Surjectivity of Euler operators on temperate distributions

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Abstract

Euler operators are partial differential operators of the form $P(\theta)$ where P is a polynomial and $\theta_j = x_j\partial/\partial x_j$. We show that every non-trivial Euler operator is surjective on the space of temperate distributions on R^d. This is in sharp contrast to the behaviour of such operators when acting on spaces of differentiable or analytic functions.

In the present note we study Euler differential operators on the space $\mathscr{S}'(\mathbb{R}^d)$ of temperate distributions on $\mathbb{R}^d$. These are operators of the form $P(\theta)$ where P is a polynomial and $\theta_j = x_j\,\partial/\partial x_j$. We show that every Euler operator is surjective on $\mathscr{S}'(\mathbb{R}^d)$ which is in sharp contrast to the behaviour in spaces of differentiable functions since the operator $P(\theta)$ is, in general, singular at the coordinate hyperplanes. Even for $d = 1$ the simple example of θ acting on $C^\infty(\mathbb{R})$ shows that surjectivity there is in general impossible. There are natural necessary conditions for a function to be in the range of an operator $P(\theta)$, solvability under these conditions has been shown in Domański-Langenbruch [2]. For real analytic functions the situation is even more complicated, see [1]. As an example our result implies the following: if g is a polynomial function on $\mathbb{R}^d$ then the equation $P(\theta)f = g$ may not have a C^∞-solution f on $\mathbb{R}^d$ but it will always have a temperate distribution f as solution on $\mathbb{R}^d$. We first study partial differential operators $P(\partial)$ with constant coefficients on the space $Y(\mathbb{R}^d)$ of C^∞-functions with exponential decay on $\mathbb{R}^d$ and on its dual the space $Y(\mathbb{R}^d)'$ the space of distributions with exponential growth. We show that every non-trivial operator $P(\partial)$ is surjective on $Y(\mathbb{R}^d)'$. By the exponential diffeomorphism this implies the surjectivity of $P(\theta)$ on the space $\mathscr{S}'(Q)$ of temperate distributions on the positive quadrant on $\mathbb{R}^d$, hence surjectivity on $\mathscr{S}'(\mathbb{R}^d)$ up to a distribution with support in the union of coordinate hyperplanes. By a method similar to the one used in [2] we then show the result by induction on the dimension.

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1 Preliminaries

We use the following notation $\partial_j = \partial/\partial x_j$, $\theta_j = x_j\,\partial_j$ and $D_j = -i\partial_j$. For a multiindex $\alpha \in \mathbb{N}_0^d$ we set $\partial^\alpha = \partial_1^{\alpha_1}..\partial_d^{\alpha_d}$, likewise for θ^α and D^α. For a polynomial $P(z) = \sum_\alpha c_\alpha z^\alpha$ we consider the *Euler operator* $P(\theta) = \sum_\alpha c_\alpha \theta^\alpha$ and also the operators $P(\partial)$ and $P(D)$, defined likewise.

$P(\theta)$ and $P(\partial)$ are connected in the following way. We set for $x \in \mathbb{R}^d$

$$\mathrm{Exp}(x) = (\exp(x_1), .., \exp(x_d)).$$

Exp is a diffeomorphism from $\mathbb{R}^d$ onto $Q := (0, +\infty)^d$. Therefore

$$C_{\mathrm{Exp}} : f \longrightarrow f \circ \mathrm{Exp}$$

is a linear topological isomorphism from $C^\infty(Q)$ onto $C^\infty(\mathbb{R}^d)$. For $f \in C^\infty(Q)$ we have $P(\partial)(f \circ \mathrm{Exp}) = (P(\theta)f) \circ \mathrm{Exp}$ that is $P(\partial) \circ C_{\mathrm{Exp}} = C_{\mathrm{Exp}} \circ P(\theta)$. In this way solvability properties of $P(\theta)$ on $C^\infty(Q)$ can be reduced to solvability properties of $P(\partial)$ on $C^\infty(\mathbb{R}^d)$. This has been done in [9]. We apply the same argument to the space $\mathscr{S}(Q)$ where $\mathscr{S}(Q) = \{f \in \mathscr{S}(\mathbb{R}^d) \,:\, \mathrm{supp}\, f \subset \overline{Q}\}$ and $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$.

Throughout the paper we use standard notation of Functional Analysis, in particular, of distribution theory, and of the theory of partial differential operators. For unexplained notation we refer to [3], [5], [6], [7], [8].

2 Distributions with exponential growth

We start with studying partial differential operators on $\mathbb{R}^d$ and we will transfer our results by the exponential diffeomorphism to results on Euler operators on Q. We set

$$\begin{aligned} Y(\mathbb{R}^d) &:= \{f \in C^\infty(\mathbb{R}^d) \,:\, \sup_x |f^{(\alpha)}(x)|\, e^{k|x|} < \infty \text{ for all } \alpha \text{ and } k \in \mathbb{N}\} \\ &= \{f \in C^\infty(\mathbb{R}^d) \,:\, \sup_x |f^{(\alpha)}(x)|\, e^{x\eta} < \infty \text{ for all } \alpha \text{ and } \eta \in \mathbb{R}^d\} \end{aligned}$$

with its natural topology. Here $x\eta = \sum_j x_j \eta_j$ and $|x| := |x|_1$.

Then $Y(\mathbb{R}^d)$ is a Fréchet space, closed under convolution and $P(\partial)$ is a continuous linear operator in $Y(\mathbb{R}^d)$ for every polynomial P. $\mathscr{D}(\mathbb{R}^d) \subset Y(\mathbb{R}^d)$ as a dense subspace, hence $Y(\mathbb{R}^d)' \subset \mathscr{D}'(\mathbb{R}^d)$. We obtain

Lemma 2.1 $C_{\mathrm{Exp}}(\mathscr{S}(Q)) = Y(\mathbb{R}^d)$.

Proof: We first claim that

$$(f \circ \mathrm{Exp})^{(\alpha)} = \sum_{\beta \le \alpha} a_\beta \, (f^{(\beta)} \circ \mathrm{Exp}) \, \mathrm{Exp}^\beta$$

with $a_\alpha = 1$ and this is shown by induction.

This implies that for $f \in \mathscr{S}(Q)$ we have

$$\sup_{x \in \mathbb{R}^d} |(f \circ \mathrm{Exp})^{(\alpha)}(x)| \, e^{k|x|} \le \sum_{\beta \le \alpha} a_\beta \sup_{\xi \in Q} |f^{(\beta)}(\xi)| |\xi|^{|\beta|+k} < +\infty$$

for all α and $k \in \mathbb{N}$.

On the other hand we have for $g = f \circ \mathrm{Exp} \in Y(\mathbb{R}^d)$

$$(f^{(\alpha)} \circ \mathrm{Exp}) \, \mathrm{Exp}^\alpha = (f \circ \mathrm{Exp})^{(\alpha)} - \sum_{\beta \le \alpha, \, \beta \ne \alpha} a_\beta \, (f^{(\beta)} \circ \mathrm{Exp}) \, \mathrm{Exp}^\beta$$

hence

$$\sup_{\xi \in Q} |f^{(\alpha)}(\xi)| \, \xi^{\alpha+\gamma} \le \sup_{x \in \mathbb{R}^d} |g^{(\alpha)}(x)| \, \mathrm{Exp}^\gamma(x) + \sum_{\beta \le \alpha, \, \beta \ne \alpha} a_\beta \, |f^{(\beta)}(\xi)| \, \xi^{\beta+\gamma}.$$

for all $\alpha \in \mathbb{N}_0^d$ and $\gamma \in \mathbb{R}^d$. From here one derives easily by induction that $f \in \mathscr{S}(\mathbb{R}^d)$. □

The space $Y(\mathbb{R}^d)$ can also be described by means of the Fourier transformation. This description might also be used to show Lemma 2.3. We will use another method. However the description exhibits that from the point of view of the Fourier transformation $Y(\mathbb{R}^d)$ is a very natural space. We define

$$H_Y(\mathbb{R}^d) = \{g \in H(\mathbb{C}^d) \; : \; \sup_{x, \, |y| \le k} |x + iy|^k |g(x+iy)| < \infty \text{ for all } k \in \mathbb{N}\}$$

and remark that $H_Y(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$, due to Cauchy's estimates. We obtain:

Proposition 2.2 *The Fourier transformation maps $Y(\mathbb{R}^d)$ isomorphically onto $H_Y(\mathbb{R}^d)$.*

Proof: For $f \in Y(\mathbb{R}^d)$ we have with $z = x + iy$

$$\hat{f}(z) \, z^\alpha = (2\pi)^{-n/2} \int z^\alpha f(\xi) e^{-iz\xi} d\xi = (2\pi)^{-n/2} \, i^{|\alpha|} \int f^{(\alpha)}(\xi) \, e^{y\xi} \, e^{-ix\xi} d\xi$$

and therefore $\hat{f} \in H_Y(\mathbb{R}^d)$.

On the other hand, for $g \in H_Y(\mathbb{R}^d)$ there is $f \in \mathscr{S}(\mathbb{R}^d)$ with $g = \hat{f}$. Then

$$\begin{aligned} f^{(\alpha)}(x) &= (2\pi)^{-n/2} i^{|\alpha|} \int g(\xi) \, \xi^\alpha e^{ix\xi} d\xi \\ &= (2\pi)^{-n/2} i^{|\alpha|} \int g(\xi + i\eta)(\xi + i\eta)^\alpha e^{-x\eta} e^{ix\xi} d\xi. \end{aligned}$$

Therefore we have for every $\eta \in \mathbb{R}^d$

$$|f^{(\alpha)}(x)|\, e^{x\eta} \leq (2\pi)^{-n/2} \int |g(\xi + i\eta)(\xi + i\eta)^\alpha| d\xi < +\infty.$$

This means that $f \in Y(\mathbb{R}^d)$. □

Our first main result is based on the following Lemma.

Lemma 2.3 *For every non-trivial polynomial the operator $P(-\partial)$ in $Y(\mathbb{R}^d)$ is an isomorphism onto its range.*

Proof: We use the construction of an elementary solution of P. Wagner [10, Proposition 1] and assume P to be nontrivial of degree m. For $\eta \in \mathbb{R}^d$ with $P_m(\eta) \neq 0$ and pairwise different real numbers $\lambda_0, .., \lambda_m$ we set with suitable coefficients a_j

$$E = \frac{1}{P_m(2\eta)} \sum_{j=0}^{m} a_j e^{\lambda_j \eta x} \mathscr{F}_\xi^{-1} \left(\frac{\overline{P(i\xi + \lambda_j \eta}}{P(i\xi + \lambda_j \eta} \right).$$

Then, by Wagner, loc. cit., E is an elementary solution for $P(\partial)$, that is, $\langle E, P(-\partial)\varphi \rangle = \varphi(0)$ for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$. By continuous extension this holds also for $\varphi \in Y(\mathbb{R}^d)$ since $E \in Y(\mathbb{R}^d)'$. That means that for any $\varphi \in Y(\mathbb{R}^d)$ the term $\varphi(y)$ is a linear combination of terms

$$\Psi_j(y) = \left\langle (P(-\partial_x)\varphi(x+y)) e^{\lambda_j \eta x}, \mathscr{F}_\xi^{-1} \left(\frac{\overline{P(i\xi + \lambda_j \eta}}{P(i\xi + \lambda_j \eta} \right) \right\rangle.$$

Then $\Psi_j(y) e^{\lambda_j \eta y}$ has the form

$$\Psi_j(y) e^{\lambda_j \eta y} = \left\langle \psi(\cdot + y), \mathscr{F}^{-1}(G) \right\rangle$$

where $\psi(x) = (P(-\partial_x)\varphi)(x)) e^{\lambda_j \eta x} \in Y(\mathbb{R}^d)$ and $G \in Ł_\infty(\mathbb{R}^d)$, $\|G\|_\infty = 1$. We obtain

$$|\Psi_j(y) e^{\lambda_j \eta y}| = \left| \int \hat{\psi}(-\xi) e^{i\xi y} G(\xi) d\xi \right| \leq \int |\hat{\psi}(\xi)|\, d\xi = \|P(-\partial)\varphi\|_{Y(\mathbb{R}^d)}$$

where $\|\cdot\|_{Y(\mathbb{R}^d)}$ is a semi-norm in $Y(\mathbb{R}^d)$ (cf. Proposition 2.2). We do that for every j and find a semi-norm $\|\cdot\|_\eta$ in $Y(\mathbb{R}^d)$ such that

$$|\varphi(y)| \leq \frac{1}{|P_m(2\eta)|} \sum_{j=0}^{m} |a_j|\, e^{-\lambda_j \eta y} \|P(-\partial)\varphi\|_\eta.$$

We may assume that our choice was so that $\lambda_j \geq 1$ for all j. We evaluate the inequality separately for every quadrant $Q_e = \{x \,:\, e_j x_j \geq 0 \text{ for all } j\}$, $e_j = \pm 1$

for all j. For given $c>0$ we choose $\eta_e \in Q_e$ such that $\eta_e y \geq c\,|y|$ for all $y \in Q_e$. This yields with a proper constant D

$$|\varphi(y)| \leq D \max_e \|P(-\partial)\varphi\|_{\eta_e} e^{-c|y|} =: \|P(-\partial)\varphi\|\, e^{-c|y|}.$$

We may apply this estimate to $\varphi^{(\alpha)}$ for any α. Since ∂^α commutes with $P(-\partial)$ and since $c>0$ was chosen arbitrarily, we obtain the result. □

As an immediate consequence we obtain by dualization of Lemma 2.3.

Theorem 2.4 *Every non-trivial operator $P(\partial)$ is surjective on $Y(\mathbb{R}^d)'$.*

3 Euler differential operators on $\mathscr{S}'(\mathbb{R}^d)$

From Lemma 2.3 we derive the following lemma which is the basis for our main result.

Lemma 3.1 *Every non-trivial Euler operator is surjective on $\mathscr{S}'(Q)$.*

Proof: To show surjectivity of $P(\theta)$ on $\mathscr{S}'(Q)$ we have to show that $P(\theta^*)$ is an isomorphism onto its range in $\mathscr{S}(Q)$. Here θ^* is the transpose of θ, that is, $\theta_j^*\varphi = -\varphi - \theta_j\varphi$ and therefore $P(\theta^*) = P(-1-\theta)$. By Lemma 2.1 and the fact that $P(-1-\partial)\circ C_{\mathrm{Exp}} = C_{\mathrm{Exp}} \circ P(\theta^*)$, we obtain from Lemma 2.3 that $P(\theta^*)$ is an isomorphism onto its range in $\mathscr{S}(Q)$ which shows the result. □

We set $Z_0 = \{x \in \mathbb{R}^d \,:\, \exists j : x_j = 0\}$ and $\mathscr{S}_0 = \{f \in \mathscr{S}(\mathbb{R}^d) \,:\, f \text{ is flat on } Z_0\}$. Since Lemma 3.1 holds, mutatis mutandis, on every 'quadrant' we obtain:

Corollary 3.2 *Every non-trivial Euler operator is surjective on $\mathscr{S}_0'$.*

Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and $P(\theta)$ be a non-trivial Euler operator. We want to solve the equation $P(\theta)U = T$. We set $T_0 = T|_{\mathscr{S}_0}$. Then $T_0 \in \mathscr{S}_0'$ and we can find $U_0 \in \mathscr{S}_0'$ such that $P(\theta)U_0 = T_0$. We extend U_0 by use of the Hahn-Banach Theorem to a distribution $U_1 \in \mathscr{S}'(\mathbb{R}^d)$ and for any solution U of our problem we have $P(\theta)(U-U_1) = 0$ on $\mathscr{S}_0$. Hence it suffices to solve the equation $P(\theta)U_2 = T_1$ with $T_1 = T - P(\partial)U_1 \in \mathscr{S}'(Z_0) = \{S \in \mathscr{S}'(\mathbb{R}^d) \,:\, S \text{ vanishes on } \mathscr{S}_0\}$.

By H_j we denote the coordinate hyperplanes $H_j = \{x \in \mathbb{R}^d \,:\, x_j = 0\}$. We set $\mathscr{S}_{j_1,..,j_k} = \{f \in \mathscr{S}(\mathbb{R}^d) \,:\, f \text{ flat on } H_{j_1} \cup .. \cup H_{j_k}\}$, then $\mathscr{S}_0 = \mathscr{S}_{1,..,d}$, and we set $\mathscr{S}'(H_{j_1} \cup .. \cup H_{j_k}) = \{S \in \mathscr{S}'(\mathbb{R}^d) \,:\, S \text{ vanishes on } \mathscr{S}_{j_1,..,j_k}\}$.

Lemma 3.3 *Every $T \in \mathscr{S}'(Z_0)$ has a decomposition $T = T_1 + \cdots + T_d$ where $T_j \in \mathscr{S}'(H_j)$ for all j.*

Proof: We act by induction and consider the canonical map

$$\Phi_k := \mathscr{S}(\mathbb{R}^d)/\mathscr{S}_{k,..,d} \to \mathscr{S}(\mathbb{R}^d)/\mathscr{S}_k \times \mathscr{S}(\mathbb{R}^d)/\mathscr{S}_{k+1,..,d}.$$

It is injective and we want to show that its image is closed. We admit that $\mathscr{S}_k + \mathscr{S}_{k+1,..,d}$ is closed in $\mathscr{S}(\mathbb{R}^d)$ and we will show this later. We consider the map

$$\Psi_k : \mathscr{S}(\mathbb{R}^d)/\mathscr{S}_k \times \mathscr{S}(\mathbb{R}^d)/\mathscr{S}_{k+1,..,d} \to \mathscr{S}(\mathbb{R}^d)/\mathscr{S}_k + \mathscr{S}_{k+1,..,d}$$

given by

$$(\varphi_1 + \mathscr{S}_k) \times (\varphi_2 + \mathscr{S}_{k+1,..,d}) \mapsto (\varphi_1 - \varphi_2) + (\mathscr{S}_k + \mathscr{S}_{k+1,..,d}).$$

We claim that $\mathrm{im}\Phi_k = \ker \Psi_k$. One inclusion is evident, for the other assume that $(\varphi_1 - \varphi_2) \in (\mathscr{S}_k + \mathscr{S}_{k+1,..,d})$. Then there are $\psi_1 \in \mathscr{S}_k$ and $\psi_2 \in \mathscr{S}_{k+1,..,d}$ such that $\varphi_1 - \varphi_2 = \psi_1 - \psi_2$, hence $\varphi_1 - \psi_1 = \varphi_2 - \psi_2 =: \varphi$. So we have $\Phi_k(\hat{\varphi}) = \hat{\varphi}_1 \times \hat{\varphi}_2$, where $\hat{}$ denotes the respective equivalence class.

Let $T_{j_1,..,j_k}$ denote a distribution in $\mathscr{S}'(H_{j_1} \cup .. \cup H_{j_k})$. Dualization of the previous shows that for every $T_{k,..,d}$ there are T_k and $T_{k+1,..,d}$ such that $T_{k,..,d} = T_k + T_{k+1,..,d}$. Starting with $T = T_{1,..,d} \in \mathscr{S}'(Z_0)$, induction over k yields the result.

It remains to show that $\mathscr{S}_k + \mathscr{S}_{k+1,..,d}$ is closed in $\mathscr{S}(\mathbb{R}^d)$. We write $\mathscr{S}(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^k)\widehat{\otimes}_\pi \mathscr{S}(\mathbb{R}^{d-k})$ and consider the map

$$A_1 \otimes A_2 : \mathscr{S}(\mathbb{R}^k)\widehat{\otimes}_\pi \mathscr{S}(\mathbb{R}^{d-k}) \to \mathscr{S}(\mathbb{R}^k)/S_k(\mathbb{R}^k)\widehat{\otimes}_\pi \mathscr{S}(\mathbb{R}^{d-k})/S_{k+1,..,d}(\mathbb{R}^{d-k}).$$

Here A_1, A_2 are the respective quotient maps and $S_k(\mathbb{R}^k)$ resp. $S_{k+1,..,d}(\mathbb{R}^{d-k})$ are self-explaining. By use of Lemma 3.4 below we get

$$\ker(A_1 \otimes A_2) = S_k(\mathbb{R}^k)\widehat{\otimes}_\pi \mathscr{S}(\mathbb{R}^{d-k}) + \mathscr{S}(\mathbb{R}^k)\widehat{\otimes}_\pi S_{k+1,..,d}(\mathbb{R}^{d-k}) = \mathscr{S}_k + \mathscr{S}_{k+1,..,d}$$

and the claim is shown. □

The following lemma is known (see [1], Theorem 2.12). We sketch a proof for the convenience of the reader.

Lemma 3.4 *Let $A_1 : E_1 \to F_1, A_2 : E_2 \to F_2$ be continuous linear maps between nuclear Fréchet spaces then for $A_1 \otimes A_2 : E_1\widehat{\otimes}_\pi E_2 \to F_1\widehat{\otimes}_\pi F_2$ we have* $\ker(A_1 \otimes A_2) = \ker A_1\widehat{\otimes}_\pi E_2 + E_1\widehat{\otimes}_\pi \ker A_2$.

Proof: By Grothendieck [4], Chap I, p. 38 and Chap II, p. 70, we obtain $\ker(A_1 \otimes \mathrm{id}_{E_2}) = \ker A_1\widehat{\otimes}_\pi E_2$ and $\ker(\mathrm{id}_{F_1} \otimes A_2) = F_1\widehat{\otimes}_\pi \ker A_2$. We calculate the kernel of the composition, which is $A_1 \otimes A_2$, and obtain the result. □

We are now ready to prove our main result. We will use the structure of distributions in $\mathscr{S}'(H_j)$ (see in analogy [7, Chap III, Théorème XXXVI]) and the fact that ∂_j and multiplication with x_j, which is up to a factor the Fourier transform of ∂_j, are surjective on $\mathscr{S}'(\mathbb{R}^d)$.

Theorem 3.5 *Every non-trivial Euler operator is surjective on $\mathscr{S}'(\mathbb{R}^d)$.*

Proof: The proof will be by induction on the dimension. We start with the induction step. Assume that $d > 1$ and the result is shown for $d-1$. We may assume that P is irreducible.

Due to Lemma 3.3 and the argument before this Lemma it is enough to assume that $T \in \mathscr{S}'(H_j)$. We may assume that $j = 1$. We set $x' = (x_2, \dots, x_d)$. Then

$$T(\varphi) = \sum_k \delta^{(k)}(x_1) T_k(x')\varphi(x_1, x').$$

and analogously for U if we try to solve $P(\theta)U = T$ with $U \in \mathscr{S}'(H_1)$. Here U_k, $T_k \in \mathscr{S}'(\mathbb{R}^{d-1})$ and the sums are finite.

We recall that for $\psi \in C^\infty(\mathbb{R})$ we have

$$\langle \theta(\delta^{(k)}), \psi\rangle = (-1)^{k+1}(x\psi)^{(k+1)}(0) = (-1)^{k+1}(k+1)\psi^{(k)}(0) = -(k+1)\langle \delta^{(k)}, \psi\rangle,$$

that is, $\theta\delta^{(k)} = (-k-1)\delta^{(k)}$.

For $P(z) = \sum_\alpha c_\alpha z^\alpha$ we obtain

$$P(\theta)U = \sum_k \sum_\alpha c_\alpha (-k-1)^{\alpha_1} \delta^{(k)} \theta'^{\alpha'} U_k = \sum_k \delta^{(k)} P(-k-1, \theta')U_k$$

and the equation we want to solve takes the form

$$\sum_k \delta^{(k)} P(-k-1, \theta')U_k = \sum_k \delta^{(k)} T_k.$$

By the induction assumption $P(-k-1, \theta')U_k = T_k$ is solvable if $P(-k-1, z') \not\equiv 0$.

So we have to consider the case where $P(-k-1, z') \equiv 0$. Then $z_1 + k + 1$ divides P. Since P is irreducible $P(z) = C\,(z_1 + k + 1)$ for some constant $C \neq 0$.

Since under the canonical isomorphism $\mathscr{S}'(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R})\widehat{\otimes}_\pi \mathscr{S}'(\mathbb{R}^{d-1})$ the operator $\theta_1 + n$ in $\mathscr{S}'(\mathbb{R}^d)$ corresponds to $(\theta + n) \otimes \mathrm{id}$ in $\mathscr{S}'(\mathbb{R})\widehat{\otimes}_\pi \mathscr{S}'(\mathbb{R}^{d-1})$ we are, due to Grothendieck's exactness theorem, reduced to the surjectivity of $\theta + n$ in $\mathscr{S}'(\mathbb{R})$ which is shown below.

For $d = 1$ we have to consider $P(z) = z - a$. It is enough to find, for all a and k, a distribution $S \in \mathscr{S}'(\mathbb{R})$ such that $(\theta - a)S = \delta^{(k)}$. Since $(\theta - a)\,\delta^{(k)} = -(k+1+a)\,\delta^{(k)}$ this is evident for $a \neq -(k+1)$. For the equation $(\theta + k + 1)S = \delta^{(k)}$ we obtain a solution as follows: choose any U with $x^{k+1}U = Y(x)$, where $Y(x) = 1$ for $x \geq 0$, $Y(x) = 0$ otherwise, then $x^k((k+1)U + \theta U) = \delta$, hence $(\theta + k + 1)U = (-1)^k\, 1/k!\, \delta^{(k)} + \sum_{j=0}^{k-1} c_j\, \delta^{(j)}$ with suitable c_j. This is easily checked on monomials. Therefore $S = (-1)^k\, k!\, \big(U - \sum_{j=0}^{k-1} \frac{c_j}{k-j}\, \delta^{(j)}\big)$ is a solution. □

Let us finally remark that for $d = 1$, that is, for the case of an ordinary Euler differential operator $P(\theta)$ the proof of surjectivity can, by the fundamental Theorem of algebra, be reduced to the case $P(\theta) = \theta - a$ and be carried out in a much more direct way. The proof then shows that $P(\theta)$ is surjective also in the space $\mathscr{D}'(\mathbb{R})$ of Schwartz distributions. Whether this is true also for higher dimensions is not known and it is an open problem.

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