_1 \longrightarrow F_1$  and  $A_2 : E_2 \longrightarrow F_2$ .

Proof: Assume  $u = \sum_{j=1}^{m} x_j \otimes y_j \in E_1 \otimes E_2$  and  $A_1 \otimes A_2 u = 0$ . We may assume additionally that the vectors  $y_1, \ldots, y_m$  are linearly independent and that  $y_1, \ldots, y_k$  span the vector space ker  $A_2 \cap \text{span}\{y_1, \ldots, y_m\}$ .

Then dim span $\{A_2y_{k+1}, \ldots, A_2y_m\} = m - k$ . Therefore

$$\{(\varphi A_2 y_{k+1}, \dots, \varphi A_2 y_m) \mid \varphi \in F_2'\} = \mathbb{R}^{m-k}.$$

Since  $0 = (\mathrm{id} \otimes \varphi)(A_1 \otimes A_2)u = \sum_{j=k+1}^m \varphi A_2 y_j A x_j$  for all  $\varphi \in F'_2$ , we have  $A_1 x_j = 0$  for  $j = k+1, \ldots, m$ , that is  $x_{k+1}, \ldots, x_m \in \ker A_1$ .

Notice that there is no need that  $\ker(A \otimes A_2) = \ker A \otimes E_2 + E_1 \otimes \ker A_2$ .

Let E and F be linear spaces, G a linear space of linear forms on E which separates points and let  $\mathscr{L}(G, F)$  denote the linear maps from E to F. Then the bilinear map  $B: E \times F \longrightarrow \mathscr{L}(G, F)$  given by B(e, f)(g) = g(e)fextends to an injective linear map  $\widetilde{B}: E \otimes F \longrightarrow \mathscr{L}(G, F)$ . Here injectivity follows easily by assuming, as in the previous proof, that for a given  $u = \sum_{j=1}^{m} x_j \otimes y_j \in E \otimes F$  the vectors  $y_1, \ldots, y_m$  are linearly independent.

We study this map now in the case of Banach spaces. Let X, Y be Banach spaces, then the continuous bilinear map  $B: X' \times Y \longrightarrow L(X,Y)$  given by B(x',y)(x) = x'(x)y extends to a continuous linear map  $\tilde{B}: X' \otimes_{\pi} Y \longrightarrow$ L(X,Y). Likewise the continuous bilinear map  $B: X \times Y \longrightarrow L(X',Y)$ given by B(x,y)(x') = x'(x)y extends to a continuous linear map  $\tilde{B}: X \otimes_{\pi} Y \longrightarrow$ L(X',Y). Both are injective.  $\tilde{B}$  again extends to a continuous linear map  $B_{L(X,Y)}: X' \otimes_{\pi} Y \longrightarrow L(X,Y)$  or  $B_{L(X',Y)}: X \otimes_{\pi} Y \longrightarrow L(X',Y)$ , respectively. Notice that neither of them needs to be injective.

As an immediate consequence of Theorem 6.5 we obtain.

**6.8 Corollary:**  $R(B_{L(X,Y)}) = \mathcal{N}(X,Y)$  topologically, i.e. the quotient norm under  $B_{L(X,Y)}$  is the nuclear norm, in particular  $\mathcal{N}(X,Y)$  is a Banach space.

If  $N(B_{L(X,Y)}) = \{0\}$  then  $B_{L(X,Y)}$  constitutes an isomorphism between  $X' \hat{\otimes}_{\pi} Y$  and  $\mathcal{N}(X,Y)$ . We will now study, when this is the case.

**Definition:** A Banach space X has the approximation property if for every compact  $K \subset X$  and  $\varepsilon > 0$  there is a finite dimensional map  $\varphi \in L(X)$  such that  $\sup_{x \in K} ||x - \varphi(x)|| < \varepsilon$ .

**Example:** (1) Every  $\ell^p$ ,  $0 \le p < +\infty$  has the approximation property.

- (2) Every Hilbert space has the approximation property.
- Proof: (1) Set  $P_n(x) = (x_1, \ldots, x_n, 0, \ldots)$  for  $x = (x_k)_{k \in \mathbb{N}}$  then  $||x P_n(x)||$  is decreasing and converges to zero for every x, so by Dini's theorem it converges uniformly for every compact set  $K \subset X$ .
  - (2) We choose an orthonormal basis  $(e_i)_{i \in I}$  and for every finite set  $e \subset I$  we put  $P_e x = \sum_{i \in e} \langle x, e_i \rangle e_i$ . Then we proceed as in (1).  $\Box$

We denote, as previously, by  $\mathscr{F}(X, Y)$  the linear space of finite dimensional maps in L(X, Y). Clearly the canonical map  $X' \otimes Y \longrightarrow L(X, Y)$  sends  $X' \otimes Y$  bijectively onto  $\mathscr{F}(X, Y)$ . We have  $\overline{\mathscr{F}(X, Y)} \subset K(X, Y)$ , the space of compact operators in L(X, Y).

6.9 Theorem: The following are equivalent

- (1) Y has the approximation property.
- (2)  $\overline{\mathscr{F}(X,Y)} = K(X,Y)$  for every Banach space X.

Proof: (1)  $\Rightarrow$  (2) Let  $U_X$  be the closed unit ball in  $X, T \in K(X, Y)$ . We set  $K = \overline{TU_X}$ . For given  $\varepsilon > 0$  we find  $\varphi \in \mathscr{F}(Y)$  such that  $||y - \varphi(y)|| < \varepsilon$  for all  $y \in K$ . Then  $||Tx - \varphi \circ T(x)|| < \varepsilon$  for all  $x \in U_X$ , hence  $||T - \varphi \circ T|| \le \varepsilon$ . Clearly  $\varphi \circ T \in \mathscr{F}(X, Y)$ .

 $(2) \Rightarrow (1)$  We choose an absolutely convex compact set  $L, K \subset L \subset Y$ , such that K is compact in  $E_L$ . Let  $j: Y_L \hookrightarrow Y$  be the identical imbedding. Since  $j \in K(Y_L, Y)$  we find for every  $\varepsilon > 0$  a map  $\varphi \in \mathscr{F}(Y_L, Y)$  with  $\sup_{y \in L} ||y - \varphi(y)|| < \varepsilon/2$ . We have

$$\varphi(y) = \sum_{j=1}^m \eta_j(y) y_j$$

with some  $m \in \mathbb{N}$ , where  $\eta_1, \ldots, \eta_m \in Y'_L, y_1, \ldots, y_m \in Y$ . We may assume  $||\eta_j||_L^* \leq 1$  for all j.

Let M be the unit ball of  $Y'_L$ , and

$$M_0 = \{\eta|_{Y_L} \mid \eta \in Y', \ |\eta(y)| \le 1 \text{ for all } y \in L\} = \{\eta|_{Y_L} \mid \eta \in L^\circ\}.$$

Here the polar is taken in Y'. From the Bipolar Theorem 2.1 for the duality of Y and Y' we obtain  $M_0^\circ = L$ , independently of whether we think of this polar being taken in Y or  $Y_L$ . Therefore we have  $(M_0^\circ)^\circ = L^\circ = M$ , where the polars are taken in the duality of  $Y_L$  and  $Y'_L$ . Hence, by the Bipolar Theorem 2.1  $M_0$  is  $\sigma(Y'_L, Y_L)$ -dense in M. Since M is bounded and an equicontinuous set of continuous functions on the  $|| \ ||_L$ -compact set K we have, due to the Arzelà-Ascoli Theorem, that  $M_0$  is dense in M in the C(K)topology, i.e. we can find  $\xi_1, \ldots, \xi_n \in Y'$  such that  $\sup_{x \in K} |\xi_j(x) - \eta_j(x)| < \delta$ where  $\delta \sum_{j=1}^m ||y_j|| < \varepsilon/2$ . Therefore  $\psi(y) = \sum_{j=1}^m \xi_j(y)y_j \in \mathscr{F}(Y)$  and  $\sup_{y \in K} ||y - \psi(y)|| < \varepsilon$ .

We prove now a series of lemmas which altogether will end up in Theorem 6.13 which gives a lot of equivalences for the approximation property among them the one we are looking for, namely the injectivity of the canonical map from the complete  $\pi$ -tensor products to the respective spaces of linear operators.

**6.10 Lemma:** If Y has the approximation property then the canonical map  $B_{L(X',Y)} \colon X \hat{\otimes}_{\pi} Y \longrightarrow L(X',Y)$  is injective for every Banach space X.

Proof: For  $u = \sum_{k=1}^{\infty} x_k \otimes y_k \in X \hat{\otimes}_{\pi} Y 8$  with  $\sum_{k=1}^{\infty} ||x_k|| ||y_k|| < \infty$  we assume  $B_{L(X',Y)}(u) = 0$ , that is

$$\sum_{k=1}^{\infty} \xi(x_k) y_k = 0, \text{ for all } \xi \in X'.$$

We may assume that  $\sum_{k} ||x_{k}|| = 1$  and  $\lim_{k} ||y_{k}|| = 0$ . We set  $K = \{y_{k} \mid k \in \mathbb{N}\}$  and find  $\varphi \in \mathscr{F}(Y)$  such that  $\sup_{k} ||y_{k} - \varphi(y_{k})|| < \varepsilon$ .

Assume that

$$\varphi(y) = \sum_{j=1}^{m} \eta_j(y) z_j,$$

where  $\eta_1, \ldots, \eta_m \in Y'$  and  $z_1, \ldots, z_m \in Y$ . For  $v = (id \otimes \varphi)u$  we obtain

$$v = \sum_{k=1}^{\infty} x_k \otimes \varphi(y_k)$$
$$= \sum_{k=1}^{\infty} x_k \otimes \sum_{j=1}^{m} \eta_j(y_k) z_j$$
$$= \sum_{j=1}^{m} \left( \sum_{k=1}^{\infty} \eta_j(y_k) x_k \right) \otimes z_j.$$

Therefore  $v \in X \otimes Y$  and

$$(B_{L(X',Y)}v)(\xi) = \sum_{j=1}^{m} \xi\left(\sum_{k=1}^{\infty} \eta_j(y_k) \, x_k\right) z_j = \sum_{j=1}^{m} \eta_j\left(\sum_k y_k \, \xi(x_k)\right) z_j = 0$$

for all  $\xi$ . As  $B_{L(X',Y)} : X' \otimes Y \hookrightarrow L(X,Y)$  is injective we have v = 0 and therefore

$$||u|| = ||u - v|| \le \sum_{k=1}^{\infty} ||x_k|| ||y_k - \varphi(y_k)|| \le \varepsilon.$$

Since this holds for every  $\varepsilon > 0$  the proof is complete.

**6.11 Lemma:** If  $B_{L(X',Y)} \colon X \hat{\otimes}_{\pi} Y \longrightarrow L(X',Y)$  is injective for every Banach space X, then also  $B_{L(X,Y)} \colon X' \hat{\otimes}_{\pi} Y \longrightarrow L(X,Y)$  is injective for every Banach space X.

Proof: For  $u = \sum_{k=1}^{\infty} \xi_k \otimes y_k \in X' \hat{\otimes}_{\pi} Y$  with  $\sum_{k=1}^{\infty} ||\xi_k|| ||y_k|| < \infty$  we assume  $B_{L(X,Y)}(u) = 0$ , i.e.

$$\sum_{k=1}^{\infty} \xi_k(x) y_k = 0, \text{ for all } x \in X.$$

Then we have for every  $\eta \in Y'$  and  $x \in X$ 

$$\left(\sum_{k=1}^{\infty} \eta(y_k)\xi_k\right)(x) = \eta\left(\sum_{k=1}^{\infty} \xi_k(x)y_k\right) = 0.$$

Therefore  $\sum_{k=1}^{\infty} \eta(y_k) \xi_k = 0$  for every  $\eta \in Y'$ . Now we take  $\zeta \in X'', \eta \in Y'$  and obtain

$$\eta\left(\sum_{k=1}^{\infty}\zeta(\xi_k)y_k\right) = \zeta\left(\sum_{k=1}^{\infty}\eta(y_k)\xi_k\right) = 0.$$

This proves that

$$\sum_{k=1}^{\infty} \zeta(\xi_k) y_k = 0$$

for every  $\zeta \in X''$ , i.e.  $B_{L(X'',Y)}u = 0$ . By the assumption, applied to the Banach space X', we get u = 0.

If Y is a Banach space then the continuous bilinear form  $(\eta, y) \mapsto \eta(y)$  on  $Y' \times Y$  generates a continuous linear form on  $Y' \hat{\otimes}_{\pi} Y$  which is called *tensor* trace.

If the tensor trace vanishes on  $N(B_{L(Y)})$  then, by Corollary 6.8, for a nuclear map  $T \in \mathcal{N}(Y)$  with

$$Tx = \sum_{k=1}^{\infty} \eta_k(x) y_k, \qquad \sum_{k=1}^{\infty} \|\eta_k\| \|y_k\| < \infty$$

the number

$$\sum_{k=1}^{\infty} \eta_k(y_k)$$

does not depend on the representation of T and defines a continuous linear form on  $\mathcal{N}(Y)$  which is called *trace*.

In this case we say that  $\mathcal{N}(Y)$  admits a trace.

If  $B_{L(X,Y)}$  is injective for every Banach space X then, in particular,  $N(B_{L(Y)}) = 0$  and therefore  $\mathcal{N}(Y)$  admits a trace.

**6.12 Lemma:** If  $\mathscr{N}(Y)$  admits a trace, then Y has the approximation property.

*Proof:* Let  $K \subset Y$  be compact. By Lemma 3.6 we may assume that there is a null sequence  $(x_n)_{n \in \mathbb{N}}$  in Y such that

$$K = \left\{ \sum_{k=1}^{\infty} \lambda_k x_k \mid \sum_{k=1}^{\infty} |\lambda_k| \le 1 \right\}.$$

We set

 $H = \{A \in L(Y_K, Y) \mid A|_K \text{ is continuous with respect to the } Y - \text{topology} \}.$ 

*H* is a closed subspace of the Banach space  $L(Y_K, Y), L(Y)|_{Y_K} \subset H$  and for every  $A \in H$  we have  $\lim_k Ax_k = 0$ .

Therefore  $\varphi(A) := (Ax_k)_{k \in \mathbb{N}}$  defines a linear map  $H \longrightarrow c_0(Y)$  which, due to

$$\sup_{x \in K} \|Ax\| = \sup_{k} \|Ax_k\| \text{ for all } A \in H,$$

is an isometric imbedding. If we identify in canonical way the dual space  $c_0(Y)'$  with  $\ell_1(Y')$ , then we obtain that  $\varphi' : \ell_1(Y') \longrightarrow H'$  is surjective. Hence for every  $\mu \in H'$  there is  $\eta = (\eta_1, \eta_2, \dots) \in \ell_1(Y')$  such that

$$\mu A = (\varphi'\eta)A = \langle \eta, \varphi A \rangle = \sum_{k=1}^{\infty} \eta_k(Ax_k).$$

Since  $\sum_{k=1}^{\infty} \|\eta_k\| \|x_k\| < \infty$  we may set

$$u := \sum_{k=1}^{\infty} \eta_k \otimes x_k \in Y' \hat{\otimes}_{\pi} Y.$$

Let us assume that  $\mu$  vanishes on  $\mathscr{F}(Y)$ , i.e.  $\mu(A) = 0$  for all  $Ax = \eta(x)y$ with  $\eta \in Y', y \in Y$ . Then

$$0 = \mu(A) = \sum_{k=1}^{\infty} \eta_k(y)\eta(x_k) = \eta\left(\sum_{k=1}^{\infty} \eta_k(y)x_k\right)$$

for all  $\eta \in Y'$  where we consider  $y \in Y$  as fixed.

Therefore we have

$$\sum_{k=1}^{\infty} \eta_k(y) x_k = 0$$

for all  $y \in Y$ , i.e.  $B_{L(Y)}(u) = 0$ .

By assumption, now the tensor trace must be zero, i.e.

$$0 = \sum_{k=1}^{\infty} \eta_k(x_k) = \mu(j),$$

where  $j: Y_K \longrightarrow Y$  denotes the identical imbedding. Notice that  $j \in H$ .

By the Bipolar Theorem 2.1, applied to H, we have proved that  $j \in \overline{\mathscr{F}(Y)}$ , where the closure is taken in H or in  $L(Y_K, Y)$  which is the same. Therefore we have for any  $\varepsilon > 0$  a map  $\varphi \in \mathscr{F}(Y)$ , such that

$$\sup_{x\in K} \|x-\varphi(x)\| < \varepsilon.$$

This completes the proof.

Putting together the previous lemmas and Theorem 6.9 we obtain:

**6.13 Theorem:** For a Banach space Y the following are equivalent:

- 1. Y has the approximation property.
- 2.  $\overline{\mathscr{F}(X,Y)} = K(X,Y)$  for every Banach space X.
- 3.  $B_{L(X',Y)}: X \hat{\otimes}_{\pi} Y \longrightarrow L(X',Y)$  is injective for every Banach space X.
- 4.  $B_{L(X,Y)}: X' \hat{\otimes}_{\pi} Y \longrightarrow L(X,Y)$  is injective for every Banach space X.
- 5.  $\mathcal{N}(Y)$  admits a trace.

### 6.1 Tensor products with function spaces

Let M be a set, L a linear space of scalar functions on M and X a linear space. Then we get from  $f \otimes x \mapsto xf(\cdot)$  a linear imbedding  $L \otimes X \hookrightarrow X^M$ . In this way we identify from now on  $L \otimes X$  with a linear subspace of  $X^M$ .

**6.14 Lemma:**  $L \otimes X$  is the set of all  $f \in X^M$  with finite dimensional range, such that for some (any) basis  $e_1, \ldots, e_m$  of span f(M) the coordinate functions are in L.

Notice that for any such basis we have a unique expansion

$$f = \sum_{j=1}^{m} f_j e_j$$

with scalar valued functions  $f_j$ . They are called coordinate functions.

The easy proof of the lemma is left to the reader.

- **Example:** (1) Let M be a topological space, X a topological vector space. Then  $\mathscr{C}(M) \otimes X$  is the space of all continuous X-valued functions on M with finite dimensional range.
  - (2) Let  $\Omega \subset \mathbb{R}^n$  be open, X again a topological vector space then for  $0 \leq p \leq +\infty$  we have:  $\mathscr{C}^p(\Omega) \otimes X$  is the space of all *p*-times continuously differentiable X-valued functions on  $\Omega$  with finite dimensional range.
  - (3) Let X be a linear space, I an index set,  $\varphi(I)$  the set of all scalar valued functions on I which are zero outside a finite set, then  $\varphi(I) \otimes X$  is the set of all X- valued functions on I which are zero outside a finite set.

If, under this identification, X carries a seminorm p and  $\varphi(I)$  is equipped with the  $\ell_1(I)$ -norm then for  $x = (x_i)_{i \in I}$  we have

$$|| ||_1 \otimes_\pi p(x) = \sum_i p(x_i).$$

**Definition:** If E is a locally convex vector space and  $\mathscr{P}$  a fundamental system of seminorms we define

$$\ell_1\{I,X\} = \left\{ x = (x_i)_{i \in I} \in \mid \hat{p}(x) := \sum_{i \in I} p(x_i) < \infty, \ p \in \mathscr{P} \right\}$$

equipped with the seminorms  $\hat{p}$ .

We obtain the

**6.15 Theorem:** If *E* is a complete locally convex vector space, then  $\ell_1(I) \hat{\otimes}_{\pi} E$  can be canonically identified with  $\ell_1\{I, X\}$ .

## 6.2 Tensor products with spaces $L_1(\mu)$

Let  $(\Omega, \mathscr{F}, \mu)$  be a  $\sigma$ -finite measure space, i.e.  $\mathscr{F}$  a  $\sigma$ -algebra on the set  $\Omega$ and  $\mu$  a positive  $\sigma$ -additive set function on  $\Omega$ .

We denote by  $\mathscr{T}$  the linear space of step functions on  $\Omega$ , i.e.  $\mathscr{T} = \operatorname{span}\{\chi_A \mid A \in \mathscr{F}, \mu A < +\infty\}$  where  $\chi_A(\omega) = 1$  for  $\omega \in A, \chi_A(\omega) = 0$  otherwise. Every  $f \in \mathscr{T}$  has a representation  $f = \sum_{j=1}^m \lambda_j \chi_{A_j}$  with disjoint sets  $A_1, \ldots, A_m, m \in \mathbb{N}$ . We call it a disjoint representation.

On  ${\mathscr T}$  we consider the seminorm

$$||f||_1 = \int_{\Omega} |f(\omega)| d\mu(\omega) = \sum_{j=1}^m |\lambda_j| \mu(A_j),$$

using a disjoint representation.

Let X be a Banach space. Then  $\mathscr{T}\otimes X$  is the set of all X-valued step functions on  $\Omega$  which admit a representation

$$f(\omega) = \sum_{j=1}^{m} x_j \chi_{A_j}(\omega)$$

where the sets  $A_j \in \mathscr{F}$  are disjoint and  $x_j \in X$ ,  $j = 1, \ldots, m$ . We call it again a disjoint representation. Notice that then  $||f(\cdot)|| = \sum_{j=1}^{m} ||x_j|| \chi_{A_j}(\cdot) \in \mathscr{T}$ .

**6.16 Lemma:** For  $f \in \mathscr{T}(X)$  we have

$$|| ||_1 \otimes_{\pi} || || = \int_{\Omega} ||f(\omega)|| d\mu(\omega) = \sum_{j=1}^{m} ||x_j|| \mu(\lambda_j),$$

using a disjoint representation.

*Proof:* We set  $p = || ||_1 \otimes_{\pi} || ||$ . For a disjoint representation

$$f(\omega) = \sum_{j=1}^{m} x_j \chi_{A_j}(\omega)$$

we obtain

$$p(f) \le \sum_{j=1}^{m} p(x_j \chi_{A_j}) = \sum_{j=1}^{m} ||x_j|| \mu(A_j) = \int_{\Omega} ||f|| d\mu.$$

For any representation

$$f = \sum_{k=1}^{n} x_k f_k \tag{(*)}$$

where  $f_k \in \mathscr{T}, k = 1, ..., n$ , we now choose disjoint sets  $A_1, ..., A_m \in \mathscr{F}$  such that for all k

$$f_k = \sum_{j=1}^m \lambda_{k,j} \chi_{A_j}.$$

Therefore

$$f = \sum_{j=1}^{m} (\sum_{k=1}^{n} \lambda_{k,j} x_k) \chi_{A_j},$$

is a disjoint representation. Hence

$$\int_{\Omega} ||f|| d\mu = \sum_{j=1}^{m} ||\sum_{k=1}^{n} \lambda_{k,j} x_k|| \mu(A_j) \le \sum_{j=1}^{m} \sum_{k=1}^{n} |\lambda_{k,j}| ||x_k|| \mu(A_j)$$
$$= \sum_{k=1}^{n} ||x_k|| \sum_{j=1}^{m} |\lambda_{k,j}| \mu(A_j) = \sum_{k=1}^{n} ||x_k|| ||f_k||_1.$$

Since this holds for every representation (\*) we have

$$\int_{\Omega}||f||d\mu\leq p(f),$$

which completes the proof.

We call an X-valued function on  $\Omega$  measurable if it is the pointwise limit of a sequence of X-valued step functions. If  $f \in X^{\Omega}$  is measurable then clearly  $||f(\cdot)||$  is measurable.

**Definition:**  $L_1(\mu, X)$  is the linear space of all measurable X-valued functions f on  $\Omega$  with  $\int_X ||f|| d\mu < +\infty$ , where we identify almost everywhere equal functions. We set

$$||f||_1 = \int_X ||f|| d\mu$$

for all  $f \in L_1(\mu, X)$ .

Obviously  $|| ||_1$  is a norm on  $L_1(\mu, X)$  and  $\mathscr{T} \otimes_{\pi} X \subset L_1(\mu, X)$  as a normed subspace.

**6.17 Lemma:**  $\mathscr{T} \otimes X$  is dense in  $L_1(\mu, X)$ .

Proof: Since  $\Omega$  is  $\sigma$ -finite we find  $\Omega_1 \subset \Omega_2 \subset \ldots$  in  $\mathscr{F}$ ,  $\mu \Omega_n < +\infty$  for all n, such that  $\Omega = \bigcup_n \Omega_n$ . We have

$$||f - \chi_{\Omega_n} f||_1 = \int_{\Omega \setminus \Omega_n} ||f|| d\mu$$

and therefore  $\chi_{\Omega_n} f \to f$  in  $L_1(\mu, X)$ . So we may assume without restriction of generality that  $\mu \Omega < +\infty$ .

f is measurable, hence pointwise limit of functions which take on only finitely many values. Therefore  $f(\Omega)$  is contained in a separable subspace of X. So we may assume that X is separable.

Let  $x_1, x_2, \ldots$  be a dense subset of X and  $\varepsilon > 0$ . Then  $X = \bigcup_n U_{\varepsilon}(x_n)$ . We set

$$\omega_1 = f^{-1}(U_{\varepsilon}(x_1)), \quad \omega_{n+1} = f^{-1}(U_{\varepsilon}(x_n)) \setminus (\omega_1 \cup \ldots \cup \omega_{n-1})$$

and

$$f_{\varepsilon} = \sum_{n=1}^{\infty} x_n \chi_{\omega_n}$$

Obviously  $f_{\varepsilon}$  is measurable and  $||f_{\varepsilon}|| \leq ||f|| + \varepsilon$ . Therefore  $f_{\varepsilon} \in L_1(\mu, X)$ . Since  $||f_{\varepsilon} - f|| \leq \varepsilon$  everywhere we obtain  $||f_{\varepsilon} - f||_1 \leq \varepsilon \mu \Omega$ .

Since

$$|f_{\varepsilon}||_{1} = \sum_{n=1}^{\infty} ||x_{n}|| \mu(\omega_{n}) < +\infty$$

we have

$$||f_{\varepsilon} - \sum_{n=1}^{m} x_n \chi_{\omega_n}||_1 = \sum_{n=m+1}^{\infty} ||x_n|| \mu(\omega_n) \longrightarrow 0,$$

which shows that  $f_{\varepsilon}$ , for every  $\varepsilon > 0$ , is a limit of step functions. So, finally, f is a limit of step functions.

#### **6.18 Theorem:** $L_1(\mu, X)$ is a Banach space.

Proof: We have to prove completeness. It suffices to prove that every absolutely convergent series in the dense subspace  $\mathscr{T} \otimes X$  converges in  $L_1(\mu, X)$ . So assume that we have  $u_1, u_2, \ldots \in \mathscr{T} \otimes X$  such that

$$\sum_{n=1}^{\infty} ||u_n||_1 < +\infty.$$

Then by the Beppo Levi Theorem we have that

$$\sum_{n=1}^{\infty} ||u_n(\omega)|| < +\infty$$

for almost every  $\omega$  and

$$g := \sum_{n=1}^{\infty} ||u_n|| \in L_1(\mu).$$

Therefore the series

$$f(\omega) := \sum_{n=1}^{\infty} u_n(\omega)$$

converges for almost every  $\omega$ , f is measurable, and in  $L_1(\mu, X)$  since  $||f(\omega)|| \le g(\omega)$ . We have

$$||f - \sum_{n=1}^{m} u_n||_1 \le \sum_{n=m+1}^{\infty} ||u_n||_1,$$

which shows that

$$f = \sum_{n=1}^{\infty} u_n$$

with convergence in  $L_1(\mu, X)$ .

Now  $\mathscr{T} \otimes_{\pi} X \subset L_1(\mu, X)$  is isometrically imbedded as a dense subspace. So we have

$$L_1(\mu, X) = \mathscr{T} \hat{\otimes}_{\pi} X = L_1(\mu) \hat{\otimes}_{\pi} X$$

which shows:

**6.19 Theorem:**  $L_1(\mu, X) = L_1(\mu) \hat{\otimes}_{\pi} X$  by means of the canonical imbedding generated by  $f \otimes x \mapsto f$ .

If  $(\Omega_1, \mathscr{F}_1, \mu_1)$ ,  $(\Omega_2, \mathscr{F}_2, \mu_2)$  are  $\sigma$ -finite measure spaces and  $(\Omega_1 \times \Omega_2, \mathscr{F}, \mu)$  denotes the product measure space, then the Fubini-Theorem says that  $L_1(\mu) = L_1(\mu_1, L_1(\mu_2))$ . Therefore we have the

**6.20 Corollary:**  $L_1(\mu) = L_1(\mu_1) \hat{\otimes}_{\pi} L_1(\mu_2)$  by means of the canonical imbedding generated by  $f(\omega_1) \otimes g(\omega_2) \mapsto f(\omega_1)g(\omega_2)$ .

**Definition:** A Banach space X has the bounded approximation property if there is a constant C and for every finite set  $e \subset X$  and  $\varepsilon > 0$  a map  $\varphi \in \mathscr{F}(X)$  such that  $||\varphi|| \leq C$  and  $\sup_{x \in e} ||x - \varphi(x)|| < \varepsilon$ .

**6.21 Lemma:** If X has the bounded approximation property, then it has the approximation property.

Proof: Let  $K \subset X$  be compact. Given  $\varepsilon > 0$  we find  $x_1, \ldots, x_m \in K$  such that  $K \subset \bigcup_{j=1}^m U_{\varepsilon}(x_j)$ .

Using the notation in the above definition, we find  $\varphi \in \mathscr{F}(X)$  such that,  $||\varphi|| \leq C$  and

$$\sup_{j=1,\dots,m} ||x_j - \varphi(x_j)|| < \varepsilon$$

For  $x \in K$  we find  $x_j$  such that  $||x - x_j|| < \varepsilon$  and therefore

$$||x - \varphi(x)|| \le ||x - x_j|| + ||x_j - \varphi(x_j)|| + ||\varphi(x_j) - \varphi(x)|| \le (2 + C)\varepsilon. \quad \Box$$

By the same argument as above we can prove:

**6.22 Lemma:** If X is Banach space,  $M \subset X$  a dense subset and for every finite set  $e \subset M$  and  $\varepsilon > 0$  we find  $\varphi \in \mathscr{F}(X)$  such that  $\|\varphi\| \leq C$  and

$$\sup_{x \in e} ||x - \varphi(x)|| < \varepsilon.$$

Then X has the bounded approximation property.

**6.23 Corollary:** If X is separable then the following are equivalent

- (1) X has the bounded approximation property.
- (2) There is a sequence  $(\varphi_n)_{n \in \mathbb{N}}$  in  $\mathscr{F}(X)$  such that  $\lim_n \varphi_n(x) = x$  for all  $x \in X$ .

Proof: (1)  $\Longrightarrow$  (2): Let  $\{x_1, x_2, \ldots\}$  be a dense subset of X. We find C > 0and for every n a map  $\varphi_n \in \mathscr{F}$  such that  $||\varphi_n|| \leq C$  and

$$\sup_{j=1,\dots,n} ||x_j - \varphi_n(x_j)|| < \frac{1}{n}.$$

By means of Lemma 6.22 we get (2).

(2)  $\implies$  (1): From the Banach-Steinhaus theorem we obtain a constant C such that  $||\varphi_n|| \leq C$  for all n.

**Remark:** The implication  $(2) \Longrightarrow (1)$  in Corollary 6.23 holds also without the assumption of separability.

- **6.24 Theorem:** (1)  $\ell_p$  has the bounded approximation property for all  $0 \le p < +\infty$ .
  - (2) Let  $(\Omega, \mathscr{F}, \mu)$  be a measure space, then  $L_p(\mu)$  has the bounded approximation property for all  $0 \le p < +\infty$ .

Proof: To prove (1) we use Corollary 6.23 with  $\varphi_n(x) = (x_1, \ldots, x_n, 0, \ldots)$ . For (2) we use Lemma 6.22 with  $M = \mathscr{T}$ . Let  $\mathcal{A} = \{A_1, \ldots, A_m\}$  denote a finite set of disjoint sets in  $\mathscr{F}, 0 < \mu A_j < +\infty$  for all j. We set for  $f \in L_p(\mu)$ 

$$P_{\mathcal{A}}f = \sum_{j=1}^{m} \frac{1}{\mu A_j} \int_{A_j} f(\omega) d\mu(\omega) \,\chi_{A_j}$$

and we obtain

$$\int_{\Omega} |P_{\mathcal{A}}f|^p d\mu = \sum_{j=1}^m \frac{1}{\mu A_j^p} \left( \int_{A_j} |f(\omega)| d\mu(\omega) \right)^p \mu A_j.$$

Because of

$$\left(\int_{A_j} |f(\omega)| d\mu(\omega)\right)^p \le (\mu A_j)^{p/q} \int_{A_j} |f(\omega)|^p d\mu(\omega)$$

and 1 + p/q - p = 0 we obtain  $||P_{\mathcal{A}}f||_p \le ||f||_p$ .

If  $\{f_1, \ldots, f_n\}$  is a finite set of step functions then we may choose  $\mathcal{A}$  such that

$$f_k = \sum_{j=1}^m \lambda_{k,j} \chi_{A_j}$$

for all k = 1, ..., n. It is easily seen that  $P_{\mathcal{A}}f_k = f_k$  for k = 1, ..., n.  $\Box$ 

From Lemma 6.21 and Theorem 6.24 we obtain:

**6.25 Corollary:**  $L_1(\mu)$  has the approximation property.

Then Corollary 6.8, Lemma 6.10 and Theorem 6.19 give:

**6.26 Theorem:** For any Banach space X we have  $\mathcal{N}(X, L_1(\mu)) \cong L_1(\mu, X')$  isometrically. Here  $f \in L_1(\mu, X')$  corresponds to the map  $T_f x = \langle f(\cdot), x \rangle$ .

Example:

$$\mathcal{N}(c_0, \ell_1) \cong \ell_1 \hat{\otimes}_{\pi} \ell_1 = \{ a = (a_{k,j})_{j,k \in \mathbb{N}} \mid ||a|| = \sum_{k,j} |a_{k,j}| < +\infty \}$$

isometrically. The matrix  $(a_{k,j})_{j,k\in\mathbb{N}}$  corresponds to the map

$$\xi = (\xi_1, \xi_2, \ldots) \mapsto (\sum_j a_{k,j} \xi_j)_{k \in \mathbb{N}}.$$

#### 6.3 The $\varepsilon$ -tensor product

Let E, F be linear spaces,  $\xi, \eta$  linear forms on E and F, respectively. Then  $(x, y) \mapsto \xi(x)\eta(y)$  is bilinear on  $E \times F$ , hence for  $u = \sum_{j=1}^{m} x_j \otimes y_j \in E \otimes F$ 

$$\langle \xi, u, \eta \rangle := \sum_{j=1}^{m} \xi(x_j) \eta(y_j)$$

does not depend on the representation of u and  $u \mapsto \langle \xi, u, \eta \rangle$  is a linear form on  $E \otimes F$ .

**Definition:** Let p on E, q on F be seminorms. Then for  $u \in E \otimes F$  we set

$$p \otimes_{\varepsilon} q(u) := \sup\{ |\langle \xi, u, \eta \rangle| \mid p^*(\xi) \le 1, q^*(\eta) \le 1 \}$$

As usually  $p^*(\xi) := \sup\{ |\xi(x)| \mid p(x) \le 1 \}$  and  $q^*(\eta)$  analogous.

**6.27 Lemma:** (1)  $p \otimes_{\varepsilon} q$  is a seminorm on  $E \otimes F$ 

- (2)  $p \otimes_{\varepsilon} q(x \otimes y) = p(x)q(y)$
- (3)  $p \otimes_{\varepsilon} q \leq p \otimes_{\pi} q$

*Proof:* (1) and (2) are immediate. To prove (3) let  $u = \sum_{j=1}^{m} x_j \otimes y_j$ . Then, because of the triangular inequality and (2), we have

$$p \otimes_{\varepsilon} q(u) \leq \sum_{j=1}^{m} p(x_j) q(y_j).$$

Since this holds for all representations of u we obtain (3).

Let E, F be locally convex, p a continuous seminorm on E, q on F, and  $U = \{x \in E \mid p(x) \leq 1\}, V = \{y \in F \mid q(y) \leq 1\}$ . We have the canonical imbeddings

$$B_{L(E',F)} \colon E \otimes F \longrightarrow L(E',F)$$
$$B_{L(F',E)} \colon E \otimes F \longrightarrow L(F',E) .$$

Then the following is quite obvious

#### 6.28 Lemma:

$$p \otimes_{\varepsilon} q(u) = \sup\{\xi \in U^o \mid q((B_{L(E',F)}u)(\xi))\}$$
$$= \sup\{\eta \in V^o \mid p((B_{L(F',E)}u)(\eta))\}.$$

From this we derive easily that for  $u \in E \otimes F$ ,  $u \neq 0$  we find a continuous seminorm such that  $p \otimes_{\varepsilon} q(u) \neq 0$ . Namely set  $\tilde{u} := B_{L(E',F)}u$ , then  $\tilde{u} \neq 0$ . Therefore we have  $\xi \in E'$ , such that  $\tilde{u}(\xi) \neq 0$ . We find p so that  $\xi \in U^o$  and q such that  $q(\tilde{u}(\xi)) \neq 0$  and obtain  $p \otimes_{\varepsilon} q(u) \neq 0$ .

Since  $p \otimes_{\varepsilon} q$  increases when p and q are increased the following definition makes sense.

**Definition:** Let E, F be locally convex. Then  $E \otimes_{\varepsilon} F$  is  $E \otimes F$  equipped with the locally convex topology given by the seminorms  $p \otimes_{\varepsilon} q$  where p runs through the continuous seminorms on E, q those on F. Its completion is denoted by  $E \otimes_{\varepsilon} F$ .

**Remark:** If  $\mathscr{P}$  is a fundamental system of seminorms on E,  $\mathscr{Q}$  a fundamental system of seminorms on F, then  $\{p \otimes_{\varepsilon} q \mid p \in \mathscr{P}, q \in \mathscr{Q}\}$  is a fundamental system of seminorms on  $E \otimes_{\varepsilon} F$ .

In particular, for E, F metrizable the space  $E \otimes_{\varepsilon} F$  is metrizable, for E, F normable the space  $E \otimes_{\varepsilon} F$  is normable.

Let M be a set, L a linear space of bounded functions on M, equipped with the norm

$$||f|| = \sup_{x \in M} |f(x)|.$$

Let E be a linear space equipped with a seminorm p, then for  $u \in L \otimes E \subset E^M$  we have

$$|| || \otimes_{\varepsilon} p(u) = \sup\{||\eta \circ u(\cdot)|| | p^{*}(\eta) \leq 1\} = \sup\{|\eta(u(x))| | x \in M, p^{*}(\eta) \leq 1\} = \sup\{p(u(x)) | x \in M\} =: \tilde{p}(u).$$

**Example:** For complete E we obtain immediately that  $c_0 \hat{\otimes}_{\varepsilon} E \cong c_0(E)$ , where  $c_0(E)$  is the space of all null sequences in E with the seminorms  $\tilde{p}(x) = \sup_n p(x_n)$ , p running through all continuous seminorms on E.

If M is a compact topological space, we set

 $\mathscr{C}(M, E) = \{ \text{ continuous } E \text{-valued functions } \}$ 

equipped with the locally convex topology given by the seminorms

$$\tilde{p}(f) := \sup\{p(f(x)) \mid x \in M\}$$

where p runs through all continuous seminorms on E (or, equivalently, through a fundamental system of seminorms on E). If E is complete, then  $\mathscr{C}(M, E)$  is complete.

6.29 Theorem: If E is complete then

$$\mathscr{C}(M, E) = \mathscr{C}(M) \hat{\otimes}_{\varepsilon} E.$$

Proof: Due to the previous we have only to show that  $\mathscr{C}(M) \otimes E$  is dense in  $\mathscr{C}(M, E)$ . We choose a seminorm p on E and  $\varepsilon > 0$ . For  $f \in \mathscr{C}(M, E)$ we find  $a_1, \ldots, a_m \in E$  such that  $M = \bigcup_{j=1}^m \{x \mid p(a_j - f(x)) < \varepsilon\}$ .

We put  $\varphi_j = \max(0, \varepsilon - p(a_j - f(x)))$  and  $\varphi = \sum_{j=1}^m \varphi_j$ . Notice that  $\varphi(x) > 0$  for all  $x \in M$ . We put

$$u = \sum_{j=1}^{m} \frac{\varphi_j}{\varphi} a_j \in \mathscr{C}(M) \otimes E.$$

Then we have for  $x \in M$ 

$$p(u(x) - f(x)) = p\left(\sum_{j=1}^{m} \frac{\varphi_j(x)}{\varphi(x)} \left(a_j - f(x)\right)\right) \le \sum_{j=1}^{m} \frac{\varphi_j(x)}{\varphi(x)} p(a_j - f(x)) < \varepsilon.$$

Therefore  $\tilde{p}(u-f) < \varepsilon$ .

If X and Y are normed spaces we consider  $X \otimes_{\varepsilon} Y$ , by means of  $|| || \otimes_{\varepsilon} || ||$ , in canonical way as a normed space.  $X \otimes_{\varepsilon} Y$  is then in canonical way a Banach space.

If X is a Banach space then  $\mathscr{C}(M,X)$  again is a Banach space by means of the norm

$$||f|| = \sup_{x \in M} ||f(x)||.$$

and we can show more precisely:

**6.30 Theorem:** For compact M and a Banach space X we have  $\mathscr{C}(M, X) \cong \mathscr{C}(M) \hat{\otimes}_{\varepsilon} X$  isometrically, by means of the continuous linear map generated by  $f \otimes x \mapsto f(\cdot)x$ .

**6.31 Corollary:** If  $M_1, M_2$  are compact, then  $\mathscr{C}(M_1 \times M_2) \cong \mathscr{C}(M_1) \hat{\otimes}_{\varepsilon} \mathscr{C}(M_2)$  isometrically.

We will now consider the space  $\ell_1(I)\hat{\otimes}_{\varepsilon} E$ , where I is an index set, E is locally convex and complete. Let p be a continuous seminorm on E,  $U = \{x \in E \mid p(x) \leq 1\}$ . For  $x = (x_i)_{i \in I} \in \ell_1(I) \otimes E$  and  $\tilde{p} = || ||_1 \otimes_{\varepsilon} p$  we have

$$\tilde{p}(x) = \sup_{\eta \in U^o} \sum_{i \in I} |\eta(x_i)|.$$

We consider the space

$$\ell_1[I, E] = \left\{ x = (x_i)_{i \in I} \in E^I \mid \sum_{i \in I} |\eta(x_i)| < +\infty \text{ for all } \eta \in E' \right\}$$

of weakly summable families over I.

**6.32 Theorem:**  $\ell_1[I, E]$  equipped with the seminorms

$$\tilde{p}(x) = \sup_{\eta \in U^o} \sum_{i \in I} |\eta(x_i)|$$

where U runs through all neighborhoods of zero in E is a locally convex space. If E is complete then also  $\ell_1[I, E]$  is complete.

*Proof:* First we have to show that for every neighborhood of zero  $U = \{x \mid p(x) \leq 1\}$  in E

$$\tilde{p}(x) := \sup_{\eta \in U^o} \sum_{i \in I} |\eta(x_i)|$$

is finite, hence is a seminorm on  $\ell_1[I, E]$ . We set for r > 0

$$V_r = \{ \eta \in E' \mid \sum_{i \in I} |\eta(x_i)| \le r \}$$
$$= \bigcap_{\substack{e \subset I \\ e \text{ finite}}} \{ \eta \in E' \mid \sum_{i \in e} |\eta(x_i)| \le r \}.$$

 $V_r$  is absolutely convex and  $\sigma(E', E)$ -closed. Since  $\bigcup_{r>0} rV_1 = \bigcup_{r>0} V_r = E'$  by definition of  $\ell_1[I, E]$  we obtain that  $V_1$  is a barrel in E'. Consequently  $E'_{U^o} \cap V_1$  is a barrel in the Banach space  $E'_{U^o}$ , and therefore there is  $\varepsilon > 0$  such that  $\varepsilon U^o \subset V_1$ . This implies that  $U^o \subset \frac{1}{\varepsilon}V_1 = V_{1/\varepsilon}$ , i.e.

$$\sup_{\eta \in U^o} \sum_{i \in I} |\eta(x_i)| \le \frac{1}{\varepsilon}$$

It remains to show that  $\ell_1[I, E]$  is complete if E is complete. Let  $(x_{\tau})_{\tau \in T}$  be a Cauchy net in  $\ell_1[I, E]$ . For any  $\tau$  we have  $x_{\tau} = (x_{\tau,i})_{i \in I}$ . For any  $i \in I$  and any continuous seminorm p on E we have for all  $\tau, \sigma \in T$ 

$$p(x_{\tau,i} - x_{\sigma,i}) = \sup_{\eta \in U^o} |\eta(x_{\tau,i} - x_{\sigma,i})| \le \tilde{p}(x_\tau - x_\sigma).$$

Therefore  $(x_{\tau,i})_{\tau \in T}$  is a Cauchy net in E and, due to completeness, there is  $x_i \in E$  such that  $\lim_{\tau} x_{\tau,i} = x_i$ .

For any continuous seminorm p and  $\varepsilon > 0$  we have  $\tau_0 = \tau_0(\varepsilon)$  such that for  $\tau, \sigma \succ \tau_0$ 

$$\tilde{p}\left(x_{\tau} - x_{\sigma}\right) \le \varepsilon$$

This means that for every  $\eta \subset U^o$ , finite  $e \in I$ ,  $\tau, \sigma \succ \tau_0$  we have

$$\sum_{i \in e} |\eta(x_{\tau,i} - x_{\sigma,i})| \le \varepsilon.$$

Taking the limit with respect to  $\sigma$  we get that for every  $\eta \in U^o$ , finite  $e \subset I$ ,  $\tau \succ \tau_0$ 

$$\sum_{i \in e} |\eta(x_{\tau,i} - x_i)| \le \varepsilon.$$
(\*)

First we choose  $\varepsilon = 1$  find  $\tau = \tau_0(1)$  and obtain

$$\sum_{i \in e} |\eta(x_i)| \le 1 + \tilde{p}(x_\tau)$$

for all finite  $e \subset I$ ,  $\eta \in U^o$ , i.e.

$$\sum_{i \in I} |\eta(x_i)| < +\infty \tag{**}$$

for all  $\eta \in U^o$ . Since we can apply this to every seminorm p, we get (\*\*) for all  $\eta \in E'$ , i.e.  $x \in \ell_1[I, E]$ .

Then we come back to (\*) which implies that for all  $\tau \succ \tau_0$ 

$$\tilde{p}(x_{\tau} - x) = \sup_{\eta \in U^o} \sum_{i \in I} |\eta(x_{\tau,i} - x_i)| \le \varepsilon.$$

Therefore  $x_{\tau} \to x$ .

By means of the consideration previous to the theorem we have

 $\ell_1(I) \otimes_{\varepsilon} E \subset \ell_1[I, E]$ 

as a topological linear subspace. To determine  $\ell_1(I)\hat{\otimes}_{\varepsilon} E$  we have to determine the closure of  $\ell_1(I) \otimes_{\varepsilon} E$  in  $\ell_1[I, E]$ .

Let  $\ell_1(I, E)$  be the linear space of all summable families  $(x_i)_{i \in I}$  in E. Summable means that there is an x such that for every seminorm p and  $\varepsilon > 0$  we have a finite set  $e_0 \subset I$  such that for all finite sets e with  $e_0 \subset e \subset I$ 

$$p(x - \sum_{i \in e} x_i) \le \varepsilon.$$

**6.33 Lemma:** If E is complete then  $\ell_1(I, E)$  is a closed subspace of  $\ell_1[I, E]$ .

Proof: Let  $(x_{\tau})_{\tau \in T}$  be a net in  $\ell_1(I, E)$ , convergent in  $\ell_1[I, E]$ ,  $x := \lim_{\tau} x_{\tau}$ . For  $U = \{x \mid p(x) \le 1\}$  and  $\varepsilon > 0$  we choose  $\tau$  such that  $\tilde{p}(x_{\tau} - x) < \varepsilon$ . We choose a finite set  $e_0 \subset \mathbb{N}$  such that for any finite set  $e \subset I$  with  $e \cap e_0 = \emptyset$  we have  $p\left(\sum_{i \in e} x_{\tau,i}\right) \le \varepsilon$ . We obtain for finite  $e \in I$ ,  $e \cap e_0 = \emptyset$ 

$$p\left(\sum_{i\in e} x_i\right) \leq p\left(\sum_{i\in e} x_{\tau,i}\right) + p\left(\sum_{i\in e} (x_i - x_{\tau,i})\right)$$
$$\leq \varepsilon + \sup_{\eta\in U^o} |\sum_{i\in e} \eta(x_i - x_{\tau,i})|$$
$$\leq \varepsilon + \tilde{p}(x - x_{\tau}) \leq 2\varepsilon.$$

Therefore  $\sum_{i \in I} x_i$  fulfills Cauchy's convergence criterion and, due to the completeness of E, is convergent.

**6.34 Lemma:**  $\ell_1(I) \otimes E \subset \ell_1(I, E)$  as a dense subspace.

Proof: Of course, we have only to show the density. For  $x = (x_i)_{i \in I} \in \ell_1(I, E)$  and finite  $e \subset I$  we define x(e) by  $x(e)_i = x_i$  for  $i \in e, x(e)_i = 0$  otherwise. Then clearly  $x(e) \in \ell_1 \otimes E$  and for any continuous seminorm p on E we have

$$\tilde{p}(x - x(e)) = \sup_{\eta \in U^o} \sum_{i \notin e} |\eta(x_i)|.$$

We choose a finite set  $e_0 \subset I$  so large that for any finite  $e \subset I$  with  $e \cap e_0 = \emptyset$ we have  $p(\sum_{i \in e} x_i) \leq \varepsilon$  and therefore  $|\sum_{i \in e} \eta(x_i)| \leq \varepsilon$  for all  $\eta \in U^o$ . Therefore we have (see the remark below)  $\sum_{i \in I \setminus e_0} |\eta(x_i)| \leq 4\varepsilon$  for all  $\eta \in U^o$ which means  $\tilde{p}(x - x(e_0)) \leq 4\varepsilon$ .

**Remark:** If  $(\xi_i)_{i \in I}$  is a family in  $\mathbb{R}$  or  $\mathbb{C}$  and

$$|\sum_{i\in e}\xi_i|\leq \rho$$

for all finite subsets  $e \subset I$ . Then

$$\sum_{i \in I} |\xi_i| \le 2\rho$$

in the real case

$$\sum_{i \in I} |\xi_i| \le 4\rho$$

in the complex case.

*Proof:* We give the proof for the real case, the complex case is, mutatis mutandis, the same. We set  $I_1 = \{i \in I \mid \xi_i \geq 0\}, I_2 = \{i \in I \mid \xi_i < 0\}$  and obtain, applying the assumption to finite sets  $e \subset I_1$ , resp.  $e \subset I_2$ ,

$$\sum_{i \in I_1} \xi_i \le \rho, \quad -\sum_{i \in I_2} \xi_i \le \rho$$

hence  $\sum_{i \in I} |\xi_i| \le 2\rho$ .

Finally we have proved:

**6.35 Theorem:** If E is complete then  $\ell_1(I) \hat{\otimes}_{\varepsilon} E = \ell_1(I, E)$ .

### 6.4 Tensor products and nuclearity

Let *E* be nuclear and  $x = (x_i)_{i \in I} \in \ell_1[I, E]$ . Let *p* be a continuous seminorm,  $U = \{x \mid p(x) \leq 1\}$ . Then we find a continuous seminorm  $q, V = \{x \mid q(x) \leq 1\}$  and a positive Radon measure  $\mu$  such that

$$p(x) \le \int_{V^o} |\eta(x)| d\mu(\eta).$$

For any finite set  $e \subset I$  we obtain

$$\sum_{i \in e} p(x_i) \leq \sum_{i \in e} \int_{V^o} |\eta(x_i)| d\mu(\eta)$$
$$= \int_{V^o} \sum_{i \in e} |\eta(x_i)| d\mu(\eta)$$
$$\leq \mu(V^o) \sup_{\eta \in V^o} \sum_{i \in I} |\eta(x_i)|$$
$$= \mu(V^o) \tilde{q}(V).$$

Therefore  $\sum_{i \in I} p(x_i) < \mu(V^o) \tilde{q}(V)$ . Since we may do that for any continuous seminorm p we have  $x \in \ell_1\{I, E\}$ , and we can find for any continuous seminorm p a continuous seminorm q and a constant C such that  $\tilde{p}(x) \leq C\tilde{q}(x)$ . We have shown:

**6.36 Theorem:** If *E* is nuclear, then  $\ell_1[I, E] = \ell_1\{I, E\}$  as topological vector spaces. In particular we have  $\ell_1(I) \otimes_{\varepsilon} E = \ell_1(I) \otimes_{\pi} E$ , therefore  $\ell_1(I) \otimes_{\varepsilon} E = \ell_1(I) \otimes_{\pi} E$ . If *E* is complete then  $\ell_1(I, E) = \ell_1\{I, E\}$ .

For this Theorem also the converse is true and this gives exactly the original definition of nuclearity by Grothendieck. To prove it we need some preparation.

**Definition:** Let X, Y be Banach spaces. A map  $A \in L(X, Y)$  is called *absolutely summing* if there is C > 0 such that for any finite set  $\{x_1, \ldots, x_m\} \subset X$  we have

$$\sum_{j=1}^{m} ||Ax_j|| \le C \sup_{\substack{||\eta|| \le 1\\ \eta \in X'}} \sum_{j=1}^{m} |\eta(x_j)|.$$

We denote by  $\pi(A)$  the infimum over all such constants C.

**6.37 Theorem:**  $A \in L(X, Y)$  is absolutely summing if and only if there is a positive Radon measure  $\mu$  on  $U^o = \{y \in X' \mid ||y|| \le 1\}$ , such that

$$||Ax|| \le \int_{U^o} |\eta(x)| d\mu(y)$$

for all  $x \in X$ .

*Proof:* Assume that

$$||Ax|| \le \int_{U^o} |\eta(x)| d\mu(y) \tag{9}$$

Then we have,

$$\sum_{j=1}^{m} ||Ax_{j}|| \leq \int_{U^{o}} \sum_{j=1}^{m} |\eta(x_{j})| d\mu(y)$$
$$\leq \mu(U^{o}) \sup_{\eta \in U^{o}} \sum_{j=1}^{m} |\eta(x_{j})|.$$

Therefore A is absolutely summing and  $\pi(A) \leq \inf\{\mu(U^o) \mid \mu \text{ satisfies } (9)\}.$ 

To prove the converse, we assume that A is absolutely summing. We put  $I := U^o$  and for  $x \in \ell_1(I, E)$  we set

$$a(x) := \sum_{\eta \in U^o} \eta(Ax_\eta).$$

We have:

$$\sum_{\eta \in U^{o}} |\eta(Ax_{\eta})| \leq \sum_{\eta \in U^{o}} ||Ax_{\eta}|| = \sup_{\substack{e \in U^{o} \\ e \text{ finite}}} \sum_{\eta \in e} ||Ax_{\eta}||$$
$$\leq \pi(A) \sup_{\substack{e \in U^{o} \\ e \text{ finite}}} \sup_{\substack{||y|| \leq 1 \\ y \in X'}} \sum_{\eta \in e} |y(x_{\eta})|$$
$$= \pi(A) \sup_{\substack{||y|| \leq 1 \\ y \in X'}} \sum_{\eta \in U^{o}} |y(x_{\eta})| = \pi(A) \tilde{p}(x)$$

where  $\tilde{p}$  is taken from  $p(\cdot) = ||\cdot||$ . Therefore the sum converges and  $|a(x)| \le \pi(A) \tilde{p}(x)$ . We have shown that  $a \in \ell_1(I, E)'$ ,  $||a|| \le \pi(A)$ .

We set  $K = [0,1]^{U^o} \times U^o$ , where  $U^o$  carries the weak\*-topology. Then K is compact. We define a map

$$\Phi\colon \ell_1(I,E)\longrightarrow C(K)$$

by

$$\Phi(x)[\lambda, y] := \sum_{\eta \in U^o} \lambda_\eta y(x_\eta)$$

for  $\lambda = (\lambda_{\eta})_{\eta \in U^o} \in [0, 1]^{U^o}$  and  $y \in U^o$ .

Clearly  $\Phi(x)$  is a function on K. We have to show that it is continuous. Let  $e \subset U^o$  be a finite set. Then

$$f_e(\lambda, y) := \sum_{\eta \in e} \lambda_\eta y(x_\eta)$$

is continuous on K. We have

$$\begin{split} \Phi(x)[\lambda, y] - f_e(\lambda, y)| &= \Big| \sum_{\eta \in U^o \setminus e} \lambda_\eta y(x_\eta) \Big| \le \sum_{\eta \in U^o \setminus e} |y(x_\eta)| \\ &\le \sup_{y \in U^o} \sum_{\eta \in U^o \setminus e} |y(x_\eta)| = \tilde{p}(x - x(e)) \le \varepsilon \end{split}$$

for e large enough. For the definition of x(e) and the last conclusion see the proof of Lemma 6.34.

Therefore  $f_e \to \Phi(x)$  uniformly on K, hence  $\Phi(x) \in C(K)$ . Since

$$\left\|\Phi(x)\right\| = \sup_{(\lambda,y)\in K} \left|\sum_{\eta\in U^o} \lambda_{\eta} y(x_{\eta})\right| = \sup_{y\in U^o} \sum_{i\in I} |y(x_i)| = \tilde{p}(x)$$

 $\Phi$  is an isometric imbedding  $\ell_1(I, E) \hookrightarrow C(K)$ . We set  $b = a \circ \Phi^{-1}$  on  $\Phi(\ell_1(I, E))$  and extend it by the Hahn-Banach Theorem to  $\tilde{\mu} \in C(K)'$  with  $||\tilde{\mu}|| = ||b|| = ||a|| \le \pi(A)$ . By the Theorem of Riesz there exist a measurable function  $\varphi_0$  on K,  $||\varphi_0||_{\infty} = 1$  and a positive Radon measure such that

$$\tilde{\mu}f = \int_K f\varphi_0 d\mu_0$$

for all  $f \in C(K)$ .

For  $\xi \in E$  and  $\eta \in U^o$  we put now

$$x_i = \begin{cases} \xi : & i = \eta \\ 0 : & \text{otherwise.} \end{cases}$$

This defines  $x \in \ell_1(I, E)$ . We have

$$a(x) = \sum_{i \in I} i(Ax_i) = \eta(A\xi)$$

$$\Phi(x)[\lambda, y] = \lambda_{\eta} y(\xi).$$

Therefore

$$\begin{split} \eta(A\xi)| =& |a(x)| = |b(\Phi(x))| = |\tilde{\mu}(\Phi(x))| \\ =& \left| \int_{K} \lambda_{\eta} y(\xi) \varphi_{0}(\lambda, y) d\mu_{0}(\lambda, y) \right| \\ \leq & \int_{K} |y(\xi)| d\mu_{0}(\lambda, y) \\ =& \int_{U^{o}} |y(\xi)| d\mu(y) \end{split}$$

where we have defined the measure  $\mu$  on  $U^o$  by

$$\int_{U^o} g(y) d\mu(y) = \int_K g(y) d\mu_0(\lambda, y).$$

Since the estimate holds for all  $\eta \in U^0$  the proof is complete.

Notice that  $\mu(U^o) = \mu_0(K) = ||\tilde{\mu}|| \leq \pi(A)$ . Together with the estimate which we have seen earlier, we have shown:

$$\pi(A) = \inf\{\mu(U^o) \mid ||Ax|| \le \int_{U^o} |y(x)| d\mu(y)\}$$

where  $\mu$  denotes a positive Radon measure on  $U^o$ .

We are now ready to prove the intended theorem.

**6.38 Theorem:** *E* is nuclear if and only if  $\ell_1 \otimes_{\varepsilon} E = \ell_1 \otimes_{\pi} E$ .

Proof: The necessity has already be shown. If  $\ell_1 \otimes_{\varepsilon} E = \ell_1 \otimes_{\pi} E$  then for every continuous seminorm p on E there is a continuous seminorm qon E such that  $\hat{p} \leq \tilde{q}$  and therefore we have for any finitely many vectors  $x_1, \ldots, x_m \in E$  that  $\hat{p}(x_1, \ldots, x_m, 0, \ldots) \leq \tilde{q}(x_1, \ldots, x_m, 0, \ldots)$ , i.e.

$$\sum_{j=1}^{m} p(x_j) \le \sup_{y \in V^o} \sum_{j=1}^{m} |y(x_j)|.$$
 (10)

Here  $V = \{x \in E \mid q(x) \leq 1\}$ . We denote by  $E_p$  and  $E_q$  the local Banach spaces of p and q and  $i_q^p \colon E_q \longrightarrow E_p$  the canonical map. Then (10) means that  $i_q^p$  is absolutely summing and therefore Theorem 6.37 implies that

$$p(x) = p(i_q^p \hat{x}) \le \int_{V^o} |\eta(x)| d\mu(y)$$

for some positive Radon measure  $\mu$  on  $V^o$ . This shows that E is nuclear.  $\Box$ 

We will extend now this investigation to a more general situation. Let E and F be locally convex spaces, E nuclear.

We assume that U, V are absolutely convex neighborhoods of zero in E,  $V \subset U$ , such that the canonical map  $i_V^U \colon E_V \longrightarrow E_U$  is nuclear. Then  $i_V^U$  can be represented in the form

$$\imath_V^U \xi = \sum_{n=1}^\infty e_n^*(\xi) \, \hat{e}_n$$

where  $e_n^* \in E_V^*$ ,  $e_n \in E$ ,  $\hat{e}_n$  its residue class in  $E_n$  and

$$\sum_{n=1}^{\infty} ||e_n^*||_V^* ||e_n||_U < \infty.$$

For  $u = \sum_{k=1}^{m} a_k \otimes b_k \in E \otimes F$  we set

$$f_n = \sum_{k=1}^m e_n^*(a_k)b_k.$$

Let W be an absolutely convex neighborhood of zero in F then

$$\begin{split} ||f_n||_W &= \sup_{y \in W^o} |y(f_n)| = \sup_{y \in W^o} |\langle e_n^*, u, y \rangle| \\ &\leq ||e_n^*||_V^* \cdot || \ ||_V \otimes_{\varepsilon} || \ ||_W(u). \end{split}$$

For any  $M \in \mathbb{N}$  we obtain

$$\sum_{n=1}^{M} e_n \otimes f_n = \sum_{n=1}^{M} e_n \otimes \left(\sum_{k=1}^{m} e_n^*(a_k)b_k\right)$$
$$= \sum_{k=1}^{m} \left(\sum_{n=1}^{M} e_n^*(a_k)e_n\right) \otimes b_k$$

and therefore

$$u - \sum_{n=1}^{M} e_n \otimes f_n = \sum_{k=1}^{m} \left( a_k - \sum_{n=1}^{M} e_n^*(a_k) e_n \right) \otimes b_k.$$

We can estimate

$$\left\|a_k - \sum_{n=1}^M e_n^*(a_k)e_n\right\|_U = \left\|\sum_{n=M+1}^\infty e_n^*(a_k)\hat{e}_n\right\|_U \le \sum_{n=M+1}^\infty ||e_n^*||_V^* ||e_n||_U ||a_k||_V$$

Finally we have

$$\begin{split} \|_{U} \otimes_{\pi} \| \|_{W}(u) &\leq \| \|_{U} \otimes_{\pi} \| \|_{W} \left( \sum_{n=1}^{M} e_{n} \otimes f_{n} \right) \\ &+ \| \|_{U} \otimes_{\pi} \| \|_{W} \left( u - \sum_{n=1}^{M} e_{n} \otimes f_{n} \right) \\ &\leq \sum_{n=1}^{M} \|e_{n}\|_{U} \|f_{n}\|_{W} + \sum_{k=1}^{m} \left( \sum_{n=M+1}^{\infty} \|e_{n}^{*}\|_{V}^{*} \|e_{n}\|_{U} \|a_{k}\|_{V} \right) \|b_{k}\|_{W} \\ &= \sum_{n=1}^{M} \|e_{n}\|_{U} \|f_{n}\|_{W} + \sum_{n=M+1}^{\infty} \|e_{n}^{*}\|_{V}^{*} \|e_{n}\|_{U} \sum_{k=1}^{m} \|a_{k}\|_{V} \|b_{k}\|_{W} \\ &\leq \sum_{n=1}^{\infty} \|e_{n}\|_{U} \|e_{n}^{*}\|_{V}^{*} \cdot \| \|_{V} \otimes_{\varepsilon} \| \|_{W}(u) \\ &+ \sum_{n=M+1}^{\infty} \|e_{n}^{*}\|_{V}^{*} \|e_{n}\|_{U} \cdot \sum_{k=1}^{m} \|a_{k}\|_{V} \|b_{k}\|_{W} \end{split}$$

Letting  $M \to +\infty$  and taking the inf over all nuclear representations of  $\imath_V^U$  we obtain

$$\| \|_U \otimes_{\pi} \| \|_W \le \nu(i_V^U) \| \|_U \otimes_{\varepsilon} \| \|_W$$

and we have proved:

**6.39 Theorem:** If *E* and *F* are locally convex spaces and *E* is nuclear then  $E \otimes_{\pi} F = E \otimes_{\varepsilon} F$ .

Combining this with Theorem 6.38 we get

6.40 Theorem: The following are equivalent:

- (1) E is nuclear
- (2)  $E \otimes_{\pi} F = E \otimes_{\varepsilon} F$  for every locally convex space F
- (3)  $E \otimes_{\pi} F = E \otimes_{\varepsilon} F$  for every Banach space F
- (4)  $E \otimes_{\pi} \ell_1 = E \otimes_{\varepsilon} \ell_1.$

The equivalence of (1) and (2) is the original definition of nuclearity given by Grothendieck.

#### 6.5 The kernel theorem

We will prove the classical kernel theorem of L. Schwartz. First we calculate some tensor products.

Using the notation of Theorem 3.21 we have:

**6.41 Theorem:**  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1})\hat{\otimes}_{\varepsilon}\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_2}) = \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1+n_2})$  by means of the map  $f(x_1) \otimes g(x_2) \mapsto f(x_1)g(x_2).$ 

*Proof:* We use the same argument as in the proof of Theorem 6.30.

For  $n \in \mathbb{N}, k \in \mathbb{N}_0$  and the canonical seminorm

$$||f||_k = \sup\{|f^{(\alpha)}(x)| \mid x \in \mathbb{R}^n, |\alpha| \le k\}$$

on  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^n)$  we set  $U_k = \{f \in \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^n) \mid ||f||_k \leq 1\}$ . Then we have for  $f \in \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1}) \otimes \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_2}) \subset \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1+n_2})$  by Lemma 6.28

 $\| \|_k \otimes_{\varepsilon} \| \|_k(f) = \sup_{\mu \in U_k^o} \| \mu_y f(\cdot, y) \|_k$ 

$$= \sup_{\mu \in U_k^o} \sup \left\{ \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \mu_y f(x, y) \right| \mid x \in \mathbb{R}^{n_1}, \, |\alpha| \le k \right\}$$
$$= \sup \left\{ \left\| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} f(x, \cdot) \right\|_k \mid x \in \mathbb{R}^{n_1}, \, |\alpha| \le k \right\}$$
$$= \sup \left\{ \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \frac{\partial^{|\beta|}}{\partial y^{\beta}} f(x, y) \right| \mid x \in \mathbb{R}^{n_1}, \, y \in \mathbb{R}^{n_2}, \, |\alpha| \le k, \, |\beta| \le k \right\}$$

Therefore

$$||f||_k \le || ||_k \otimes_{\varepsilon} || ||_k(f) \le ||f||_{2k}$$

for all k, which shows that

$$\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1}) \otimes_{\varepsilon} \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_2}) \subset \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1+n_2})$$

as a topological linear subspace.

It remains to show that  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1}) \otimes \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_2})$  is dense in  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1+n_2})$ . For  $f \in \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1+n_2})$  let

$$f(x,y) = \sum_{k \in \mathbb{Z}^{n_1}, l \in \mathbb{Z}^{n_2}} c_{k,l} e^{2\pi i \frac{1}{r} \sum_{\nu=1}^{n_1} k_{\nu} x_{\nu}} e^{2\pi i \frac{1}{r} \sum_{\nu=1}^{n_2} l_{\nu} y_{\nu}}$$

be the Fourier expansion. The partial sums are in  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1}) \otimes \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_2})$  and the series converges in  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^{n_1+n_2})$ . For compact  $K \subset \mathbb{R}^n$  we set

$$\mathscr{D}(K) = \{ \varphi \in \mathscr{C}^{\infty}(\mathbb{R}^n) \mid \operatorname{supp} \varphi \subset K \}$$

with the induced topology of  $\mathscr{C}^{\infty}(\mathbb{R}^n)$ . A canonical fundamental system of seminorms are the same norms  $\| \|_k$  as above in  $\mathscr{C}^{\infty}_{(r)}(\mathbb{R}^n)$ .  $\mathscr{D}(K)$  is a closed subspace of  $\mathscr{C}^{\infty}(\mathbb{R}^n)$ , hence a Fréchet space. For any set  $M \subset \mathbb{R}^n$  we put

 $\mathscr{D}(M) = \{ \varphi \in \mathscr{C}^\infty(\mathbb{R}^n) \ | \ \mathrm{supp}\, \varphi \ \mathrm{compact}, \ \mathrm{supp}\, \varphi \subset M \},$ 

without assigning a topology to it.

**6.42 Lemma:** For any compact  $K \subset \mathbb{R}^n$  the space  $\mathscr{D}(\overset{\circ}{K})$  is dense in  $\mathscr{D}(K)$ .

Proof: Let  $\varphi_1 \in \mathscr{D}(\mathbb{R}^n)$ ,  $\operatorname{supp} \varphi_1 \subset \{x \mid |x| < 1\}$ ,  $\int \varphi_1(x) dx = 1$  and set  $\varphi_{\varepsilon}(x) = \varepsilon^{-n} \varphi_1(x/\varepsilon)$ . We put  $K_{\varepsilon} = \{x \in K \mid \operatorname{dist}(x, \partial K) \ge \varepsilon\}$  and

$$\chi_{\varepsilon}(x) = \int_{K_{\varepsilon}} \varphi_{\varepsilon}(x-\xi) d\xi.$$

Then  $\chi_{\varepsilon} \in \mathscr{D}(\overset{\circ}{K}), \ \chi_{\varepsilon}(x) = 1$  for  $x \in K_{2\varepsilon}$  and  $\|\chi_{\varepsilon}\|_{k} \leq C_{k}\varepsilon^{-k}$  for  $\varepsilon \leq 1$ . For  $\varphi \in \mathscr{D}(K)$  and  $0 < \varepsilon < 1$  we obtain

$$\begin{aligned} \|\varphi - \varphi \chi_{\varepsilon}\|_{k} &= \|\varphi (1 - \chi_{\varepsilon})\|_{k} \leq C_{k}' \sup_{\substack{x \notin K_{2\varepsilon} \\ |\alpha| \leq k}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\varphi^{(\beta)}(x)| \varepsilon^{|\beta| - |\alpha|} \\ &\leq C_{k}'' \|\varphi\|_{k+1} \varepsilon^{k+1 - |\beta| + |\beta| - k} = C_{k}'' \|\varphi\|_{k+1} \varepsilon. \end{aligned}$$

For the second inequality we used that

$$|\varphi^{(\beta)}(x)| \le (2\varepsilon)^{k+1-|\beta|} \|\varphi\|_{k+1}$$

on  $K \setminus K_{2\varepsilon} = \{x \in K \mid \operatorname{dist}(x, \partial K) < 2\varepsilon\}.$ Since  $\varphi \chi_{\varepsilon} \in \mathscr{D}(\overset{\circ}{K})$  we obtain the result.

We use the previous Lemma to show:

**6.43 Theorem:** For any compact sets  $K_1 \subset \mathbb{R}^{n_1}$ ,  $K_2 \subset \mathbb{R}^{n_2}$  we have

$$\mathscr{D}(K_1)\hat{\otimes}_{\varepsilon}\mathscr{D}(K_2) = \mathscr{D}(K_1 \times K_2)$$

by canonical identification.

Proof: As in the case of periodic functions we have  $\mathscr{D}(K_1) \otimes_{\varepsilon} \mathscr{D}(K_2) \subset \mathscr{D}(K_1 \times K_2)$  as a linear topological subspace. It remains to show that  $\mathscr{D}(K_1) \otimes \mathscr{D}(K_2)$  is dense in  $\mathscr{D}(K_1 \times K_2)$ .

For that it suffices, due to Lemma 6.42, to show that

$$\overline{\mathscr{D}(K_1)\otimes\mathscr{D}(K_2)}\supset\mathscr{D}(\overset{\circ}{K_1}\times\overset{\circ}{K_2})$$

For  $\varphi \in \mathscr{D}(\mathring{K_1} \times \mathring{K_2})$  we choose  $\psi \in \mathscr{D}(K_1 \times K_2)$ ,  $\psi(x, y) = \psi_1(x)\psi_2(y)$ ,  $x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}$ , such that  $\psi(x, y) = 1$  for  $(x, y) \in \text{supp } \varphi$ .

Since, in the notation of Lemma 6.42, we have supp  $\varphi \subset K_{1,2\varepsilon} \times K_{2,2\varepsilon}$  for suitable  $\varepsilon > 0$ , we may use  $\psi_1(x) = \chi_{\varepsilon}^1(x)$ ,  $\psi_2(y) = \chi_{\varepsilon}^2(y)$  as in the proof of Lemma 6.42 where  $\chi_s^j$  is defined on  $\mathbb{R}^{n_j}$ , j = 1, 2.

We choose  $r > \text{diam}(K_1 \times K_2)$  and set with  $n = n_1 + n_2$ 

$$\phi(\xi) := \sum_{k \in \mathbb{Z}^n} \varphi(\xi + rk)$$

for  $\xi \in \mathbb{R}^n$ . Then  $\phi \in \mathscr{C}^{\infty}_{(r)}(\mathbb{R}^n)$  and  $\phi \psi = \varphi$ . Therefore, for  $\xi = (x, y)$  as before,

$$\varphi(x,y) = \sum_{k \in \mathbb{Z}^{n_1}, l \in \mathbb{Z}^{n_2}} c_{k,l} \psi_1(x) \psi_2(y) e^{2\pi i \frac{1}{r} \sum_{\nu=1}^{n_1} k_\nu x_\nu} e^{2\pi i \frac{1}{r} \sum_{\nu=1}^{n_2} k_\nu y_\nu}$$

where  $c_{k,l}$  are the Fourier coefficients of  $\phi$ . Arguing as in the proof of Theorem 6.41 we obtain the result.

Since for compact  $K \subset \mathbb{R}^n$  the space  $\mathscr{D}(K)$  is a closed subspace of  $\mathscr{C}^{\infty}(\mathbb{R}^n)$  it is nuclear (see Theorem 3.24). From Theorem 6.40 and Theorem 6.43 we conclude

**6.44 Corollary:** For any compact set  $K_1 \subset \mathbb{R}^{n_1}$ ,  $K_2 \subset \mathbb{R}^{n_2}$  we have  $\mathscr{D}(K_1 \times K_2) = \mathscr{D}(K_1) \hat{\otimes}_{\pi} \mathscr{D}(K_2)$  by canonical identification.

To formulate the classical kernel theorem we need the following concept. For  $\Omega \subset \mathbb{R}^n$  and any locally convex space G we define  $L(\mathscr{D}(\Omega), G)$  to be the linear space of all continuous linear maps  $A \colon \mathscr{D}(\Omega) \longrightarrow G$  such that  $A|_{\mathscr{D}(K)} \in L(\mathscr{D}(K), G)$  for every compact set  $K \subset \Omega$ .

We set  $\mathscr{D}'(\Omega) := L(\mathscr{D}(K), \mathbb{K})$  and call it the space of *Schwartz distributions*. It is a locally convex space with the fundamental system of semi-norms

$$p_B(T) := \sup_{\varphi \in B} |T\varphi|,$$

where B runs through the following sets: there is a compact set  $K \subset \Omega$  such that  $B \subset \mathscr{D}(K)$  and bounded there.

For any locally convex space a linear map  $A: G \longrightarrow \mathscr{D}'(\Omega)$  is continuous if, and only if,  $x \mapsto Ax|_{\mathscr{Q}(K)}$  is continuous for every compact  $K \subset \Omega$ .

**6.45 Theorem:** Let  $\Omega_1 \subset \mathbb{R}^{n_1}, \Omega_2 \subset \mathbb{R}^{n_2}$  be open. For every continuous linear map  $A \in L(\mathscr{D}(\Omega_1), \mathscr{D}'(\Omega_2))$  there is a distribution  $T \in \mathscr{D}'(\Omega_1 \times \Omega_2)$  such that  $(A\varphi)\psi = T(\varphi(x)\psi(y))$  for every  $\varphi \in \mathscr{D}(\Omega_1), \psi \in \mathscr{D}(\Omega_2)$ .

Proof: The bilinear form  $B(\varphi, \psi) := (A\varphi)\psi$  on  $\mathscr{D}(\Omega_1) \times \mathscr{D}(\Omega_2)$  extends to a linear form  $T_0$  on  $\mathscr{D}(\Omega_1) \otimes \mathscr{D}(\Omega_2)$ . Let  $K_1 \subset \Omega_1, K_2 \subset \Omega_2$  be compact and call  $S_{K_1,K_2}$  the restriction of  $T_0$  to  $\mathscr{D}(K_1) \otimes \mathscr{D}(K_2)$ . If we can show that  $S_{K_1,K_2}$  is continuous on  $\mathscr{D}(K_1) \otimes_{\pi} \mathscr{D}(K_2)$ , then it extends to a continuous linear form  $T_{K_1,K_2}$  on  $\mathscr{D}(K_1) \otimes_{\pi} \mathscr{D}(K_2) = \mathscr{D}(K_1 \times K_2)$ .  $\bigcup_{K_1,K_2} \mathscr{D}(K_1 \times K_2) = \mathscr{D}(\Omega_1 \times \Omega_2)$  and for  $L_1 \subset K_1, L_2 \subset K_2$  we have  $T_{K_1,K_2}|_{\mathscr{D}(L_1 \times L_2)} = T_{L_1,L_2}$ , because the restriction of both to  $\mathscr{D}(L_1) \otimes \mathscr{D}(L_2)$  is  $S_{L_1,L_2}$ . Therefore we have a linear form T on  $\mathscr{D}(\Omega_1 \times \Omega_2)$  such that  $T|_{\mathscr{D}(K_1 \times K_2)} = T_{K_1,K_2}$  for all  $K_1, K_2$ .

If  $K \subset \Omega_1 \times \Omega_2$  is compact we find compact  $K_1 \subset \Omega_1$ ,  $K_2 \subset \Omega_2$  such that  $K \subset K_1 \times K_2$  and therefore  $T|_{\mathscr{D}(K)} = (T|_{\mathscr{D}(K_1 \times K_2)})|_{\mathscr{D}(K)}$  is continuous.

So it remains to show that  $S_{K_1,K_2}$  is continuous on  $\mathscr{D}(K_1) \otimes_{\pi} \mathscr{D}(K_2)$ , or, that  $B|_{\mathscr{D}(K_1) \times \mathscr{D}(K_2)}$  is a continuous bilinear form. If  $\psi \in \mathscr{D}(K_2)$  is fixed and  $\varphi_n \to 0$  in  $\mathscr{D}(K_1)$  then  $A\varphi_n \to 0$  in  $\mathscr{D}'(\Omega)$ , in particular  $(A\varphi_n)\psi \to 0$ . This shows that for every  $\psi \in \mathscr{D}(K_2)$  the linear form  $\varphi \mapsto B(\varphi, \psi)$  is continuous on  $\mathscr{D}(K_1)$ .

If  $\varphi \in \mathscr{D}(K_1)$  is fixed and  $\psi_n \to 0$  in  $\mathscr{D}(K_2)$  then  $(A\varphi)\psi_n \to 0$ , since  $A\varphi \in \mathscr{D}'(\Omega)$ . So for every  $\varphi \in \mathscr{D}(K_1)$  the linear form  $\psi \mapsto B(\varphi, \psi)$  is continuous on  $\mathscr{D}(K_2)$ .

We have show that  $B|_{\mathscr{D}(K_1)\times\mathscr{D}(K_2)}$  is separately continuous bilinear form on  $\mathscr{D}(K_1)\times\mathscr{D}(K_2)$ . The following Theorem completes the proof.  $\Box$ 

**6.46 Theorem:** If E and F are Fréchet spaces then every separately continuous bilinear form B on  $E \times F$  is continuous.

*Proof:* For every  $y \in F$  and bounded  $M \subset E$  we have by assumption

$$\sup_{x \in M} |B(x,y)| < +\infty.$$
(11)

We fix M and set

$$L_n = \{ y \in F \mid \sup_{x \in M} |B(x, y)| \le n \}$$
$$= \bigcap_{x \in M} \{ y \in F \mid |B(x, y)| \le n \}.$$

 $L_n$  is an intersection of absolutely convex closed sets, hence absolutely convex and closed. Obviously  $L_n = n \cdot L_1$ .

By (11) we have  $\bigcup_n nL_1 = \bigcup_n L_n = F$ , i.e.  $L_1$  is a barrel, hence by Theorem 1.3 a neighborhood of zero in F. We have shown that  $\sup_{x \in M} |B(x, \cdot)|$  is a continuous seminorm on F.

Now assume  $x_n \to x$  in  $E, y_m \to y$  in F. Then

$$|B(x_n, y_n) - B(x, y)| \le |B(x_n, y_n) - B(x_n, y)| + |B(x_n, y) - B(x, y)|$$
  
= |B(x<sub>n</sub>, y<sub>n</sub> - y)| + |B(x<sub>n</sub> - x, y)|  
$$\le \sup_{\nu} |B(x_{\nu}, y_n - y)| + |B(x_n - x, y)|.$$

Since, due to the previous,  $\sup_{\nu} |B(x_{\nu}, \cdot)|$  is a continuous seminorm on F and  $B(\cdot, y)$  is continuous, both terms on the right hand side converge to zero.

## 7 Interpolational Invariants

Throughout this section  $E, F, \ldots$  will be Fréchet spaces with a fundamental system  $\| \|_0 \le \| \|_1 \le \| \|_2 \ldots$  of seminorms.

**Definition:** E has property (DN) (or:  $E \in (DN)$ ) if the following holds

 $\exists p \,\forall k \,\exists K, C \colon \| \|_k^2 \leq C \| \|_p \| \|_K.$ 

 $\| \|_p$  is called a *dominating norm*.

**Remark:** A dominating norm is a norm on E.

**7.1 Lemma:** (1) If  $E \cong F$  and  $E \in (DN)$  then  $F \in (DN)$ .

(2) If  $F \subset E$  is a subspace and  $E \in (DN)$  then also  $F \in (DN)$ .

Examples for (DN) and non (DN) spaces are given in the following proposition.

**7.2 Proposition:**  $\Lambda_r(\alpha) \in (DN)$  if, and only if,  $r = +\infty$ .

Proof: If  $r = +\infty$  then the norms  $(| k_k|_{k \in \mathbb{N}_0})$  are a fundamental system of seminorms and we have, due to the Cauchy-Schwarz inequality:

$$\begin{aligned} |x|_{k}^{2} &= \sum_{j} |x_{j}|^{2} e^{2k\alpha_{j}} \\ &\leq \left(\sum_{j} |x_{j}|^{2}\right)^{\frac{1}{2}} \left(\sum_{j} |x_{j}|^{2} e^{4k\alpha_{j}}\right)^{\frac{1}{2}} \\ &= |x|_{0} |x|_{2k}. \end{aligned}$$

So  $\Lambda_{\infty}(\alpha)$  has property (DN) with  $| \mid_0$  as a dominating norm. Now assume  $\Lambda_r(\alpha) \in (DN)$  and let  $t_k \nearrow r$ . Assume  $| \mid_{t_0}$  is a dominating norm. Then for every k there is K, C such that

$$|x|_{t_k}^2 \le C|x|_{t_0}|x|_{t_K}$$

for all  $x \in \Lambda_r(\alpha)$ . We apply that to  $e_i$  and obtain

$$e^{2t_k\alpha_j} \le C e^{(t_0 + t_K)\,\alpha_j}$$

and therefore for large  $\boldsymbol{j}$ 

$$t_k \le \frac{\log C}{2\alpha_j} + \frac{1}{2}(t_0 + t_K).$$

Letting  $j \to +\infty$  we get for all k

$$t_k \le \frac{1}{2}(t_0 + r),$$

which is possible only for  $r = +\infty$ .

This can, of course, also be derived from the following theorem.

**7.3 Theorem:**  $\lambda(A) \in (DN)$  if, and only if, the following holds

$$\exists p \,\forall k \,\exists K, C \,\forall j \colon a_{j,k}^2 \leq C \,a_{j,p} \,a_{j,K}.$$

*Proof:* To get the necessity of the condition we apply the inequality in the definition of (DN) to the canonical basis vectors  $e_j$ . Sufficiency follows from the Cauchy-Schwarz inequality:

$$\begin{aligned} \|x\|_{k}^{2} &= \sum_{j} |x_{j}|^{2} a_{j,k}^{2} \\ &\leq C \sum_{j} |x_{j}| a_{j,p} |x_{j}| a_{j,K} \\ &\leq C \|x\|_{p} \|x\|_{K}. \end{aligned}$$

**Remark:** In the previous theorem  $|| ||_p$  is a dominating norm.

If  $\lambda(A) \in (DN)$ , we find a dominating norm  $|| ||_p$  and set

$$b_{j,k} := \frac{a_{j,k+p}}{a_{j,p}}.$$

Then  $y \mapsto (x_j a_{j,p})_j$  defines an isomorphism from  $\lambda(A)$  onto  $\lambda(B)$ . For B we have  $b_{j,0} = 1$  for all  $j, \parallel \parallel_0$  is a dominating norm on  $\lambda(B)$  and the condition in Theorem 7.3 takes the form

$$\forall k \exists K, C \,\forall j \colon b_{j,k}^2 \le C b_{j,K}.$$

Therefore the multipliers of  $\lambda(B)'$  are easily seen to be the set of all sequences t such that there is k with  $\sup_j \frac{|t_j|}{b_{j,k}} < +\infty$ . By Theorem 5.6 this yields:

**7.4 Theorem:** If  $\lambda(A) \in (DN)$  is a regular Schwartz space and B as above then

$$\Delta(\lambda(A)) = \{t \mid \text{ there is } k \text{ such that } \sup_{j} \frac{|t_j|}{b_{j,k}} < +\infty\}.$$

**7.5 Corollary:** If  $\lambda(A)$  and  $\lambda(\tilde{A})$  are regular Schwartz spaces with property (DN) then  $\Delta(\lambda(A)) = \Delta(\lambda(\tilde{A}))$  implies that  $\lambda(A)$  and  $\lambda(\tilde{A})$  are isomorphic by a diagonal transformation.

Here diagonal transformation means a transformation whose matrix is diagonal, i.e. of the form  $x = (x_j)_{j \in \mathbb{N}} \mapsto (d_j x_j)_{j \in \mathbb{N}}$ .

Proof: Let B and  $\tilde{B}$  be as above then for every l we have  $(\tilde{b}_{j,l})_{j\in\mathbb{N}} \in \Delta(\lambda(\tilde{B})) = \Delta(\lambda(B))$  which means the existence of k and C such that

$$b_{j,l} \le C b_{j,k}.$$

In the same way we obtain,  $\tilde{k}, \tilde{C}$  such that

$$b_{j,l} \leq \tilde{C}b_{j,\tilde{k}}.$$

Therefore  $\lambda(B) = \lambda(\tilde{B})$  which implies the assertion, since  $\lambda(B)$  and  $\lambda(\tilde{B})$  are isomorphic to  $\lambda(A)$  and  $\lambda(\tilde{A})$ , respectively, by means of diagonal transformations.

# References

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