NON-NATURAL TOPOLOGIES ON SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract

It is shown that every proper Fréchet space with weak*-separable dual admits uncountably many inequivalent Fréchet topologies. This applies, in particular, to spaces of holomorphic functions so solving in the negative a problem of JARNICKI and PFLUG. For this case an example with a short self-contained access is added.

It is a well known and often used fact, following from the closed graph theorem, that for any Fréchet space of continuous functions the following is true: if any convergent sequence of functions in $E$ also converges pointwise (or locally in $L_1$) then convergence in $E$ implies uniform convergence on compact sets. This is of particular interest if $E$ is a Fréchet space of holomorphic functions (see KRANTZ [2]). In this connection the question has been raised whether this might be true for any Fréchet space $E$ of holomorphic functions (see JARNICKI AND PFLUG [1, Remark 1.10.6, (b), p. 66]), that is, if every convergent sequence in $E$ converges uniformly on compact sets. In the present note this question is solved in the negative in a very strict sense. For functional analytic tools and unexplained notation see [3].

The author thanks Peter Pflug for drawing his attention to this problem.

Lemma 1 All proper (that is: not Banach) Fréchet spaces with weak*-separable dual are linearly isomorphic.

Proof: By dim $E$ we denote the linear dimension of a linear space $E$ and we set $\omega := \mathbb{C}^\mathbb{N}$ with the product topology. By Eidelheit’s Theorem we know that there is a linear surjective map from $E$ onto $\omega$ (see [3, 26.28]). This implies that $\dim \omega \leq \dim E$.

Let $\{y_1, y_2, \ldots\}$ be a weak*-dense set in $E'$. Then $x \mapsto (y_1(x), y_2(x), \ldots)$ is a linear, injective map $E \to \omega$. Therefore $\dim E \leq \dim \omega$.

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Hence we obtain \( \dim E = \dim \omega \) which shows the result.

**Remark 2** In the above proof it is enough that \( \text{span}\{y_1, y_2, \ldots\} \) is weak*-star dense, that is \( \{x : y_j(x) = 0 \text{ for all } j\} = \{0\} \). In fact, the existence of a sequence \((y_j)_{j \in \mathbb{N}}\) in \( E' \) with \( \{x : y_j(x) = 0 \text{ for all } j\} = \{0\} \) is equivalent to the weak*-separability of \( E' \).

For any scalar sequence \( 0 < \alpha_1 \leq \alpha_2 \leq \cdots \to +\infty \) we set (see [3, §29])

\[
\Lambda_\infty(\alpha) := \{x \in \omega : |x|^2_t = \sum_{j=1}^{\infty} |x_j|^2 t^{\alpha_j} < \infty \text{ for all } t > 0\}.
\]

Equipped with the norms \( |\cdot|_k, k \in \mathbb{N} \), this is a Fréchet space. It is known (see [3, 29.1]) that there is a linear topological isomorphism \( \Lambda_\infty(\alpha) \cong \Lambda_\infty(\beta) \) if, and only if, there is \( C > 0 \) such that \( \frac{1}{C} \alpha_j \leq \beta_j \leq C \alpha_j \) for all \( j \). Therefore there are uncountably many non-isomorphic spaces \( \Lambda_\infty(\alpha) \) and all satisfy the assumptions of Lemma 1.

**Theorem 3** Let \( E \) be a proper Fréchet space with weak*-separable dual, then on \( E \) there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of \( E \).

**Proof:** For any \( \alpha \) we have a linear isomorphism \( E \cong \Lambda_\infty(\alpha) \) and this linear isomorphism induces a Fréchet topology on \( E \). So we obtain uncountably many inequivalent Fréchet topologies on \( E \). Only one of these can possibly coincide with the given topology of \( E \). This shows the result. \( \square \)

This has several consequences. At first the assumptions of Theorem 3 are satisfied for many spaces in analysis.

**Corollary 4** Let \( X \) be a \( \sigma \)-compact manifold and \( E \subset C(X) \) a continuously imbedded proper Fréchet space, then on \( E \) there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of \( E \).

**Proof:** Let \( \{x_n : n \in \mathbb{N}\} \) be a dense subset of \( X \), then \( \{\delta x_n : n \in \mathbb{N}\} \) has the properties like in Remark 2. Hence \( E' \) is weak*-separable. \( \square \)

As a special case of this we can solve in the negative the above-mentioned problem of JARNECKI and PFLUG (see [1, Remark 1.10.6, (b), p. 66]). Let \( \Omega \subset \mathbb{C}^n \) be open and \( \mathcal{O}(\Omega) \) the space of holomorphic functions on \( \Omega \) with the compact-open topology. Let \( E \subset \mathcal{O}(\Omega) \) be a continuously imbedded proper Fréchet space. Then we have:

**Corollary 5** On \( E \) there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of \( E \).
And we obtain:

**Theorem 6** If $E$ is an infinite dimensional Fréchet-Schwartz space, then on $E$ there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of $E$.

**Proof:** In this case $E'$ is even separable in the strong topology. □

For $E = \mathcal{O}(\Omega)$ we could have used also this theorem to show Corollary 5.

Since the question originates from complex analysis we give finally a function-theoretic example with a direct and self-contained proof:

Let $G \subset \mathbb{C}$ a domain, that is an open and connected subset of $\mathbb{C}$, $\mathcal{O}(G)$ the Fréchet space of all holomorphic functions on $G$ equipped with the compact-open topology.

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence without accumulation point in $G$. Then $f \mapsto (f(z_n))_{n \in \mathbb{N}}$ is a linear surjective map from $\mathcal{O}(G)$ onto $\omega$. Therefore $\dim \omega \leq \dim \mathcal{O}(G)$.

On the other hand: fix $z \in G$. Then $f \mapsto (f^{(p^{-1})}(z))_{n \in \mathbb{N}}$ is a linear injective map from $\mathcal{O}(G)$ into $\omega$. Therefore $\dim \mathcal{O}(G) \leq \dim \omega$.

Consequently $\mathcal{O}(G)$ and $\omega$ are linearly isomorphic. They are not isomorphic as Fréchet spaces, since $\mathcal{O}(G)$ admits a continuous norm, but $\omega = \mathbb{C}^\mathbb{N}$ does not. Due to the open mapping theorem the linear isomorphism is continuous in neither direction.

So the space $\mathcal{O}(G)$ with the topology $\tau$ induced from $\omega$ by the linear isomorphism is a Fréchet space such that convergence with respect to $\tau$ does not imply uniform convergence on compact sets, that is, it is not *natural* in the sense of [1].

**References**

