A Fundamental System of Seminorms for $A(K)$

Dietmar Vogt

Let $K \subset \mathbb{R}^d$ be compact and $A(K)$ the space of germs of real analytic functions on K with its natural (LF)-topology (see e.g. [1], 24.38, (2)). This topology can also be given by $A(K) = \lim \operatorname{ind}_{k\to+\infty} A_k$ where

$$A_k = \{(f_\alpha)_{\alpha\in\mathbb{N}_0^d} \in C(K)^{\mathbb{N}_0^d} \;:\; \|f\|_k := \sup_{x\in K} \frac{|f^{(\alpha)}(x)|}{\alpha!} k^{-|\alpha|} < +\infty\}.$$

Based on this description we give in the present note an explicit fundamental system of seminorms for $A(K)$.

We start with a modified problem.

Let X be a Banach space. We put

$$F_k = \{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in X^{\mathbb{N}_0^d} \;:\; \|x\|_k := \sup_\alpha \|x_\alpha\| k^{-|\alpha|} < +\infty\}$$

and

$$F = \lim \operatorname{ind}_{k\to+\infty} F_k.$$

On F we consider for any positive null-sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ the continuous norm

$$|x|_\delta = \sup_\alpha \|x_\alpha\| \delta_{|\alpha|}^{|\alpha|}.$$

Lemma 1 *The norms $|\cdot|_\delta$ are a fundamental system of seminorms on F.*

Proof: It is sufficient to show that for every positive sequence ε_k, $k\in\mathbb{N}$, there exists δ such that

$$U_\delta := \{x \;:\; |x|_\delta \le 1\} \subset \sum_k \varepsilon_k B_k$$

where B_k denotes the unit ball of F_k. Without restriction of generality we may assume, that $\varepsilon_k \le 1$ for all k.

Fot every k we choose $n_k > n_{k-1}$, such that $k-1 < \varepsilon_k^{1/n_k} k$. We put $\delta_n^{-1} = \varepsilon_k^{1/n_k} k$ for $n_k \le n < n_{k+1}$. We obtain for these n

$$\delta_n^{-n} = \varepsilon_k^{n/n_k} k^n \le \varepsilon_k k^n.$$

Due to the construction $\delta = (\delta_n)_n$ is a null-sequence. For $x \in U_\delta$ and $n_k \leq |\alpha| < n_{k+1}$ we have $x_\alpha \in \varepsilon_k B_k$ and therefore

$$\xi_k = \sum_{n_k \leq |\alpha| < n_{k+1}} x_\alpha \in \varepsilon_k B_k.$$

Since $x = \sum_k \xi_k$ the proof is complete. □

Theorem 2 *If $K \subset \mathbb{R}^d$ is compact, then the norms*

$$|f|_\delta = \sup_\alpha \sup_{x \in K} \frac{|f^{(\alpha)}(x)|}{\alpha!} \delta_{|\alpha|}^{|\alpha|},$$

where δ runs through all positive null-sequences, are a fundamental system of seminorms in $A(K)$.

Proof: Let F be as above with $X = C(K)$. We define a map $A : A(K) \to F$ by $A(f) = \left(\frac{f^{(\alpha)}(x)}{\alpha!}\right)_{\alpha \in \mathbb{N}_0^d}$. The map A is obviously continuous and $A^{-1}(B)$ is bounded in $A(K)$ for every bounded set B in F. From Baernstein's Lemma (see [1], 26.26) it follows, that A is an injective topological homomorphism. Hence Lemma 1 proves the result. □

References

[1] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford (1997).

Bergische Universität Wuppertal,
FB Math.-Nat., Gauß-Str. 20,
D-42119 Wuppertal, Germany
e-mail: dvogt@math.uni-wuppertal.de