HADAMARD TYPE OPERATORS ON SPACES OF REAL
ANALYTIC FUNCTIONS IN SEVERAL VARIABLES

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Abstract

We consider multipliers on the space of real analytic functions of several variables $A(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, i.e., linear continuous operators for which all monomials are eigenvectors. If zero belongs to $\Omega$ these operators are just multipliers on the sequences of Taylor coefficients at zero. In particular, Euler differential operators of arbitrary order are multipliers. We represent all multipliers via a kind of multiplicative convolutions with analytic functionals and characterize the corresponding sequences of eigenvalues as moments of suitable analytic functionals. Moreover, we represent multipliers via suitable holomorphic functions with Laurent coefficients equal to the eigenvalues of the operator. We identify in some standard cases what topology should be put on the suitable space of analytic functionals in order that the above mentioned isomorphism becomes a topological one when the space of multipliers inherits the topology of uniform convergence on bounded sets from the space of all endomorphisms on $A(\Omega)$. We also characterize in the same cases when the discovered topology coincides with the classical topology of bounded convergence on the space of analytic functionals. We provide several examples of multipliers and show surjectivity results for multipliers on $A(\Omega)$ if $\Omega \subset \mathbb{R}^d$.

1 Introduction

By a (Hadamard type) multiplier on the space of real analytic functions $A(\Omega)$ we mean each linear continuous map $M : A(\Omega) \to A(\Omega)$ for which the monomials $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ are eigenvectors with a corresponding multiple sequence of eigenvalues $(m_\alpha)_{\alpha \in \mathbb{N}^d}$. Here $\Omega$ is an open nonempty subset of $\mathbb{R}^d$. It can be easily seen that if zero $0 := (0, \ldots, 0)$ belongs to $\Omega$ then the map just multiplies the sequence of Taylor coefficients $(f_\alpha)_{\alpha \in \mathbb{N}^d}$ at zero of the function $f$ by the multiplier sequence $(m_\alpha)_{\alpha \in \mathbb{N}^d}$. Nevertheless multipliers are not simply diagonal operators since the monomials never form a basis of $A(\Omega)$ for any open set $\Omega \subset \mathbb{R}^d$, see [12]. There are several natural examples of such operators, among them variable coefficient linear partial differential operators of Euler type (for more examples, see Section 7). Let us observe that the class of multipliers $M(\Omega)$ forms a closed commutative subalgebra of the algebra of all linear continuous operators on $A(\Omega)$ equipped with the topology of uniform convergence on bounded subsets (= the strong topology).

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In the holomorphic case our multipliers are often called Hadamard multipliers since the
holomorphic function whose Taylor coefficients sequence is just the coefficientwise product of the
Taylor coefficient sequences of two other holomorphic functions \( f \) and \( g \) is called the Hadamard
product \( f \ast g \). Moreover, such product is related to the famous Hadamard multiplication theorem
[3, Ch. 1.4].

In the present paper we consider three main problems.

First, we find a representation of all multipliers on the space \( \mathcal{A}(\Omega) \) for arbitrary open non-
empty sets \( \Omega \subset \mathbb{R}^d \) via analytic functionals \( T \in \mathcal{A}(V(\Omega))' \), i.e., those analytic functionals \( T \in \mathcal{A}(\mathbb{R}^d)' \) with
\[
\text{supp } T \subset V(\Omega) := \{ x \in \mathbb{R}^d \mid x\Omega \subset \Omega \},
\]
here multiplication is meant coordinatewise, see Theorem 3.4. An analogous result was proved
for the one variable case (i.e., \( \Omega \subset \mathbb{R} \)) in [7] but the several variable case is essentially different.

In the one variable case it was proved in [7] that
\[
\mathcal{B} : \mathcal{A}(V(\Omega))' \to M(\Omega), \quad \mathcal{B}(T)(g)(y) := \langle g(y), T \rangle
\]
is surjective observing that every \( M \in M(\Omega) \) corresponds to a functional \( T \in \mathcal{A}(\mathbb{R})' \) with
\[
\text{supp } T \subset \tilde{V}(\Omega) := \{ x \in \mathbb{R}^d \mid x(\Omega \cap N\mathbb{Z}) \subset \Omega \},
\]
where
\[
N\mathbb{Z} := \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_j \neq 0 \text{ for } j = 1, \ldots, d \}.
\]
In the one variable case this is enough since \( \tilde{V}(\Omega) = V(\Omega) \) (we say that \( \Omega \) is acceptable) but in
the several variable case the latter is no longer true, see Remark 3.7 (b). Moreover, in the one-
variable proof it was used extensively the so-called Köthe-Grothendieck duality which represents
an analytic functional as a holomorphic function on the complement of its support. This is not
available for the several variable case in general and that is why in the present proof of the
representation theorem (i.e. bijectivity of \( \mathcal{B} \)) we substitute this by a new elementary approach
which allows to avoid any representation of the analytic functional.

The second problem considered here is to characterize multiplier sequences corresponding to
multipliers in \( M(\Omega) \), \( \Omega \subset \mathbb{R}^d \) an arbitrary open nonempty set. Please note that since polynomials
are dense in \( \mathcal{A}(\Omega) \) the multiplier sequence uniquely determines the multiplier. In fact, the result
mentioned in the previous paragraph gives a required description. Namely a sequence \( (m_\alpha)_{\alpha \in \mathbb{N}^d} \)
is a multiplier sequence if and only if it is a moment sequence of some analytic functional \( T \in \mathcal{A}(V(\Omega))' \) (see Theorem 3.4). In the one variable case we found also a characterization
as coefficient sequences of Laurent series representations at zero of holomorphic functions on
the complement of \( V(\Omega) \) (see [7, Theorem 2.8]) using Köthe-Grothendieck representation of
functionals. An analogous representation via Taylor coefficients was also given there. In the
several variable case this does not work. Nevertheless in Section 4 (Theorem 4.5 and 4.8) we
provide another representation of multipliers in \( M(\Omega) \) with a multiplier sequence \( (m_\alpha)_{\alpha} \) in terms
of holomorphic functions with Laurent or Taylor coefficients \( (m_\alpha)_{\alpha} \).

The third problem is the question which topology on \( \mathcal{A}'(V(\Omega)) \) is induced by \( \mathcal{B} \) from \( M(\Omega) \subset
\mathcal{L}_b(\mathcal{A}(\Omega)) \), i.e., for which topology \( t \) on \( \mathcal{A}'(V(\Omega)) \) the map
\[
\mathcal{B} : (\mathcal{A}'(V(\Omega)), t) \to M(\Omega)
\]
is a topological isomorphism, where \( M(\Omega) \) is always equipped with the topology of uniform
convergence on bounded sets inherited from the space of all linear continuous operators \( \mathcal{L}_b(\mathcal{A}(\Omega)) \)
on \( \mathcal{A}(\Omega) \). In [7] considering the one variable case we propose a somehow naive conjecture that
\[
(1) \quad \mathcal{B} : \mathcal{A}(V(\Omega))'_b \to M(\Omega)
\]
is always a topological isomorphism where \( b \) means the natural topology of uniform convergence on bounded sets. Here we prove that this conjecture is false in general (even in the one-dimensional case!), see Theorem 6.12 and the remarks below. In fact, we show that a more promising candidate is a weaker topology: the so-called \( k \)-topology on \( \mathcal{A}'(V(\Omega)) \), i.e.,

\[
\mathcal{A}'(V(\Omega))_k := \text{proj}_{K \in \Omega} \mathcal{A}'(V_K(\Omega))_b
\]

where \( K \) runs through all compact subsets of \( \Omega \) and

\[
V_K(\Omega) := \{ x \in \mathbb{R}^d \mid xK \subset \Omega \}.
\]

We proved (Theorem 5.11) that

\[
\mathcal{B} : \mathcal{A}'(V(\Omega))_k \to M(\Omega)
\]

is always continuous. In the natural cases like if either \( \Omega \) is convex or \( \Omega \subset NZ \) or \( \dim \Omega = 1 \) then \( \mathcal{B} \) as above (2) is even a topological isomorphism (Theorem 5.20) but the conjecture that this is always the case remains open. Instead of the Köthe-Grothendieck duality so useful in the one-dimensional case we have to use here the so-called Tillmann-Grothendieck duality, i.e., a representation of analytic functionals via suitable chosen harmonic functions (or, more precisely, classes of such functions). A surprising consequence of the above theory is the observation (Proposition 5.14) that \( M(\mathbb{R}^d) \) is complemented in \( L_b(\mathbb{R}^d) \! \)!

In general, the \( b \)-topology, the \( k \)-topology and the topology induced by \( \mathcal{B} \) from \( M(\Omega) \) are very close to each other: they have the same bounded sets and convergent sequences (Theorem 5.15). In Section 6 we identify cases when these topologies are identical, i.e., when \( \mathcal{B} \) in (1) is a topological isomorphism. If either \( \dim \Omega = 1 \) or \( \Omega \subset NZ \) or \( \Omega \) is convex it holds if and only if \( \partial V(\Omega) \cap V(\Omega) \) is compact and in the latter two cases if and only if \( V(\Omega) \) is either compact or open (Theorem 6.12 and 6.15). Section 6 is completed by several examples and simple results explaining when this is the case.

In Section 7, we collect examples of multipliers on spaces of analytic functions of several variables (Euler differential operators, integral multiplier operators, dilation operators and superposition multipliers) together with their basic properties and explain the role they play in the theory (see, for example, Proposition 7.1 or Theorem 7.3).

Finally, in Section 8, we consider the case \( \Omega \subset \mathbb{R}^d_+ \) where one can translate the problems on multipliers to problems on classical convolution operators. After explaining this translation we get some results on surjectivity of finite order Euler partial differential operators following the results of Hörmander and Langenbruch on convolution operators.

The one variable case of multipliers on spaces of real analytic functions was studied in [7] (and further analyzed in [8], [9]). In [21], [22], Euler differential operators (which are special cases of multipliers) were considered also on \( \mathcal{A}(\Omega) \). Korobeinik considered this type of variable coefficients linear differential operators in [24], [25]. The topic of Taylor coefficient multipliers is in fact very classical, already Hadamard considered such operators in [18, page 158 ff.]. There is an extensive literature on Hadamard type multipliers acting on spaces of holomorphic functions on open complex sets: see, for instance, [4], [5], [16], a series of papers of Müller and Pohlen [31], [32], [33] as well as a series of papers of Render (where the algebraic structure of \( M(\Omega) \) is studied) see for example the survey paper [36].

Recently, the third named author [40] (comp. also [41]) considered the analogon of multipliers on the space of smooth functions and some of the ideas explained there are clearly inspiring for the present paper.
Let us recall that the space of real analytic functions $\mathcal{A}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^d$ (or an arbitrary set $\Omega$!) is endowed with its natural topology $\text{ind}_U H(U)$, i.e., the locally convex inductive limit topology, where $U \subset \mathbb{C}^d$ runs through all complex neighborhoods of $\Omega$. With this topology it is clear that for any set $S \subset \mathbb{R}^d$ the space $\mathcal{A}(S)'$ is really equal to the dual of $\mathcal{A}(S)$. For more information on the space of real analytic functions see the survey [6].

In the present paper we always use the coordinatewise multiplication:

$$xy := (x_1y_1, \ldots, x_dy_d), \quad \text{if } x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d.$$ 

We define

$$NZ := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_j \neq 0, \text{ for every } j = 1, \ldots, d\}.$$ 

Clearly, if $y \in NZ$ we may define

$$\frac{1}{y} \Omega = \left\{ \left(\frac{x_1}{y_1}, \ldots, \frac{x_d}{y_d}\right) : (x_1, \ldots, x_d) \in \Omega \right\}.$$ 

By $0, \mathbb{I} \in \mathbb{R}^d$ we denote $0 := (0, \ldots, 0)$ and $\mathbb{I} := (1, \ldots, 1)$. For a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we set

$$|x| := \left(x_1^2 + \cdots + x_d^2\right)^{1/2} \quad \text{and} \quad d(x, y) := |x - y|.$$ 

Moreover,

$$B_\varepsilon(x) = \{y|d(x, y) < \varepsilon\}.$$ 

We will use multiindices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and set

$$|\alpha| = \alpha_1 + \cdots + \alpha_d \quad \text{and} \quad \alpha! = \alpha_1! \cdots \alpha_d!.$$ 

For non-explained notions from Functional Analysis and Harmonic Function Theory see [30] and [1], respectively.

## 2 Elements of Harmonic Function Theory

We will need the following elements of the harmonic function theory. For a compact set $K \subset \mathbb{R}^n$ let $C_\Delta(K)$ (and $C_{\Delta,0}(\mathbb{R}^n \setminus K)$, respectively) denote the family of all harmonic germs near $K$ (and the harmonic functions on $\mathbb{R}^n \setminus K$ vanishing at $\infty$, respectively). It is well known that every continuous linear functional $T$ on $C_\Delta(K)$ corresponds to a harmonic function $f_T \in C_{\Delta,0}(\mathbb{R}^n \setminus K)$ via the so-called Tillmann-Grothendieck duality (TG duality, see [38, Satz 6] and also [2]: a general version for zero solutions of hypoelliptic partial differential operators is contained in [26]). To be precise, let

$$G(\xi) := \frac{-1}{c_n(n - 2)}|\xi|^{2-n}, \quad \xi \neq 0,$$ 

be the canonical elementary solution for the Laplacian in $n$ variables (for $n \geq 3$, see e.g. [1, p. 193], $c_n$ is the area of the unit sphere). The function $f_T$ corresponding to $T$ is then defined by

$$f_T(\xi) := \langle G(\xi - \cdot) , T \rangle, \xi \in \mathbb{R}^n \setminus K.$$ 

(3)

The correspondence of $f_T$ and $T$ is given by the TG duality (see [2, (4)])

$$T(h) = (f_T, h) := \langle \Delta(\varphi h) , f_T \rangle := \int \Delta(\varphi h)(\xi)f_T(\xi)\,d\xi, \quad h \in C_\Delta(K),$$ 

(4)
where \( \varphi \in C_0^\infty(U) \) is chosen such that \( \varphi = 1 \) near \( K \) if \( h \in C_\Delta(U) \) — the class of all harmonic functions on \( U \) and \( U \) is an open neighbourhood of \( K \). By Green’s formula, we also get (compare [38, (48)])

\[
(f_T, h) = \int_{\partial A} f_T(\xi) \frac{\partial}{\partial \nu} h(\xi) - h(\xi) \frac{\partial}{\partial \nu} f_T(\xi) \, d\sigma(\xi), \quad h \in C_\Delta(K),
\]

where \( A \) is a compact set with smooth boundary such that \( K \subset A^0 \subset A \subset U \) if \( h \in C_\Delta(U) \) and \( \sigma \) is the Lebesgue-surface measure.

We will apply the duality (4) to represent analytic functionals supported in a compact set \( K \subset \mathbb{R}^d \) as harmonic functions on \( \mathbb{R}^{d+1} \setminus K \). For this we write the points in \( \mathbb{R}^{d+1} \) as \( (x, t) \in \mathbb{R}^d \times \mathbb{R} \). For \( K \subset \mathbb{R}^d \) compact let \( \tilde{\mathcal{C}}_\Delta(K) \) denote the class of all harmonic germs (in \( (d+1) \) variables) near \( K \) which are even with respect to the variable \( t \). Notice that \( \mathcal{A}(K) \) is topologically isomorphic to \( \tilde{\mathcal{C}}_\Delta(K) \) via the mapping

\[
S : \mathcal{A}(K) \to \tilde{\mathcal{C}}_\Delta(K),
\]

where \( S(g) \) is the harmonic function near \( K \) with Cauchy data (existing also by the Cauchy-Kovalevska theorem)

\[
S(g)(x, 0) = g(x), \quad \partial_t S(g)(x, 0) = 0.
\]

An explicit formula for \( S(g) \) is provided by

\[
S(g)(x, t) := \sum_{k=0}^{\infty} (-\Delta_x)^k g(x) \frac{t^{2k}}{(2k)!}.
\]

It is easily seen that

\[
S : H(C^d) \to \tilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}) \text{ is continuous}.
\]

We also need the following Cauchy type estimate: there is \( C > 0 \) such that for any \( \delta > 0 \) and any \( \beta \in \mathbb{N}^{d+1} \) the following holds if \( f \) is harmonic near \( B_\delta(0) \subset \mathbb{R}^{d+1} \)

\[
|\partial^\beta f(0)| \leq \beta!(C/\delta)^{|\beta|} \sup_{\xi \in B_\delta(0)} |f(\xi)|
\]

(see [14, Theorem 2.2.7]). From (8) we obtain the following precise estimate for the derivatives of \( G \): there is \( C > 0 \) such that

\[
\sup_{(x, t) \neq 0, \alpha \in \mathbb{N}^{d+1}} \frac{|\partial^\alpha G(x, t)|||_{(x, t)}|}{C^{||\alpha||+d-1}} < \infty.
\]

Indeed, this estimate follows for \(|(x, t)| = 1\) from (8) (with \( \delta := 1/2 \)), and for general \((x, t) \neq 0\) by the homogeneity of \( \Delta \) (consider \( G_\tau(\xi) := G(\tau \xi) \) for \( \tau > 0 \)).

For \( K \subset \mathbb{R}^d \) let \( \tilde{\mathcal{C}}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K) \) denote the class of harmonic functions on \( \mathbb{R}^{d+1} \setminus K \) which are even with respect to the variable \( t \) and vanish at \( \infty \). The TG duality (4) then shows that \( \mathcal{A}(K)' \) is topologically isomorphic to \( \tilde{\mathcal{C}}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K) \) via

\[
T(g) = (f_T, S(g)) = \langle \Delta(\varphi S(g)), f_T \rangle, \quad T \in \mathcal{A}(K)', g \in \mathcal{A}(K),
\]

where

\[
f_T(x, t) = \langle G(x - \cdot), T \rangle, \quad (x, t) \in \mathbb{R}^{d+1} \setminus K,
\]

by (3). Notice that \( f_T \in \tilde{\mathcal{C}}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K) \).
3 The Representation Theorem

First, we introduce the so-called dilation sets (see [7, Sec. 2 and 3]). Let $\Omega \subset \mathbb{R}^d$ be an open set. Then we define the dilation set as follows:

$$V(\Omega) := \{x : x\Omega \subset \Omega\} = \bigcap_{y \in \Omega} \{x : xy \in \Omega\}.$$  

It is very useful to have the following notation

$$S_\eta = \{x \mid x\eta \in S\}$$

hence $V(\Omega) = \bigcap_{\eta \in \Omega} \Omega_\eta$. Clearly, for non-empty $\Omega$ holds $1 \in V(\Omega)$ so $V(\Omega)$ is non-empty.

Let us formulate the following observation.

**Proposition 3.1** For $\eta \in \mathbb{R}^d$ we set $I_\eta := \{j \leq d \mid \eta_j = 0\}$. Then

$$S_\eta = \left\{y = (y_1, \ldots, y_d) \mid \forall j \not\in I_\eta : y_j = \frac{x_j}{\eta_j} \text{ where } x = (x_1, \ldots, x_d) \in S \text{ and } x_j = 0 \ \forall \ j \in I_\eta \right\}.$$  

Hence $S_\eta$ is a product of a subset $A \subset \mathbb{R}^{I_\eta'}$, $I_\eta' = \{1, \ldots, d\} \setminus I_\eta$, and of the space $\mathbb{R}^{I_\eta}$ such that $A$ is open (closed, compact, resp.) whenever $S$ is open (closed, compact, resp.). In particular, for open (closed) $S$ also $S_\eta$ is open (closed).

To give some feeling let us collect several easy facts for dilation sets.

**Proposition 3.2** If $\Omega \subset \mathbb{R}^d$ is open bounded then $V(\Omega)$ is bounded. If $V(\Omega)$ is bounded then $V(\Omega) \subset [-1,1]^d$. If $\Omega$ is convex then $V(\Omega)$ is convex as well. If $\Omega$ is convex bounded and symmetric with respect to all hyperplanes of the coordinate system then $V(\Omega) = [-1,1]^d$.

**Proof:** The first statement is obvious. The second follows from the obvious fact that $V(\Omega) \cdot V(\Omega) \subset V(\Omega)$. To prove the convexity statement we define the linear map $h_y : \mathbb{R}^d \to \mathbb{R}^d$ by $h_y(x) = yx$. The set $h_y^{-1}(\Omega)$ is convex. Hence $V(\Omega) = \bigcap_{y \in \Omega} h_y^{-1}(\Omega)$ is also convex.

Symmetry with respect to all hyperplanes means that every vector of the form

$$x = (\pm 1, \pm 1, \ldots, \pm 1)$$

belongs to $V(\Omega)$. Hence

$$[-1,1]^d = \text{conv}\{(\pm 1, \pm 1, \ldots, \pm 1)\} \subset \text{conv}(V(\Omega)) = V(\Omega) \subset [-1,1]^d.$$  

This proves the final claim. \qed

The above proposition implies immediately that if $\Omega$ is an open ball of a finite dimensional space $\ell_p$ for $1 \leq p \leq \infty$ then $V(\Omega)$ is the closed unit ball of the finite dimensional space $\ell_\infty$.

The following instructive examples are verified by direct calculation.

**Example 3.3** (Menagerie of dilation sets).

1. If $\Omega = \{x \in \mathbb{R}^2 : d(x, (1,10)) < 2\}$ then $V(\Omega) = \{1\}$.
2. If $\Omega = \{x \in \mathbb{R}^2 : 1 < x_1 < 2\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : x_1 = 1\}$.
3. If $\Omega = \{x \in \mathbb{R}^2 : (1/2)x_1 < x_2 < 3x_1, 0 < x_1\}$ then $V(\Omega) = \{(t,t) \in \mathbb{R}^2 : 0 < t < \infty\}$. 


4. If $\Omega = \{x \in \mathbb{R}^2 : d((x, 0, 3)) < 2\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, x_2 = 1\}$.

5. If $\Omega = \{x \in \mathbb{R}^2 : -x_1 < x_2 < -x_1 + (1/2)\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : 0 < x_1 = x_2 \leq 1\}$.

6. If $\Omega = \{x \in \mathbb{R}^2 : 0 < x_2 < 2x_1\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : 0 < x_2 \leq x_1\}$.

The following is the fundamental representation theorem for multipliers via analytic functionals. Notice that multipliers are a kind of convolution over $\mathbb{R}^d$ with coordinatewise multiplication.

**The Representation Theorem 3.4** Let $\Omega \subset \mathbb{R}^d$ be an open set. The map

$$\mathcal{B} : \mathcal{A}(V(\Omega))'_b \to M(\Omega) \subset L_b(\mathcal{A}(\Omega)), \quad \mathcal{B}(T)(g)(y) := \langle g(y \cdot), T \rangle, T \in \mathcal{A}(V(\Omega))', g \in \mathcal{A}(\Omega),$$

is a bijective linear map and the multiplier sequence of $\mathcal{B}(T)$ is equal to the sequence of moments of the analytic functional $T$, i.e. to $(\langle x^\alpha, T \rangle)_{\alpha \in \mathbb{N}^d}$.

Moreover, if $M$ is a multiplier on $\mathcal{A}(\Omega)$ then for any $y \in \Omega \cap N\mathbb{Z}$, the analytic functional $T \in \mathcal{A}(\mathbb{R}^d)'$ defined as

$$T = \delta_y \circ M \circ M_{1/y} : \mathcal{A}((1/y)\Omega) \to \mathbb{C},$$

(where $M_y(g)(\xi) := g(y \xi)$ and $\delta_y$ denotes the point evaluation at $y$) does not depend on $y$, its support is contained in $V(\Omega)$ and $M = \mathcal{B}(T)$.

In Section 5 we will show that $\mathcal{B} : \mathcal{A}(V(\Omega))'_b \to M(\Omega) \subset L_b(\mathcal{A}(\Omega))$ is always continuous and in Section 6 we will discuss the problem when this map is a topological isomorphism.

The proof of the representation theorem will be contained in the following two lemmas.

**Lemma 3.5** Let $\Omega \subset \mathbb{R}^d$ be an open set. For any $T \in \mathcal{A}(V(\Omega))'$ and $f \in \mathcal{A}(\Omega)$ we set:

$$M_T f(x) := \langle f(x \cdot), T \rangle.$$ 

Then $M_T \in M(\Omega)$ and the multiplier sequence $(m_\alpha)_{\alpha \in \mathbb{N}^d}$ is equal to the sequence of moments of $T$, i.e., $m_\alpha = \langle y^\alpha, T_y \rangle$.

**Proof:** Fix an open subset $\Omega' \subset \Omega$ which is relatively compact in $\Omega$ and choose the open set

$$U := \{x \mid x\overline{\Omega} \subset \Omega\}.$$

For any $f \in \mathcal{A}(\Omega)$ the function $(x, y) \mapsto f(xy)$ is real analytic on $\Omega' \times U$. It is well-known that then the map $x \mapsto f(x \cdot)$ is real analytic from $\Omega'$ to $\mathcal{A}(U)$. Therefore for any $T \in \mathcal{A}(V(\Omega))' \subset \mathcal{A}(U)'$ the map

$$x \mapsto \langle f(x \cdot), T \rangle$$

is real analytic on $\Omega'$. Since $\Omega' \subset \Omega$ was chosen arbitrarily we have proved that $M_T f \in \mathcal{A}(\Omega)$.

It is easy to observe that

$$M_T x^\alpha = \langle y^\alpha, T_y \rangle x^\alpha.$$

Now, it remains to show that $M_T : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is continuous. We apply de Wilde's closed graph theorem which holds for $\mathcal{A}(\Omega)$ ([30, 24.31], comp. [6, Cor. 1.28]). Let us fix $x \in \Omega$ then we get that

$$f \mapsto f(x \cdot)$$
is a continuous linear map \( \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega_x) \), where \( \Omega_x = \{ y \mid xy \in \Omega \} \subset \mathbb{R}^d \). Clearly, \( \Omega_x \) is an open neighborhood of \( V(\Omega) \), thus \( T \in \mathcal{A}(\Omega_x)' \) and

\[
 f \mapsto \langle f(x \cdot), T \rangle \in \mathcal{A}(\Omega)'.
\]

This shows that the graph of \( M_T : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega) \) is closed. \( \square \)

We denote by \( C(\Omega) \) the space of continuous functions on \( \Omega \) with the compact open topology.

Let us recall \( \text{NZ} := \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_j \neq 0, \text{ for every } j = 1, \ldots, d \} \).

and

\[
 \frac{1}{y} \Omega = \left\{ \left( \frac{x_1}{y_1}, \ldots, \frac{x_d}{y_d} \right) \mid (x_1, \ldots, x_d) \in \Omega \right\} \text{ if } y \in \text{NZ}.
\]

Let \( I \subset \{1, \ldots, d\} \) we set

\[
 H_I := \{ x = (x_1, \ldots, x_d) \mid x_j = 0 \text{ for every } j \in I \},
\]

and

\[
 \text{NZ}_I := \{ x = (x_1, \ldots, x_d) : x_j = 0 \text{ for every } j \in I \text{ and } x_j \neq 0 \text{ for every } j \notin I \}.
\]

**Lemma 3.6** Let \( \Omega \subset \mathbb{R}^d \) be an open set. For every continuous linear map \( M : \mathcal{A}(\Omega) \rightarrow C(\Omega) \) with all monomials as eigenvectors there is \( T \in \mathcal{A}(V(\Omega))' \) such that \( M = M_T \) as defined in Lemma 3.5. In particular, \( M \in M(\Omega) \).

**Proof:** For any \( \eta \in \Omega \cap \text{NZ} \) we define

\[
 T_\eta \in \mathcal{A}(\mathbb{R}^d)', \quad T_\eta f := (Mf_\eta)(\eta),
\]

where \( f_\eta(x) := f \left( \frac{x}{\eta} \right) \).

Let us note that

\[
 T_\eta x^\alpha = m_\alpha \quad \text{ for every } \alpha \in \mathbb{N}^d \text{ and } \eta \in \Omega \cap \text{NZ}.
\]

Since polynomials are dense in \( \mathcal{A}(\Omega) \) (see \([6, \text{Th. 1.16}]\)) the functional \( T_\eta \) does not depend on \( \eta \) and \( T_\eta \) \( T \in \mathcal{A}(\mathbb{R}^d)' \) for every \( \eta \in \Omega \cap \text{NZ} \). Moreover, if we show that \( \text{supp } T \subset V(\Omega) \) then

\[
 Mx^\alpha = M_T x^\alpha \quad \text{ for any } \alpha \in \mathbb{N}^d
\]

and so \( M = M_T : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega) \) continuously by Lemma 3.5.

Now, we start to prove \( \text{supp } T \subset V(\Omega) \). Since \( \mathcal{A}(\Omega) = \overline{\text{proj}}_{K \Subset \Omega} H(K) \) for every compact set \( L \Subset \Omega \) there is a compact set \( K \Subset \Omega \) such that

\[
 M : H(K) \rightarrow C(L)
\]

is continuous. Since

\[
 M_{1/\eta} : H(K_\eta) \rightarrow H(K), \quad f \mapsto f_\eta,
\]

is a linear continuous map thus for \( \eta \in L \cap \text{NZ} \) we have

\[
 (13) \quad T_\eta \in \mathcal{A}(K_\eta)' \quad \text{i.e., supp } T_\eta \subset K_\eta.
\]
Take any $\eta \in \Omega$, $\eta \neq 0$. Clearly, for some $I \subset \{1, \ldots, d\}$ we have $\eta \in \Omega \cap NZ_I$. Without loss of generality we assume that $I = \{j + 1, \ldots, d\}$. Thus

$$\eta = (\eta_1, \ldots, \eta_j, 0, \ldots, 0), \quad \text{where } \eta_1, \ldots, \eta_j \neq 0$$

and for some $\varepsilon > 0$ and every $0 < \varepsilon < \varepsilon_0$ we have

$$\eta_{\varepsilon} := (\eta_1, \ldots, \eta_j, \varepsilon, \ldots, \varepsilon) \in L \cap NZ_I$$

for a fixed compact set $L \in \Omega$. We take $K \in \Omega$ such that $M : H(K) \to C(L)$ is continuous.

If $x \notin K_\eta$ there is a compact ball $B \subset \mathbb{R}^d \setminus K_\eta$ with the center $x$ since $K_\eta$ is closed (see Proposition 3.1). Then $\eta B \cap K = \emptyset$ and for $\varepsilon > 0$ small enough also $\eta_\varepsilon B \cap K = \emptyset$, i.e., $B \cap K_\eta = \emptyset$. By (13)

$$\operatorname{supp}T_{\eta_\varepsilon} \cap B \subset K_\eta \cap B = \emptyset.$$ 

Since $x \notin K_\eta$ was chosen arbitrarily we have proved

$$\operatorname{supp}T \subset K_\eta \subset \Omega_\eta.$$ 

If $0 \in \Omega$ this holds also for $\eta = 0$ since then $\Omega_\eta = \mathbb{R}^d$, and so $\operatorname{supp}T \subset V(\Omega)$. \qed

**Remark 3.7** (a) In fact, we have proved that in the definition of the multiplier we can relax some conditions. In fact every linear continuous map $M : \mathcal{A}(\Omega) \to C(\Omega)$ with monomials as eigenvectors is automatically a multiplier since it maps $\mathcal{A}(\Omega)$ continuously into $\mathcal{A}(\Omega)$.

(b) One of the difficulties in the proof of Theorem 3.4 is to show that the analytic functional $T$ has a support contained in $V(\Omega)$. It is relatively easy to show that $\operatorname{supp}T \subset \bigcap_{y \in \Omega \cap NZ} \{y \Omega =: \tilde{V}(\Omega)\}$. Now, in the one dimensional case it holds always that $V(\Omega) = \tilde{V}(\Omega)$. In the many variable case this is true for convex sets (see Proposition 3.9 below) but in general it is true neither for open $V(\Omega)$ (take $\Omega = \mathbb{R}^d \setminus \{0\}$, then $\tilde{V}(\Omega) = \mathbb{R}^d \setminus \{0\} \neq NZ = V(\Omega)$) nor for compact $V(\Omega)$ (see Example 6.16).

Let us call sets $\Omega$ *acceptable* if $V(\Omega) = \tilde{V}(\Omega)$ . Notice that the sets $V(\Omega)$ and $\tilde{V}(\Omega)$ in general may differ drastically (see Example 6.16).

To understand the situation, first some general observations: Let $x \in \tilde{V}(\Omega)$. We set $L_x := \{y : y_j = 0 \text{ whenever } x_j = 0\}$. Since $x(\Omega \cap NZ) \subset \Omega \cap L_x$ we have $x\Omega \subset \Omega \cap L_x$. Therefore $x\Omega$ is open in $L_x$. Therefore $x\Omega \subset \text{interior}_{L_x}(\Omega \cap L_x)$.

An open subset $U$ of a topological space is called *regular open* if $\text{interior} U = U$. Examples are convex open subsets of a locally convex space.

We have shown:

**Lemma 3.8** If $\Omega \cap L_x$ is regular open for every $x \in \tilde{V}(\Omega)$ then $\Omega$ is acceptable.

**Proposition 3.9** If $\Omega$ is a non-empty open convex set then it is acceptable.

**Proof:** For any linear subspace $L$ of $\mathbb{R}^d$ the set $\Omega \cap L$ is convex and open in $L$, hence regular open in $L$. \qed

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In Theorem 3.4 we have established a linear isomorphism \( \mathcal{B} : \mathcal{A}(V(\Omega))' \to M(\Omega) \). By this isomorphism the algebra \( M(\Omega) \) induces the following multiplication on \( \mathcal{A}(V(\Omega))' \)

\[(T * S)f = T_s(S_y f(xy)).\]

This is true, because by definition \((T*S)f = (MT\circ MS)f\). Therefore \( \mathcal{B} : (\mathcal{A}(V(\Omega))', *) \to M(\Omega) \) is an algebra isomorphism.

So the situation is the following: \( \mathcal{A}(\mathbb{R}^d)' \) is a commutative algebra and \( \mathcal{A}(V(\Omega)) \) is a subalgebra for any open \( \Omega \subset \mathbb{R}^d \). If \( m_\alpha(T) \) and \( m_\alpha(S) \) are the moment sequences of \( T \) and \( S \), then \( m_\alpha(T*S) = m_\alpha(T)m_\alpha(S) \).

This observation will help us in the following section, where we represent the algebras \( M(\Omega) \cong A(V(\Omega)) \) by algebras of holomorphic functions equipped with Hadamard multiplication of Laurent, resp. Taylor coefficients.

## 4 Representation via Hadamard Multiplication of Holomorphic Functions

In this section we will first represent \( \mathcal{A}(V(\Omega))' \cong M(\Omega) \) by an algebra of holomorphic functions with Hadamard multiplication of Laurent coefficients. This will be done by the Cauchy transform, so the essential content of the following will be an exact description of the Cauchy transforms of elements in \( \mathcal{A}(V(\Omega))' \). Most of it is well known (see e.g. [37]) but we will make it as self-contained as possible for the convenience of the reader.

For \( T \in \mathcal{A}'(\mathbb{R}^d) \) and \( z \in \mathbb{C}^d, z_j \neq 0 \) for all \( j \) we set \( \mathcal{C}(z) = \prod_{j=1}^d \frac{1}{z_j} \). For any subset \( B \subset \mathbb{R}^d \) we define

\[\mathcal{W}(B) = \{ z \in \mathbb{C}^d : \xi_j \neq z_j \text{ for all } \xi \in B \text{ and } j = 1, \ldots, d \}.\]

For \( z \in \mathcal{W}(\text{supp } T) \) we define \( \mathcal{C}_T(z) = T_\xi(\mathcal{C}(\xi - z)) = (T * \mathcal{C})(z) \).

If \( \text{supp } T \subset \{ \xi \in \mathbb{R}^d : |\xi|_\infty \leq R \} \) then \( (\mathbb{C} \setminus [-R, +R])^d \subset \mathcal{W}(\text{supp } T) \) and \( \mathcal{C}_T \) extends to a holomorphic function on \( (\mathbb{C} \setminus [-R, +R])^d, \mathbb{C} \) denoting the Riemann sphere.

For \( \min_j |z_j| > R \) the function \( \mathcal{C}_T(z) \) is defined and holomorphic and it has the expansion

\[\mathcal{C}_T(z) = \frac{1}{z_1 \cdots z_d} \sum_{\alpha \in \mathbb{N}_0^d} T_\xi(\xi^\alpha) \frac{1}{z^\alpha} = \sum_{\alpha \in \mathbb{N}_0^d} m_\alpha \frac{1}{\xi^\alpha + 1}.\]

We have proved the following

**Proposition 4.1** \( T \mapsto \mathcal{C}_T \) is an algebra isomorphism from the algebra \( (\mathcal{A}'(\mathbb{R}^d), *) \) to the algebra of all holomorphic functions on \( (\mathbb{C} \setminus [-R, +R])^d \) for some \( R > 0 \), regular with value 0 in all infinite points of \( \mathbb{C}^d \), equipped with Hadamard multiplication of the coefficients of the Laurent expansion around \( (\infty, \ldots, \infty) \).

**Proof:** Only surjectivity has to be shown. Let a function \( g \) of this type be given. In each variable \( z_j \) we use the path \( \gamma_j \) defined by \( r + i\varepsilon \to -r + i\varepsilon \to -r - i\varepsilon \to r - i\varepsilon \to r + i\varepsilon \) and set \( S_j \) to be the convex hull of \( \gamma_j \). Here \( r > R \), and \( \varepsilon > 0 \) is chosen so small that \( \prod_{j \leq d} S_j \) is contained in the domain of definition of a given function \( f \in \mathcal{A}'(\mathbb{R}^d) \). We define

\[(14)\quad T(f) := \left( \frac{1}{2\pi i} \right)^d \int_{\gamma_1} \cdots \int_{\gamma_d} f(\zeta_1, \ldots, \zeta_d) g(\zeta_1, \ldots, \zeta_d) \, d\zeta_1 \cdots d\zeta_d.\]
Since this can be done for every $\varepsilon > 0$ and is independent of $\varepsilon$ we have defined $T \in \mathcal{A}([−r, +r]^d)$. Clearly $\mathcal{C}_T = g$. □

To determine the Cauchy transforms of the subalgebras $\mathcal{A}(V(\Omega))$ of $\mathcal{A}(\mathbb{R}^d)$ we need the following definition:

**Definition 4.2** For any holomorphic function $f$ on $(\mathbb{C} \setminus \mathbb{R})^d$ we define the closed set $\sigma(f) \subseteq \mathbb{R}^d$ in the following way: $x \in \mathbb{R}^d$ is not in $\sigma(f)$ if there exist neighborhoods $U_j \subseteq \mathbb{R}$ of $x_j$ and holomorphic functions $f_j$ on $(\mathbb{C} \setminus \mathbb{R})^d \cup ((\mathbb{C} \setminus \mathbb{R})^{d−1} \times U_j \times (\mathbb{C} \setminus \mathbb{R})^{d−j})$ such that $f = f_1 + \cdots + f_d$ on $(\mathbb{C} \setminus \mathbb{R})^d$.

Let us remark that $\sigma(T)$ is the support of the hyperfunction determined by $f$ (see [37]).

**Proposition 4.3** $\text{supp } T = \sigma(\mathcal{C}_T)$.

**Proof**: Let $Q := \prod_{j=1}^{d}[a_j, b_j]$ and $Q \cap \text{supp } T = \emptyset$. By [37, Théorème 121,(4)], there are $T_j \in \mathcal{A}((x \in \mathbb{R}^d : x_j \not\in [a_j, b_j])$ such that $T = T_1 + \cdots + T_d$. So we have $\{x : x_j \in [a_j, b_j]\} \subseteq \mathcal{H}(\text{supp } T_j)$ and $\mathcal{C}_T = \mathcal{C}_{T_1} + \cdots + \mathcal{C}_{T_d}$ where $\mathcal{C}_{T_j}$ is holomorphic on $(\mathbb{C} \setminus \mathbb{R})^d \cup ((\mathbb{C} \setminus \mathbb{R})^{d−1} \times [a_j, b_j] \times (\mathbb{C} \setminus \mathbb{R})^{d−j})$ which shows that $Q \cap \sigma(\mathcal{C}_T) = \emptyset$. Therefore $\sigma(\mathcal{C}_T) \supseteq \text{supp } T$.

It remains to show that $\text{supp } T \subseteq \sigma(\mathcal{C}_T)$. Assume $x \not\in \sigma(\mathcal{C}_T)$. Then there is a neighborhood $Q$ of $x$ such that $Q \cap \sigma(\mathcal{C}_T) = \emptyset$ and we may assume that $T = T_1 + \cdots + T_d$ where $T_j$ is as above. We choose $r$ large enough and replace $\gamma_j$ in (14) with $r + i\varepsilon \to b_j + i\varepsilon \to b_j - i\varepsilon \to r - i\varepsilon \to r + i\varepsilon$ and $a_j + i\varepsilon \to -r + i\varepsilon \to -r - i\varepsilon \to a_j - i\varepsilon \to a_j + i\varepsilon$ which shows that $\{x : x_j \in [a_j, b_j]\} \cap \text{supp } T_j = \emptyset$ for all $j$. Hence $Q \cap \text{supp } T = \emptyset$. □

**Definition 4.4** Let $X \subseteq \mathbb{R}^d$ be closed under multiplication. Then let $\mathcal{H}_C(X)$ denote the algebra of all holomorphic functions $f$ on $(\hat{\mathbb{C}} \setminus [−R, +R])^d$ for some $R > 0$, regular with value 0 in all infinite points of $\hat{\mathbb{C}}^d$, such that $\sigma(f) \subseteq X$, equipped with Hadamard multiplication of the coefficients of the Laurent expansion around $(\infty, \ldots, \infty)$.

We have shown:

**Theorem 4.5** $M_T \mapsto \mathcal{C}_T$ defines an algebra isomorphism $M(\Omega) \to \mathcal{H}_C(V(\Omega))$.

In a next step we want to change the equivalence into one with Hadamard multiplication of power series. We use the automorphism $\tau : (z_1, \ldots, z_d) \to (1/z_1, \ldots, 1/z_d)$ of $\hat{\mathbb{C}}$. We set

$$ \mathcal{R}f(z) := \frac{1}{z_1 \cdots z_d} f\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) $$

and

$$ \mathcal{H} := \text{ind}_U \mathcal{H}((\mathbb{C} \setminus \mathbb{R}) \cup U)^d) $$

where $U$ runs through all open neighborhoods of zero in $\mathbb{C}$. Then $\mathcal{R}$ is a linear isomorphism from $\mathcal{H}_C$ onto $\mathcal{H}$. If $f \in \mathcal{H}_C$ and $f(z) = \sum_{a \in \mathbb{N}_0} m_a z^{-a}$ its Laurent expansion around $(\infty, \ldots, \infty)$ then $\mathcal{R}f(z) = \sum_{a \in \mathbb{N}_0} m_a z^{-a}$ is the Taylor expansion of $\mathcal{R}f$ around $(0, \ldots, 0)$.

In particular $\mathcal{H}$ is an algebra with respect to Hadamard multiplications, which means the following: if $f, g \in \mathcal{H}$ and $f(z) = \sum_a b_a z^a$, $g(z) = \sum_a c_a z^a$ are their Taylor expansions, then there is a unique $f * g \in \mathcal{H}$, such that $(f * g)(z) = \sum_a b_a c_a z^a$ is its Taylor expansion. For $T \in \mathcal{A}(\mathbb{R}^d)^d$ we set $C_T = \mathcal{R}(\mathcal{C}_T)$ then we obtain by obvious calculations

$$ C_T(z) = T_{\xi}(\prod_j \frac{1}{1 - \xi_j z_j}). $$

We have shown:
Proposition 4.6 $T \mapsto C_T$ defines an algebra isomorphism from $(\mathcal{A}(\mathbb{R}^d)', \star)$ onto $(\mathcal{H}, \star)$.

Since $r$ is an automorphism of $\hat{\mathbb{C}}^d$ which maps $\mathbb{R}^d$ to $\mathbb{R}^d$ we have $\sigma(\mathcal{B}(f)) = r(\sigma(f))$ and therefore

$$\supp T = \sigma(\mathcal{E}_T) = r(\sigma(C_T)).$$

This leads to the definition:

Definition 4.7 Let $X \subset \mathbb{R}^d$ be closed under multiplication. Then we define

$$\mathcal{H}(X) = \{ f \in \mathcal{H} : r(\sigma(f)) \subset X \}.$$ 

Finally obtain:

Theorem 4.8 The map $M_T \mapsto C_T$ is an algebra isomorphism from $M(\Omega)$ to $\mathcal{H}(V(\Omega))$ equipped with Hadamard multiplication. If $0 \in \Omega$ and $\Omega$ is connected, then $M_T \in M(\Omega)$ acts on $\mathcal{A}(\Omega)$ by Hadamard multiplication with $C_T$.

Proof: Only the last part has to be shown. It is enough to show it on monomials. But there it is obvious by definition. $\square$

5 Topological Representation

We will study the topological aspects of the Representation Theorem 3.4, in particular, for which topologies on $\mathcal{A}(V(\Omega))'$ the map $\mathcal{B} : \mathcal{A}(V(\Omega))' \to M(\Omega) \subset L_b(\mathcal{A}(\Omega))$ is continuous and, more sophisticated, which topology is induced on $\mathcal{A}(V(\Omega))'$ via $\mathcal{B}$ from $L_b(\mathcal{A}(\Omega))$. Before we start our investigation we need some information about the topologies on $\mathcal{A}(V(\Omega))$ and $\mathcal{A}(V(\Omega))'$.

For that let $S$ be a an arbitrary subset of $\mathbb{R}^d$. Then there are two natural ways to define topologies on $\mathcal{A}(S)$:

$$\mathcal{A}_I(S) = \text{ind}_U H(U) \text{ or } \mathcal{A}_P(S) = \text{proj}_K A(K),$$

where $U$ runs through the complex neighborhoods of $S$ and $H(U)$ is the space of holomorphic functions on $U$, and $K$ runs through the compact subsets of $S$. These topologies coincide (see [29, Théorème 1.2]) and define, what we consider to be the natural topology on $\mathcal{A}(S)$.

$\mathcal{A}(S)$ is nuclear and, by the first version, ultrabornological, by the second version, complete. $B \subset \mathcal{A}(S)$ is bounded if there is an open neighborhood $U$ of $S$ such that $B$ is bounded in $H(U)$, see [29, Th. 1.2, Proposition 1.2], comp. [6, Fact 1.21, Th. 1.27]. Therefore topologically

$$\mathcal{A}(S)'_b = \text{proj}_U H(U)'_b.$$ 

This implies immediately that topologically

$$\mathcal{A}(S)'_b = \text{proj}_{W \supset S} \mathcal{A}(W)'_b$$

where $W$ runs through all real open neighbourhoods of $S$. Since $\mathcal{A}(S)$ is ultrabornological $\mathcal{A}(S)'_b$ is complete [30, 24.11]. Since $\mathcal{A}(S)$ is a complete Schwartz space its dual is ultrabornological by [30, 24.23].

As linear space we have $\mathcal{A}(S)'_b = \text{ind}_{K \subset S} \mathcal{A}(K)'_b$ and the topology of this inductive limit is finer than $\mathcal{A}(S)'_b$. If $S$ is hemicompact, in particular if $S$ is locally closed then $\text{ind}_{K \subset S} \mathcal{A}(K)'_b$ is an (LF)-space, hence webbed, and, by de Wilde’s Theorem (see [30, 24.30]), we have $\mathcal{A}(S)'_b = \text{ind}_{K \subset S} \mathcal{A}(K)'_b$ and $\mathcal{A}(S)'_b$ is an (LF)-space.

In particular, we have shown:
Proposition 5.1 For every nonempty open set $\Omega \subset \mathbb{R}^d$ the space $\mathcal{A}(V(\Omega))'_b$ is an ultrabornological complete space with $\mathcal{A}(V(\Omega))'_b = \text{proj}_{W \supset V(\Omega)} \mathcal{A}(W)'_b$, where $W \subset \mathbb{R}^d$ runs through all open neighbourhoods of $V(\Omega)$.

We will need some more definitions. Let $\Omega \subset \mathbb{R}^d$ be a non void open set. For a compact set $K \subset \Omega$ let us define

$$V_K(\Omega) := \{ \xi \mid \xi K \subset \Omega \}.$$ 

Notice that $V_K(\Omega)$ is open since for $\xi \in V_K(\Omega)$

$$(\xi + B_\gamma(0))K \subset \xi K + B_\gamma(0)K \subset \xi K + B_\varepsilon(0) \subset \Omega$$

if $\varepsilon > 0$ and then $\gamma > 0$ are chosen suitably. Clearly, $V_K(\Omega)$ contains $V(\Omega)$ and

$$\bigcap_{K \in \Omega} V_K(\Omega) = \{ \xi \mid \xi \Omega \subset \Omega \} = V(\Omega).$$

The second “natural” topology on $\mathcal{A}(V(\Omega))'$ we call $k$-topology and it is by definition:

$$\mathcal{A}(V(\Omega))'_k := \text{proj}_{K \in \Omega} \mathcal{A}(V_K(\Omega))'_b,$$

where $K$ runs through all compact subsets of $\Omega$. This topology is analogous to the $t$-topology introduced in [40].

Proposition 5.2 For every open nonempty set $\Omega \subset \mathbb{R}^d$ the space $\mathcal{A}(V(\Omega))'_k$ is a complete countable projective limit of LFN-spaces (i.e., countable locally convex inductive limits of nuclear Fréchet spaces). In particular, $\mathcal{A}(V(\Omega))'_k$ is webbed.

Clearly, the $k$-topology is not stronger than the $b$-topology. We will show that

$$\mathcal{B} : \mathcal{A}(V(\Omega))'_k \to L_b(\mathcal{A}(\Omega))$$

is continuous and there are reasons to conjecture that $L_b(\mathcal{A}(\Omega))$ induces via $\mathcal{B}$ the $k$-topology on $\mathcal{A}(V(\Omega))'$ (see Theorem 5.20, Example 5.13 and cf. [40]).

We will need the following simple remark. For $K \in \Omega$ we denote by $V_K^0(\Omega)$ the union of all connected components of $V_K(\Omega)$ which have a nonempty intersection with $V(\Omega)$, and we have

$$\mathcal{A}(V(\Omega))'_k := \text{proj}_{K \in \Omega} \mathcal{A}(V_K^0(\Omega))'_b.$$ 

We can now compare the $b$- and the $k$-topologies. We recall the fact, which is due to the Cartan-Grauert Theorem in the version of [13, Lemma 1.1.(b)], that for any open $U \subset \mathbb{R}^d$ there is a real analytic function on $U$ which cannot be extended beyond $U$.

Proposition 5.3 Let $\Omega \subset \mathbb{R}^d$ be on open nonempty set. The $k$-topology and the $b$-topology on $\mathcal{A}(V(\Omega))$ are equal if and only if the sets $V_K^0(\Omega)$ form a basis of neighborhoods for $V(\Omega)$.

**Proof:** Since one implication is trivial, it remains to show that from equality of the topologies follows: if $U$ is an open neighborhood of $V(\Omega)$ then there is $K \subset \Omega$ such that $V_K^0 \subset U$.

We choose a function $f \in \mathcal{A}(U)$ which cannot be extended beyond $U$. $y_f : T \mapsto T(f)$ is a linear form on $\mathcal{A}(V(\Omega))'_b = \mathcal{A}(V(\Omega))'_k$. Therefore there exists $K \in \Omega$ such that $y_f \in (\mathcal{A}(V_K^0(\Omega))'_b)'$ which means that there is $g \in \mathcal{A}(V_K^0(\Omega))$ such that $y_f(T) = T(g)$ for all $T \in \mathcal{A}(V_K^0(\Omega))'$. Applying this to $T = \delta_x^{(n)}$ for all $x \in V(\Omega)$ and $n \in \mathbb{N}_0^d$, we obtain that $f = g$ in a neighborhood of $V(\Omega)$. If $V_K^0(\Omega) \not\subset U$ there must be $x \in \partial U \cap V_K(U)$, that means $g$ extends $f$ into a neighborhood of $x$, which contradicts the choice of $f$. $\square$

Notice that the sets $V_K^0(\Omega)$ can be very different from the sets $V_K(\Omega)$ and, with the latter, Proposition 5.3 would be false, as the following example shows.
Example 5.4 Let $\Omega := \mathbb{R}_+^2 \setminus \{1\}$. Then $V(\Omega) = \{1\}$ and the b- and k-topologies coincide by Proposition 5.3 while all $V_K(\Omega)$ are unbounded.

Proof: Let $K_n := \{(x, y) : 1/n \leq x, y \leq n\} \setminus \{(x, y) : n/(n + 1) < x, y < (n + 1)/n\}$ then $V_{K_n}(\Omega) = \{(x, y) : n/(n + 1) < x, y < (n + 1)/n\} \cup (\mathbb{R}^2 \setminus \{(x, y) : 1/n \leq x, y \leq n\})$ hence $V_{K_n}^b(\Omega) = \{(x, y) : n/(n + 1) < x, y < (n + 1)/n\}$. □

Now, we analyze continuity of the map $\mathcal{B}$. We assume without restriction of generality that $1 \in \Omega$. We set for compact $K \subset \Omega$

\[M(\Omega, K) = \{M \in L_b(\mathcal{A}(\Omega), \mathcal{A}(K)) : M \text{ admits all monomials as 'eigenvectors'}\},\]

\[MC(\Omega, K) = \{M \in L_b(\mathcal{A}(\Omega), C(K)) : M \text{ admits all monomials as 'eigenvectors'}\}.

Here $C(K)$ carries the sup-norm topology. Obviously $M(\Omega, K) \subset MC(\Omega, K)$ with continuous embedding.

We assume from now on that $1 \in K$, then $V_K(\Omega) \subset \Omega$, and we assume that $K \cap NZ$ is dense in $K$.

Proposition 5.5 $M(\Omega, K) = MC(\Omega, K)$ as sets, their equicontinuous sets coincide. $T \mapsto M_T$ defines a continuous isomorphism from $\mathcal{A}(V_K(\Omega))^\prime_b$ onto $M(\Omega, K)$. Its inverse map is $M \mapsto T_M$ where $T_M f = (M f)(1)$. Both maps send equicontinuous sets to equicontinuous sets.

Proof: We fix a compact set $L \subset V_K(\Omega)$. By the result of the third named author [39], a standard seminorm on $\mathcal{A}(L)$ is given by

\[\|f\|_{L, \delta} := \sup_{\alpha \in \mathbb{N}_0^d, y \in L} \frac{|f^{(\alpha)}(y)|}{\delta^{[\alpha]}},\]

where $\delta_k$ is a strictly positive sequence tending to 0, which, without restriction of generality, may be assumed to be decreasing. We assume $|T f| \leq \|f\|_{L, \delta}$.

We fix $\delta$ and set $r = \sup\{|x| + |y| + \delta_0 : x \in K, y \in L\}$. We obtain for $M = M_T$

\[\|M f\|_{K, \delta} = \sup_{\alpha \in \mathbb{N}_0^d, x \in K} \left|T_y \left(\frac{y^\alpha f^{(\alpha)}(xy)}{\alpha!}\right)\delta^{[\alpha]}\right| \leq \sup_{\alpha \in \mathbb{N}_0^d, x \in K} \sup_{\beta \in \mathbb{N}_0^d, y \in L} \frac{|\beta^\alpha y^\beta f^{(\alpha)}(xy)|}{\alpha!} \delta^{[\alpha]} \frac{\gamma^{[\beta]}}{\delta^{[\beta]}},\]

We estimate the derivatives in the last term:

\[\left|\frac{1}{\alpha! \beta!} \partial^\beta_y \left(\frac{y^\alpha f^{(\alpha)}(xy)}{\alpha!}\right)\right| = \left|\frac{1}{\alpha! \beta!} \sum_{0 \leq \gamma \leq \min(\alpha, \beta)} \left(\begin{array}{c} \beta \\ \gamma \end{array}\right) \frac{\alpha!}{(\alpha - \gamma)!} y^{\alpha - \gamma} x^{\beta - \gamma} f^{(\alpha + \beta - \gamma)}(xy)\right| \leq 2^{[\beta]} \sup_{0 \leq \gamma \leq \min(\alpha, \beta)} y^{[\alpha + \beta - 2\gamma]} \left(\frac{\alpha + \beta - \gamma}{\beta}\right) \frac{|f^{(\alpha + \beta - \gamma)}(xy)|}{(\alpha + \beta - \gamma)!} \leq \sup_{0 \leq \gamma \leq \min(\alpha, \beta)} (4r)^{[\alpha + \beta - \gamma]} \frac{|f^{(\alpha + \beta - \gamma)}(xy)|}{(\alpha + \beta - \gamma)!} r^{-[\gamma]}.

To show continuity of $\mathcal{B}$ we may assume $\delta \geq \tilde{\delta}$ and $\delta$ decreasing. We put

\[\tilde{c}_M = \sup_{n+m=M} \frac{\delta^{n[M]} \delta^{m[M]}}{n+m}.\]

Then $\tilde{c}_M$ is a strictly positive null-sequence. We denote by $(c_M)_M$ its decreasing majorant.

Then we have for $\alpha, \beta \in \mathbb{N}_0^d$ and $0 \leq \gamma \leq \min(\alpha, \beta)$ the estimate $r^{-[\gamma]} \delta^{[\alpha]} \delta^{[\beta]} \leq c^{[\alpha + \beta - \gamma]}$. Since $KL \in \Omega$ we obtain:

\[\|M f\|_{K, \delta} \leq \|f\|_{KL, 4c}.\]
We have shown that \( T \mapsto M_T \) maps equicontinuous subsets of \( \mathcal{A}'(V_K(\Omega)) \) into equicontinuous, hence bounded, subsets of \( M(\Omega, K) \). Since \( \mathcal{A}'(V_K(\Omega)) \) is bornological the map \( T \mapsto M_T \) is continuous from \( \mathcal{A}'(V_K(\Omega)) \) to \( M(\Omega, K) \).

To show the reverse direction, we assume that \( M \subset MC(\Omega, K) \) is equicontinuous, that is, we find a compact set \( L \subset \Omega \) and a null-sequence \( \delta \) such that

\[
\sup_{\eta \in K} |(Mf)(\eta)| \leq \|f\|_{L, \delta} = \sup_{\alpha, x \in L} \frac{|f^{(\alpha)}(x)|}{\alpha!} \delta^{(\alpha)}
\]

for all \( M \in M \) and \( f \in \mathcal{A}(\Omega) \), in particular, for all \( f \in \mathcal{A}(\mathbb{R}^d) \).

Given \( f \in \mathcal{A}(\mathbb{R}^d) \), this applies to all \( f_\eta \), \( f_\eta(x) = f \left( \frac{x}{\eta} \right), \eta \in K \cap NZ \), and we obtain for all these \( \eta \)

\[
|Tf| = |(Mf)(1)| = |(Mf_\eta)(\eta)| \leq \|f_\eta\|_{L, \delta(\eta)} = \|f\|_{L, \delta(\eta)}
\]

where

\[
\delta(\eta)_m := \frac{\delta_m}{\min_{j=1, \ldots, d} |\eta(j)|}, \quad \eta = (\eta^{(1)}, \ldots, \eta^{(d)}).
\]

We conclude, using the assumption that \( K \cap NZ \) is dense in \( K \), that

\[
\text{supp } T \subset \bigcap_{\eta \in K \cap NZ} L_\eta = \{ y \in \mathbb{R}^d : yK \subset L \} \in V_K(\Omega)
\]

and we can find \( \eta_1, \ldots, \eta_m \in K \cap NZ \) such that

\[
\text{supp } T \subset \hat{L} := \bigcap_{j=1}^m L_{\eta_j} \in V_K(\Omega).
\]

This holds for all \( M \in M \). From (15), (16) and [37], p. 47 f. we conclude that there is a null-sequence \( \gamma \) such that

\[
|Tf| \leq \|f\|_{L, \gamma}
\]

for all \( M \in M \) and \( f \in \mathcal{A}(U) \).

Finally, we have continuous maps \( \mathcal{A}'(V_K(\Omega))'_b \to M(\Omega, K) \to MC(\Omega, K) \) and the composition is surjective. Therefore \( M(\Omega, K) = MC(\Omega, K) \) as sets. \( \Box \)

**Remark 5.6** (a) Let us recall that \( \mathcal{A}'(\Omega)'_b \) and \( \mathcal{A}'(L)'_b\), \( L \Subset \Omega \), are nuclear, thus (see [23, Ch. 21]) we have topological isomorphisms:

\[
L_b(\mathcal{A}(\Omega), C(K)) \cong \mathcal{A}(\Omega)'_b \hat{\otimes} C(K), \quad L_b(\mathcal{A}(L), C(K)) \cong \mathcal{A}(L)'_b \hat{\otimes} C(K).
\]

Algebraically

\[
L_b(\mathcal{A}(\Omega), C(K)) = \text{ind}_{\Omega \Subset \Omega} L_b(\mathcal{A}(L), C(K))
\]

and, by [17, I §1, no. 3, Cor. p. 47], the topologies coincide as well. Hence \( L_b(\mathcal{A}(\Omega), C(K)) \) is an LF-space.

(b) Also the spaces \( \mathcal{A}'(V_K(\Omega))'_b \) are (LF)-spaces and the step spaces are \( \mathcal{A}'(L)'_b \), \( L \) compact in \( V_K(\Omega) \). \( \mathcal{B} : \mathcal{A}'(V_K(\Omega))'_b \to L_b(\mathcal{A}(\Omega), C(K)) \) is a continuous, injective map, its range \( M(\Omega, K) \) is closed in \( L_b(\mathcal{A}(\Omega), C(K)) \). It is a correspondence between the bounded (= equicontinuous) sets in \( \mathcal{A}'(V_K(\Omega))'_b \) and \( M(\Omega, K) \). For every compact \( L \subset \Omega \) there is a compact \( \hat{L} \subset V_K(\Omega) \) such that \( \mathcal{B}^{-1}(L(\mathcal{A}(L), C(K))) \subset \mathcal{A}'(\hat{L}) \). To show the last assertion we use the proof of Proposition 5.5 with \( \delta \) and \( \gamma \) being constant or Grothendieck’s Factorization Theorem.
The question whether \( B : A(V_k(\Omega))_b^* \to M(\Omega, K) \) is a topological isomorphism is, by Remark 5.6, a classical problem of well-locatedness (see e.g., [15]), i.e., the question if closed subspace \( M(\Omega, K) \) in \( L_b(A(\Omega), C(K)) \) is a topological inductive limit of \( B(A(L))_b^* \), \( L \in V_k(\Omega) \) (or, equivalently, of \( M(\Omega, K) \cap L_b(A(L), C(K)) \), \( L \in \Omega \)). For a stronger assumption on \( K \), however, we can show it.

**Proposition 5.7** If \( K \subset \Omega \cap N \), then \( B : A(V_k(\Omega))_b^* \to MC(\Omega, K) \) is a linear topological isomorphism. In particular, \( M(\Omega, K) = MC(\Omega, K) \) as topological linear spaces.

The proof of this needs some preparation. It is based on the harmonic representation of analytic functionals combined with some ideas and results from [39] which are introduced now:

**Lemma 5.8** [39] Let \((X, \| \|)\) be a Banach space and let \( F := \text{ind}_{k \to \infty} F_k \) where
\[
F_k := \{(x_\alpha)_{\alpha \in \mathbb{N}_0^d} \in X_{\mathbb{N}_0^d}^N \mid \|x\|_k := \sup_{\alpha} \|x_\alpha\|k^{-|\alpha|} < \infty\}.
\]
Then a fundamental system of seminorms on \( F \) is given by
\[
|x|_\delta := \sup_{\alpha} \|x_\alpha\|\delta^{[\alpha]},
\]
where \( \delta = (\delta_j)_{j \in \mathbb{N}} \) is any strictly positive sequence tending to 0.

Notice that a fundamental system of bounded sets in \( F \) is given by the closed unit balls \( B_k \) in \( F_k \). Indeed, a fundamental system is provided by the closures \( B_k \) (in \( F \), see [30, 25.16]) which coincide with \( B_k \) since the identity mapping \( id : F \to X_{\mathbb{N}_0^d}^N \) is continuous.

Let us introduce the following space:
\[
\tilde{C}_{\Delta, c}(\tilde{V}) := \text{ind}_{K \in \tilde{V}} \tilde{C}_{\Delta, 0}(\mathbb{R}^{d+1} \setminus K).
\]
In [39] the Lemma 5.8 was used to determine a canonical fundamental system on the space \( A(K) \) for compact \( K \). A variant of this proof gives the following basic result:

**Theorem 5.9** Let \( \tilde{V} \subset \mathbb{R}^{d+1} \) be open such that \( t \to 0 \) if \( \tilde{V} \ni (x, t) \to \infty \). Then a fundamental system of seminorms on \( \tilde{C}_{\Delta, c}(\tilde{V}) \) is given by
\[
|f|_\delta := \sup_{\xi \in \partial \tilde{V}, \alpha \in \mathbb{N}_0^{d+1}} \frac{|f^{(\alpha)}(\xi)|}{\alpha!}\delta^{[\alpha]},
\]
where \( \delta = (\delta_j)_{j \in \mathbb{N}} \) is any positive sequence tending to zero.

**Proof:** We define \( F \) as in Lemma 5.8 using
\[
X := C_0(\partial \tilde{V}) := \{f \in C(\partial \tilde{V}) \mid \lim_{\tilde{V} \ni \xi \to \infty} f(\xi) = 0\}
\]
edowed with the sup-norm. Let \( A(f) := (\frac{1}{\alpha!} f^{(\alpha)}|_{\partial \tilde{V}})_{\alpha \in \mathbb{N}_0^{d+1}} \). Then \( A : \tilde{C}_{\Delta, c}(\tilde{V}) \to F \) is defined and continuous by Lemma 5.8. We will prove that \( A \) is an injective topological homomorphism using Baernstein’s Lemma [30, 26.26]. By Lemma 5.8 this will show the theorem.

Notice that \( \tilde{C}_{\Delta, c}(\tilde{V}) \) is a (DFS)-space. \( F \) is a (DF)-space by [30, 25.16]. By the remark after Lemma 5.8 we have to show that \( A^{-1}(B_k) \) is bounded in \( \tilde{C}_{\Delta, c}(\tilde{V}) \). Clearly, the functions
in $A^{-1}(B_k)$ are uniformly bounded on $U_\varepsilon := \partial \tilde{V} + B_\varepsilon(0)$ for some $\varepsilon > 0$ by Taylor expansion. $K := V \setminus U_\varepsilon$ is a compact set contained in $\tilde{V}$ since for any $\varepsilon > 0$ there is $\gamma > 0$ such that
\[ |t| < \varepsilon \text{ if } (x, t) \in \tilde{V} \text{ and } |x| \geq \gamma \]
by the assumption on $\tilde{V}$. Hence $A^{-1}(B_k)$ is bounded in $\tilde{C}_{\Delta,\varepsilon}(\tilde{V})$ since it is well known that for compact $K \subset \tilde{V}$ the topology of $\tilde{C}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K)$ is induced by $\tilde{C}_{\Delta}(\tilde{V} \setminus K)$.

**Remark 5.10** Let us observe that $\tilde{K}^2 > 0$ or
\[ \text{Notice that we have } (19) \]
\[ C_L \]
Since $(18)$, $\partial V$ is compact by (17) the right hand side of (18) is a continuous seminorm on $\tilde{C}_{\Delta,\varepsilon}(\tilde{V})$. Since for any $\varepsilon > 0$ there is $\gamma > 0$ such that $|t| < \varepsilon$ if $(x, t) \in \tilde{V}$ and $|x| \geq \gamma$ the set $\tilde{V}$ satisfies the assumption of Theorem 5.9. Since $\tilde{V} \neq \mathbb{R}^d$ we may assume that $\partial V \neq \emptyset$. Let $\Omega = V \setminus \partial V$. For $x \in \tilde{V}$ choose $\hat{x} \in \partial V$ such that $|x - \hat{x}| = \text{dist}(x, \partial V)$ (especially, $x = \hat{x}$ if $x \in \partial V$) and set $y_x := y_{\hat{x}} \in K$ chosen for $\hat{x}$ by (17).

Let $M_T := \mathcal{B}(T)$ for $T \in \mathcal{A}(\Omega, K)$ and let $f_T \in \tilde{C}_{\Delta,\varepsilon}(\tilde{V})$ be the representation of $T$ by the TG duality. Let $\delta = (\delta_j)_{j \in \mathbb{N}}$ be a strictly positive sequence tending to $0$. By the definition of $\tilde{V}$ we have
\[ \partial \tilde{V} = \{(x, +/ - t(x)) \mid x \in V \}. \]
Notice that $f_T$ is even in $t$. By Theorem 5.9 we thus have to estimate
\[ |f_T|_\delta = \sup_{x \in \tilde{V}, \alpha \in \mathbb{N}^{d+1}} \frac{|f_T^{(\alpha)}(x, t(x))|}{\alpha!} \delta_{[\alpha]} = \sup_{x \in \tilde{V}, \alpha \in \mathbb{N}^{d+1}} \frac{|\xi T, G^{(\alpha)}(x - y_x, (\xi / y_x), t(x))|}{\alpha!} \delta_{[\alpha]} \]
\[ = \sup_{x \in \tilde{V}, \alpha \in \mathbb{N}^{d+1}} |M_T(h_{x, \alpha})(y_x)| \leq \sup_{y \in K, h \in B} |M_T(h)(y)| \]
where
\[ B := \{ h_{x, \alpha} := \frac{G^{(\alpha)}(x - \xi / y_x, t(x))}{\alpha!} \delta_{[\alpha]} \mid x \in \tilde{V}, \alpha \in \mathbb{N}^{d+1} \}. \]
Since $K \subset \Omega$ is compact by (17) the right hand side of (18) is a continuous seminorm on $L_b(\mathcal{A}(\Omega), C(K))$ if we show that $B$ is bounded in $\mathcal{A}(\Omega)$. To prove this, let $J \subset \Omega$ be compact. Notice that
\[ C_J := \inf_{x \in \tilde{V}, \xi \in J} |(x - \xi / y_x, t(x))| > 0. \]
If not, then there are sequences \( x_n \in V \) and \( \xi_n \in J \) such that \( |(x_n - \xi_n/y_{\xi_n}, t(x_n))| \to 0 \). Since \( K \subset NZ \) and \( J \) are compact we may assume that \( y_{\xi_n} \to y \in K \subset NZ \) and \( \xi_n \to \xi \in J \), hence

\[
(20) \quad x_n y_{\xi_n} - \xi_n = y_{\xi_n} (x_n - \xi_n/y_{\xi_n}) \to 0.
\]

If the sequence \((x_n)_n\) is unbounded then \( x_n y_{\xi_n} \to \infty \) for a subsequence since \( y_{\xi_n} \to y \in NZ \). This is a contradiction to (20). Hence \((x_n)_n\) is bounded and we can assume that \( x_n \to x_0 \in \partial V \) since the function \( t \) is continuous on \( V \) and strictly positive on \( V \). Hence we get by (20) that

\[
\Omega \supset J \ni \xi = \lim \xi_n = \lim x_n y_{\xi_n} = \lim (x_n y_{\xi_n} + (x_n - x_n)y_{\xi_n}) = \lim x_n y_{\xi_n} \notin \Omega
\]

by the definition of \( y_{\xi_n} \) since \( |x_n - x_n| = \text{dist}(x_n, \partial V) \to 0 \), a contradiction.

For any \( \gamma > 0 \) we may choose \( C_\gamma \) such that \( \delta^j_\gamma \leq C_\gamma \delta^j \) for any \( j \) since \( \delta_j \to 0 \). Also notice that \( \eta := \min(1, \inf \{ |y_j| \mid y \in K, j \leq d \}) > 0 \) since \( K \subset NZ \) is compact. Using (19) and (9) we thus get

\[
\sup_{x \in V, \alpha \in \mathbb{N}^{d+1}} \sup_{\xi \in J, \beta \in \mathbb{N}^{d+1}} |\partial^\beta h(x, \alpha)(\xi)|_\beta \leq \sup_{x \in V, \alpha \in \mathbb{N}^{d+1}} \left( |\partial^\alpha \partial_\xi^\beta G(x - \frac{\xi}{y_x}, t(x))| \frac{\gamma |\beta| \delta^{[\alpha]}_{|\alpha|}}{|\partial^\beta \alpha!|} \frac{\gamma |\beta| \delta^{[\alpha]}_{|\alpha|}}{|\partial^\beta \alpha!|} \right)
\]

\[
\leq C_0 C_\gamma \sup_{x \in V, \alpha \in \mathbb{N}^{d+1}} \sup_{\xi \in J, \beta \in \mathbb{N}^{d+1}} |(x - \xi/y_x, t(x))|^{-|\beta|-|\alpha|-d} \left( \frac{2C_\gamma}{\eta} \right)^{|\beta|+|\alpha|}
\]

\[
\leq C_0 C_\gamma C_j^{1-d} \sup_{j \in \mathbb{N}_0} \left( \frac{2C_\gamma}{C_j \eta} \right)^j \leq C_0 C_\gamma C_j^{1-d}
\]

if \( \gamma < C_j \eta/(2C) \). The theorem is proved.

As a direct consequence of Propositions 5.5 and 5.7 we obtain:

**Theorem 5.11** For every open \( \Omega \subset \mathbb{R}^d \) the map \( B : \mathcal{A}(V(\Omega))_k \to M(\Omega) \) is continuous. If \( \Omega \subset NZ \) it is a topological isomorphism.

**Proof:** The first part follows from Proposition 5.5 by going to the projective limit over \( K \), where \( K_n = \mathbb{R}^n \) and \( \omega_1 \subset \omega_2 \subset \omega_3 \ldots \) is an open exhaustion of \( \Omega \). The second part follows in the same way from Proposition 5.7.

From Proposition 5.3 and Theorem 5.11 and the fact that the \( k \)-topology is weaker than the \( b \)-topology we obtain:

**Corollary 5.12** \( B : \mathcal{A}(V(\Omega))_b \to L_k(\mathcal{A}(\Omega)) \) is continuous. If it is open onto its image, then the \( b \)- and the \( k \)-topology on \( \mathcal{A}(V(\Omega))_b \) coincide and, in consequence, \( V(\Omega) \) has a countable neighborhood basis.

This shows that, in general, the \( b \)-topology is not the “natural” topology, induced via \( B \) on \( \mathcal{A}(V(\Omega)) \).

**Example 5.13** Let \( \Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, 1 < y < 2\} \) then \( V(\Omega) = \{(x, 1) \in \mathbb{R}^2 : x > 0\} \).

By Theorem 5.11 the topology induced on \( \mathcal{A}(V(\Omega))_b \) via \( B \) is the \( k \)-topology. The \( b \)- and the \( k \)-topology do not coincide on \( \mathcal{A}(V(\Omega)) \) since \( V(\Omega) \) does not admit a countable neighborhood basis. However \( \mathcal{A}(V(\Omega))_b \) is an \( (LF) \)-space since \( V(\Omega) \) is locally closed. Then, due to de Wilde’s Theorem, \( \mathcal{A}(V(\Omega))_k \) cannot be bornological (see Theorem 5.15 below).

A positive case is given in the following
Proposition 5.14 Let $\Omega \subset \mathbb{R}^d$ be an open set. If $\Omega = V(\Omega)$ then $M(\Omega)$ is a complemented subalgebra in $L_b(\mathcal{A}(\Omega))$ with the following continuous projection:

$$P : L_b(\mathcal{A}(\Omega)) \to M(\Omega); \quad P(L) := M_T \text{ where } T = \delta_k \circ L.$$ 

In particular, $\mathcal{B}$ is a topological homomorphism.

Next we show that all topologies under consideration have the same bounded sets.

Theorem 5.15 A set $B \subset \mathcal{A}(V(\Omega))_b'$ is bounded if and only if $\{M_T := \mathcal{B}(T) \mid T \in B\}$ is bounded in $L_b(\mathcal{A}(\Omega))$. In particular:

(a) a sequence $(T_n)_{n \in \mathbb{N}} \subset \mathcal{A}(V(\Omega))_b'$ is convergent if and only if $(\mathcal{B}(T_n))_{n \in \mathbb{N}} \subset M(\Omega)$ is convergent;

(b) the $b$-topology and the $k$-topology on $\mathcal{A}(V(\Omega))'$ have the same bounded sets and the same convergent sequences.

Hence $\mathcal{B} : \mathcal{A}(V(\Omega))_b' \to M(\Omega)$ is a topological isomorphism if and only if $M(\Omega)$ equipped with the topology inherited from $L_b(\mathcal{A}(\Omega))$ is (ultra)bornological.

Let us note that if $V(\Omega)$ is compact then $\mathcal{B} : \mathcal{A}(V(\Omega))_b' \to M(\Omega)$ is a topological isomorphism if and only if $M(\Omega)$ is metrizable (since metrizable spaces are bornological [30, 24.13]).

Proof: As we explained in the introduction to the present Section in $\mathcal{A}(V(\Omega))_b'$ all bounded sets are relatively compact. Hence by Corollary 5.12 we only have to show the "if" part of the first statement.

If $\mathcal{M} \subset M(\Omega)$ is bounded and $\omega \subset \subset \Omega$ open and $K = \mathcal{M}$, then $\mathcal{M}$ is bounded in $M(\Omega, K)$ and therefore equicontinuous. By Proposition 5.5 the set $T := \{T \in \mathcal{A}(V_K(\Omega))' : M_T \in \mathcal{M}\}$ is equicontinuous.

The proof of [37, Th. 1.11] can be easily adapted to show that for any equicontinuous set $B$ of analytic functionals on $\mathcal{A}(U)$, $U \subset \mathbb{R}^d$ open, there is a minimal compact set $L$ such that $B$ is also equicontinuous in $\mathcal{A}(L)'$. If we choose $L$ as above for $T$ then $L \subset V_K(\Omega)$ for every $K \subset \Omega$. Hence $L \subset \bigcap V_K(\Omega) = V(\Omega)$ and this shows the claim. 

The second part of Theorem 5.11 is a special case of a more general observation. For an open set $\Omega \subset \mathbb{R}^d$ we define another topology on $\mathcal{A}(V(\Omega))'$

$$\mathcal{A}(V(\Omega))_{knz} := \text{proj}_{K \in \Omega \cap NZ} \mathcal{A}(V_K(\Omega))_b'$$

By Proposition 5.7 it is weaker than the topology induced on $\mathcal{A}(V(\Omega))'$ via $\mathcal{B}$.

If $\Omega$ is acceptable (see Section 3) we have $V(\Omega) = \bigcap_{K \in \Omega \cap NZ} V_K(\Omega)$ and therefore the $knz$-topology is a complete locally convex topology on $\mathcal{A}(V(\Omega))'$ which, of course, is weaker than the $k$-topology. By $M(\Omega, \Omega') \subset L_b(\mathcal{A}(\Omega), \mathcal{A}(\Omega'))$ we denote as previously the subspace of all maps admitting all monomials as ‘eigenvectors’. From Proposition 5.7 we obtain:

Proposition 5.16 If $\Omega$ is acceptable then $M(\Omega) = M(\Omega, \Omega \cap NZ)$ and the map $\mathcal{B} : \mathcal{A}(V(\Omega))_{knz}' \to M(\Omega, \Omega \cap NZ)$ is a topological linear isomorphism.

Remark 5.17 We are considering the following four topologies ordered from the strongest to the weakest:

- the $b$-topology, i.e., $\text{proj}_{U \supset V(\Omega)} \mathcal{A}(U)'_b$, where $U$ runs through all open neighbourhoods of $V(\Omega)$;
• the $k$-topology, i.e., $\text{proj}_K \mathcal{A}(V_K(\Omega))'_b$, where $K$ runs through all compact subsets of $\Omega$;
• the topology inherited from $M(\Omega)$ via $\mathcal{B}$;
• the $knz$-topology, i.e., $\text{proj}_K \mathcal{A}(V_K(\Omega))'_b$, where $K$ runs through all compact subsets of $\Omega \cap NZ$.

For the relation between these topologies we know the following:

• The $b$- and the $k$-topologies coincide if, and only if, the sets $V^0_K(\Omega)$, $K \subset \Omega$ compact, form a neighborhood basis for $V(\Omega)$.

• The $k$- and the $knz$-topologies coincide if, and only if, for every compact $L \subset \Omega$ there is a compact $K \subset \Omega \cap NZ$ such that $V^0_K(\Omega) \subset V_L(\Omega)$.

• The $b$- and the $knz$-topologies coincide if and only if the sets $V^0_K(\Omega)$, $K \subset \Omega \cap NZ$ compact, form a neighborhood basis for $V(\Omega)$.

The first item is Proposition 5.3, the second and the third follow by an analogous proof.

**Definition 5.18** An open nonempty set $\Omega \subset \mathbb{R}^d$ is called:

(a) pretty nice if the $b$- and $knz$-topologies coincide,

(b) fine if the $b$- and $k$-topologies coincide,

(c) pleasant if the $k$- and $knz$-topologies coincide.

The property of being fine and pleasant are completely independent as we will see later, see Example 6.16 and Theorems 5.20 and 6.15.

The notion of pretty nice sets should be compared with the notion of nice sets introduced in [7] for open sets $\Omega \subset \mathbb{R}$. There we assumed that $V_K(\Omega) \subset V$ for some finite set $K \subset \Omega \cap NZ$. Clearly, nice sets are pretty nice. In fact, the class of pretty nice sets is strictly larger already in the case of one variable: all examples of not nice sets given in [7] are pretty nice (see [7, Examples 2.5 and 3.2] and Example 5.19 below).

The following example shows that in the description of pretty nice sets we cannot put finite sets instead of compact ones.

**Example 5.19** Let $\Omega := \mathbb{R}^d \setminus \{1\}$ for $a \in \mathbb{R}^d$. Then $V(\Omega) = \{1\}$ and $\Omega$ is pretty nice but not nice for $d = 1$ while $\Omega$ is pretty nice but the $K$'s in the definition of the $knz$-topology cannot be chosen finite for $d \geq 2$.

**Proof:** The set $\Omega$ is pretty nice by the argument in Example 5.4. i) Let $d = 1$. Then $\Omega$ is not nice by [7, Examples 3.2]. ii) Let $d \geq 2$ (see Example 5.4). If $K$ is finite then the complement (in $\mathbb{R}^d_+$) of the $V_K(\Omega)$ finite. 

On the basis of the information collected up to now we can identify the topology induced by $M(\Omega)$ on $\mathcal{A}(V(\Omega))'$ via $\mathcal{B}$ for many important cases.

**Theorem 5.20** Let $\Omega \subset \mathbb{R}^d$ be an open nonempty set. In the following cases $\mathcal{B}: \mathcal{A}(V(\Omega))'_k \rightarrow M(\Omega)$ is a linear topological isomorphism, that is $M(\Omega)$ induces on $\mathcal{A}(V(\Omega))'$ via $\mathcal{B}$ the $k$-topology:

(a) $\Omega \subset NZ$;
(b) $d = 1$;

(c) $\Omega$ is convex.

In fact, in these cases $\Omega$ is pleasant.

**Proof:** For (a) see Theorem 5.11. In the other cases we show (using the criterion in Remark 5.17) that, as in case (a), $\Omega$ is pleasant.

(b): By [7, Proposition 2.1], if $0 \in V(\Omega)$ then either $V(\Omega)$ is bounded or $V(\Omega) = \Omega = \mathbb{R}$. In the latter case obviously the set $\Omega$ is pleasant. In the former we can choose $K \subset \Omega$ such that $V := V^0_K(\Omega)$ is bounded so $\partial V$ is compact. Clearly $V^0_K(\Omega) \subset \mathbb{R} \setminus \partial V$.

Take $J := K \setminus B_\delta(0)$. Choose $\delta > 0$ so small that $\partial V \cdot B_\delta(0) \subset \Omega$ (this is possible since $\partial V$ is bounded and $\Omega$ is an open neighbourhood of 0). Now, for any $\xi \in \partial V$ holds $\xi K \not\subset \Omega$ but $\xi B_\delta(0) \subset \Omega$. Hence $\xi J \not\subset \Omega$ and $V_J(\Omega) \subset \mathbb{R} \setminus \partial V$, so $V^0_J(\Omega) \subset V$.

If $0 \not\in V(\Omega)$ the set $\Omega \subset NZ$ so we apply (a).

(c): If $\xi J \subset \Omega$ then $\xi \text{conv} J \subset \Omega$. Thus $V_J(\Omega) \subset \text{conv} J(\Omega)$.

It is easy to see that for any compact $K \subset \Omega$ there is compact $L \subset \Omega \cap NZ$ such that $K \subset \text{conv} L$. Hence $V_L(\Omega) \subset \text{conv} L(\Omega) \subset V_K(\Omega)$.

Based on this and also on the analogous results in [40] we make the

**Conjecture.** For every open non-empty set $\Omega \subset \mathbb{R}^d$ the map $\mathcal{B} : \mathcal{A}(V(\Omega))_b^k \to M(\Omega)$ is a topological isomorphism.

6 Topological Representation in Terms of $\mathcal{A}(V(\Omega))_b^k$

In Theorem 5.20 we have solved for many important cases the problem of topological representation of $M(\Omega)$ via $\mathcal{B}$ and shown that the induced topology is the $k$-topology. It remains the question under which conditions the induced topology is the $b$-topology, that is, the map $\mathcal{B} : \mathcal{A}(V(\Omega))_b^k \to M(\Omega)$ is a linear topological isomorphism. For all the cases covered by Theorem 5.20, in particular for all convex sets, we will solve this problem completely.

Since in all these cases the sets are shown in Theorem 5.20 to be pleasant, our problem means the question, when $\Omega$ is fine. This is, as we already have remarked earlier (see Corollary 5.12), a rather restrictive property.

**Proposition 6.1** If $\Omega \subset \mathbb{R}^d$ is fine then $V(\Omega)$ has a countable basis of open neighbourhoods or, equivalently, $\partial V(\Omega) \cap V(\Omega)$ is compact.

**Proof:** The first part follows from Proposition 5.3, the equivalence of the second assertion appears to be well known, we present a proof for the convenience of the reader.

Let a set $S \subset \mathbb{R}^d$ satisfy $\partial S \cap S$ is compact. Then we write $S = \text{Int} S \cup (\partial S \setminus S)$. The second summand has a countable neighbourhood basis $(V_n)_{n \in \mathbb{N}}$, so $(U_n)_{n \in \mathbb{N}}$ is a countable neighbourhood basis for $S$ where $U_n = V_n \cup \text{Int} S$.
On the other hand, assume that \( \partial S \cap S \) is not compact but \( S \) has a countable open neighbourhood basis \((U_n)_{n \in \mathbb{N}}\). Then there is a sequence \((x_n)_{n \in \mathbb{N}}\) in \( \partial S \cap S \) such that \( x_n \to x \in \partial S \setminus S \) or \((x_n)_{n \in \mathbb{N}}\) has no accumulation point. Let us take \( y_n \in U_n \setminus S \) such that \( d(x_n, y_n) < \frac{1}{n} \) and the interval \([x_n, y_n]\) is contained in \( U_n \). Now, the open set

\[
U := \mathbb{R}^d \setminus \left( \{y_n \mid n \in \mathbb{N} \} \cup \{x\} \right) \quad \text{(if } x_n \to x) \quad \text{or}
U := \mathbb{R}^d \setminus \{y_n \mid n \in \mathbb{N} \} \quad \text{(if } (x_n)_{n \in \mathbb{N}}\text{ has no accumulation point)}
\]
does not contain any \( U_n \) but it is a neighbourhood of \( S \); a contradiction. \( \square \)

Before we go on, we summarize some basic properties of dilation sets (for the one variable case comp. [7, Proposition 2.1]).

We need some more notation.

Let \( I \subset \{1, \ldots, d\} \) then \( H_I := \{x = (x_1, \ldots, x_d) \mid x_j = 0 \text{ for every } j \in I\} \), \( V_I(\Omega) := V(\Omega) \cap H_I \) and

\[
NZ_I := \{x = (x_1, \ldots, x_d) : x_j = 0 \text{ for every } j \in I \text{ and } x_j \neq 0 \text{ for every } j \notin I\}.
\]

Clearly \( NZ_\emptyset = NZ \). We denote by \( \partial_I A \) the boundary in \( H_I \) of a subset \( A \subset H_I \).

**Proposition 6.2** For every non-empty open set \( \Omega \subset \mathbb{R}^d \) and every set \( I \subset \{1, \ldots, d\} \) we have \( V_I(\Omega) \cdot V(\Omega) \subset V_I(\Omega) \) and \( 1 = (1, \ldots, 1) \in V(\Omega) \). Moreover, \( V(\Omega) \cap NZ_I \) is closed in \( NZ_I \). If \( \Omega \) is convex then \( V_I(\Omega) \) is convex as well.

**Proof:** The first statement is obvious.

For the second claim, let \((x_n) \subset V(\Omega) \cap NZ_I\) be a sequence convergent to \( x \in NZ_I, x = (\tilde{x}_1, \ldots, \tilde{x}_d), x_n = (x_{n1}, \ldots, x_{nd})\). For any \( y = (y_1, \ldots, y_d) \in \Omega \) we define \( y_n := (y_{n1}, \ldots, y_{nd}),\)

\[
y_{nj} := \begin{cases} y_j & \text{for } j \in I; \\ y_j \cdot \frac{\tilde{x}_j}{x_{nj}} & \text{for } j \notin I. \end{cases}
\]

Clearly, \( y_n \to y \), hence for \( n \) sufficiently big \( y_n \in \Omega \). On the other hand, since \( x_n \in V(\Omega), y_n x_n \in \Omega \) but \( x_n y_n = xy \). We have proved that \( xy \in \Omega \) for any \( y \in \Omega \) hence \( x \in V(\Omega) \). \( \square \)

The third statement follows from Proposition 3.2.

A very special case is the case of open \( V(\Omega) \).

**Proposition 6.3** For open \( \Omega \subset \mathbb{R}^d \) the following are equivalent:

1. \( V(\Omega) \) is open.
2. \( \mathbb{R}^d_+ \subset V(\Omega) \).
3. There is \( 0 \leq k \leq d \) and a subset \( J \) of \( \{+1, -1\}^{d-k} \) such that, after a permutation of variables,

   \[
   V(\Omega) = \mathbb{R}^k \times \bigcup_{e \in G} e \cdot \mathbb{R}_{+}^{d-k},
   \]

   where \( G \) is a subgroup of \( \{-1, +1\}^{d-k} \).
4. There is a finite set \( K \subset \Omega \) such that \( V_K(\Omega) = V(\Omega) \).
Proof: (1) $\Rightarrow$ (2): Assume there is $a \in \mathbb{R}^d$, $a \notin V(\Omega)$. Then the interval $[1, a]$ meets $\partial V(\Omega)$ in a point in $\mathbb{R}^d_+ \subset \mathbb{NZ}$, which by Proposition 6.2 must belong to $V(\Omega)$; a contradiction with (1).

(1) $\land$ (2) $\Rightarrow$ (3): We set $\sigma = \{ j : \exists a \in V(\Omega), a_j = 0 \}$. We may assume $\sigma = \{ 1, \ldots, k \}$. Since $V(\Omega)$ is multiplicatively closed (in particular, square closed) and $\mathbb{R}^d_+ \subset V(\Omega)$ we obtain $(0, \ldots, 0, 1, \ldots, 1) \in V(\Omega)$ with $k$ zeros. Since $V(\Omega)$ is open there is $\varepsilon > 0$ such that $[-\varepsilon, +\varepsilon]^k \times \{ 1_{d-k} \} \in V(\Omega)$. We use again that $\mathbb{R}^d_+$ operates on $V(\Omega)$ and obtain $\mathbb{R}^k \times \{ 1_{d-k} \} \in V(\Omega)$.

Let now $a = (a', a'') \in V(\Omega)$ where $a' = (a_1, \ldots, a_k)$, $a'' = (a_{k+1}, \ldots, a_d)$. Then, by the same openness argument as before, we may assume that $a_j \neq 0$ for $j = 1, \ldots, k$. Since $\mathbb{R}^k \times \{ 1_{d-k} \} \in V(\Omega)$ operates on $V(\Omega)$ we obtain $\mathbb{R}^k \times e \cdot R^d_{d-k} \subset V(\Omega)$ where $e_j = \text{sign } a_j$ for $j = k + 1, \ldots, d$.

If $G = \{ e \in \{ -1, +1 \}^{d-k} : (1_k, e) \in V(\Omega) \}$ then $G$ is a group and, with this group we get the representation (21).

(3) $\Rightarrow$ (1): Obvious.

(2) $\Rightarrow$ (4): We set $K = \Omega \cap \{-1, 0, +1\}^d$. Since $\mathbb{R}^d_+ \subset V(\Omega)$ every $x \in \Omega$ can be written as $x = e x_+$ where $e \in K$ and $x_+ \in \mathbb{R}^d_+$. That is, $\Omega = K \cdot \mathbb{R}^d_+$. Since $\mathbb{R}^d_+$ operates on $V_K(\Omega)$ we see that $V_K(\Omega) = V(\Omega)$.

(4) $\Rightarrow$ (1). This is again obvious since $V_K(\Omega)$ is open. $\square$

This gives us more information for the case treated in Proposition 5.14.

Corollary 6.4 Let $\Omega \subset \mathbb{R}^d$ be open, then $V(\Omega) = \Omega$ if, and only if $\Omega$ has the form (21).

From Proposition 6.3, (4) we get:

Corollary 6.5 If $V(\Omega)$ is open, then $\Omega$ is fine, that is, the b- and the k-topology coincide.

Assume still that $V(\Omega)$ is open. We set $K_0 = \Omega \cap \{-1, +1\}^d$ then for any compact $K$ with $K_0 \subset K \subset \Omega \cap \mathbb{NZ}$ we have $V_K(\Omega) = \{ x \in \mathbb{R}^d : x \cdot (\Omega \cap \mathbb{NZ}) \subset \Omega \} = \hat{V}(\Omega)$ and we have shown:

Remark 6.6 If $V(\Omega)$ is open, then $\Omega$ is pretty nice, that is, the b- and the knz-topology coincide, if and only if $\Omega$ is acceptable.

This leads to the following simple example:

Example 6.7 Let $\Omega = \mathbb{R}^2 \setminus (\{ 0 \} \times [0, +\infty))$. Then $V(\Omega) = (\mathbb{R}_+ \cup \mathbb{R}_-) \times \mathbb{R}_+$ and $\hat{V}(\Omega) = V_{K_0}(\Omega) = (\mathbb{R}_+ \cup \mathbb{R}_-) \times \mathbb{R}$. Therefore $\Omega$ is not acceptable.

For $d > 1$ we show that the condition in Proposition 6.1 has strong consequences.

Proposition 6.8 Let $d > 1$. If $\partial V(\Omega) \cap V(\Omega)$ is compact and not empty, then $V(\Omega)$ is bounded.

Proof: Since $V(\Omega)$ is square closed it suffices to show that $V(\Omega) \cap (\mathbb{R}_+ \setminus \{ 0 \})^d$ is bounded. If $V(\Omega) \cap \mathbb{R}^d_+$ is bounded but $V(\Omega) \cap (\mathbb{R}_+ \cup \{ 0 \})^d$ is unbounded then $\partial V(\Omega) \cap V(\Omega)$ is unbounded. So it suffices to show that $V(\Omega) \cap \mathbb{R}^d_+$ is bounded.

First, observe that $V(\Omega) \cap \mathbb{R}^d_+$ and $\mathbb{R}^d_+ \setminus V(\Omega)$ cannot be both unbounded. Indeed, let $(x_n)_{n \in \mathbb{N}} \subset V(\Omega) \cap \mathbb{R}^d_+$ and $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d_+ \setminus V(\Omega)$ be unbounded sequences. Then on the interval $[x_n, y_n]$ there is a point $z_n \in \partial V(\Omega) \cap \mathbb{R}^d_+$. Clearly, the sequence $(z_n)_{n \in \mathbb{N}}$ is unbounded. Since $V(\Omega) \cap \mathbb{R}^d_+$ is closed in $\mathbb{R}^d_+$ (Proposition 6.2), we get $(z_n)_{n \in \mathbb{N}} \subset \partial V(\Omega) \cap V(\Omega)$; a contradiction.

Secondly, observe that for $d > 1$ if $V(\Omega) \cap \mathbb{R}^d_+$ is unbounded then $V(\Omega) \supset \mathbb{R}^d_+$, hence $V(\Omega)$ is open by Proposition 6.3. Indeed, by the previous observation $\mathbb{R}^d_+ \setminus V(\Omega)$ is bounded. Take any point $(x_1, \ldots, x_d) \in \mathbb{R}^d_+$. Define

$$a_\varepsilon := (\varepsilon, \ldots, \varepsilon, x_d/\varepsilon), \quad b_\varepsilon := (x_1/\varepsilon, \ldots, x_{d-1}/\varepsilon, \varepsilon),$$

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where $\varepsilon > 0$. For $\varepsilon > 0$ small enough we have $a_\varepsilon b_\varepsilon \in V(\Omega) \cap \mathbb{R}_+^d$ and therefore $a_\varepsilon \cdot b_\varepsilon = (x_1, \ldots, x_d) \in V(\Omega)$. \hfill $\square$

It is easy to see that Proposition 6.8 does not hold for $d = 1$ (take $\Omega = \mathbb{R} \setminus ([0,1/2] \cup \{1\})$ and hence $V(\Omega) = \{1\} \cup [2, \infty[$, compare [7, Proposition 2.1]). We cannot improve Proposition 6.8 to show that $V(\Omega)$ must be compact, see the following example.

**Example 6.9** An example of a pretty nice (hence fine, pleasant and nice) set $\Omega \subset \mathbb{R}^2$ with bounded but non-compact $V(\Omega)$.

**Proof:** Let us take $\Omega$ as the union of the following sets

$$
\{(x,y) \mid -1 < y \leq -1/2, -1 < x < 1\},
\{(x,y) \mid -1/2 < y < 0, -1 < x < \varphi(y)\},
\{(x,y) \mid 0 \leq y < 1/4, -1 < x\},
\{(x,y) \mid 1/4 \leq y \leq 3/8, 0 < x\},
\{(x,y) \mid 3/8 < y < 1/2, -1 < x\}
$$

with the sequence of sets

$$
\Omega_0 := \{(x,y) \mid 1 < x < 2, 3/4 < y < 1\}, \quad \Omega_n := \{(x,y) \mid 3^n - 1 < x < 3^n + 1, 3/4 < y < 1\}
$$

for $n = 1, 2, \ldots$ and where $\varphi : (-1/2, 0) \to (1, +\infty)$, $\varphi(-1/2) = 1$, is a strictly increasing function tending to infinity at zero from below.

It needs some calculations but one see that

$$
V(\Omega) = \{(x,y) \mid 0 < x \leq 1, 0 \leq y \leq 1/2\} \cup \{(0,y) \mid -1/4 \leq y \leq 1/4\} \cup \{(0,1/2)\} \cup \{(1,1)\}.
$$

Moreover, $\Omega$ is pretty nice — the somehow tedious calculations are left to the reader. \hfill $\square$

In view of Proposition 6.8 it is interesting to know when $V(\Omega)$ is bounded.

**Proposition 6.10** Let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set containing zero. Then $V(\Omega)$ is bounded if and only if $\Omega$ contains no axis.

**Proof:** If $\Omega$ contains an axis then $V(\Omega)$ contains the same axis.

If $V(\Omega)$ is unbounded there is a coordinate $j$ such that

$$
\{x_j : x = (x_1, \ldots, x_j, \ldots, x_d) \in V(\Omega)\}
$$

is unbounded. Since $0 \in \Omega$ so there is $\varepsilon > 0$ such that for any $|x_j| < \varepsilon$ the vector $\hat{x}_j := (0, \ldots, 0, x_j, 0, \ldots, 0)$ belongs to $\Omega$. Multiplying the elements of $V(\Omega)$ and $\hat{x}_j$ we will get all elements of the $j$-th axis. So $\Omega$ contains this axis. \hfill $\square$

The necessary condition in Proposition 6.1 need not be sufficient, in fact we have the following example:

**Example 6.11** An open set $\Omega \subset \mathbb{R}^2$ such that $V(\Omega)$ is compact but $\Omega$ is not fine.

**Proof:** Let us take $\Omega$ to be the union of the following sets:

$$
\{(x,y) \mid -1 < y < 1/2, -1 < x, x \neq 1\},
\{(x,y) \mid 1/2 \leq y \leq 3/4, 0 < x, x \neq 1, x \neq 2\},
\{(x,y) \mid 3/4 < y < 1/2, 0 < x, x \neq 2\}.
$$
Then
\[ V(\Omega) = \{(0,y) : 0 \leq y \leq 1/2\} \cup \{(1,1)\}. \]

Take \( y \) slightly bigger than \( 1/2 \). Then \((x,y) \not\in V(\Omega)\) for \( x \) close to 0. But the only \((\tilde{x},\tilde{y}) \in \Omega\) such that \((\tilde{x},\tilde{y})(x,y) \not\in \Omega\) are
\[ \tilde{x} = \frac{1}{x}, \text{ or } \tilde{x} = \frac{2}{x}. \]

Hence the set of \((\tilde{x},\tilde{y})\) cannot be chosen compact if \( x \to 0 \). \( \square \)

However, it turns out that in the one dimensional case the necessary condition in Proposition 6.1 is indeed sufficient, which leads to an explicit description in terms of \( V(\Omega) \).

**Theorem 6.12** If \( \Omega \subset \mathbb{R} \) is a nonempty open set then the following assertions are equivalent:

(a) the map \( \mathcal{A} : \mathcal{A}(V(\Omega))' \to M(\Omega) \) is a topological isomorphism;

(b) the set \( \Omega \) is pretty nice (or, equivalently, fine);

(c) \( \partial V(\Omega) \cap V(\Omega) \) is compact;

(d) one of the following conditions hold:

- 0 \( \in V(\Omega) \) (i.e., 0 \( \in \Omega \));

- \( V(\Omega) \subset \{1,-1\} \);

- \( V(\Omega) \) has a non-empty interior.

The result above improves [7, Th. 2.6]. In fact, by Proposition 6.2 and [7, Proposition 2.1], if 0 \( \in \Omega \) then the conditions above are always satisfied. In case 0 \( \not\in \Omega \), the conditions are not satisfied if and only if \( \partial V(\Omega) \) contains either an unbounded sequence or a zero sequence (for instance the condition above is not satisfied for the set \( \Omega = (0, +\infty) \setminus \{2^n \mid n \in \mathbb{Z}\} \) where \( V(\Omega) = \{2^n \mid n \in \mathbb{Z}\} \).

Before we prove the result we need the following useful lemma showing in some cases that the crucial condition (17) holds locally for \( K \subset \Omega \cap NZ \):

**Lemma 6.13** Let \( \Omega \subset \mathbb{R}^d \) be a non-empty open set. If \( x_0 \in NZ \setminus \tilde{V}(\Omega) \) or \( x_0 \not\in \Omega \) then there are a neighbourhood \( U_0 \) of \( x_0 \) and a compact set \( K \subset \Omega \cap NZ \) such that
\[ \forall x \in U_0 \exists y_x \in K : xy_x \not\in \Omega. \]

**Proof:** i) Let \( x_0 \in NZ \setminus \tilde{V}(\Omega) \) and choose \( y_0 \in \Omega \cap NZ \) such that \( x_0y_0 \not\in \Omega \). For \( |x| \leq \delta \) we set \( y(x) := -x y_0/(x + x_0) \). For sufficiently small \( \delta > 0 \), \( y(x) \) is defined (since \( x_0 \in NZ \)) and \( \{y_0 + y(x) \mid |x| \leq \delta\} \) is a compact subset of \( \Omega \cap NZ \). Obviously, \((x_0 + x)(y_0 + y(x)) = (x_0 + x)[y_0 - x y_0/(x + x)] = x_0y_0 \not\in \Omega \).

ii) By assumption there is \( y_0 \in \Omega \) such that \( x_0y_0 \not\in \Omega \). Then \((x_0 + y_0 + y_1) \not\in \Omega \) and \((y_0 + y_1) \in \Omega \) for small \( |y_1| \) and we may choose \((y_0 + y_1) \in NZ \). Hence \( x_0 \not\in V(\Omega) \) and \((x_0 + x)(y_0 + y_1) \not\in \Omega \) for small \( x \). The Lemma is proved. \( \square \)

**Proof of Theorem 6.12:** (a) and (b) are equivalent by Theorem 5.20, and (b) implies (c) by Proposition 6.1.

(c) \( \Rightarrow \) (b): If \( V(\Omega) \) is closed and \( \partial V(\Omega) \) is compact then for every open neighbourhood \( U \) of \( V(\Omega) \) there is a smaller open neighbourhood \( V \) of \( V(\Omega) \) such that \( 0 \not\in \partial V \) and \( \partial V \) is compact. By Remark 3.7 (b), \( \Omega \) is acceptable so \( \Omega \) is pretty nice by Lemma 6.13.
If $V(\Omega)$ is not closed so $0 \in \partial V(\Omega) \setminus V(\Omega)$ (see Proposition 6.2) but then $0 \notin \Omega$. Moreover, $\partial V(\Omega)$ splits into the two disjoint compact sets $\{0\}$ and $\partial V(\Omega) \cap V(\Omega)$. For any neighbourhood $U$ of $V(\Omega)$ there is a smaller open neighbourhood $V$ of $V(\Omega)$ such that $\partial V$ is compact and splits into two disjoint compact sets $\{0\}$ and $A$. By Lemma 6.13, there is a compact set $K \subset \Omega \cap NZ$ such that $V_K(\Omega)$ is disjoint with $A$. Moreover, $0 \cdot K = \{0\}$ and $0 \notin \Omega$. Hence $0 \notin V_K(\Omega)$. This completes the proof (see Remark 5.17).

(c)⇒(d): If $V(\Omega)$ has an empty interior, $0 \notin V(\Omega)$ and $V(\Omega)$ contains an element $x > 0$ with $|x| \neq 1$ then $\partial V(\Omega) \cap \Omega$ contains $x^n$ for every $n \in \mathbb{N}$. Then clearly $\partial V(\Omega) \cap \Omega$ is not compact.

(d)⇒(c): If $0 \in V(\Omega)$ then $V(\Omega)$ is closed. so it is either bounded (then compact and (c) is satisfied) or $\Omega = \mathbb{R}$ (see [7, Prop. 2.1 (c)]). If $V(\Omega) \subset \{1, -1\}$ then (c) is satisfied. If $V(\Omega)$ has a nonempty interior then using semigroup property it is easily seen that its boundary is finite. □

Also in the other cases covered by Theorem 5.20, in particular for convex sets, the necessary condition in Proposition 6.1 turns out to be sufficient and this leads to an explicit description in terms of $V(\Omega)$.

We need some preparation which has its own value.

**Proposition 6.14** If $\Omega$ is open, convex and $0 \in \Omega$ then $V(\Omega)$ is closed.

**Proof:** We denote $V(A, B) := \{x \in \mathbb{R}^d : xA \subset B\}$. We need two elementary facts:
1. If $\gamma \in \mathbb{R}_+$ (more general $\gamma \in NZ$) and $A, B \subset \mathbb{R}^d$ then $V(\gamma A, \gamma B) = V(A, B)$.
2. If $\omega \subset \mathbb{R}^d$ is open, convex and $0 \in \omega$. If $0 < \gamma < 1$ then $\gamma \omega \subset \omega$.

For $0 < s < 1$ and $R > 0$ we set $K_{s,R} = s\Omega \cap B_R$ where $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$. Then for $0 < \tau < t < 1$ we obtain

$$V(K_{s,R}, \Omega) = V(K_{t_s,tR}, t\Omega) \supset V(K_{t_s,tR}, t\Omega) \supset V(K_{t_s,tR}, t\Omega) = V(K_{t_s,tR}, \Omega).$$

Since $V(\Omega) = \bigcap_{0<s<1,R>0}^\cup V(K_{s,R}, \Omega)$ and the middle term in (22) is closed we see that $V(\Omega)$ is closed.

Now we are ready to prove:

**Theorem 6.15** Let $\Omega \subset \mathbb{R}^d$, $d > 1$, be an open nonempty set and either $\Omega \subset NZ$ or $\Omega$ convex. Then the following are equivalent:

(a) the map $\mathcal{B} : \mathcal{A}(V(\Omega))_p \to M(\Omega)$ is a topological isomorphism;

(b) the set $\Omega$ is pretty nice (equivalently, fine);

(c) the set $\partial V(\Omega) \cap V(\Omega)$ is compact;

(d) the set $V(\Omega)$ is either open or compact;

**Proof:** (a) ⇔ (b): By Theorem 5.20 in both cases $\Omega$ is pleasant. So (a) holds if and only if $\Omega$ is fine and this implies that $\Omega$ is pretty nice.

(b) ⇒ (c): Proposition 6.1,

(c) ⇒ (d): From Proposition 6.8 we know that $V(\Omega)$ is either open or bounded.

If it is bounded and $\Omega \subset NZ$, also $V(\Omega) \subset NZ$ and, by Proposition 6.2, $\partial V(\Omega) \cap NZ \subset V(\Omega)$. Thus for any $I \subset \{1, \ldots, d\}$ the boundary of $V(\Omega) \cap NZ_I$ in $NZ_I$ belongs to the closure of $\partial V(\Omega) \cap V(\Omega)$ but does not belong to $V(\Omega)$, hence it is empty. Thus the set $\overline{V(\Omega)} \cap (\mathbb{R}^d \setminus NZ)$ is either empty or unbounded. The second case is impossible because $V(\Omega)$ is bounded. Summarizing, $\partial V(\Omega) \subset V(\Omega)$ so $V(\Omega)$ is compact.

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It remains to show that $V(\Omega)$ is closed, if it is convex and bounded. By Proposition 3.2, $V(\Omega)$ is contained in the closed unit ball of the $d$-dimensional space $\ell_\infty$.

Let $x \in \partial V(\Omega) \cap V(\Omega)$. By Proposition 6.2, $x \in H_I$ for some set $I \subset \{1, \ldots, d\}$ but there is a sequence $(x_n)_{n \in \mathbb{N}} \subset (V(\Omega) \cap \partial V(\Omega) \cap NZ_j) \cap H_I$ for some $J \subseteq I$ tending to $x$. Choose an axis $Y$ contained in $H_J$ but not in $H_I$.

Now, we take a line $\ell_n$ parallel to $H_I$ going through $x_n$ and crossing $Y$ at some point $v_n$. Except the point $v_n$ the whole $\ell_n$ is contained in $NZ_j$. Thus there is a point $w_n \in \ell_n \cap \partial V(\Omega) \cap NZ_j$ (and thus by Proposition 6.2, $w_n \in V(\Omega)$) such that $x_n \in [w_n, v_n]$. Choosing a subsequence of $(x_n)_{n \in \mathbb{N}}$ without loss of generality we may assume that $w_n \to w \in H_I$. Since $\partial V(\Omega) \cap V(\Omega)$ is closed, $w \in V(\Omega)$.

Again without loss of generality we may assume that either for every $n \in \mathbb{N}$ the interval $[w_n, v_n]$ is contained in $V(\Omega)$ or for every $n \in \mathbb{N}$ there exists $\tilde{v}_n \in (x_n, v_n)$ such that $\tilde{v}_n \in \partial V(\Omega)$.

In the first case, we take a line $p_n$ parallel to $Y$, perpendicular to $H_I$ going through a point $z_n \in [x_n, v_n)$, $d(z_n, v_n) < 1/n$. Then there is a point $u_n \in p_n \cap \partial V(\Omega)$. Again by Proposition 6.2, $u_n \in \partial V(\Omega) \cap V(\Omega)$ and their accumulation point $u \in \partial V(\Omega) \cap V(\Omega) \cap Y$. Clearly, 

$$0 = u \cdot w \in V(\Omega) \cdot V(\Omega) \subset V(\Omega)$$

and therefore $0 \in \Omega$. By Proposition 6.14, $V(\Omega)$ is closed.

In the second case, by Proposition 6.2, $\tilde{v}_n \in \partial V(\Omega) \cap V(\Omega)$. The sequence $(\tilde{v}_n)_{n \in \mathbb{N}}$ has an accumulation point $\tilde{v} \in H_I \cap \partial V(\Omega) \cap V(\Omega)$, since $\partial V(\Omega) \cap V(\Omega)$ is closed. It is clear that $x \in [w, \tilde{v}]$, so by convexity of $V(\Omega)$ holds $x \in V(\Omega)$.

(d) $\Rightarrow$ (b): If $V(\Omega)$ is open then, by Corollary 6.5, $\Omega$ is fine. So here it remains to show that for convex $\Omega$ with compact $V(\Omega)$ the set $\Omega$ is fine.

Now, let $V(\Omega)$ be compact. By Proposition 6.2, $V(\Omega)$ is convex. Let $U$ be an arbitrary open convex neighbourhood of $V(\Omega)$ where $\partial U$ is a compact surface in $\mathbb{R}^d$. Let $x \in \partial U$ and let $x_1 \notin V(\Omega)$ be an internal point of some interval connecting $x$ and some point in $V(\Omega)$. By Proposition 3.9, there is $y \in \Omega \cap NZ$ such that $x_1 \notin \frac{y}{\|y\|} \Omega$. It is easily seen that there is some open neighbourhood of $x$ on $\partial U$ disjoint from $\frac{y}{\|y\}} \Omega$.

We have proved that for any point $\frac{y}{\|y\}} \Omega$ there is $y_x \in \Omega \cap NZ$ such that for some neighbourhood $U_x$ of $y_x$ in $\partial U$, $U_x \cap \frac{y}{\|y\}} \Omega = \emptyset$. Since $(U_x)$ is a covering of $\partial U$, there are finitely many $x_1, \ldots, x_n$ so $U_{x_1} \cup \cdots \cup U_{x_n}$ covers $\partial U$, and then

$$\bigcap_{j=1}^n \frac{1}{y_{x_j}} \Omega \subset U.$$

Hence $\Omega$ is pretty nice (and even nice).

It was just shown that for convex $\Omega$ and compact $V(\Omega)$ the set $\Omega$ is pretty nice. This is not true for general $\Omega$. The following example is a variant of Example [7, Example 3.2]. It shows that $\Omega$ need not be pleasant even for compact $V(\Omega)$.

**Example 6.16 Swiss cross.** For $0 < a < b$ let $\Omega := \{x \in \mathbb{R}^2 \mid ||x||_\infty < b\} \setminus \left([[-a, a] \times \{0\}) \cup (\{0\} \times [-a, a])\right)$. Then $\tilde{V}(\Omega) = \{x \in \mathbb{R}^2 \mid 0 < |x_1|, |x_2| \leq 1\}$ while $V(\Omega) = \{\pm1\}^2$, hence $V(\Omega) \cap NZ \neq \tilde{V}(\Omega) \cap NZ$ and $\Omega$ is not pretty nice but fine — hence not pleasant.

**Proof:** Indeed, if $K = \{(x, y) \in \Omega \mid d((x, y), \partial \Omega) > \varepsilon\}$ then for small $\varepsilon > 0$ the set $V_K(\Omega)$ is a small neighbourhood of $V(\Omega)$. On the other hand, $\partial B_{\varepsilon}(1) \cap \tilde{V}(\Omega) \neq \emptyset$ for any $\varepsilon > 0$. This completes the proof, comp. Remark 5.17.

Substituting the interval $[-a, a]$ by $[0, a]$ in the above example we obtain $V(\Omega) = \{1\}$.  

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Corollary 6.17 A nonempty open set $\Omega \subset \mathbb{R}_+^d$, $d > 1$, is pretty nice if and only if either $\Omega = \mathbb{R}_+^d$ or $V(\Omega) = \{1\}$.

Proof: By Theorem 6.15, sufficiency follows. For necessity, observe, again by Theorem 6.15, that if $V(\Omega)$ is not open and $\Omega \subset \mathbb{R}_+^d$ is pretty nice then $V(\Omega)$ must be compact. If $\eta \neq 1$ belongs to $V(\Omega)$ then $\eta^n \in V(\Omega)$ for all $n \in \mathbb{N}$, and this sequence is either unbounded or converges to some point outside $\mathbb{R}_+^d$, a contradiction.

In case $\Omega$ contains zero a description of pretty nice sets is even more straightforward. From Proposition 6.14, Proposition 6.10 and Theorem 6.15 it follows immediately:

Corollary 6.18 A non-empty open convex set $\Omega$ containing zero in $\mathbb{R}^d$, $d > 1$, is pretty nice if and only if it contains no axis (then $V(\Omega)$ is compact) or it is equal to the whole space $\mathbb{R}^d$.

It is easily seen that there are plenty of unbounded convex open sets $\Omega$ containing zero with compact $V(\Omega)$.

Proposition 6.19 Let $\Omega \subset \mathbb{R}^d$ be an open nonempty set. Then $V(\Omega) \subset \mathbb{N}^d$ is compact if and only if

$$V(\Omega) \subset \{\pm 1\}^d.$$  \hspace{1cm} (23)

Proof: $V(\Omega) \subset [-1,1]^d$ by Proposition 3.2. For $x = (x_1, \ldots, x_d) \in V(\Omega)$ the sequence $y_n := x^n \in V(\Omega)$ by Proposition 6.2 and it has a subsequence converging to $y \in V(\Omega)$ since $V(\Omega)$ is closed. Then $y \notin \mathbb{N}^d$ if $|x_j| < 1$ for some $j$, a contradiction.

The condition (23) holds for instance, if each of the intersections $\Omega_j$ of $\Omega$ with the $j$th coordinate axis is non void and satisfies

$$\Omega_j \subset [-C_2, -C_1] \cup [C_1, C_2]$$

for some $0 < C_1 < C_2 < \infty$

or if $\Omega$ is bounded and $\overline{\Omega} \subset \mathbb{N}^d$.

The criterion of Lemma 6.13 can also be applied for many open sets with $C^1$-boundary with $V(\Omega)$ not necessarily contained in $\mathbb{N}^d$.

7 Special Classes of Multipliers

In this section we present four important classes of multipliers, the Euler operators, the integral operators, the dilation operators and the superposition operators. We define $\eta_\alpha$ by

$$\eta_\alpha(x) = x^\alpha$$

for any $\alpha \in \mathbb{N}^d$.

Euler operators. First, we present the so-called Euler partial differential equations (of finite or infinite order). The one variable theory is classical (see [21], [22], [24], [25], [16], for a survey see Section 4 in [7]) but the authors could not find its several variables analogue in the literature. We present the theory in details in the forthcoming paper [10].

We call an entire function $f \in H(\mathbb{C}^d)$ to be of exponential type zero if for any $\varepsilon > 0$ there is a constant $C$ such that for any $z \in \mathbb{C}^d$ holds

$$|f(z)| \leq C \exp(\varepsilon |z|),$$

\hspace{1cm} 28
Let $\alpha \in \mathbb{N}^d$. Equivalently, if

$$f(z) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha$$

then the entire function $f$ is of exponential type zero if and only if

$$\forall \varepsilon > 0 : \sup_{\alpha} |a_\alpha| \frac{\alpha!}{\varepsilon^{\alpha}} < \infty.$$  

(24)

The class of entire functions of exponential type zero will be denoted by $\text{Exp}(\{0\})$.

We define the partial differential operators (the Euler differentials) with variable coefficients by $\theta_j(f)(x) := x_j \frac{\partial f}{\partial x_j}(x)$ for any $j = 1, \ldots, d$.

It is proved in [10] that for every open non-empty set $\Omega \subset \mathbb{R}^d$ and every entire function $E$, $E(z) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha$, of exponential type zero the map

$$E(\theta) : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega), \quad E(\theta)(g)(z) := \sum_{\alpha \in \mathbb{N}^d} a_\alpha \theta^\alpha(g)(z), \quad \text{where } \theta^\alpha := \theta_1^{\alpha_1} \theta_2^{\alpha_2} \cdots \theta_d^{\alpha_d},$$

is a continuous linear operator (multiplier) with the multiplier sequence $(E(\alpha))_{\alpha \in \mathbb{N}^d}$. Moreover, the corresponding analytic functional via the Representation Theorem 3.4 is equal to $T$ defined by

$$\langle g, T \rangle = \sum_{\alpha \in \mathbb{N}^d} a_\alpha \frac{\partial^{\alpha}}{\partial x^\alpha}(1),$$

hence $T \in \mathcal{A}(\{1\})_b^\prime$. On the other hand, every multiplier whose corresponding analytic functional has support concentrated at $1$ (or, equivalently, a multiplier which acts on $\mathcal{A}(\Omega)$ for every open set $\Omega \subset \mathbb{R}^d$) is equal to some $E(\theta)$ for some entire function $E$ of exponential type zero.

If $V(\Omega) = \{1\}$ then the Euler differential operators are the only multipliers on $\mathcal{A}(\Omega)$. In many cases they form a big subset of all multipliers:

**Proposition 7.1** Let $\Omega$ be any open set with $V(\Omega)$ connected. Then the linear span of all multipliers $\theta^\alpha := \theta_1^{\alpha_1} \cdots \theta_d^{\alpha_d}$, $\alpha \in \mathbb{N}^d$, (the algebra of Euler differential operators of finite order) is dense in the space $M(\Omega)$ of all multipliers on $\mathcal{A}(\Omega)$. Hence in that case, the class of multipliers is just the closure of the linear span of the Euler differential operators of finite order.

**Proof:** By Theorem 3.4 and Corollary 5.12, the map $\mathcal{B} : \mathcal{A}(V(\Omega))_b^\prime \rightarrow M(\Omega)$ is a continuous bijective map which maps $\text{lim}(\delta_{1}^{(\alpha)} : |\alpha| \leq m}$ onto $\text{lim}(\theta^\alpha : |\alpha| \leq m)$. So it suffices to show that the linear span of $\delta_{1}^{(\alpha)}$ is dense in $\mathcal{A}(V(\Omega))_b^\prime$. Since $\mathcal{A}(V(\Omega))$ is reflexive, this will follow from the weak-star density. The latter is a consequence of the fact that if $f^{(\alpha)}(1) = 0$ for every $\alpha \in \mathbb{N}^d$ then $f \equiv 0$ since $V(\Omega)$ is connected.

Euler differentials $\theta_j$ determine multipliers also in another way (for the one variable case see [7, Th.2.13]).

**Proposition 7.2** Let $\Omega \subset \mathbb{R}^d$ be an open connected non-empty set. A continuous linear operator $T : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ is a multiplier if and only if it commutes with all operators $\theta_j$, $j = 1, \ldots, d$.

**Proof:** Since $\theta_j$ is a multiplier, it commutes with every multiplier.

Let us assume that $T$ commutes with all $\theta_j$. Thus

$$\theta_j T(\eta_\alpha)(x) = T(\theta_j(\eta_\alpha))(x) = T({\alpha_j\eta_\alpha})(x) = {\alpha_j T(\eta_\alpha)}(x).$$
We fix \( x^0 \in \Omega \). In a neighborhood of \( x^0 \) the solution \( T(\eta_\alpha) \) of this system of differential equations has the form
\[
T(\eta_\alpha)(x) = C\eta_\alpha(x)
\]
which then, due to connectedness of \( \Omega \) and unique analytic continuations hold in all of \( \Omega \). We have proved that \( T \) is a multiplier.

**Integral operators.** There are plenty of multipliers which are integral operators with multiplier sequences \((m_\alpha)_{\alpha \in \mathbb{N}^d}\). We mention only two typical examples:
\[
M^{(1)}(g)(y) = \int_0^1 g(ty) dt, \quad m_\alpha = \frac{1}{|\alpha| + 1};
\]
\[
M^{(2)}(g)(y) = \int_{Q_d} g(x \cdot y) dx, \quad m_\alpha = \prod_{j=1}^d \frac{1}{\alpha_j + 1},
\]
where \( Q_d = [0,1]^d \). This type of multipliers for the one variable case appear already in [18].

**Dilation operators.** We define the dilation operator \( M_a \) with the dilation factor \( a \in \mathbb{R}^d \) as follows:
\[
M_a(g)(y) := g(a \cdot y).
\]
Clearly, \((m_\alpha)_{\alpha \in \mathbb{N}^d}, m_\alpha = a^\alpha = a_1^{\alpha_1} \cdots a_d^{\alpha_d} \), is the corresponding multiplier sequence. This operator acts on \( \mathcal{A}(\Omega) \) if and only if \( a \in V(\Omega) \). Dilation operators play an important role in the Representation Theorem 3.4, since this theorem somehow shows that every multiplier is a “combination” of dilation operators with factors belonging to \( V(\Omega) \). In that sense dilation operators determine multipliers but there is another reason why dilations determine multipliers:

**Theorem 7.3** Let \( \Omega \) be an open convex non-empty set. The following assertions are equivalent:

(a) \( V(\Omega) \) has non-empty interior;

(b) a continuous linear operator \( T : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega) \) is a multiplier if and only if it commutes with all dilations \( M_a \) for every \( a \in V(\Omega) \).

The proof is more complicated than in the one dimensional case [7, Th. 2.15]. Before we present this proof we need the following proposition — this result has no non-trivial one-dimensional analogue.

**Proposition 7.4** Let \( \Omega \) be an open convex non-empty subset of \( \mathbb{R}^d \). Then \( V(\Omega) \) has empty interior if and only if one of the following two conditions holds:

(i) there is \( j \) such that for every \( x = (x_1, \ldots, x_d) \in V(\Omega) \) holds \( x_j = 1 \);

(ii) there are \( j, k, j \neq k \), such that for every \( x = (x_1, \ldots, x_d) \in V(\Omega) \) holds \( x_j = x_k \).

**Remark 7.5** Let us observe that for a convex open set \( \Omega \subset \mathbb{R}^d \) the dilation set \( V(\Omega) \) has a non-empty interior, for instance, in the following cases:

- \( \Omega \) contains zero and is bounded;

- \( V(\Omega) \) is open (a description of all open sets \( V(\Omega) \) was given in the proof of Theorem 6.15).
Proof: It is enough to prove necessity only. Assume that $V(\Omega)$ has empty interior. Since it is a convex set it is contained in a hyperplane given by

$$a_1 x_1 + \cdots + a_d x_d = b$$

with suitable $a_j$ not all zero and $b$.

If $V(\Omega) \subset \bigcup_{j<k} \{ x : x_j = x_k \} \cup \{ x : x_j = -x_k \}$ then it is contained in one of the hyperplanes. Since $1 \in V(\Omega)$ it must be of the form $\{ x : x_j = x_k \}$.

Otherwise there is $x \in V(\Omega)$ such that all $|x_j|$ are different. We may assume $|x_1| < \cdots < |x_d|$. Then we have

$$a_1 x_1^n + \cdots + a_d x_d^n = b$$

for all $n \in \mathbb{N}_0$. Dividing through $x_d^n$ and letting $n \to +\infty$ we obtain $a_d = 0$. Repeating this we end up with $a_1 x_1 = b$ with $a_1 \neq 0$. Since $1 \in V(\Omega)$ we get $a_1 = b$ and therefore $x_1 = 1$ for all $x \in V(\Omega)$.

□

Proof of Theorem 7.3. (a)$\Rightarrow$(b): We may assume that $1 \in \Omega$. Clearly, $M_a$ are multipliers so every multiplier commutes with every $M_a$. Assume now that $T$ commutes with all dilations $M_a$ for $a \in V(\Omega)$. Then

$$T(\eta_a)[a] = (M_a \circ T)(\eta_a)[1] = T((a\eta)^a)[1] = m_a a^a.$$  

with $m_a = (T\eta_a)[1]$. This holds for all $a \in V(\Omega) \subset \Omega$. Since $\Omega$ is connected and $V(\Omega)$ has non-void interior, it holds for all $a \in \Omega$.

(b)$\Rightarrow$(a): Assume that $V(\Omega)$ has empty interior. By Proposition 7.4, one of the following two cases holds:

1. there exists $j \in \{1, \ldots, d\}$ such that $V(\Omega) \subset \{ a : a_j = 1 \}$;
2. there exist $j, k \in \{1, \ldots, d\}$, $j \neq k$, such that $V(\Omega) \subset \{ a : a_j = a_k \}$.

Case 1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an analytic function. Then

$$T_\varphi(f)(x) := \varphi(x_j) f(x)$$

is a linear continuous map $T_\varphi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ which is not necessarily a multiplier. On the other hand,

$$M_a T_\varphi(f)(x) = \varphi(x_j) f(ax) = T_\varphi(M_a(f))(x).$$

Case 2. The map

$$T_{j,k}(f)(x) = x_j \frac{\partial}{\partial x_k} f(x)$$

is a linear continuous map $T_{j,k} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ which is not a multiplier. On the other hand,

$$M_a T_{j,k}(f)(x) = a_j x_j \frac{\partial}{\partial x_k} f(ax) \quad \text{and} \quad T_{j,k}(M_a f)(x) = x_j a_k \frac{\partial}{\partial x_k} f(ax).$$

Since $a_j = a_k$ we have

$$T_{j,k} M_a = M_a T_{j,k}$$

for any $a \in V(\Omega)$.

□

As in the proof of Proposition 7.1 we can prove that if $V(\Omega)$ is connected with non-empty interior then the multipliers $M(\Omega)$ are the closure of the linear span of the dilation operators on $\mathcal{A}(\Omega)$.
**Superposition operators.** Take any distribution \( T \in \left( C^\infty \left( (-1,1)^d \right) \right)' \) and a smooth function \( f \in C^\infty \left( [-1,1]^d \right) \). Then we define a multiplier by

\[
S_{f,T}(g)(y) := \sum_{\alpha \in \mathbb{N}^d} \frac{\partial^\alpha g(0)}{\alpha!} y^\alpha \langle R_p^0(f) \circ \tilde{\eta}_\alpha, T \rangle + \sum_{|\beta| \leq p} \frac{\partial^\beta f(0)}{\beta!} \langle g \circ (y \tilde{\eta}_\beta), T \rangle
\]

with the multiplier sequence \((m_\alpha)_{\alpha \in \mathbb{N}^d}\) given by

\[
m_\alpha = \langle f \circ \tilde{\eta}_\alpha, T \rangle,
\]

where the natural number \( p \) is chosen so big that the series is absolutely convergent and where \( R_p^0 \) means the Taylor remainder of order \( p \) at zero, \( \tilde{\eta}_\alpha(x) := (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}) \). The obtained multiplier sequence explains the name superposition multiplier.

Especially interesting is the case when \( T \) is a Dirac distribution \( \delta_\varepsilon \) concentrated at \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \subset (-1,1)^d \). Then the multiplier sequence is of the form

\[
m_\alpha = f(\varepsilon_1^{\alpha_1}, \ldots, \varepsilon_d^{\alpha_d}).
\]

If \( f \) does not vanish at \( (\varepsilon_1^{\alpha_1}, \ldots, \varepsilon_d^{\alpha_d}) \) for any \( \alpha \in \mathbb{N}^d \) and at zero this multiplier is invertible: its inverse is just \( S_{f,f,\delta_\varepsilon} \), where \( f \) is a smooth function not vanishing on \([-1,1]^d\) with the same values as \( f \) at all points \( (\varepsilon_1^{\alpha_1}, \ldots, \varepsilon_d^{\alpha_d}), \alpha \in \mathbb{N}^d \).

8 Multipliers on \( \mathcal{A}(\mathbb{R}^d_+) \)

Since the Euler differential \( \theta_j \) is singular for \( x_j = 0 \) it is to be expected that the behaviour of Euler differential operators is quite different depending whether \( \Omega \subset NZ \) and \( \Omega \cap NZ \neq \emptyset \). To see this we start with considering \( \Omega \subset \mathbb{R}^d \) open and connected with \( 0 \in \Omega \).

Let \( P(\theta) \) be an Euler differential polynomial, then \( m_\alpha = P(\theta)(\xi^\alpha)[1] = P(\alpha) \). Therefore, for \( f \in \mathcal{A}(\Omega) \) with Taylor expansion \( f(x) = \sum_\alpha c_\alpha x^\alpha \) we know that \( P(\theta)f(x) = \sum_\alpha c_\alpha P(\alpha)x^\alpha \) around 0. This and the same argument applied to the dual map \( \langle P(\theta) \rangle^* \), acting on \( \mathcal{A}(\Omega)' \cong \mathcal{H}_c(\Omega) \) by Hadamard multiplication, yields for kernel and range of \( P(\theta) \):

**Lemma 8.1** We have:

\[
\ker P(\theta) = \{ f \in \mathcal{A}(\Omega) : f^{(\alpha)}(0) = 0 \text{ whenever } P(\alpha) \neq 0 \},
\]

\[
\text{im } P(\theta) \subset \{ f \in \mathcal{A}(\Omega) : f^{(\alpha)}(0) = 0 \text{ whenever } P(\alpha) = 0 \}.
\]

If \( P(\alpha) = 0 \) has only finitely many integer solutions then the converse of the latter holds as well.

If \( n \) is the (finite) number of integer solutions of the equation \( P(\alpha) = 0 \) then \( n = \dim \ker P(\theta) \) and \( \text{codim } \text{im } P(\theta) = n \).

**Corollary 8.2** (1) If \( P(x) = \sum_j x_j^n \) for \( m \in \mathbb{N} \) then \( \ker P(\theta) = \mathbb{C}\{ f \equiv 1 \} \).

(2) If \( P \) is an elliptic polynomial, then \( \dim \ker P(\theta) \) is finite dimensional.
To have a concrete example let us consider \( P(x) = \sum_j x_j^2 \), then \( \ker P(\theta) = \mathbb{C}\{f \equiv 1\} \). However, if \( \Omega \subset (\mathbb{R}^d_+)^d \) then \( \ker P(\theta) \) is the set of all functions of the form \( f(x) = g(\log x_1, \ldots, \log x_d) \) where \( g \) is analytic on \( \log \Omega \). This can easily be verified, see also below.

Building a solution theory for Euler differential operators even for \( \Omega = \mathbb{R}^d \) may lead to deep problems as is seen by the following example:

**Example 8.3** Let \( P(x) := (x_1+1)^m + (x_2+1)^m - (x_3+1)^m \) with \( m \geq 3 \). Then \( P(\theta) \) is injective.

**Proof:** By Lemma 8.1 this is equivalent to Fermat’s Last Theorem. \( \square \)

While surjectivity of Euler differential operators on arbitrary open subsets of \( \mathbb{R}^d \) will be studied in the forthcoming paper [11], we will in this section treat the case of \( \Omega \subset N \mathbb{Z} \) and it is easily seen that it suffices to concentrate on open sets \( \Omega \subset (\mathbb{R}^d_+)^d \).

We define the analytic diffeomorphisms \( \log : \Omega \to \log \Omega \) and \( \exp : \log \Omega \to \Omega \) by

\[
\log(x) = (\log x_1, \ldots, \log x_d), \quad \exp(x) = (\exp x_1, \ldots, \exp x_d).
\]

Then we have for \( f \in \mathscr{A}(\Omega) \)

\[
(P(\theta)f) \circ \exp = P(\partial)(f \circ \exp).
\]

An immediate consequence is:

**Lemma 8.4** We have for any open \( \Omega \subset (\mathbb{R}^d_+)^d \):

\[
\ker P(\theta) = \{g \circ \log : P(\partial)g = 0\}, \quad \text{im } P(\theta) = \{g \circ \log : g \in \text{im } P(\partial)\}.
\]

**Corollary 8.5** \( P(\theta) \) is surjective on \( \mathscr{A}(\Omega) \) if and only if \( P(\partial) \) is surjective on \( \mathscr{A}(\log \Omega) \).

Surjectivity of partial differential operators \( P(D) \) with constant coefficients on \( \mathscr{A}(\omega) \) for convex open \( \omega \) was characterized by Hörmander [19] by conditions of Phragmén-Lindelöf type valid for plurisubharmonic functions \( PSH(V) \) on the characteristic variety \( Z \) of the polynomial \( P \) (or its principal part \( P_m \), respectively). These results can immediately be applied to Euler differential operator by setting \( \omega = \log \Omega \). Notice that \( \log \Omega \) is convex if and only if \( \Omega \) is multiplicatively convex, that is, with \( x, y \in \Omega \) and \( 0 < t < 1 \) also \( x^t y^{1-t} \in \Omega \).

**Example 8.6** (Surjective and non-surjective Euler operators of second order)

- (Piccinini 72, [34], [35]) \( \sum_{j=1}^d \theta_j^2 \) is not surjective on \( \mathscr{A}(\mathbb{R}^{d+1}_+) \) for \( d \geq 2 \).

- The “Laplace-Euler” operator \( \sum_{j=1}^d \theta_j^2 \) and the “wave-Euler” operator \( \theta_1^2 - \sum_{j=2}^d \theta_j^2 \) are surjective on \( \mathscr{A}(\mathbb{R}_+^d) \) for \( d \geq 2 \).

- The “heat-Euler” operator \( \theta_1 - \sum_{j=2}^d \theta_j^2 \) is not surjective on \( \mathscr{A}(\mathbb{R}_+^d) \) for \( d \geq 3 \).

- (Hörmander 73, [19]) A second order pdo \( P(\theta) \) is surjective on \( \mathscr{A}(\mathbb{R}_+^d) \) iff the principal part \( P_m \) is either elliptic, or proportional to a real indefinite quadratic form or to the product of two real linear forms.

For general open \( \omega \) a characterization of surjective partial differential operators \( P(D) : \mathscr{A}(\omega) \to \mathscr{A}(\omega) \) was obtained by Langenbruch [27] using shifted elementary solutions which are real analytic on relatively compact subsets of \( \omega \). Also this result can be directly applied to Euler differential operators by direct transfer.

Using [19] and [28] we get:

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Corollary 8.7 For any non-empty open $\Omega \subseteq \mathbb{R}_+^d$, if $P(\theta)$ is surjective on $\mathscr{A}(\Omega)$ then it is surjective on $\mathscr{A}(\mathbb{R}_+^d)$.

Problem 8.8 Let $P(\theta) : \mathscr{A}(\Omega) \rightarrow \mathscr{A}(\Omega)$ be surjective for $\Omega \subseteq \mathbb{R}_+^d$. Is $P_m(\theta)$ surjective as well?

More results on the inheritance of surjectivity for arbitrary (not necessarily convex) open sets $\Omega$ were proved by Langenbruch [27], [28] which implies the results for Euler-type operators.

Corollary 8.9 Let $\Omega \subseteq \mathbb{R}_+^d$ be open.

- If $P(\theta)$ is surjective on $\mathscr{A}(\mathbb{R}_+^d)$ then for any non-empty $\Omega \subseteq \mathbb{R}_+^d$ there exists the smallest $\tilde{\Omega} \supset \Omega$ such that $P(\theta)$ is surjective on $\mathscr{A}(\tilde{\Omega})$.

- If $P(\theta)$ is surjective on every $\mathscr{A}(\Omega_j)$, $\Omega_j \subseteq \mathbb{R}_+^d$ then $P(\theta)$ is surjective on $\mathscr{A}(\Omega)$ when $\Omega$ is the interior of the intersection $\bigcap \Omega_j$.

- If $\Omega \subseteq \mathbb{R}_+^d$ has $C^1$-boundary, $P$ is homogeneous and $P(\theta)$ is surjective on $\mathscr{A}(\Omega)$ then $P(\theta)$ is surjective on $\mathscr{A}(\log \text{conv} \Omega)$, where $\log \text{conv} \Omega$ is the smallest superset of $\Omega$ with convex image under the function log.

The same argument as for Lemma 8.4 can be used for general multipliers on $\mathscr{A}(\Omega)$ where $\Omega \subseteq \mathbb{R}_+^d$. By the Theorem 5.20, $\Omega$ is always pleasant and

$$\mathcal{B} : \mathscr{A}(V(\Omega))' \rightarrow M(\Omega)$$

is a topological isomorphism. Moreover, as in the one-dimensional case (comp. [7, Th. 6.1, 5.3]), we can represent multipliers as convolution operators and, via Fourier-Laplace transform, as entire functions of restricted growth (in particular, of order one).

Define the composition operator $C_\varphi(f) = f \circ \varphi$ for a real analytic map $\varphi$. The map

$$\mathcal{E} : L_0(\mathscr{A}(\Omega)) \rightarrow L_0(\mathscr{A}(\log \Omega)),$$

$$\mathcal{E}(M) = C_{\exp} \circ M \circ C_{\log}$$

is a topological isomorphism onto. Clearly, $\mathcal{E}$ maps $\partial_j$ onto $\partial_j$. Therefore:

Theorem 8.10 The following conditions are equivalent for connected open $\Omega \subseteq \mathbb{R}_+^d$:

(a) $M$ is a multiplier on $\mathscr{A}(\Omega)$.

(b) $\mathcal{E}(M)$ is a convolution operator on $\mathscr{A}(\log \Omega)$.

It is not so obvious to get a topological isomorphism of $\mathscr{A}(\log V(\Omega))'_b$ with the set of all convolution operators on $\mathscr{A}(\log \Omega)$ (i.e., operators commuting with all partial derivatives). If $\mathcal{B} : \mathscr{A}(V(\Omega))'_b \rightarrow M(\Omega)$ is a topological isomorphism this is so. By Corollary 6.17, this holds if and only if either $\Omega = \mathbb{R}_+^d$ (i.e., $\log \Omega = \mathbb{R}^d$) or $V(\Omega) = \{1\}$ (i.e., $\log V(\Omega) = \{0\}$).

It is of great interest to describe the transfer via $\mathcal{E}$ of the description of analytic functionals by means of the Laplace transform (see [20, Section 4.5]). For that we need convexity of $\log \Omega$. If $\log \Omega$ is convex then $\log V(\Omega)$ is convex as well. Let us denote the support function of a convex compact set $K$ by

$$H_K(y) := \sup_{z \in K} \text{Re}(z_1y_1 + \cdots + z_dy_d)).$$

Then for any convex compact set $K$ and any convex set $\Omega$ we define

$$\text{Exp}(K) := \{ f \in H(\mathbb{C}^d) : \forall \varepsilon > 0 : \|f\|_{K,\varepsilon} < \infty \}, \quad \text{Exp}(\Omega) := \bigcup_{K \subseteq \Omega} \text{Exp}(K),$$

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where \( \|f\|_{K, \varepsilon} := \sup_{z \in \mathbb{C}} |f(z)| \exp(-H_K(z) - \varepsilon|z|) \). Finally, we recall the definition of the Laplace transform of an analytic functional \( \mu \): 

\[
\mathcal{L}(\mu)(z) = \langle \exp(z \cdot), \mu \rangle, \quad z \in \mathbb{C}^d.
\]

Assume that \( \log V(\Omega) \) is convex, then \( \mathcal{L} : \mathcal{S}(\log V(\Omega))'_b \to \text{Exp}(\log V(\Omega)) \) is an algebra isomorphism (see [20, Th. 4.5.3]), a topological algebra isomorphism if \( V(\Omega) \) is open or closed. Here \( \text{Exp}(\log V(\Omega)) \) is an algebra with respect to pointwise multiplication. Define

\[
\eta_z(x) := \exp(z_1 \log x_1 + \cdots + z_d \log x_d) = x_1^{z_1} \cdots x_d^{z_d}.
\]

Summarizing we have (using Corollary 6.17):

**Theorem 8.11** Let \( \Omega \subset \mathbb{R}^d_+ \) be an open set and \( \log V(\Omega) \) be convex. Then the map \( \mathcal{M} : M(\Omega) \to \text{Exp}(\log V(\Omega)) \),

\[
\mathcal{M}(M)(z) := \mathcal{L}(\mathcal{S}(M))(z) = \langle \eta_z, \mathcal{B}^{-1}(M) \rangle = \text{eigenvalue of } M \text{ for the eigenvector } \eta_z
\]

is an algebra homomorphism onto such that

\[
\mathcal{M}(M)(\alpha) = m_\alpha \quad \text{for every } \alpha \in \mathbb{N}^d
\]

and \( \mathcal{M} \) is a topological isomorphism if and only if \( \Omega \) is either \( \Omega = \mathbb{R}^d_+ \) or \( V(\Omega) = \{1\} \).

**Problem 8.12** Characterize surjective multipliers on \( \mathcal{A}(\Omega) \) for open \( \Omega \subset \mathbb{R}^d_+ \).

Of course, this problem is equivalent to the surjectivity problem for convolution operators on the sets \( \log \Omega \subset \mathbb{R}^d \).

**References**


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