Non-natural topologies on spaces of holomorphic functions

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Abstract

It is shown that every proper Fréchet space with weak*-separable dual admits uncountably many inequivalent Fréchet topologies. This applies, in particular, to spaces of holomorphic functions so solving in the negative a problem of Jarnicki and Pflug. For this case an example with a short self-contained access is added.

It is a well known and often used fact, following from the closed graph theorem, that for any Fréchet space of continuous functions the following is true: if any convergent sequence of functions in E also converges pointwise (or locally in L_1) then convergence in E implies uniform convergence on compact sets. This is of particular interest if E is a Fréchet space of holomorphic functions (see Krantz [2]). In this connection the question has been raised whether this might be true for any Fréchet space E of holomorphic functions (see Jarnicki and Pflug [1, Remark 1.10.6, (b), p. 66]), that is, if every convergent sequence in E converges uniformly on compact sets. In the present note this question is solved in the negative in a very strict sense. For functional analytic tools and unexplained notation see [3].

The author thanks Peter Pflug for drawing his attention to this problem.

Lemma 1 *All proper (that is: not Banach) Fréchet spaces with weak*-separable dual are linearly isomorphic.*

Proof: By $\dim E$ we denote the linear dimension of a linear space E and we set $\omega := \mathbb{C}^{\mathbb{N}}$ with the product topology. By Eidelheit's Theorem we know that there is a linear surjective map from E onto ω (see [3, 26.28]). This implies that $\dim \omega \leq \dim E$.

Let $\{y_1, y_2, \dots\}$ be a weak*-dense set in E'. Then $x \mapsto (y_1(x), y_2(x), \dots)$ is a linear, injective map $E \hookrightarrow \omega$. Therefore $\dim E \leq \dim \omega$.

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Hence we obtain $\dim E = \dim \omega$ which shows the result. $\square$

Remark 2 In the above proof it is enough that $\operatorname{span}\{y_1, y_2, \dots\}$ is weak*-star dense, that is $\{x : y_j(x) = 0 \text{ for all } j\} = \{0\}$. In fact, the existence of a sequence $(y_j)_{j\in\mathbb{N}}$ in E' with $\{x : y_j(x) = 0 \text{ for all } j\} = \{0\}$ is equivalent to the weak*-separability of E'.

For any scalar sequence $0 < \alpha_1 \le \alpha_2 \le \cdots \nearrow +\infty$ we set (see [3, §29])

$$\Lambda_\infty(\alpha) := \{x \in \omega : |x|_t^2 = \sum_{j=1}^{\infty} |x_j|^2 e^{t\alpha_j} < \infty \text{ for all } t > 0\}.$$

Equipped with the norms $|\cdot|_k$, $k \in \mathbb{N}$, this is a Fréchet space. It is known (see [3, 29.1]) that there is a linear topological isomorphism $\Lambda_\infty(\alpha) \cong \Lambda_\infty(\beta)$ if, and only if, there is $C > 0$ such that $\frac{1}{C}\,\alpha_j \le \beta_j \le C\,\alpha_j$ for all j. Therefore there are uncountably many non-isomorphic spaces $\Lambda_\infty(\alpha)$ and all satisfy the assumptions of Lemma 1.

Theorem 3 *Let E be a proper Fréchet space with weak*-separable dual, then on E there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of E.*

Proof: For any α we have a linear isomorphism $E \cong \Lambda_\infty(\alpha)$ and this linear isomorphism induces a Fréchet topology on E. So we obtain uncountably many inequivalent Fréchet topologies on E. Only one of these can possibly coincide with the given topology of E. This shows the result. $\square$

This has several consequences. At first the assumptions of Theorem 3 are satisfied for many spaces in analysis.

Corollary 4 *Let X be a σ-compact manifold and $E \subset C(X)$ a continuously imbedded proper Fréchet space, then on E there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of E.*

Proof: Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of X, then $\{\delta_{x_n} : n \in \mathbb{N}\}$ has the properties like in Remark 2. Hence E' is weak*-separable. $\square$

As a special case of this we can solve in the negative the above-mentioned problem of Jarnicki and Pflug (see [1, Remark 1.10.6, (b), p. 66]). Let $\Omega \subset \mathbb{C}^n$ be open and $\mathscr{O}(\Omega)$ the space of holomorphic functions on Ω with the compact-open topology. Let $E \subset \mathscr{O}(\Omega)$ be a continuously imbedded proper Fréchet space. Then we have:

Corollary 5 *On E there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of E.*

And we obtain:

Theorem 6 *If E is an infinite dimensional Fréchet-Schwartz space, then on E there exist uncountably many mutually inequivalent Fréchet topologies which are inequivalent to the given topology of E.*

Proof: In this case E' is even separable in the strong topology. □

For $E = \mathscr{O}(\Omega)$ we could have used also this theorem to show Corollary 5.

Since the question originates from complex analysis we give finally a function-theoretic example with a direct and self-contained proof:

Let $G \subset \mathbb{C}$ a domain, that is an open and connected subset of $\mathbb{C}$, $\mathscr{O}(G)$ the Fréchet space of all holomorphic functions on G equipped with the compact-open topology.

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence without accumulation point in G. Then $f \mapsto (f(z_n))_{n\in\mathbb{N}}$ is a linear surjective map from $\mathscr{O}(G)$ onto ω. Therefore $\dim \omega \leq \dim \mathscr{O}(G)$.

On the other hand: fix $z \in G$. Then $f \mapsto (f^{(p-1)}(z))_{n\in\mathbb{N}}$ is a linear injective map from $\mathscr{O}(G)$ into ω. Therefore $\dim \mathscr{O}(G) \leq \dim \omega$.

Consequently $\mathscr{O}(G)$ and ω are linearly isomorphic. They are not isomorphic as Fréchet spaces, since $\mathscr{O}(G)$ admits a continuous norm, but $\omega = \mathbb{C}^{\mathbb{N}}$ does not. Due to the open mapping theorem the linear isomorphism is continuous in neither direction.

So the space $\mathscr{O}(G)$ with the topology τ induced from ω by the linear isomorphism is a Fréchet space such that convergence with respect to τ does not imply uniform convergence on compact sets, that is, it is not *natural* in the sense of [1].

References

[1] M. Jarnicki, P. Pflug: *First Steps in Several Complex Variables: Reinhardt Domains,* EMS Textbooks in Mathematics, (2008).

[2] S. G. Krantz, *Topologies on the space of holomorphic functions (REVISED),* arXiv:0707.1876v2, 23 Jul 2007.

[3] R. Meise, D. Vogt: *Introduction to functional analysis,* Clarendon Press, Oxford, (1997).