Operators of Hadamard type on spaces of smooth functions

Dietmar Vogt

Abstract

For an open set $\Omega \subset \mathbb{R}^d$ we study the algebra $M(\Omega)$ of continuous linear operators admitting the monomials as eigenvectors. We give a concrete representation of these operators. In classical cases they coincide with operators considered by Hadamard in [5]. We also study the topology of $M(\Omega)$ and the algebra of eigenvalue sequences.

In the present paper we study the continuous linear operators M on $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, which admit all monomials x^α as eigenvectors. We give a representation of those operators in the spirit of Domański-Langenbruch [1], that is, we show that they are given as $(Mf)x = T_\eta F(\eta x)$ where $T \in \mathscr{E}'(V(\Omega))$ and $V(\Omega)$ is the unital semigroup of all diagonal matrices on $\mathbb{R}^d$ which leave Ω invariant. For $\Omega = \mathbb{R}^d$ these are exactly the operators which were considered by Hadamard in [5], page 158 f. We study, and determine, the topology of the algebra $M(\Omega)$ of such operators, inherited from $L(C^\infty(\Omega))$, in terms of the topology of $\mathscr{E}'(V(\Omega))$ and specify all this for special Ω. Finally we collect some information on the algebra of 'multipliers', that is the eigenvalue sequences of operators in $M(\Omega)$.

The paper was motivated by the papers [1, 2, 3] of Domański and Langenbruch, where they studied the analogous problem for spaces of real analytic functions in one variable. In the meantime they also extended there investigation to the multidimensional case (see [4]). The situation in the smooth case, however, is quite different, due to the

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existence of cutoff-functions and to the different topological structure of the relevant spaces. It deserves its own interest and it seems to be obvious that other spaces with partition of unity and similar topological structure can be treated by the same methods.

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1 Basics

Let $\Omega \subset \mathbb{R}^d$ be an open subset, $C^\infty(\Omega)$ the Fréchet space of infinitely differentiable functions on Ω. For any subset $X \subset \mathbb{R}^d$ we set $\mathscr{E}'(X) := \{T \in \mathscr{E}'(\mathbb{R}^d) : \operatorname{supp} T \subset X\}$. The proper topology, for our purposes, of $\mathscr{E}'(X)$ we discuss later.

We want to study the algebra $M(\Omega) \subset L(C^\infty(\Omega))$ of continuous linear operators on $C^\infty(\Omega)$ which admit all monomial functions on Ω as eigenvectors, that is, for $M \in M(\Omega)$ we have $Mx^\alpha = m_\alpha x^\alpha$ where $m_\alpha \in \mathbb{C}$ are the eigenvalues. $T \in M(\Omega)$ if, and only if, $Tx^\alpha \in \operatorname{span}\{x^\alpha\}$ for all α. Since the polynomials are dense in $C^\infty(\Omega)$ the operator T is uniquely determined by the Tx^α and, if $T \in M(\Omega)$, by the m_α. From this the following is obvious:

Proposition 1.1 *$M(\Omega)$ is a closed commutative subalgebra of $L_\sigma(C^\infty(\Omega))$ and therefore also of $L_b(C^\infty(\Omega))$. In the topology inherited from $L_b(C^\infty(\Omega))$ it is complete.*

Here $L_\sigma(C^\infty(\Omega))$ denotes $L(C^\infty(\Omega))$ equipped with the pointwise weak convergence and $L_b(C^\infty(\Omega))$ the same equipped with the uniform convergence on bounded sets. By $M_\sigma(\Omega)$ or $M_b(\Omega)$ we denote $M(\Omega)$ equipped with the topology inherited from $L_\sigma(C^\infty(\Omega))$ or $L_b(C^\infty(\Omega))$, respectively.

From the Banach-Steinhaus Theorem and the fact that $C^\infty(\Omega)$ is a Montel space follows that the sequential convergence in $M_\sigma(\Omega)$ and in $M_b(\Omega)$ coincide.

Lemma 1.2 *The map which assigns to every $M \in M(\Omega)$ the family $(m_\alpha)_{\alpha\in\mathbb{N}_0^d}$ is a continuous algebra isomorphism from $M_\sigma(\Omega)$ to*

$$\Lambda(\Omega) := \{(m_\alpha)_{\alpha\in\mathbb{N}_0^d} : M \in M(\Omega)\} \subset \mathbb{C}^{\mathbb{N}_0^d}$$

equipped with pointwise multiplication.

We follow [1] and set

$$V(\Omega) := \{y \in \mathbb{R}^d : xy \in \Omega \text{ for all } x \in \Omega\} = \bigcap_{\eta\in\Omega} \Omega_\eta.$$

Here $xy := (x_1y_1, \ldots, x_dy_d)$ for $x, y \in \mathbb{R}^d$ and $\Omega_\eta = \{y \in \mathbb{R}^d : \eta y \in \Omega\}$, $\frac{1}{\eta} := (\frac{1}{\eta_1}, \ldots, \frac{1}{\eta_d})$ for $\eta \in (\mathbb{R} \setminus \{0\})^d$. Obviously $V(\Omega)$ is multiplicatively closed with respect to the multiplication $(x, y) \mapsto xy$ in $\mathbb{R}^d$ and $\mathbf{1} := (1, \ldots, 1) \in V(\Omega)$.

The definition can, of course, be applied to any subset of $\mathbb{R}^d$.

Examples: 1. For any $y \in V(\Omega)$ we define the operator V_y by $(V_yf)(x) := f(xy)$. Then $V_y \in M(\Omega)$.
2. For $\alpha \in \mathbb{N}_0^d$ we set $D_\alpha f(x) := x^\alpha \partial^\alpha f(x)$. Then $D_\alpha \in M(\Omega)$.
3. For $\alpha \in \mathbb{N}_0^d$ and $y \in V(\Omega)$ we set $D_{\alpha,y}f(x) := x^\alpha f^{(\alpha)}(xy)$. Then $D_{\alpha,y} \in M(\Omega)$.

Further examples we will obtain in the next section.

2 Representation

An open neighborhood U of $V(\Omega)$ we call a *distinguished neighborhood* if $UV(\Omega) \subset U$. All Ω_η are distinguished neighborhoods of $V(\Omega)$. For every distinguished neighborhood of $V(\Omega)$ we have $V(\Omega) \subset V(U)$.

Lemma 2.1 *For any $T \in \mathscr{E}'(V(\Omega))$ and $f \in C^\infty(\Omega)$ we set $M_Tf(x) := T_yf(xy)$. Then $M_T \in M(\Omega)$ and $m_\alpha = T_y y^\alpha$.*

Proof: For compact sets $B \subset \Omega$ the set $\{y : yB \subset \Omega\}$ is a distinguished neighborhood of $V(\Omega)$. Therefore there is a distinguished neighborhood U of $V(\Omega)$ such that $BU \subset \Omega$.

We fix $\Omega' \subset\subset \Omega$ and choose a distinguished neighborhood U of $V(\Omega)$ such that $\Omega' U \subset \Omega$. For any $f \in C^\infty(\Omega)$ the function $(x, y) \mapsto f(xy)$ is a C^∞-function on $\Omega' \times U$. Therefore for any $T \in \mathscr{E}'(U)$ the function $x \mapsto T_yf(xy)$ is defined on Ω' and $x \mapsto T_yf(xy)$ is in $C^\infty(\Omega')$. For $T \in \mathscr{E}(V(\Omega))'$ this means that $M_Tf(x)$ is defined for every $x \in \Omega$ and $M_Tf \in C^\infty(\Omega)$. $\square$

Example: For $T = \delta_y^{(\alpha)}$ we obtain $M_T = D_{\alpha,y}$. In particular we obtain for $y = \mathbf{1}$ that $M_T = D_\alpha$.

We set $MC(\Omega) := \{M \in L(C^\infty(\Omega), C(\Omega)) : M$ admits all monomials as eigenvectors$\}$.

Lemma 2.2 *For every $M \in MC(\Omega)$ there is $T \in \mathscr{E}'(V(\Omega))$ that $M = M_T$. In particular $M \in M(\Omega)$.*

Proof: Let $\Omega' \subset\subset \Omega$ be open. Then we find a compact $K \subset \Omega$, C and p such that

$$\sup_{\eta \in \Omega'} |(Mf)(\eta)| \leq C\, \|f\|_{K,p} = C\, \sup\{|f^{(\alpha)}(x)| : x \in K,\ |\alpha| \leq p\}.$$

For any $\eta \in \Omega' \cap (\mathbb{R} \setminus \{0\})^d$ we define $T_\eta \in \mathscr{E}'(\mathbb{R}^d)$ by $T_\eta f := (Mf_\eta)(\eta)$ where $f_\eta(x) = f(\frac{1}{\eta}x)$. Our continuity estimate implies that with some K, C and p depending only on Ω' we have

$$|T_\eta f| \leq C \, \|f_\eta\|_{K,p}. \tag{1}$$

We apply the formula to $f(x) = x^\alpha$ and obtain $T_\eta x^\alpha = m_\alpha$ where $Mx^\alpha = m_\alpha x^\alpha$. Since the polynomials are dense in $C^\infty(\mathbb{R}^d)$ and all T_η coincide on the polynomials, T_η does not depend on η, that is, all T_η define one $T \in \mathscr{E}'(\mathbb{R}^d)$ which in the moment still depends on Ω'. We have

$$|Tf| \leq C \, \|f_\eta\|_{K,p} \tag{2}$$

for all $\eta \in \Omega' \cap (\mathbb{R} \setminus \{0\})^d$.

Let now $\eta \in \Omega'$ be arbitrary. For sake of simplicity and without restriction of generality we assume that $\eta = (\eta_1, \ldots, \eta_\nu, 0, \ldots, 0)$ and $\eta_j \neq 0$ for $j = 1, \ldots, \nu$. We set $\eta_\varepsilon = (\eta_1, \ldots, \eta_\nu, \varepsilon, \ldots, \varepsilon)$ and for $0 < \varepsilon \leq \varepsilon_0$ we have $\eta_\varepsilon \in \Omega'$.

We consider now $f \in \mathscr{D}(\mathbb{R}^d)$ such that $\operatorname{supp} f \cap K_\eta = \emptyset$. We set $H := \{x \in \mathbb{R}^d \,:\, x_{\nu+1} = \cdots = x_d = 0\}$ and notice that $K_\eta = \{(\frac{x_1}{\eta_1}, \ldots, \frac{x_\nu}{\eta_\nu}) \,:\, x \in K \cap H\} \times \mathbb{R}^{d-\nu}$. Then we can find $0 < \varepsilon \leq \varepsilon_0$ such that $\operatorname{supp} f_{\eta_\varepsilon} \cap K = \emptyset$. From estimate (2) now follows that $Tf = 0$ and we have shown that $\operatorname{supp} T \subset K_\eta \subset \Omega_\eta$.

By the same argument as used before we see that T does not depend on the set Ω' from which the construction starts. Therefore $\operatorname{supp} T \subset \bigcap_{\eta \in \Omega} \Omega_\eta = V(\Omega)$. □

Theorem 2.3 $M(\Omega) = MC(\Omega) = \{M_T \,:\, T \in \mathscr{E}'(V(\Omega))\}$.

Proof: By Lemmas 2.1 and 2.2 we have $M(\Omega) \subset MC(\Omega) \subset \{M_T \,:\, T \in \mathscr{E}'(V(\Omega))\} \subset M(\Omega)$. □

By the isomorphism of Theorem 2.3 there is an algebra structure on $\mathscr{E}'(V(\Omega))$ defined. To understand this structure we make the following definition: for T, $S \in \mathscr{E}'(\mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}^d)$ we set $(T \star S)f = T_x(S_y f(xy))$. This makes $\mathscr{E}'(\mathbb{R}^d)$ a commutative algebra with $\mathbf{1}$. We have $\operatorname{supp}(T \star S) \subset \operatorname{supp} T \cdot \operatorname{supp} S$.

Let us recall that the sets $V(\Omega)$ are of special nature: If $V = V(\Omega)$ for some open $\Omega \subset \mathbb{R}^d$ then $\mathbf{1} \in V$ and V is closed under multiplication. Now it is clear from the previous that for any set V which contains $\mathbf{1}$ and is multiplicatively closed the space $\mathscr{E}'(V)$ is an subalgebra with $\mathbf{1}$ of $\mathscr{E}'(\mathbb{R}^d)$.

Now, returning to the set $V(\Omega)$ for some open Ω, we see from the very definition that for T, $S \in \mathscr{E}'(\Omega)$ we have $M_{T \star S} = M_T \circ M_S$. Collecting all this information we have:

Proposition 2.4 *The map $T \mapsto M_T$ is an algebra isomorphism from $(\mathscr{E}'(V(\Omega)), \star)$ to $M(\Omega)$.*

3 Topologies

We will now study more precisely the topology of $M(\Omega)$. We remark that for every distinguished neighborhood of $V(\Omega)$ we have $V(\Omega) \subset V(U)$ and therefore, due to Theorem 2.3 and with natural identification, $M(\Omega) \subset M(U)$.

We define for $\Omega' \subset\subset \Omega$

$$U = U_{\Omega'} := \{y \in \mathbb{R}^d \,:\, y\,\Omega' \subset\subset \Omega\}$$

and obtain

Lemma 3.1 *U is a distinguished open neighborhood of $V(\Omega)$ such that $U\Omega' \subset \Omega$.*

Proof: To show that U is open we fix $y \in U$, then $y\,\Omega' \subset\subset \Omega$, hence there is $\delta > 0$ such that $y\,\Omega' + B_\delta \subset\subset \Omega$. Here $B_\delta = \{x \,:\, |x| \leq \delta\}$. Choose $\varepsilon > 0$ such that $\Omega' B_\varepsilon \subset B_\delta$, then $(y + B_\varepsilon)\Omega' \subset y\,\Omega' + B_\varepsilon\Omega' \subset y\,\Omega' + B_\delta \subset\subset \Omega$. So U is open. Clearly $V(\Omega) \subset U$ and U is distinguished. $\square$

We set for $\Omega' \subset\subset \Omega$

$$M(\Omega, \Omega') = \{M \in L(C^\infty(\Omega), CB(\Omega')) \,:\, M \text{ admits all monomials as 'eigenvectors'}\}.$$

Here $CB(\Omega')$ is the space of all bounded continuous functions on Ω' with the sup-norm topology.

Proposition 3.2 *$T \mapsto M_T$ defines a topological isomorphism from $\mathscr{E}'(U)$ onto $M(\Omega, \Omega')$.*

Proof: We fix a compact set $K \subset U$ and assume that

$$|Tf| \leq C \sup_{|\alpha| \leq p,\, x \in K} |f^{(\alpha)}(x)|.$$

From that we obtain for $M = M_T$

$$\|Mf\|_{\Omega'} = \sup_{x \in \Omega'} |T_y f(xy)| \leq CC_1 \sup_{|\alpha| \leq p,\, z \in \Omega' K} |f^{(\alpha)}(z)|$$

where C_1 depends only on Ω' and p. Clearly $\Omega' K \subset\subset \Omega$.

We have shown that $T \mapsto M_T$ maps equicontinuous subsets of $\mathscr{E}'(U)$ into equicontinuous, hence bounded, subsets of $M(\Omega, \Omega')$. Since $\mathscr{E}'(U)$ is bornological the map $T \mapsto M_T$ is continuous from $\mathscr{E}'(U)$ to $M(\Omega, \Omega')$.

To show the reverse direction, we assume that $\mathcal{M} \subset M(\Omega, \Omega')$ is equicontinuous. Then equation (2) in the proof of Lemma 2.2 holds with the same C, K, P for all $M \in \mathcal{M}$. We fix one $\eta \in \Omega' \cap (\mathbb{R} \setminus \{0\})^d$ and see that $\{T : M_T \in \mathcal{M}\}$ is equicontinuous in $\mathscr{E}'(\mathbb{R}^d)$.

On the other hand we see from the proof of Lemma 2.2 that for all these T we have $\operatorname{supp} T \subset \bigcap_{\eta \in \Omega'} K_\eta = \{y \in \mathbb{R}^d : y\,\Omega' \subset K\}$ which is a compact subset of U. So we have shown that for any equicontinuous set $\mathcal{M} \subset M(\Omega, \Omega')$ the set $\{T \in \mathscr{E}(U) : M_T \in \mathcal{M}\}$ is equicontinuous, hence bounded, in $\mathscr{E}'(U)$.

We consider now the map $T \mapsto M_T$ as a continuous linear map from $\mathscr{E}'(U)$ to $L(C^\infty(\Omega), CB(\Omega'))$. The latter space is a (DF)-space and the bounded and the equicontinuous sets coincide. For any equicontinuous set $\mathcal{M} \subset L(C^\infty(\Omega), CB(\Omega'))$ we know by the previous, that $\{T \in \mathscr{E}'(U) : M_T \in \mathcal{M}\} = \{T \in \mathscr{E}'(U) : M_T \in \mathcal{M} \cap M(\Omega, \Omega')\}$ is equicontinuous, hence bounded, in $\mathscr{E}'(U)$. By Baernstein's Lemma [6, 26.26], it follows that $T \mapsto M_T$ is an injective topological homomorphism into $L(C^\infty(\Omega), CB(\Omega'))$ hence an isomorphism onto $M(\Omega, \Omega')$. $\square$

Let now $\omega_1 \subset\subset \omega_2 \subset\subset \ldots$ be an exhaustion of Ω. We set $U_n = U_{\omega_n}$. Then $U_1 \supset U_2 \supset \ldots$ is a decreasing sequence of open neighborhoods of $V(\Omega)$ and it is clear that $\bigcap_n U_n = V(\Omega)$. We set

$$\mathscr{E}'_t(V(\Omega)) := \lim \operatorname{proj}_n \mathscr{E}'(U_n).$$

Here $\mathscr{E}'(U_n)$ carries its usual topology. It is clear that the topology of $\mathscr{E}'_t(V(\Omega)$ does not depend on the choice of the exhaustion of Ω.

Now we obtain immediately from Proposition 3.2

Proposition 3.3 *$T \mapsto M_T$ establishes a topological isomorphism $MC(\Omega) \cong \mathscr{E}'_t(V(\Omega))$.*

Finally we have:

Theorem 3.4 *$M(\Omega) = MC(\Omega) \cong \mathscr{E}'_t(V(\Omega))$ topologically, the last isomorphism is established by $T \mapsto M_T$.*

Proof: It is shown like in the proof of Proposition 3.2 that $T \mapsto M_T$ is a continuous linear map $\mathscr{E}'_t(V(\Omega)) \to M(\Omega)$. Hence we have a chain of injective continuous linear maps $\mathscr{E}'_t(V(\Omega)) \to M(\Omega) \to MC(\Omega)$ the combination of which is, by Proposition 3.3, a topological isomorphism. Therefore all maps are topological isomorphisms. $\square$

The U_n are, in general, not a neighborhood basis of $V(\Omega)$ as shown by the following example:

EXAMPLE: Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 1 < y < 2\}$. Then it is easily seen that $V(\Omega) = \{(x,1) : x \in \mathbb{R}\}$. We choose e sequence $\sqrt{2} > r_1 > r_2 >, \cdots \searrow 1$ and set $\omega_n = \{(x,y) : |x| < n, r_n < y < \frac{2}{r_n}\}$. For this exhaustion of Ω we obtain $U_n = \{(x,y) \in \mathbb{R}^2 : \frac{1}{r_n} < y < r_n\}$, that is, a basis for all neighborhoods $U \supset V(\Omega)$ which contain a strip $\mathbb{R}\times]1-\varepsilon, 1+\varepsilon[$.

In this case $V(\Omega)$ is closed. Another example is:

EXAMPLE: Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x, 1 < y < 2\}$. Then $V(\Omega) = \{(x,1) : 0 < x\}$ and we may choose the exhaustion $\omega_n = \{(x,y) : \frac{1}{n} < x < n, r_n < y < \frac{2}{r_n}\}$. For this exhaustion of Ω we obtain $U_n = \{(x,y) \in \mathbb{R}^2 : 0 < x, \frac{1}{r_n} < y < r_n\}$.

In this case $V(\Omega)$ is not closed, but it is closed in all U_n.

In these cases the description of the topology is much simpler. We need a definition:

Definition 1 *Let $X \subset \mathbb{R}^d$. We say that $\mathscr{E}'(X)$ carries the standard topology, if X is locally compact, σ-compact and $\mathscr{E}'(X) = \lim \operatorname{ind}_n \mathscr{E}'(K_n)$, where $K_1 \subset\subset K_2 \subset\subset \dots$ is some/any compact exhaustion of X. In this case we write $\mathscr{E}'_s(X)$ for $\mathscr{E}'(X)$ equipped with this topology.*

Lemma 3.5 *If $\Omega \subset \mathbb{R}^d$ is open and $X \subset \Omega$ is closed in Ω then X is locally compact, σ-compact and $\mathscr{E}'(\Omega)$ induces the standard topology on $\mathscr{E}'(X)$.*

Proof: Since $\mathscr{E}'(X)$ is a closed subspace of the (DFS)-space $\mathscr{E}'(\Omega)$, it is a (DFS)-space, hence bornological in the induced topology. Since the topology of $\lim \operatorname{ind}_n \mathscr{E}'(K_n)$ is clearly stronger, the result follows from the open mapping theorem (see [6], 24.30). □

Theorem 3.6 *1. If $V(\Omega)$ is closed then $M(\Omega) \cong \mathscr{E}'(V(\Omega))$ equipped with the canonical (DF)-topology inherited from $\mathscr{E}'(\mathbb{R}^d)$.*

2. If $V(\Omega)$ is closed in some U_n, then in all $V(U_m)$ for $m \geq n$, and $M(\Omega) \cong \mathscr{E}'(V(\Omega))$ equipped with the canonical (DF)-topology inherited from $\mathscr{E}'(U_n)$.

In both cases $V(\Omega)$ is locally compact, σ-compact and $M(\Omega) \cong \mathscr{E}'_s(V(\Omega))$.

Proof: It suffices to show 2. $\mathscr{E}'_t(V(\Omega)) = \lim \operatorname{proj}_{n=m}^{\infty} \mathscr{E}'_{(n)}(V(\Omega)))$ where (n) denotes the topology induced by $\mathscr{E}'(U_n)$. By Lemma 3.5 all these topologies coincide with the standard topology. □

Theorem 3.6 applies, of course, if $V(\Omega)$ is finite. A little more sophisticated is the following case:

Lemma 3.7 *If $\overline{V(\Omega)} \setminus V(\Omega)$ is finite, then $V(\Omega)$ is closed in some U_n, hence Theorem 3.6, 2. applies.*

Proof: Since $\bigcap_n U_n = V(\Omega)$ there is U_n such that $(\overline{V(\Omega)} \setminus V(\Omega)) \cap U_n = \emptyset$. Since $V(\Omega) \subset U_n$ for all n this shows the result. $\square$

It is a quite interesting problem, when $V(\Omega)$ is closed in some U_n. In this connection the last example is somehow typical. Before we discuss it we state that $\overline{V(\Omega)} \cdot \overline{\Omega} \subset \overline{\Omega}$ and, by definition, $V(\Omega) \cdot \Omega \subset \Omega$. If now $y \in \overline{V(\Omega)} \setminus V(\Omega)$, then $y\Omega \subset \overline{\Omega}$ but $y\Omega \not\subset \Omega$. Therefore the map $x \mapsto yx$ is not open, that is, there is j such that $y_j = 0$. If we denote the coordinate hyperplanes by $H_j = \{x : x_j = 0\}$ and set $H = \bigcup_j H_j$, then we have shown that $\overline{V(\Omega)} \setminus V(\Omega) \subset H$ and therefore $(\overline{V(\Omega)} \setminus V(\Omega)) \cdot \overline{\Omega} \subset H \cap \overline{\Omega}$ (cf. [4]).

For $d = 1$ we have $H = \{0\}$ and therefore $\overline{V(\Omega)} \setminus V(\Omega) \subset \{0\}$. Therefore Lemma 3.7 applies and we have shown:

Theorem 3.8 *In the one-dimensional case $V(\Omega)$ is locally compact, σ-compact and $T \mapsto M_T$ establishes an isomorphism $M(\Omega) \cong \mathscr{E}_s'(V(\Omega))$.*

4 Examples

We consider now some concrete cases. We remark that in all these cases (except, of course, the general cases 1., 2. in Lemma 4.2) Theorem 3.6, resp. Lemma 3.7, applies, that is, $V(\Omega)$ is locally compact, σ-compact and $M(\Omega) \cong \mathscr{E}_s'(V(\Omega))$.

We obtain immediately:

Theorem 4.1 *If $V(\Omega)$ is finite then $M(\Omega) = \operatorname{span}\{D_{\alpha,y} : \alpha \in \mathbb{N}_0^d, y \in V(\Omega)\}$. If, in particular, $V(\Omega) = \{\mathbf{1}\}$ then $M(\Omega) = \operatorname{span}\{D_\alpha : \alpha \in \mathbb{N}_0^d\}$. $M(\Omega)$ carries the standard topology of a countably dimensional space.*

To obtain better knowledge of this and similar cases, we study now $V(\Omega)$ for special types of sets. For more such examples of sets $V(\Omega)$ see also [4]. The following cases are easy:

Lemma 4.2 *1. If Ω is bounded, then $V(\Omega) \subset \{x : |x|_\infty \leq 1\}$.*

2. If Ω is bounded away from 0 then $V(\Omega) \subset \{x : |x|_\infty \geq 1\}$.

3. If Ω is bounded and bounded away from 0 then there is j such that $V(\Omega)$ is contained in the union of the two hyperplanes $\{x : x_j = 1\}$ and $\{x : x_j = -1\}$.

4. If Ω is bounded, bounded away from 0 and invariant under permutations of the variables then $V(\Omega) \subset \{x : |x_j| = 1$ for all $j\}$.

Proof: 1. Assume there is $x \in V(\Omega)$ with $|x|_\infty = r > 1$. Choose j such that $|x_j| = r$ and $\xi \in \Omega$ such that $x_j \xi_j \neq 0$. Set $R = \sup_{\xi \in \Omega} |\xi|_\infty$. Then we have for every $n \in \mathbb{N}$ that $x^{2n}\xi \in \Omega$ and therefore

$$R \geq |x^{2n}\xi|_\infty \geq r^{2n}|\xi_j|$$

which yields a contradiction.

2. Assume that $x \in V(\Omega)$ with $|x|_\infty < 1$ and $\xi \in \Omega$. Then $x^n \xi \in \Omega$ for all $n \in \mathbb{N}$ and $\lim_n x^n \xi = 0$ contradicting the assumption on Ω. Therefore $|x|_\infty \geq 1$ for all $x \in V(\Omega)$.

3. From 1. and 2. we conclude that $|x|_\infty = 1$ for all $x \in V(\Omega)$. Since $V(\Omega)$ is multiplicatively closed this implies that there is j such that $|x_j| = 1$ for all $x \in V(\Omega)$, which is the same as our claim.

4. is an immediate consequence of 3. $\square$

Easy consequences of 3. are the following. They show that the case of Theorem 4.1 really occurs.

Example: If Ω is bounded, connected and bounded away from all coordinate planes $\{x : x_j = 0\}$ then $V(\Omega) = \{\mathbf{1}\}$.

Example: If $\Omega = \{x : r < |x|_p < R\}$ where $0 < p \leq \infty$ and $0 < r < R$ then $V(\Omega) = \{x : |x_j| = 1$ for all $j\}$ hence finite.

A little more sophisticated are the following examples.

Lemma 4.3 *Let $f \in C[0,1]$ be non-negative, $f(0) = 0$, and $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, |y| < f(x)\}$. If $\lim_{\xi \to 0} f(x\xi)/f(\xi) = 0$ for all $0 < x < 1$ then $V(\Omega) =]0,1[\times\{0\} \cup \{1\} \times [-1,+1]$ and $\overline{V(\Omega)} \setminus V(\Omega) = \{(0,0)\}$.*

Proof: The implication $\supset$ is obvious. Assume $(x,y) \in V(\Omega)$ and $0 < x < 1$, then $|y|\eta < f(x\xi)$ for all $0 \leq \eta < f(\xi)$. Therefore $|y| \leq f(x\xi)/f(\xi)$ for all $0 < \xi < 1$ which implies $y = 0$. Since $x \leq 0$ cannot happen, this proves the result. $\square$

Example: If $\Omega = \{(x,y) \in \mathbb{R}^2 : |y| < e^{-1/x}\}$ then $V(\Omega) =]0,1[\times\{0\} \cup \{1\} \times [-1,+1]$.

By an analogous argument one shows:

Example: If $\Omega = \{(x,y) \in \mathbb{R}^2 : |y| < e^{-x^2}\}$ then $V(\Omega) = \mathbb{R} \times \{0\} \cup [-1,+1] \times [-1,+1]$.

The most relevant examples of sets Ω are the open unit ball and the whole of $\mathbb{R}^d$. The first case is contained in the following result:

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with the following properties:*
1. If $x \in \Omega$ and $|y_j| \leq |x_j|$ for all j then also $y \in \Omega$.
2. Ω is invariant under permutations of the variables.

Then $V(\Omega) = Q := \{x \,:\, |x|_\infty \leq 1\}$ and every $M \in M(\Omega)$ has the following form:

$$Mf(x) = \sum_{|\alpha| \leq m} x^\alpha \int_Q g_\alpha(y) f^{(\alpha)}(xy)dy,$$

with $g_\alpha \in L_1(Q)$ for all α.

On the other hand every such formula defines an operator $M \in M(\Omega)$.

For $\Omega = \mathbb{R}^d$ we obtain:

Theorem 4.5 *$V(\mathbb{R}^d) = \mathbb{R}^d$ and every $M \in M(\mathbb{R}^d)$ has the following form:*

$$Mf(x) = \sum_{|\alpha| \leq m} x^\alpha \int_{\mathbb{R}^d} g_\alpha(y) f^{(\alpha)}(xy)dy,$$

with $g_\alpha \in L_1(\mathbb{R}^d)$ with compact support for all α.

On the other hand every such formula defines an operator $M \in M(\mathbb{R}^d)$.

The operators in Theorems 4.4 and 4.5 have been considered in Hadamard [5], page 158 f.

5 Multiplier sequences

An interesting question is the description of $\Lambda(\Omega)$, that is, of the multiplier sequences. From Theorem 2.3 we get the following result:

Proposition 5.1 $\Lambda(\Omega) := \{(T_y y^\alpha)_{\alpha \in \mathbb{N}_0^d} \,:\, T \in \mathscr{E}(V(\Omega))'\}$.

That means, the problem of determining $\Lambda(\Omega)$ is a multidimensional, distributional moment problem (of course without positivity request). One has to determine the set of all scalar families $(m_\alpha)_{\alpha \in \mathbb{N}_0^d}$ for which there is a distribution with support in $V(\Omega)$ with $T_y y^\alpha = m_\alpha$ for all $\alpha \in \mathbb{N}_0^d$.

Corollary 5.2 *$\Lambda(\Omega)$ depends only on $V(\Omega)$. If $V(\Omega) \subset V(\Omega')$ then $\Lambda(\Omega)$ is a subalgebra of $\Lambda(\Omega')$. In particular $\Lambda(\Omega) \subset \Lambda(\mathbb{R}^d)$ for all Ω.*

By definition we obtain for $M = M_T \in M(\Omega)$

$$M(\exp)(x) = \sum_\alpha m_\alpha \frac{x^\alpha}{\alpha!}.$$

On the other hand we obtain by use of Proposition 5.1 and Theorem 2.3:

$$M(\exp) = T_y \exp(\cdot\, y) = \mathscr{L}(T)$$

where $\mathscr{L}$ denotes the Laplace transform. We have shown:

Theorem 5.3 *For $M = M_T \in M(\Omega)$ we have that $M(\exp) = \mathscr{L}(T)$ is an entire function, $m_\alpha = \mathscr{L}(T)^{(\alpha)}(0)$ for all α.*

Let φ denote the finite scalar sequences. We obtain:

Corollary 5.4 *$\varphi \subset \Lambda(\Omega)$ if, and only if, $0 \in \Omega$.*

Proof: $\varphi \subset \Lambda(\Omega)$ iff all polynomials are contained in $\{\mathscr{L}(T) : T \in \mathscr{E}'(V(\Omega))\}$ iff $\delta \in \mathscr{E}'(V(\Omega))$ iff $0 \in V(\Omega)$ iff $0 \in \Omega$. $\square$

Example: If $V(\Omega) = \{\mathbf{1}\}$ (see example before Theorem 4.4) then we have $\mathscr{E}'(V(\Omega)) = \text{span}\{\delta_{\mathbf{1}}^{(\beta)} : \beta \in \mathbb{N}_0^d\}$ hence

$$\mathscr{L}(\mathscr{E}'(V(\Omega))) = \text{span}\{x^\beta e^x : \beta \in \mathbb{N}_0^d\} = \{p(x)e^x : p \text{ polynomial}\}.$$

So, with $p(x) = \sum_\beta c_\beta \frac{x^\beta}{\beta!}$ we obtain $\Lambda(\Omega) = \{(\sum_{\beta \le \alpha} c_\beta \binom{\alpha}{\beta})_\alpha : \alpha \in \mathbb{N}_0^d\}$ where always only finitely many c_β are not zero.

To get another description we set for a subset $B \subset \mathbb{R}^d$

$$W(B) := \{z \in \mathbb{C}^d : x_j z_j \neq 1 \text{ for all } x \in B \text{ and } j = 1, \ldots, d\}.$$

If B is compact then $W(B)$ is open and contains a neighborhood of zero. We set on $W(\{\mathbf{1}\})$

$$C(x) = \prod_1^d \frac{1}{1 - x_j}.$$

Then $T_y C(zy)$ is defined on $W(\operatorname{supp} T)$ and holomorphic there. For small $|z|$ its convergent power series expansion is

$$T_y C(zy) = \sum_\alpha m_\alpha z^\alpha.$$

Therefore we have:

Proposition 5.5 $\limsup_{|\alpha|\to\infty} m_\alpha^{\frac{1}{|\alpha|}} < \infty$.

This implies that every M_T defines a tame diagonal operator on $H(\mathbb{C}^d) \cong \Lambda_\infty(n^{1/d})$.

Theorem 5.6 *If $0 \in \Omega$, $T \in \mathscr{E}'(V(\Omega))$ and $m_\alpha = Tx^\alpha$ then the Taylor series of $M_T f$ (whether convergent or not) is $\sum_\alpha c_\alpha m_\alpha x^\alpha$ if $\sum_\alpha c_\alpha x^\alpha$ is the Taylor series of f (whether convergent or not).*

Proof: Since $f \mapsto m_\alpha f^{(\alpha)}(0)$ and $f \mapsto (M_T f)^{(\alpha)}(0)$ are continuous linear functionals on $C^\infty(\Omega)$ which coincide on the monomials, hence on the polynomials which are dense in $\mathbb{C}^\infty(\Omega)$, they coincide on $C^\infty(\Omega)$. $\square$

In contrast to the case of real analytic function the property of M_T in Theorem 5.6 does not determine an operator uniquely, because, given such an operator M the operator $f \mapsto \varphi M f$ has the same property for any $\varphi \in C^\infty(\Omega)$ which is constant near 0. We have even more.

Proposition 5.7 *Let $0 \in \Omega$. For every numerical family $(m_\alpha)_{\alpha\in\mathbb{N}_0^d}$ there is an operator $M \in L(C^\infty(\Omega))$ such that the Taylor series of $M(f)$ (whether convergent or not) is $\sum_\alpha c_\alpha m_\alpha x^\alpha$ if $\sum_\alpha c_\alpha x^\alpha$ is the Taylor series of f (whether convergent or not).*

Proof: We choose a closed ball B around 0, such that $B \subset \Omega$. We set

$$G := \{(f,g) \in C^\infty(\Omega) \times C^\infty(\Omega) \text{ with } g^{(\alpha)}(0) = m_\alpha f^{(\alpha)}(0) \text{ for all } \alpha \in \mathbb{N}_0^d\}.$$

Then the maps $\pi_1 : (f,g) \mapsto f|_B$ and $\pi_2 : (f,g) \mapsto g$ are continuous and surjective linear maps from G to $C^\infty(B)$ and $C^\infty(\Omega)$ respectively.

We consider the exact sequence

$$0 \longrightarrow C^\infty(\Omega, B) \times C^\infty(\Omega, \{0\}) \longrightarrow G \xrightarrow{\pi_1} C^\infty(B) \longrightarrow 0.$$

Here $C^\infty(\Omega, M)$ denotes the set of all C^∞-functions on Ω which vanish on M with all their derivatives. These spaces have property (Ω), $C^\infty(B)$ has property (DN). Therefore the exact sequence splits and π_1 has a continuous linear right inverse $\rho : C^\infty(B) \to G$. We set $Mf := (\pi_2 \circ \rho(f|_B)$ for $f \in C^\infty(\Omega)$. Then M is the desired map. $\square$

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Bergische Universität Wuppertal,
FB Math.-Nat., Gauß-Str. 20,
D-42119 Wuppertal, Germany
e-mail: dvogt@math.uni-wuppertal.de