The tensor algebra of power series spaces

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Abstract

The linear isomorphism type of the tensor algebra $T(E)$ of Fréchet
spaces and, in particular, of power series spaces is studied. While for
nuclear power series spaces of infinite type it is always $s$ the situation
for finite type power series spaces is more complicated. The linear
isomorphism $T(s) \cong s$ can be used to define a multiplication on $s$
which makes it a Fréchet m-algebra $s_*$. This may be used to give an
algebra analogue to the structure theory of $s$, that is, characterize
Fréchet m-algebras with $(\Omega)$ as quotient algebras of $s_*$ and Fréchet
m-algebras with $(\text{DN})$ and $(\Omega)$ as quotient algebras of $s_*$ with respect
to a complemented ideal.

In [8] there was calculated the linear isomorphism type of the space $s$
of rapidly decreasing sequences and all of its complemented subspaces $E$. It
was shown that $T(E) \cong s$ whenever $\dim E \geq 2$. This includes all of so-
called power series spaces of infinite type, including the space $H(\mathbb{C}^d)$ of
entire functions for any dimension $d$. In the present work we study the
tensor algebra of Fréchet spaces in a more general context.

We use these results to give another proof for the isomorphism theorem in
the case of infinite type power series spaces. The isomorphism $T(s) \cong s$
defines a multiplication on $s$ which turns it into a Fréchet m-algebra which
we call $s_*$. Its quotient algebras are all nuclear Fréchet m-algebras with
property $(\Omega)$ or, equivalently, which are linearly isomorphic to a quotient of
$s$, and its quotient algebras with respect to a complemented ideal are the
nuclear Fréchet m-algebras with properties $(\text{DN})$ and $(\Omega)$. This gives an
“algebra equivalent” to the structure theory of $s$ as developed in [9].

In the last section we extend our study to the finite type power series
spaces, including e. g. the spaces $s_0$ of very slowly increasing functions
and the spaces $H(\mathbb{D}^d)$ of holomorphic functions on the $d$-dimensional poly-
disc. There the situation is much more complicated. But it is interesting to
observe that for $E = s_0$ and $E = H(\mathbb{D}^d)$ the results are different, however

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in the second case the result does not depend on $d$. In fact we obtain coincidence of $T(E)$ on the same class of power comparable $E$ (or $E$ of type (1)) as in [2], where the symmetric tensor algebras of infinite type power series spaces have been investigated.

It should be finally mentioned that Fréchet m-algebras which are linearly isomorphic to $s$ play a prominent role in [3] from where some of the notation below is taken. It might also be interesting to ask for the continuity of multiplicative forms on $s_*$, which would solve Michael’s conjecture (for a similar approach see [6]).

## 1 The tensor algebra of a Fréchet space

A Fréchet space is a complete metrizable locally convex space, its topology can be given by a fundamental system of seminorms $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_3 \ldots$. A Fréchet m-algebra is an algebra over $\mathbb{C}$ which is a Fréchet space and admits a fundamental system of submultiplicative seminorms.

For a Fréchet space $E$ we set

$$E^\otimes n := E \hat{\otimes} \ldots \hat{\otimes} E$$

the n-fold complete $\pi$-tensor product, and for any continuous seminorm $p$ on $E$ we denote by $p^\otimes n$ the n-fold $\pi$-tensor product of $p$.

With this notation we define

$$T(E) = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \prod_{n=1}^{\infty} E^\otimes n : \|x\|_k = \sum_{n=1}^{\infty} e^{kn}\|x_n\|_{E^\otimes n}^n < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$  

(1)

By

$$x_1 \otimes \cdots \otimes x_n \times y_1 \otimes \cdots \otimes y_m := x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m$$

$T(E)$ becomes a Fréchet m-algebra. In a natural way $E \subset T(E)$ and every continuous linear map $E \rightarrow A$, where $A$ is a Fréchet m-algebra, extends to a uniquely determined continuous algebra homomorphism $T(E) \rightarrow A$. If $E$ and $F$ are Fréchet spaces then every continuous linear map $\varphi : E \rightarrow F$ extends to a continuous algebra homomorphism $T(\varphi) : T(E) \rightarrow T(F)$.

Obviously the definition of $T(E)$ is independent of the fundamental system of seminorms on $E$ and $T(E)$ is determined, up to bicontinuous algebra isomorphism, by its universal property. It is called the tensor algebra of $E$.

If $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_3 \ldots$ is a fundamental system of seminorms for $E$ then we denote by $E_k$ the respective local Banach spaces and by $J_m^n : E_m \rightarrow E_n$
the canonical linking maps for \( m \geq n \) (see [5]). The local Banach spaces of \( T(E) \) have the following representation:

\[
T(E)_k = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \prod_{n=1}^{\infty} E_k^n : \|x\|_k = \sum_{n=1}^{\infty} e^{kn} \|x_n\|_k < +\infty \right\}. \tag{2}
\]

The following facts are easy consequences of the definition of the tensor algebra and its universal property:

1. If \( E \cong F \) then \( T(E) \cong T(F) \) with a bicontinuous algebra isomorphism.

2. If \( q : E \to F \) is a continuous surjective map, then \( T(q) : T(E) \to T(F) \) is a continuous surjective algebra homomorphism.

3. If \( F \) is a complemented subspace of \( E \) and \( P \) the projection, then \( T(F) \) is a complemented subalgebra of \( T(E) \) with the algebra homomorphism \( T(P) \) as projection.

If \( A \) is a Fréchet m-algebra then the identity \( \text{id}_A \) extends to a surjective algebra homomorphism \( q_A : T(A) \to A \) and we have an exact sequence

\[
0 \to J_A \to T(A) \xrightarrow{q_A} A \to 0
\]

where \( J_A \) is an ideal in \( T(A) \). Obviously \( q_A \) has \( A \hookrightarrow T(A) \) as a continuous linear right inverse.

If therefore the Fréchet m-algebras \( A \) and \( B \) are linearly isomorphic they can be considered as quotient algebras of the same algebra \( T(A) \) with respect to different complemented ideals. The same holds for algebras \( B \) which are linearly isomorphic to a quotient of \( A \). However there the ideals need not to be complemented.

We can improve (1) of the above assertions.

**Proposition 1.1** Let \( E \) and \( F \) be Fréchet spaces, then the following are equivalent:

1. \( T(E) \cong T(F) \) with a bicontinuous algebra isomorphism.
2. \( E \cong F \).

**Proof.** We have only to show (1) \(\Rightarrow\) (2). Let \( \varphi : T(E) \to T(F) \) be a bicontinuous algebra isomorphism. We refer to formula (1) and set

\[
T_2(E) = \{ x \in T(E) : x_1 = 0 \} = \overline{T(E)^2}.
\]
Likewise for $F$. Then it is obvious that $\varphi$ maps $T_2(E)$ bijectively onto $T_2(F)$. Let $\pi_F$ be the projection in $T(F)$ onto $F$ with kernel $T_2(F)$. Then $\pi_F \circ \varphi|_E$ is an isomorphism from $E$ onto $F$.

We close this section by two simple examples, which we take from [8]:

1. $T(C) = H_0(C)$, the space of all entire functions on $\mathbb{C}$ which vanish in 0.
2. $T(C^2) \cong s$, the space of all rapidly decreasing sequences (see below).

While the first example is the representation (1), the second is only a linear isomorphism. Because we will be using the second assertion we repeat, for the convenience of the reader, its simple proof.

Proof. We think of $C^2$ as equipped with the $\ell_1$-norm. Then $(C^2)^{\otimes n} = C_2^n$ again equipped with the $\ell_1$-norm (see [4, Chap. I, §2, no.2, Cor. 4, p. 61]). So we obtain counting the natural basis of $C_2^n$ from $2^n$ to $2^{n+1} - 1$:

$$T(C^2) = \left\{ x = (x_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \|x\|_k = \sum_{n=1}^{\infty} 2^{kn} \left( \sum_{j=2^n}^{2^{n+1}-1} |x_j| \right) \text{ for all } k \right\}.$$ 

For $2^n \leq j < 2^{n+1}$ we have $2^{kn} \leq j^k < 2^k \cdot 2^{kn}$. Therefore $T(C^2) \cong s$. □

2 Power series spaces

For any sequence $\alpha$ with $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots < +\infty$ and for $r = 0$ or $r = +\infty$ we set

$$\Lambda_r(\alpha) = \{ x = (x_j)_{j \in \mathbb{N}} : |x|_t = \sum_{j=0}^{\infty} |x_j|e^{t\alpha_j} < +\infty \text{ for all } t < r \}.$$ 

$\Lambda_r(\alpha)$ with the norms $| \cdot |_t$ is a Fréchet space. It is called a power series space, for $r = 0$ of finite type, for $r = +\infty$ of infinite type.

Of particular importance is the case $\alpha_n = \log n$. In this case we set $\Lambda_\infty(\log n) =: s$ and $\Lambda_0(\log n) =: s_0$. $s$ is the space of all rapidly decreasing sequences and $s_0$ is a space of very slowly increasing sequences.

Moreover we note that for $\alpha_n = n$ we have $\Lambda_\infty(n) \cong H(\mathbb{C})$, the space of all entire function in one complex variable and $\Lambda_0(n) \cong H(\mathbb{D})$ the space of holomorphic functions on the open unit disc.

We set $r_k = k$ if $r = \infty$ and $r_k = -\frac{1}{k}$ if $r = 0$. For the tensor algebra $T(\Lambda_r(\alpha))$ we obtain

$$T(\Lambda_r(\alpha)) = \{ x = (x_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}} : \|x\|_k = \sum_{n=1}^{\infty} \sum_{j\in\mathbb{N}} |x_{n,j}|e^{k^{n+r_k(\alpha_{j_1} + \cdots + \alpha_{j_n})}} < \infty \text{ for all } k \in \mathbb{N} \}.$$


We set \( \nu_n(j) = \alpha_{j_1} + \ldots + \alpha_{j_n} \) for \( j \in \mathbb{N}^n \) and denote by \( \beta_n(\nu) \), \( \nu = 1, 2, \ldots \) an increasing enumeration of the numbers \( \nu_n(j) \), \( j \in \mathbb{N}^n \).

Then we obtain

\[
T(\Lambda_r(\alpha)) \cong \{ x = (x_{n,\nu})_{n,\nu \in \mathbb{N}} : \|x\|_k = \sum_{n,\nu} |x_{n,\nu}| e^{kn + tr_k \beta_n(\nu)} < +\infty \text{ for all } k \in \mathbb{N} \}. \tag{3}
\]

From this representation we derive immediately:

**Lemma 2.1** The tensor algebra of an infinite type power series space is again an infinite type power series space.

We remark that \( \Lambda_r(\alpha) \cong \Lambda_r(\beta) \) if and only if \( \Lambda_r(\alpha) = \Lambda_r(\beta) \) and this is equivalent to the existence of a constant \( C > 0 \) such that \( \frac{1}{C} \alpha_j \leq \beta_j \leq C \alpha_j \) for large \( j \) (see [5, Proposition 29.1]).

At some point we will assume that \( \log n = O(\alpha_n) \), that is

\[
\limsup_n \frac{\log n}{\alpha_n} < +\infty. \tag{4}
\]

For \( r = \infty \) this is equivalent to the nuclearity of \( \Lambda_\infty(\alpha) \). All infinite type power series series spaces which are relevant in analysis belong to this class. It is not difficult to show that under this condition there is a subsequence \( (n_k)_{k \in \mathbb{N}} \) of \( \mathbb{N} \) such that \( \Lambda_r(\alpha) = \Lambda_r(\log n_k) \).

### 3 Linear topological properties of tensor algebras of Fréchet spaces

Throughout this section \( \cong \) will always denote a linear topological isomorphism. First we see that nuclearity of a Fréchet space is inherited by its tensor algebra.

**Theorem 3.1** For any Fréchet space \( E \) we have: \( E \) is nuclear if and only if \( T(E) \) is nuclear.

**Proof.** If \( T(E) \) is nuclear then also \( E \) as an (even complemented) subspace. To prove the converse we assume that \( E \) is a nuclear Fréchet space. This means that for every \( k \) we find \( p \) such that \( J_{k+p}^k : E_{k+p} \to E_k \) is nuclear. We have to show that for some \( q \) also the linking map \( J_{k+q}^k : T(E)_{k+q} \to T(E)_k \) is nuclear.
Now with $j_{k+p}^k$ also $(j_{k+p}^k)^{\otimes n} : E_{k+p}^{\otimes n} \to E_k^{\otimes n}$ is nuclear and $\nu((j_{k+p}^k)^{\otimes n}) \leq \nu(j_{k+p}^k)^n$, where $\nu(\cdot)$ denotes the nuclear norm of an operator. We choose $q \geq p$ such that $\nu(j_{k+p}^k) < e^q$ and remark that $\nu(j_{k+q}^k) \leq \nu(j_{k+p}^k)$. Then a straightforward calculation shows that $J_{k+q}^k$ is nuclear. 

For our general discussion we need the following lemma.

**Lemma 3.2** For any two Fréchet spaces $E$ and $F$ the space $T(E)\hat{\otimes}T(F)$ is isomorphic to a complemented subspace of $T(E \oplus F)$.

**Proof.** We have

$$T(E)\hat{\otimes}T(F) = \{ u = (u_{l,k}) \in \prod_{l,k \in \mathbb{N}} E^{\otimes l} \otimes F^{\otimes k} : \|u\|_m = \sum_n e^{mn} \sum_{l+k=n} \|u_{l,k}\|_m < +\infty \}.$$ 

This space arises in a natural way as a complemented subspace of $T(E \oplus F)$ by decomposing each summand $(E \oplus F)^{\otimes n}$ into $2^n$ direct summands following from the direct decomposition $E \oplus F$. 

We call a Fréchet space $E$ shift-stable if $E \cong \mathbb{C} \oplus E$. A power series space is shift-stable if and only if $\limsup_n \frac{\|a_{n+1}\|}{\|a_n\|} < +\infty$. In this case we call $\alpha$ shift-stable. Notice that all concrete spaces we will be considering are shift-stable.

**Lemma 3.3** If $E$ is shift-stable then $T(E) \cong s\hat{\otimes}T(E)$.

**Proof.** By assumption we have $E \cong \mathbb{C}^2 \oplus E$. Since, by Theorem 4.2, $T(\mathbb{C}^2) \cong s$, we get from Lemma 3.2 that $s\hat{\otimes}T(E) \cong T(\mathbb{C}^2)\hat{\otimes}T(E)$ is a complemented subspace of $T(\mathbb{C}^2 \oplus E) \cong T(E)$.

On the other hand $T(E)$ is obviously isomorphic to a complemented subspace of $s\hat{\otimes}T(E)$. [7, Proposition 1.2] then yields the result. 

By using again [7, Proposition 1.2] Lemma 3.3 implies:

**Proposition 3.4** If $E$ and $F$ are Fréchet spaces, $E$ shift-stable, $F$ isomorphic to a complemented subspace of $T(E)$ and $T(E)$ isomorphic to a complemented subspace of $F$, then $F \cong T(E)$.

We will use the following simple remark:

**Lemma 3.5** For any Fréchet space $E$ we have

$$T(E) \cong E \oplus T(E\hat{\otimes}E) \oplus E\hat{\otimes}T(E\hat{\otimes}E).$$
This shows, in particular, that $T(E\hat{\otimes}E)$ is a complemented subspace of $T(E)$. We will use Proposition 3.4 to show:

**Theorem 3.6** For any shift-stable Fréchet space $E$ we have $T(E) \cong T(E\hat{\otimes}E)$.

**Proof.** By Lemma 3.5 $T(E\hat{\otimes}E)$ is isomorphic to a complemented subspace of $T(E)$. Since $E$ is clearly isomorphic to a complemented subspace of $E\hat{\otimes}E$, the space $T(E)$ is isomorphic to a complemented subspace of $T(E\hat{\otimes}E)$. As $E$ is shift stable Proposition 3.4 yields the result.

Moreover, we get from Lemma 3.5 and Theorem 3.6 the following coincidences:

**Corollary 3.7** For any shift-stable Fréchet space $E$ we have

$$T(E) \cong E \oplus T(E) \cong E\hat{\otimes}T(E) \cong T(E) \oplus T(E).$$

**Proof.** By Theorem 3.6 we can write the identity in Lemma 3.5 as

$$T(E) \cong E \oplus T(E) \oplus E\hat{\otimes}T(E).$$

So $T(E)$ contains $E \oplus T(E)$ and $E\hat{\otimes}T(E)$ as complemented subspaces. As both contain $T(E)$ as complemented subspace, the first two isomorphisms follow from Proposition 3.4. The last one then follows from the first two and equation (5).

Finally, we call a Fréchet space $E$ stable if $E \oplus E \cong E$. A power series space is stable if and only if $\limsup_n \frac{\alpha_n}{\alpha_1} < +\infty$. In this case we call $\alpha$ stable. Clearly every stable power series space is shift-stable. Notice that all concrete spaces we will be considering are even stable.

**Theorem 3.8** For any stable Fréchet space $E$ we have

$$T(E)\hat{\otimes}T(E) \cong T(E).$$

**Proof.** By Lemma 3.2 $T(E)\hat{\otimes}T(E)$ is isomorphic to a complemented subspace of $T(E \oplus E) \cong T(E)$. As $T(E)$ is clearly isomorphic to a complemented subspace of $T(E)\hat{\otimes}T(E)$ Proposition 3.4 gives the result.

4 The tensor algebra of an infinite type power series space

The following theorem is already contained in [8]. However its proof here will be different.
Theorem 4.1 If $\log n = O(\alpha_n)$, that is $\Lambda_\infty(\alpha)$ nuclear, then $T(\Lambda_\infty(\alpha)) \cong s$.

Proof. By Lemma 2.1 and Theorem 3.1 we know that there is $\beta$ such that $T(\Lambda_\infty(\alpha)) \cong \Lambda_\infty(\beta)$ and $\log n = O(\beta_n)$.

On the other hand, let $\pi$ be the canonical projection in $\Lambda_\infty(\alpha)$ onto the span $F$ of the first two natural basis vectors $\{e_1, e_2\}$ in $\Lambda_\infty(\alpha)$, then $T(\pi)$ is a projection in $T(\Lambda_\infty(\alpha))$ onto $T(F) \cong T(\mathbb{C}^2) \cong s$. We remark that the basis vectors in $T(\mathbb{C}^2)$ which constitute the isomorphism to $s$ are products of $e_1$ and $e_2$, therefore a subset of the basis vectors in $T(\Lambda_\infty(\alpha))$ which constitute the isomorphism to $\Lambda_\infty(\beta)$. This implies that there is a subsequence $\beta_{n_k}$ which is equivalent to $\log k$. In particular we have $\beta_k \leq \beta_{n_k} = O(\log k)$. Therefore $\Lambda_\infty(\beta) = s$.

In particular we have:

Corollary 4.2 $T(s) \cong s$.

We obtain a complete characterization of the Fréchet spaces $E$ with $T(E) \cong s$.

Theorem 4.3 For a Fréchet space the following are equivalent:
1. $T(E) \cong s$.
2. $E$ is isomorphic to a complemented subspace of $s$ and $\dim E \geq 2$.

Proof. 1. implies 2. because $E$ is a complemented subspace of $T(E)$. If, on the other hand, 2. is satisfied, then we may assume that $E$ is a complemented subspace of $s$. Let $P$ be the projection. Then $T(P)$ is a projection in $T(s) \cong s$ onto $T(E)$. So $T(E)$ is a complemented subspace of $s$. We choose a 2-dimensional subspace $F \subset E$. Let $\pi$ be a projection in $E$ onto $F$. Then $T(\pi)$ is a projection in $T(E)$ onto $T(F) \cong s$. Therefore $s$ is isomorphic to a complemented subspace of $T(E)$. By [7, Proposition 1.2] we conclude that $T(E) \cong s$.

Clearly Theorem 4.3 contains Theorem 4.1. However the latter gives in its special case a more precise description of an isomorphism which we will now use.

The isomorphism in Theorem 4.1 hence also in Corollary 4.2 is of a special type: the basis vectors in $T(s)$ which constitute the isomorphism to $s$ are pure (tensor-)products of canonical basis vectors in $s$. They are ordered in a way, that the norms are increasing. We fix now once forever such an ordering, that is a special such isomorphism. Then this isomorphism equips the space $s$ with a multiplication which turns it into a Fréchet $m$-algebra, which we call $s_\cdot$. The multiplication in $s_\cdot$ is of special form: products of basis vectors are basis vectors.
In the sequel isomorphisms with \( s \) and its quotients or complemented subspaces are understood as linear isomorphisms, isomorphisms with \( s_\bullet \) and its quotients as Fréchet algebra isomorphisms.

The multiplicative equivalent of Theorem 4.3 is:

**Theorem 4.4** For a Fréchet space \( E \) the following are equivalent:
1. \( T(E) \cong s_\bullet \).
2. \( E \cong s \).

*Proof.* Consequence of Proposition 1.1 and Corollary 4.2.

The considerations at the end of Section 1 and the characterization of the quotient spaces and complemented subspaces of \( s \) in [9] (see also [5, §31]) now lead immediately to the following results, which give a Fréchet algebra equivalent to these characterizations.

**Theorem 4.5** For a Fréchet \( m \)-algebra the following are equivalent:
1. \( A \) is isomorphic to a quotient of \( s \).
2. \( A \) is nuclear and has property \((\Omega)\).
3. \( A \) is isomorphic to a quotient of \( s_\bullet \) with respect to a closed ideal.

**Theorem 4.6** For a Fréchet \( m \)-algebra the following are equivalent:
1. \( A \) is isomorphic to a complemented subspace of \( s \).
2. \( A \) is nuclear and has properties \((\text{DN})\) and \((\Omega)\).
3. \( A \) is isomorphic to a quotient of \( s_\bullet \) with respect to a complemented ideal.

For the definition and basic properties of the invariants \((\text{DN})\) and \((\Omega)\) see e. g. [5, §29].

### 5 The tensor algebra of finite type power series spaces

Now we will determine the linear isomorphism type of the tensor algebra for certain power series spaces of finite type.

**Theorem 5.1** \( T(s_0) \cong H(\mathbb{C})\hat{\otimes}s_0 \).

*Proof.* We refer to formula (3) and want to estimate the numbers \( \beta_n(\nu) \). For that we put for \( r \geq 0 \)

\[
m_n(r) = \# \{ j \in \mathbb{N}^n : \nu_n(j) \leq r \} = \# \{ j \in \mathbb{N}^n : \sum_{\nu=1}^n \log j_\nu \leq r \}.
\]
Counting only the $j$ with $j_2 = \ldots = j_n = 1$ we see that
\[ m_n(r) \geq [e^r] \geq e^r - 1. \]

For a reverse estimate we refer to the proof of [8, Theorem 5] and obtain
\[ m_n(r) \leq e^r \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{r^k}{k!} \leq 2^{n-1} e^r \sum_{k=0}^{n-1} \frac{r^k}{k!} \leq e^{n+2r}. \]

This yields
\[ \nu \leq m_n(\beta_n(\nu)) \leq e^{n+2\beta_n(\nu)} \]

hence
\[ \beta_n(\nu) \geq \frac{1}{2} (\log \nu - n). \]

On the other side we have for all $r < \beta_n(\nu)$
\[ \nu \geq m_n(r) \geq e^r - 1. \]

Hence $\nu \geq e^{\beta_n(\nu)} - 1$ and therefore
\[ \beta_n(\nu) \leq \log(\nu + 1) \leq \log 2 + \log \nu. \]

Therefore we have
\[ kn - \frac{1}{k} \beta_n(\nu) \leq (k + \frac{1}{2}) n - \frac{1}{2k} \log \nu \leq 2kn - \frac{1}{2k} \log \nu \]

and
\[ kn - \frac{1}{k} \beta_n(\nu) \geq - \frac{1}{k} \log 2 + kn - \frac{1}{k} \log \nu. \]

If we set for $x = (x_{n,\nu})_{n,\nu \in \mathbb{N}}$
\[ |x|_k = \sum_{n,\nu} |x_{n,\nu}| e^{kn - \frac{1}{k} \log \nu}, \]

i.e. the standard norms which constitute $H(\mathbb{C}) \hat{\otimes}_\pi s_0$, then we have
\[ \frac{1}{2} |x|_k \leq \|x\|_k \leq |x|_{2k}. \]

This completes the proof. \hfill \Box

Due to Lemma 3.3 we can put Theorem 5.1 in a more symmetric form.
**Corollary 5.2** \( T(s_0) \cong s \hat{\otimes} s_0 \).

It is impossible for \( T(s_0) \) to be the common tensor algebra for all complemented subspaces of \( s_0 \). This has nuclearity reasons and is due to Theorem 3.1.

We set now \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and we will study the tensor algebra of \( H(\mathbb{D}) \cong \Lambda_0(n) \). Proceeding as in the proof of Theorem 5.1 we have to estimate

\[
m_n(r) = \# \{ j \in \mathbb{N}^n : \sum_{\nu=1}^n j_\nu \leq r \}.
\]

Comparing with the volume of the standard simplex in \( \mathbb{R}^n \) we obtain

\[
m_n(r) \leq \frac{r^n}{n!} \leq m_n(r + \sqrt{n})
\]

and therefore

\[
\nu \leq m_n(\beta_n(\nu)) \leq \frac{\beta_n(\nu)^n}{n!}
\]

which implies

\[
\frac{1}{e} n\nu^{\frac{1}{n}} \leq (n!\nu)^{\frac{1}{n}} \leq \beta_n(\nu)
\]

as a lower estimate. An upper estimate we get from

\[
\nu \geq m_n(r) \geq \frac{1}{e} n\nu^{\frac{1}{n}} \leq \frac{(r - \sqrt{n})^n}{n!}
\]

for all \( r < \beta_n(\nu) \) which implies, replacing \( r \) with \( \beta_n(\nu) \),

\[
\beta_n(\nu) \leq (n!\nu)^{\frac{1}{n}} + \sqrt{n} \leq 2n\nu^{\frac{1}{n}}.
\]

So we have

\[
\frac{1}{3} n\nu^{\frac{1}{n}} \leq \beta_n(\nu) \leq 2n\nu^{\frac{1}{n}}.
\]

**Definition.** We set:

\[
TA_0 := \{ x = (x_{n,\nu})_{n,\nu \in \mathbb{N}} : \| x \|_k = \sum_{n,\nu} |x_{n,\nu}| e^{\eta(k - \frac{1}{2} \nu^{\frac{1}{n}})} < +\infty \text{ for all } k \in \mathbb{N} \}.
\]

Hence we have proved

**Lemma 5.3** \( T(H(\mathbb{D})) \cong TA_0 \).
Since for the \(d\)-dimensional polydisc \(D^d\) we have \(H(D^d) \cong H(D)^{\otimes d}\) we deduce immediately from Theorem 3.6 and Lemma 5.3:

**Theorem 5.4** \(T(H(D^d)) \cong TA_0\) for all dimensions \(d\).

We will generalize this result to a larger class of finite type power series spaces.

**Definition.** We call an exponent sequence \(\alpha\) **power comparable** if there are \(c > 0\) and \(0 < a \leq b\) so that
\[
\frac{1}{c} n^a \leq \alpha_n \leq c n^b
\]
for all \(n \in \mathbb{N}\).

To exploit this condition we need the following well known Lemma (cf. [1]) for which we give the simple proof for the convenience of the reader.

**Lemma 5.5** If \(\alpha\) is stable and \(\alpha_n \leq C \beta_n\) for all \(n\) and some \(C > 0\) then there is a subsequence \(k_n\) of \(N\) and \(D > 0\) so that
\[
\frac{1}{D} \alpha_{k_n} \leq \beta_n \leq D \alpha_{k_n}
\]
for all \(n \in \mathbb{N}\). In particular \(\Lambda_r(\beta)\) is a complemented subspace of \(\Lambda_r(\alpha)\) for \(r = 0, \infty\).

**Proof.** We set
\[
m_n := \sup\{m : \alpha_m \leq C \beta_n\},
\]
then \(m_n \geq n\). We put \(k_n = m_n + n\) and obtain
\[
\alpha_{k_n} \leq \alpha_{m_n} \leq \lambda \alpha_{m_n} \leq \lambda C \beta_n \leq \lambda \alpha_{m_n+1} \leq \lambda \alpha_{k_n}
\]
which completes the proof. \(\square\)

**Theorem 5.6** If \(\alpha\) is stable and power comparable then \(T(\Lambda_0(\alpha)) \cong T A_0\).

**Proof.** First we note that for any \(\nu \in \mathbb{Z}\) we have
\[
\Lambda_0(n^{2\nu}) \hat{\otimes} \Lambda_0(n^{2\nu}) \cong \Lambda_0(n^{2\nu-1})
\]
and that \(H(D) \cong \Lambda_0(n^{2\nu})\) with \(\nu = 0\).

From Theorem 3.6 we deduce that \(T(\Lambda_0(n^{2\nu})) \cong T(\Lambda_0(n)) \cong TA_0\) for all \(\nu \in \mathbb{Z}\). Since \(\alpha\) is power comparable we may assume that for some \(\nu \in \mathbb{N}\) and \(c > 0\) we have
\[
\frac{1}{c} n^{2-\nu} \leq \alpha_n \leq c n^{2\nu}
\]
for all $n \in \mathbb{N}$.

From Lemma 5.5 applied to the stable sequences $n^{2^{-\nu}}$ and $\alpha_n$ we obtain that $T(\Lambda_0(\alpha))$ is isomorphic to a complemented subspace of $T(\Lambda_0(n^{2^{-\nu}})) \cong T\Lambda_0$ and $T\Lambda_0 \cong T(\Lambda_0(n^{2^\nu}))$ is a isomorphic to a complemented subspace of $T(\Lambda_0(\alpha))$. From Proposition 3.4 we obtain the result. 

We consider also the following condition:

**Definition.** $\alpha$ is of type (I) if there is $p \in \mathbb{N}$ so that

$$1 < \inf_n \frac{\alpha_{pn}}{\alpha_n} \leq \sup_n \frac{\alpha_{pn}}{\alpha_n} < +\infty.$$ 

This condition was introduced in [2, Proposition 4.7] and there it is shown that it is equivalent to $\alpha$ being stable and $(n^{-\beta}\alpha_n)_n$ increasing for some $\beta > 0$.

To consider exponent sequences of type (I) is insofar interesting in our context as for those sequences the linear isomorphism type of the symmetric tensor algebra $S(\Lambda_\infty(\alpha))$ has been calculated in [2, Theorem 5.5] as $S(\Lambda_\infty(\alpha)) \cong \Lambda_\infty(\beta)$ where $\beta_n = \alpha_{[\log(n+1)]} \cdot \log(n+1)$.

**Lemma 5.7** If $\alpha$ is of type (I) then it is stable and power comparable.

**Proof.** We may assume that $\alpha_1 = 1$. If

$$1 < q = \inf_n \frac{\alpha_{pn}}{\alpha_n} \leq \sup_n \frac{\alpha_{pn}}{\alpha_n} = Q$$

then we have

$$q^{\nu} \leq \alpha_{p^{\nu}} \leq Q^{\nu}$$

for all $\nu \in \mathbb{N}_0$. For $p^{\nu} \leq n \leq p^{\nu+1}$ this gives with $a = \frac{\log q}{\log p}$ and $b = \frac{\log Q}{\log p}$

$$\frac{1}{q} n^a \leq q^{\nu} \leq \alpha_{p^{\nu}} \leq \alpha_n \leq \alpha_{p^{\nu+1}} \leq Q^{\nu+1} \leq Q n^b$$

which completes the proof. 

Hence we obtain

**Corollary 5.8** If $\alpha$ is of type (I) then $T(\Lambda_0(\alpha)) \cong T\Lambda_0$. 

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6 The space \( T\Lambda_0 \)

The space \( T\Lambda_0 \) is a nuclear (see Theorem 3.1) double indexed sequence space of the class of power spaces of mixed type. This class was introduced by Zahariuta in [10] (see also Zahariuta [11, §2]). They are Köthe spaces given by matrices of the form \( b_{j,k} = e^{(\lambda_j k - \frac{1}{2}) a_j} \) with \( \limsup \lambda_j > 0 \), \( \liminf \lambda_j = 0 \). In our case, with \( j = (n, \nu) \), we have \( a_{n,\nu} = n\nu \frac{1}{k} \) and \( \lambda_{n,\nu} = \nu^{-\frac{1}{2}} \).

To understand the asymptotics of the \( a_{n,\nu} \) and to write the space in a single indexed form we estimate the increasing arrangement \( \gamma_{\mu}, \mu \in \mathbb{N} \), of the set \( \{n\nu^{\frac{1}{k}} : n, \nu \in \mathbb{N}\} \).

For this we need to estimate

\[
M(t) = \#\{(n, \nu) : n\nu^{\frac{1}{k}} \leq t\} = \#\{(n, \nu) : \nu \leq (t/n)^n\}.
\]

For every \( n \leq t \) we have \([(t/n)^n]\) pairs \((n, \nu)\) fulfilling the estimates. Therefore

\[
M(t) = \sum_{n=1}^{[t]} [(t/n)^n].
\]

This immediately gives \( M(t) \leq e^t \). For a lower estimate we use the summand with \( n = [t/e] \). We obtain

\[
M(t) \geq [e^n] \geq e^{\frac{1}{e}-1} - 1.
\]

Proceeding like in the calculations in the previous section we obtain

\[
\log \mu \leq \gamma_{\mu} \leq \log (\mu + 1) + (e + 1).
\]

Therefore we may replace \( \gamma_{\mu} \) by \( \log \mu \) and write \( T\Lambda_0 \) in the following form:

\[
T\Lambda_0 = \{ x = (x_{\mu})_{\mu \in \mathbb{N}} : \|x\|_k = \sum_{\mu=1}^{\infty} |x_{\mu}| e^{kn_{\mu} - \frac{1}{2} \log \mu} < \infty \}.
\]

\((n_{\mu})_{\mu \in \mathbb{N}}\) is a sequence of integers which takes on every integer infinitely often. Moreover, without the \( kn_{\mu}\)-term we would have \( s_0 \) which is not nuclear. The \( kn_{\mu}\)-term provides nuclearity.

Finally let us remark that all the finite type power series spaces in question are complemented subspaces of \( s_0 \), hence \( T\Lambda_0 \) is a complemented subspace of \( T(s_0) \cong H(\mathbb{C}) \hat{\otimes} s_0 \). If we visualize the elements of this space as matrices, then \( T\Lambda_0 \) in the above representation consists of matrices which have in every column exactly one entry and in every row infinitely many sparsely distributed entries.
References


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