Exposé
on a paper of Dronov and Kaplitskii

Dietmar Vogt

In [1] Dronov and Kaplitzki showed that every complemented subspace of a nuclear Köthe space E with a regular basis of type (d_1) has a basis so, in particular, solving the long standing problem whether any complemented subspace of the space (s) of rapidly decreasing sequences has a basis. We present a slightly modified version of their proof which shows that the range of every closed-range operator in E has a basis.

Let $\lambda(A)$ be a nuclear Köthe space a regular basis of type (d_1). The latter means that E has property (DN). Without restriction of generality we may assume:

1. $a_{1,n} = 1$ for all n.
2. $a_{k,n}^2 \leq a_{k+1,n}$ for all k, n.
3. $\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{k+1,n}} \leq 1$ for all k.
4. $\frac{a_{k,n+1}}{a_{k+1,n+1}} \leq \frac{a_{k,n}}{a_{k+1,n}}$ for all k, n.

Due to nuclearity we may use the following two equivalent norm systems

- $|x|_k = \sup_n |x_n| a_{k,n}$, $k \in \mathbb{N}$.
- $\|x\|_k = \left(\sum_{n=1}^{\infty} |x_n|^2 a_{k,n}^2\right)^{1/2}$, $k \in \mathbb{N}$.

Because of 3. above we obtain:

- $|x|_k \leq \|x\|_k \leq |x|_{k+1}$, $k \in \mathbb{N}$.

The respective local Banach spaces are

$$G_k = c_0(a_k) \text{ and } H_k = \ell_2(a_k),$$

We consider an operator $T \in L(E)$ and we set $F = T(E) \subset E$, We want to study properties of F. T is given in the form

$$Tx = \left(\sum_{j=1}^{\infty} t_{i,j} x_j\right)_{i \in \mathbb{N}}.$$

We define an operator $|T|$ by

$$|T|x = \left(\sum_{j=1}^{\infty} |t_{i,j}| x_j \right)_{i \in \mathbb{N}}.$$

To see that this defines an operator $|T| \in L(E)$ we recall the explicit description of the matrices of operators in c_0. $(t_{i,j})_{i,j\in\mathbb{N}}$ defines an operator in c_0 iff 1. $\sup_i \sum_{j=1}^{\infty} |t_{i,j}| < \infty$ and 2. $\lim_i t_{i,j} = 0$ for all j. This description depends on $|t_{i,j}|$ only.

Without restriction of generality we may assume

5. $\||T|x\||_k \leq \frac{1}{2}|x|_{k+1}$, $k \in \mathbb{N}$.

Next we define three "dead-end spaces" that is continuously imbedded Banach spaces, given by two weights

$$a_{\infty,n}^2 := \sum_{k=1}^{\infty} \delta_k^2 a_{k,n}^2,$$

where the δ_k will be determined later, and

$$b_{\infty,n} := a_{n,n}.$$

We set $H_\infty = \ell_2(a_\infty)$ with the norm

$$\|x\|_\infty^2 := \|x\|_{H_\infty}^2 = \sum_{k=1}^{\infty} \delta_k^2 \|x\|_k^2.$$

We set $G_\infty = c_0(a_\infty)$ with the norm

$$|x|_\infty := |x|_{G_\infty} = \sup_n |x_n| a_{\infty,n}.$$

Moreover we set $G_{\infty,0} = c_0(b_\infty)$ with the norm

$$|x|_{\infty,0} := |x|_{G_{\infty,0}} = \sup_n |x_n| b_{\infty,n}.$$

We obtain

$$G_{\infty,0} \subset H_k \subset G_k, \qquad k \in \mathbb{N}.$$

The second inclusion is obvious, the first one we get from

$$\sum_{n=k+1}^{\infty} a_{k,n}^2 |x_n|^2 = \sum_{n=k+1}^{\infty} \frac{a_{k,n}^2}{a_{n,n}^2} a_{n,n}^2 |x_n|^2 \leq \left(\sum_{n=k+1}^{\infty} \frac{a_{k,n}^2}{a_{n,n}^2} \right) |x|_{\infty,0}^2 \leq |x|_{\infty,0}^2.$$

With some constant D_k we have

$$\|x\|_k \leq D_k |x|_{\infty,0}.$$

We may assume $D_k \leq D_{k+1}$ for all k. We obtain

$$\|Tx\|_\infty^2 \;=\; \sum_{k=1}^\infty \delta_k^2 \|Tx\|_k^2 \leq \sum_{k=1}^\infty \delta_k^2 \|x\|_{k+1}^2 \leq \left(\sum_{k=1}^\infty \delta_k^2 D_{k+1}^2\right) |x|_{\infty,0}^2 \leq |x|_{\infty,0}^2,$$

where we have chosen, also under consideration of later application, $\delta_k \leq 1/(2^k D_{k+2})$. So we have shown

$$\|Tx\|_\infty \leq |x|_{\infty,0} \tag{1}$$

This shows that $L := \{Tx \,:\, x \in G_{\infty,0}\} \subset H_\infty\}$, We set F_k the completion of L with respect to $\|\cdot\|_k$ and F_∞ the completion with respect to $\|\cdot\|_\infty$. The embedding $J : F_\infty \hookrightarrow F_1$ is clearly nuclear, hence we can expand it as

$$J(x) = \sum_{j=1}^\infty \langle x, f_j\rangle_{H_1} f_j$$

where $(f_j)_{j\in\mathbb{N}}$ is orthogonal in H_∞, orthonormal in H_1. We set

$$T_n(x) = \sum_{j=1}^n \langle Tx, f_j\rangle_{H_1} f_j.$$

For every $x \in G_{\infty,0}$ we get $T_n(x) \longrightarrow T(x)$ in H_∞. We want to show that the family of maps $\{T_n \,:\, n \in \mathbb{N}\}$ is equicontinuos in $L(E)$. This will imply

$$Tx = \sum_{j=1}^\infty \langle Tx, f_j\rangle_{H_1} f_j$$

for all $x \in E$, the series converging in E.

Theorem: *If $T(E)$ is closed, then $F_\infty \subset T(E)$, hence all $f_j \in T(E)$. So the f_j are a basis in $T(E)$.*

Because of orthogonality we have

- $\|T_n x\|_1 \leq \|Tx\|_1$.
- $\|T_n x\|_\infty \leq \|Tx\|_\infty$.

For $x \in \varphi^+$ we have $\|Tx\|_1 \leq \||T|x\|_1$ and therefore we have

- $\|T_n x\|_1 \leq \||T|x\|_1$.

To get an estimate between the $\|\cdot\|_\infty$ and the $|\cdot|_\infty$ norm, we fix some $r \in \mathbb{N}$ and obtain

$$
\begin{aligned}
(2) \quad \|x\|_\infty &= \|x\|_{\ell_2(a_\infty)} \le \|x\|_{\ell_1(a_\infty)} = \sum_{n=1}^\infty |x_n| a_{\infty,n} = \sum_{n=1}^\infty \frac{a_{r,n}}{a_{r+1,n}} \left(\frac{a_{r+1,n}}{a_{r,n}} |x_n| a_{\infty,n} \right) \\
&\le \sum_{n=1}^\infty \frac{a_{r,n}}{a_{r+1,n}} \sup_{n\in\mathbb{N}} \left\{ \frac{a_{r+1,n}}{a_{r,n}} |x_n| a_{\infty,n} \right\} \\
&\le \left| \frac{a_{r+1}}{a_r} x \right|_\infty
\end{aligned}
$$

for all $x \in G_{\infty,0}$.

To see that $\frac{a_{r+1}}{a_r} x \in G_\infty$ for all $x \in G_{\infty,0}$ we use the estimate:

$$
\|\frac{a_{r+1}}{a_r} x\|_k^2 \le \|a_k x\|_k^2 = \sum_{n=1}^\infty a_{k,n}^4 |x_n|^2 \le \sum_{n=1}^\infty a_{k+1,n}^2 |x_n|^2 = \|x\|_{k+1}^2
$$

for all $k > r$. From that we obtain

$$
\begin{aligned}
\|\frac{a_{r+1}}{a_r} x\|_\infty^2 &= \sum_{k=1}^\infty \delta_k^2 \|\frac{a_{r+1}}{a_r} x\|_k^2 \\
&\le \sum_{k=1}^r \delta_k^2 \|\frac{a_{r+1}}{a_r} x\|_k^2 + \sum_{k=r+1}^\infty \delta_k^2 \|\frac{a_{r+1}}{a_r} x\|_k^2 \\
&\le \|\frac{a_{r+1}}{a_r} x\|_r^2 \sum_{k=1}^r \delta_k^2 + \sum_{k=r+1}^\infty \delta_k^2 \|x\|_{k+1}^2 \\
&= \|x\|_{r+1}^2 \sum_{k=1}^r \delta_k^2 + \sum_{k=r+1}^\infty \delta_k^2 \|x\|_{k+1}^2 \\
&\le (D_{r+1}^2 + \sum_{k=r+1}^\infty \delta_k^2 D_{k+1}^2) |x|_{\infty,0}^2 \\
&\le (D_{r+1}^2 + 1) |x|_{\infty,0}^2
\end{aligned}
$$

Therefore $\frac{a_{r+1}}{a_r} x \in H_\infty \subset G_\infty$ for all $x \in G_{\infty,0}$.

We apply the previous to $|T|x$ and obtain:

$$
\begin{aligned}
\|\frac{a_{r+1}}{a_r} |T| x\|_\infty^2 &\le \||T|x\|_{r+1}^2 \sum_{k=1}^r \delta_k^2 + \sum_{k=r+1}^\infty \delta_k^2 \||T|x\|_{k+1}^2 \\
&\le \|x\|_{r+2}^2 \sum_{k=1}^r \delta_k^2 + \sum_{k=r+1}^\infty \delta_k^2 \|x\|_{k+2}^2 \\
&\le (D_{r+2}^2 + \sum_{k=r+1}^\infty \delta_k^2 D_{k+2}^2) |x|_{\infty,0}^2 \\
&\le (D_{r+2}^2 + 1) |x|_{\infty,0}^2.
\end{aligned}
$$

We define $J_r x := \frac{a_{r+1}}{a_r} x$ and we have shown

$$\| J_r |T| x \|_\infty \leq M_r |x|_{\infty,0} \tag{3}$$

with $M_r^2 = D_{r+2}^2 + 1$.

We will use the canonical projection in the sequence space E:

$$\left(Q^{(N)} x\right)_n = \begin{cases} x_n, & n = 1, 2, \ldots, N \\ 0, & n > N. \end{cases}$$

We obtain for all $x \in G_1$

$$\begin{aligned} \left| \frac{a_{r+1}}{a_r} |T| \frac{a_1}{a_{r+2}} Q^{(N)} x \right|_r &= \left| |T| \frac{a_1}{a_{r+2}} Q^{(N)} x \right|_{r+1} \\ &\leq \frac{1}{2} \left| \frac{a_1}{a_{r+2}} Q^{(N)} x \right|_{r+2} = \frac{1}{2} |Q^{(N)} x|_1 \leq \frac{1}{2} |x|_1. \end{aligned} \tag{4}$$

We set $J_r' = a_1 / a_{r+2}$ and define:

$$A_r = J_r |T| J_r' \quad \text{and} \quad A_r^{(N)} = A_r \circ Q^{(N)}.$$

The operator acts $G_1 \to G_r$, hence also $G_r \to G_r$ and $\|A_r^{(N)}\|_{G_r \to G_r} \leq \frac{1}{2}$.

For all $x \in G_{\infty,0}^+ = \omega^+ \cap G_{\infty,0}$ we obtain

$$|T_n x|_\infty \leq \|T_n x\|_\infty \leq \|T x\|_\infty \leq \| \, |T| x \|_\infty \leq \left| \frac{a_{r+1}}{a_r} |T| \, x \right|_\infty = |J_r |T| x|_\infty.$$

The last inequality comes from (2).

For $x \in \varphi^+$ we get the estimates:

$$|T_n J_r' x|_\infty \leq |J_r |T| J_r' x|_\infty = |A_r x|_\infty$$

$$|T_n J_r' x|_1 \leq |J_r |T| J_r' x|_1 \leq |x|_1$$

In the second line the first estimate comes from

$$\|x\|_1^2 = \sum_n |x_n|^2 = \sum_n \frac{a_r^2}{a_{r+1}^2} \frac{a_{r+1}^2}{a_r^2} |x_n|^2 \leq |J_r x|_1^2$$

which implies

$$|T_n x|_1 \leq \|T_n x\|_1 \leq \|T x\|_1 \leq \| \, |T| x \|_1 \leq |J_r |T| x|_1.$$

The second estimate is (4).

We define

$$S_{n,r} = T_n J_r' \quad \text{and} \quad S_{n,r}^{(N)} = S_{n,r} Q^{(N)}.$$

The first of the inequalities above is not applicable for interpolation, but it becomes applicable when we restrict it to a suitable cone.

We set

$$Q_{r,N} = \{x \in \omega^+ \, : \, x \geq A_r^{(N)} x\}.$$

Then $Q_{r,N}$ is a cone and we have:

$$|S_{n,r}^{(N)} x|_1 \leq |x|_1 \text{ for } x \in G_1^+,$$

$$|S_{n,r}^{(N)} x|_\infty \leq |x|_\infty \text{ for } x \in Q_{r,N} \cap G_\infty^+.$$

By use of the interpolation theorem for cones [1, Theorem 1] we obtain that for every r there is a constant $C(r)$ such that

$$|S_{n,r}^{(N)} x|_r \leq C(r)\, |x|_r \text{ for } x \in Q_{r,N} \cap G_r^+.$$

Since $A_r^{(N)} \in L(G_r)$ with $\|A_r^{(N)}\| \leq 1/2$ the operator $B := I - A_r^{(N)}$ is invertible in $L(G_r)$ and $\|B^{-1}\| \leq 2$. We can write for $x \in G_r$

$$x = B^{-1}Bx = B^{-1}(Bx)_+ - B^{-1}(Bx)_-.$$

Clearly $B^{-1}(Bx)_+, B^{-1}(Bx)_- \in Q_{r,N} \cap G_r^+$. Positivity is seen by the Neumann series. Since $\|B^{-1}B\| \leq 4$, we have shown that every $x \in G_r$ can be written as $x = y - z$ here $y, z \in Q_{r,N} \cap G_r^+$ and $\|y\|_r \leq 4\|x\|_r$ and $\|z\|_r \leq 4\|x\|_r$.

It follows that

$$|S_{n,r}^{(N)} x|_r \leq 8C(r)\, |x|_r \text{ for } x \in G_r.$$

With $N \to \infty$ we conclude that

$$|T_n J_r' x|_r \leq 8C(r)\, |x|_r \text{ for } x \in G_r.$$

Finally we have

$$|T_n x|_r \leq 8C(r)\, |a_{r+2} x|_r \leq 8C(r) |x|_{r+3}.$$

Since that can be done for all r the family $(T_n)_{n\in\mathbb{N}}$ is equicontinuous in E which had to be shown.

We have to verify the conditions of the cone interpolation theorem [1, Theorem 1].

1. $Q_{r,N}$ is a lower semi-lattice. This follows immediately from the fact that $A_r^{(N)}$ is monotone on ω^+.

2. $Q_{r,N} \cap G_\infty^+$ is total in G_∞. We have seen that $I - A_r^{(N)}$ is invertible in G_r, hence injective on G_∞. We show that $A_r^{(N)} \in L(G_\infty)$. Then, as $A_r^{(N)}$ is finite dimensional, $I - A_r^{(N)}$ is bijective in G_∞. Arguing as above we obtain that $Q_{r,N} \cap G_\infty^+ - Q_{r,N} \cap G_\infty^+ = G_\infty$.

We have the following chain of inequalities, where the second one is (3) the last one finite dimensionality of $Q^{(N)}$.

$$|A_r^{(N)} x|_\infty = |J_r|P|J_r' Q^{(N)} x|_\infty \leq \|J_r|P|J_r' Q^{(N)} x\|_\infty \leq |J_r' Q^{(N)} x|_{\infty,0} \leq |Q^{(N)} x|_{\infty,0} \leq C(N)|x|_\infty.$$

This completes the argument for 2.

3. Finally we have to show that $Q_{r,N} \cap G_\infty^+$ contains a strictly positive element. To show that we choose a strictly positive element $x_0 \in G_\infty$ and put $x = (I - A_r^{(N)})^{-1} x_0$. x can be calculated by means of the Neumann series in G_r. Since $A_r^{(N)}$ is positive this shows that $x \geq x_0$ which shows the claim.

References

[1] A. K. Dronov, V. M. Kaplitzkii, On the existence of a basis in a complemented subspace of a nuclear Köthe space of class *(d1)*. (Russian), Mat. Sb.209(2018), no.10, 50–70; translation in Sb. Math. 209 (2018), no.10, 1463–1481.

Bergische Universität Wuppertal,
FB Math.-Nat., Gauß-Str. 20,
D-42119 Wuppertal, Germany
e-mail: dvogt@math.uni-wuppertal.de