ON THE SPLITTING RELATION FOR
FRECHET-HILBERT SPACES

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Dedicated to the memory of Susanne Dierolf

Abstract

A shorter proof is given for a theorem of Domański and Mastyło characterizing the pairs \((E, F)\) of Fréchet-Hilbert spaces with the property that every exact sequence \(0 \rightarrow F \rightarrow G \rightarrow E \rightarrow 0\) of Fréchet-Hilbert spaces splits. The results on acyclicity of inductive spectra of metrizable locally convex spaces which we use are also presented with proofs.

In their paper [3] Domański and Mastyło showed that the splitting condition \((S)\) (see definition in Section 3) characterizes the pairs \((E, F)\) of Fréchet-Hilbert spaces with the property that every exact sequence

\[
0 \rightarrow F \rightarrow G \rightarrow E \rightarrow 0
\]

of Fréchet-Hilbert spaces splits. The crucial part had been to show the sufficiency of \((S)\) and this had been open for quite a while. In the present paper we give a simplified proof for the sufficiency of \((S)\).

It had been known that \((S)\) characterizes the pairs \((E, F)\) of Fréchet spaces such that every exact sequence (1) splits, under the assumption that one of the spaces is nuclear or both spaces are Köthe sequence spaces (for the latter see Krone-Vogt [6]). In the case of one of the spaces being nuclear, necessity of \((S)\) and sufficiency of a stronger splitting condition, which had been proposed by Apiola [1], had been shown in Vogt [13], sufficiency of \((S)\) in this case was shown in Frerick [4] and Frerick-Wengenroth [5]. In the present paper we use the equivalent but more elegant formulation of \((S)\) given by Langenbruch [7].

\[1\] 2000 Mathematics Subject Classification. Primary: 46A04 Secondary: 46M18, 46M40

Key words and phrases: Fréchet-Hilbert space, exact sequence, splitting condition, inductive spectrum, acyclic
The core of the present paper is Section 1, where we give a short proof of the basic result in [3]. The idea of the proof of Lemma 1.1 is similar to the proof of [3, Theorem 3.1] and based on ideas from Ovchinnikov [9]. We believe that also the subsequent presentation of the proof of the characterization deserves some interest. As we use the characterization of acyclicity of certain inductive spectra we include an excursion on this topic which might also be of interest.

In Section 2 we present the essential results, as far as we need them, on the characterization of acyclic and weakly acyclic inductive spectra of metrizable locally convex spaces with full proofs. The presentation is in the spirit of [14], [17] where a version of (P) appeared as a necessary condition. The first characterization of acyclicity in terms of a condition (M) had been given by Palamodov [10] and Retakh [12], it was essentially improved by Wengenroth [19]. His argument has in consequence also the sufficiency of (P) and appears here (in different language) in Lemma 2.2. Of course, this is directly connected to the proofs of the characterization of Proj$^1 = 0$ for a projective spectrum of (LB)-spaces as given in [5] and [2].

In Section 3 we give a direct and self contained proof of the sufficiency part of the splitting theorem in the Fréchet-Hilbert case, the part which had been open and was shown in [3], and in Section 4 we give a proof of the complete characterization of Fréchet-Hilbert splitting pairs. Here we don’t try to be self-contained.

We use standard terminology and results of the theory of locally convex spaces, see e. g. [8]. For the homological background see [20].

1 Fundamental lemma

A pair $(X_0, X_1)$ of Hilbert spaces which are both subspaces of a linear space $V$ we call a couple of Hilbert spaces. In particular, $X_0 \cap X_1$ is then well defined. The following is our version of [3, Theorem 3.1].

**Lemma 1.1** Let $s,t > 0$ and $(X_0, X_1), (Y_0, Y_1)$ be two couples of Hilbert spaces. On $X = X_0 \cap X_1$ and $Y = Y_0 \cap Y_1$ we assume to have Hilbert norms such that

$$\|y\|_Y \|x\|_X \leq s\|y\|_{Y_0} \|x\|_{X_0} + t\|y\|_{Y_1} \|x\|_{X_1}$$

for all $x \in X$, $y \in Y$. Then we have

$$\|u\|_{X \otimes Y} \leq s\|u\|_{Y_0 \otimes X_0} + t\|u\|_{Y_1 \otimes X_1}$$

for all $u \in Y \otimes X$. 


Proof. If \( u = \sum_{\nu=1}^{n} y_{\nu} \otimes x_{\nu} \) then we may assume that the \( y_{\nu} \) as well as the \( x_{\nu} \) are linearly independent. The various tensor norms can be calculated in \( \text{span}\{y_1, \ldots, y_n\} \otimes \text{span}\{x_1, \ldots, x_n\} \) equipped with the respective norms. Therefore we may assume that \( X = X_0 = X_1 \) and \( Y = Y_0 = Y_1 \), both of the same finite dimension.

\( u \) defines linear maps \( T_j = Y_j \rightarrow X_j, j = 0, 1 \) by

\[
T_j y = \sum_{\nu=1}^{n} \langle y, y_{\nu} \rangle Y_j x_{\nu}.
\]

By the above remarks we may assume \( T_j \) invertible.

Let \( T_j = U_j \circ |T_j| \) be the polar decomposition of \( T_j \). We set \( A_j = U_j \circ + \sqrt{|T_j|} \) and \( B_j = + \sqrt{|T_j|} \). On \( X \) we define scalar products by

\[
(x, y) := \langle A_j^{-1} x, A_j^{-1} y \rangle_{Y_j}
\]

with norm \( | \cdot |_j \) and we choose an orthonormal basis \( e_1, \ldots, e_n \) in \( X \) with respect to \( (\cdot, \cdot)_0 \) which is orthogonal with respect to \( (\cdot, \cdot)_1 \). We set \( e_\nu^0 = e_\nu \) and \( e_\nu^1 = \frac{1}{|e_\nu^0|} e_\nu \). Then we have

\[
T_j y = \sum_{\nu=1}^{n} (T_j y, e_\nu^j) e_\nu^j = \sum_{\nu=1}^{n} \langle B_j y, A_j^{-1} e_\nu^j \rangle_{Y_j} e_\nu^j = \sum_{\nu=1}^{n} \langle y, B_j A_j^{-1} e_\nu^j \rangle_{Y_j} e_\nu^j
\]

for all \( y \in Y \) and therefore

\[
u = \sum_{\nu=1}^{n} (B_0 A_0^{-1} e_\nu) \otimes e_\nu = \sum_{\nu=1}^{n} \left( \frac{1}{|e_\nu^0|^2} B_1 A_1^{-1} e_\nu \right) \otimes e_\nu.
\]

This implies \( B_0 A_0^{-1} e_\nu = \frac{1}{|e_\nu^0|^2} B_1 A_1^{-1} e_\nu =: g_\nu \) for all \( \nu \).

By use of our assumption we have

\[
\|u\|_{X \otimes Y} \leq \sum_{\nu=1}^{n} \|g_\nu\|_Y \|e_\nu\|_X
\]

\[
\leq s \sum_{\nu=1}^{n} \|g_\nu\|_Y \|e_\nu\|_{X_0} + t \sum_{\nu=1}^{n} \|g_\nu\|_Y \|e_\nu\|_{X_1}.
\]

We set \( f_\nu^j = A_j^{-1} e_\nu^j \). Then \( f_1^j, \ldots, f_n^j \) is an orthonormal basis of \( Y_j \) and we obtain, using that \( U_j \) is unitary,

\[
\sum_{\nu=1}^{n} \|g_\nu\|_{Y_j} \|e_\nu\|_{X_j} = \sum_{\nu=1}^{n} \|B_j f_\nu^j\|_{Y_j} \|A_j f_\nu^j\|_{X_j}
\]

\[
= \sum_{\nu=1}^{n} \|B_j f_\nu^j\|_{Y_j}^2 = \sigma(B_j)^2 = \nu(T_j)
\]

\[
= \|u\|_{Y_j \otimes X_j}.
\]

Here \( \sigma(\cdot) \) denotes the Hilbert-Schmidt norm and \( \nu(\cdot) \) denotes the nuclear norm. \( \square \)
2 Acyclic inductive spectra

Let $X$ be linear space, $X_1 \subset X_2 \subset X_3 \subset \ldots$ an increasing sequence of linear spaces with $\bigcup_{n=1}^{\infty} X_n = X$. We assume that every $X_n$ is equipped with an increasing sequence $\| \|_{n,k}, k \in \mathbb{N}$ of seminorms, such that $\| \|_{n,k} \geq \| \|_{n+1,k}$ for all $n, k$. Without restriction of generality we may assume $2\| \|_{n,k} \leq \| \|_{n,k+1}$ for all $n, k$.

We define a map $\tau : \bigoplus_{n=1}^{\infty} X_n \longrightarrow \bigoplus_{n=1}^{\infty} X_n$ by $\tau : (u_n)_{n \in \mathbb{N}} \mapsto (u_n - u_{n-1})_{n \in \mathbb{N}}$, where $u_0 := 0$. Then we have $R(\tau) = \{(v_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} v_n = 0\}$, the sums, of course, being finite. Obviously $\tau$ is injective. Due to our assumptions, the seminorms of the form $\sum_{n=1}^{\infty} \| u_n \|_{n,m(n)}$ are a fundamental system of seminorms on $\bigoplus_{n=1}^{\infty} X_n$.

The inductive spectrum $\mathcal{X} : X_1 \subset X_2 \subset X_3 \subset \ldots$ is called acyclic $\tau$ is topologically injective, i.e. $\tau^{-1} : R(\tau) \longrightarrow \bigoplus_{n=1}^{\infty} X_n$ is continuous. It is called weakly acyclic if $\tau^{-1}$ is weakly continuous. If $\mathcal{X}$ is (weakly) acyclic then also every equivalent spectrum is (weakly) acyclic. The spectrum $\tilde{X}_1 \subset \tilde{X}_2 \subset \tilde{X}_3 \subset \ldots$ is equivalent, if for every $n$ there is $N$ such that $X_n \subset X_N$ and $\tilde{X}_n \subset X_N$ with continuous imbeddings.

**Definition.** $\mathcal{X}$ satisfies condition (P) if the following holds:

$$\forall \mu \exists k \forall K \exists \nu \forall m, \varepsilon > 0 \exists N, S \forall u \in X_\mu :$$

$$\|u\|_{k,m} \leq \varepsilon \|u\|_{\mu,\nu} + S \|u\|_{N,N}.$$ 

We wish to show that (P) is equivalent to acyclicity. While necessity of (P) is relatively easy the crucial point is sufficiency of (P).

Let (P) be satisfied. First we replace $\| \cdot \|_{n,k}$ by $2^k \| \cdot \|_{n,k}$. Then, by inductively choosing subsequences of the $n$ and $k$, we get an equivalent inductive spectrum $\tilde{\mathcal{X}} = (X_{n,\nu})_{\nu \in \mathbb{N}}$ which satisfies the following assumptions:

1. $2\| \cdot \|_{n,m} \leq \| \cdot \|_{n,m+1}$ for all $n$ and $m$

2. For every $n > 1, m \in \mathbb{N}$ and $\varepsilon > 0$ there is $N = N(n, m, \varepsilon)$ with

$$\|u\|_{n,m} \leq \varepsilon (\|u\|_{n-1,n-1} + \|u\|_{n+1,N})$$

for all $u \in X_{n-1}$.

3. $N(n, n, 1/4) = n + 1$.

We will use the following notation: Let $\| \cdot \|_j, j = 1, \ldots, m$ be seminorms on subspaces $X_j$ of a linear space $X$ then we set on $X_1 + \cdots + X_m$

$$\bigwedge_{j=1}^{m} \|u\|_j := \inf \left\{ \sum_{j=1}^{m} \|u_j\|_j : u_j \in X_j \text{ for all } j \text{ and } u = \sum_{j=1}^{m} u_j \right\}.$$
We will need the following simple lemma:

**Lemma 2.1** Let \( \| \|_1, \| \|_2 \leq \| \|_2, \| \|_3 \) be seminorms on linear spaces \( X_1 \subset X_2 \subset X_3 \), with \( \| \|_2' \leq \| \|_1 + \| \|_3 \) on \( X_1 \) and \( \| \|_3 \leq \| \|_2 \) on \( X_2 \). Then we have

\[
\| \|_2' \leq 2 \bigwedge_{j=1}^{2} \| \|_j + \| \|_3
\]
on \( X_2 \).

**Proof.** For \( u = u_1 + u_2, \ u_j \in X_j \), we obtain

\[
\| u \|_2' \leq \| u_2 \|_2' + \| u_1 \|_1 + \| u_1 \|_3 \\
\leq \| u_2 \|_i + \| u_1 \|_1 + \| u \|_3 + \| u_2 \|_3 \\
\leq 2(\| u_1 \|_1 + \| u_2 \|_2) + \| u \|_3.
\]

For the first inequality we used the first assumption, for the third one \( \| u_2 \|_2 \leq \| u_2 \|_2 \) and the second assumption. \( \square \)

**Lemma 2.2** Let \( X \) satisfy assumptions 1., 2., 3. above. Then for every \( n > 1, m \in \mathbb{N} \) and \( \varepsilon > 0 \) there is \( M = M(n, m, \varepsilon) \) with

\[
\| u \|_{n,m} \leq \varepsilon(\bigwedge_{k=1}^{n-1} \| u \|_{k,k+1} + \| u \|_{n+1,n})
\]

for all \( u \in X_{n-1} \). Moreover we may choose \( M(n, n, 1) = n + 1 \).

**Proof.** We proceed by induction on \( n \). For \( n = 2 \) this follows from the assumption.

So we assume the assertion for \( n - 1 \) and this implies

\[
\| u \|_{n-1,n-1} \leq \bigwedge_{k=1}^{n-2} \| u \|_{k,k+1} + \| u \|_{n,n}, \quad u \in X_{n-2}.
\]

From Lemma 2.1 we obtain

\[
\| u \|_{n-1,n-1} \leq 2 \bigwedge_{k=1}^{n-1} \| u \|_{k,k+1} + \| u \|_{n,n}, \quad u \in X_{n-1}.
\]

For given \( m \geq n \) and \( \varepsilon \leq 1/2 \) we insert this in the inequality 2. This yields

\[
\| u \|_{n,m} \leq \varepsilon(2 \bigwedge_{k=1}^{n-1} \| u \|_{k,k+1} + \| u \|_{n,n} + \| u \|_{n+1,n}), \quad u \in X_{n-1}.
\]
Putting $\varepsilon \|u\|_{n,n} \leq \frac{1}{2} \|u\|_{n,m}$ to the left and multiplying by 2 we obtain
\[
\|u\|_{n,m} \leq 4\varepsilon \bigg(\sum_{k=1}^{n-1} \|u\|_{k,k+1} + 2\varepsilon \|u\|_{n+1,N}\bigg), \quad u \in X_{n-1}.
\]
Since $0 < \varepsilon \leq 1/2$ was arbitrary this completes the induction in the general case. The special case then follows from $N(n, n, 1/4) = n + 1$. \qed

**Theorem 2.3** $\mathcal{X}$ is acyclic if and only if it satisfies (P).

**Proof.** First we prove sufficiency of (P). We may assume $\mathcal{X}$ to be of the special form as before. We have to show that $\tau^{-1} : R(\tau) \rightarrow \bigoplus_{k \in \mathbb{N}} X_k$ is continuous. Since $\tau^{-1}(u_k)_{k \in \mathbb{N}} = (\sum_{k=1}^{n} u_k)_{n \in \mathbb{N}}$, this means that for every sequence $(m(n))_{n \in \mathbb{N}}$ we should find a sequence $(M(n))_{n \in \mathbb{N}}$ such that
\[
\sum_{n=1}^{\infty} \sum_{k=0}^{n} \|u_k\|_{n,m(n)} \leq \sum_{n=1}^{\infty} \|u_n\|_{n,M(n)}
\]
for all $(u_n)_{n \in \mathbb{N}} \in R(\tau)$.

So let $u_n \in X_n$ be a sequence, $m \in \mathbb{N}$ and $M = M(n, m, \varepsilon)$ as in Lemma 2.2 which we will use for the second estimate of the following. We may assume $M \geq m$.

\[
\|\sum_{k=0}^{n} u_k\|_{n,m} \leq \|u_n\|_{n,m} + \|\sum_{k=0}^{n-1} u_k\|_{n,m}
\]
\[
\leq \|u_n\|_{n,m} + \varepsilon \left(\sum_{k=1}^{n-1} \|u_k\|_{k,k+1} + \sum_{k=0}^{n-1} \|u_k\|_{n,M}\right)
\]
\[
\leq \|u_n\|_{n,m} + \varepsilon \left(\sum_{k=1}^{n-1} \|u_k\|_{k,k+1} + \|u_n + u_{n+1}\|_{n+1,M} + \sum_{k=0}^{n+1} \|u_k\|_{n+1,M}\right)
\]
\[
\leq 2\|u_n\|_{n,M} + \|u_{n+1}\|_{n+1,M} + \varepsilon \sum_{k=1}^{n-1} \|u_k\|_{k,k+1} + \varepsilon \sum_{k=0}^{n+1} \|u_k\|_{n+1,M}.
\]

We use this to choose inductively a strictly increasing sequence $M(n) \geq
$m(n) + 1$ of integers, such that

$$\left\| \sum_{k=0}^{n} u_k \right\|_{n,M(n)} \leq 2\left\| u_n \right\|_{n,M(n+1)} + \left\| u_{n+1} \right\|_{n+1,M(n+1)} +$$

$$+ 2^{-n} \sum_{k=1}^{n-1} \left\| u_k \right\|_{k,k+1} + 1/2 \sum_{k=0}^{n+1} \left\| u_k \right\|_{n+1,M(n+1)}.$$ 

Summing up over $n$ we obtain

$$\sum_{n=1}^{\infty} \left\| \sum_{k=0}^{n} u_k \right\|_{n,M(n)} \leq 3 \sum_{n=1}^{\infty} \left\| u_n \right\|_{n,M(n+1)} + \sum_{k=1}^{\infty} \left\| u_k \right\|_{k,k+1} +$$

$$+ 1/2 \sum_{n=1}^{\infty} \sum_{k=0}^{n+1} \left\| u_k \right\|_{n+1,M(n+1)}.$$ 

Putting the last term to the left and multiplying by 2 we get

$$\sum_{n=1}^{\infty} \left\| \sum_{k=0}^{n} u_k \right\|_{n,m(n)} \leq \sum_{n=1}^{\infty} \sum_{k=0}^{n} \left\| u_k \right\|_{n,M(n)} \leq 8 \sum_{n=1}^{\infty} \left\| u_n \right\|_{n,M(n+1)}$$

which completes the proof of sufficiency.

To prove necessity of (P) we assume that (P) does not hold. Then there exists $\mu$, such that for all $k$ there are $m(k), K(k) \geq k, \varepsilon(k) > 0$ such that for every $N, S > 0$ there is $u \in X_{\mu}$ with

$$\left\| u \right\|_{k,m(k)} \geq \varepsilon(k) \left\| u \right\|_{\mu,k} + S \left\| u \right\|_{K(k),N}.$$ 

Continuity of $\tau^{-1}$ yields sequences $n(k)$ and $D(k)$ such that

$$\sum_{k=1}^{\infty} \frac{k}{\varepsilon(k)} \left\| u_k \right\|_{k,m(k)} \leq \sum_{k=1}^{\infty} D(k) \left\| u_k - u_{k-1} \right\|_{k,n(k)}.$$ 

For $u \in X_{\mu}$ we set $u_{\mu} = \cdots = u_{K(k)} = u$, $u_k = 0$ otherwise and obtain

$$k \left\| u \right\|_{k,m(k)} \leq \varepsilon(k) D(\mu) \left\| u \right\|_{\mu,n(\mu)} + \varepsilon(k) D(K(k)) \left\| u \right\|_{K(k),n(k)}.$$ 

for all $u \in X_{\mu}$ and $k \geq \mu$. We choose $k$ so large that $k \geq D(\mu), k \geq n(\mu)$ and obtain a contradiction.

**Remark.** In fact we showed even necessity of the following condition, which is formally stronger than (P):

$$\forall \mu \exists k, \nu \exists m, K, \varepsilon > 0 \exists N, S \forall u \in X_{\mu}^*: \left\| u \right\|_{k,m} \leq \varepsilon \left\| u \right\|_{\mu,\nu} + S \left\| u \right\|_{K,N}.$$
This means, of course, that both conditions are equivalent. In fact, the
quantifiers in the stronger version of \((P)\) are those of the earlier version \((P^*_2)\)
in [14, 17]. The version we are using in our definition is due to Langenbruch
[7].

**Lemma 2.4** As the proof of necessity shows, to show \((P)\) it is enough that
\(\tau^{-1}\) is continuous on every countably dimensional subspace of \(X\).

For the case of weak acyclicity we will need only the necessity part.

**Proposition 2.5** If \(X\) is weakly acyclic then it satisfies the following condi-
tion \((WP)\):

\[
\forall \mu \exists k \forall K \exists \nu \forall m \exists N, S \forall u \in X_\mu : \|u\|_{k,m} \leq S(\|u\|_{\mu,\nu} + \|u\|_{K,N}).
\]

**Proof.** In precisely the same way as in the necessity proof of Theorem 2.3
(write down contraposition, apply to same element as in this proof, will
contradict continuity of \(y\) with respect to topology induced by \(\bigoplus_{k=1}^{\infty} X_k\))
we show

\[
\forall \mu \exists k \forall K \exists \nu \forall y \in X_k^* \exists N, S \forall u \in X_\mu : |y(u)| \leq S(\|u\|_{\mu,\nu} + \|u\|_{K,N}).
\]

The result then follows by a simple application of Baire’s theorem to the
canonical Banach subspaces of \(X_k^*\). \(\square\)

### 3 Splitting theorem, rapid access

Let \(E\) and \(F\) be Fréchet-Hilbert spaces, \(\|\|_1 \leq \|\|_2 \leq \ldots\) fundamental
systems of Hilbert-seminorms and \(E_k, F_k\) the respective local Hilbert spaces
with linking maps \(j_{k+1}^k : F_{k+1} \rightarrow F_k\) (in the case of \(F\)).

We set \(X_n = E \otimes_\pi F_n^*\). Then \(X_n\) is locally convex, metrizable with the
seminorms \(\|\|_{n,k} = \|\|_{E_k \otimes F_n^*}\). We obtain an inductive spectrum \(\mathcal{X}\).

**Definition.** \((E, F)\) satisfies condition \((S)\) if the following holds:

\[
\forall \mu \exists k \forall K \exists \nu \forall m, \varepsilon > 0 \exists N, S \forall x \in E, y \in F_\mu^* : \|x\|_m \|y\|_K^* \leq \varepsilon \|x\|_\nu \|y\|_\mu^* + S \|x\|_N \|y\|_K^*.
\]

**Lemma 3.1** \(\mathcal{X}\) satisfies \((P)\) if and only if \((E, F)\) satisfy \((S)\).
Proof. The if -part is an immediate consequence of Lemma 1.1, the only if -part is trivial. □

We define a linear map $\sigma : \prod_n L(E, F_n) \rightarrow \prod_n L(E, F_n)$ by $\sigma(\varphi_n)_{n \in \mathbb{N}} = (j_{n+1}^n \circ \varphi_{n+1} - \varphi_n)_{n \in \mathbb{N}}$ and observe that $\sigma = \tau^*$ where $\tau$ is the map defined in the previous section applied to $X$ as defined before. Here we identified $L(E, F_n) = (E \otimes \pi F_n^*)^*$. From Lemma 3.1 and Theorem 2.3 we obtain:

**Lemma 3.2** If $(E, F)$ satisfy (S) then $\sigma$ is surjective.

A standard argument now leads to our main result, the sufficiency part of Theorem 4.2 in [3].

**Theorem 3.3** If $(E, F)$ satisfy (S), then every exact sequence

$$0 \rightarrow F \rightarrow G \rightarrow^q E \rightarrow 0$$

of Fréchet-Hilbert spaces splits.

Proof. We may assume $F = \ker q$ and choose a fundamental system of hilbertian seminorms on $G$. Then the trace, resp. quotient seminorms define a fundamental system of hilbertian seminorms on $F$ and $E$ and we get for every $n$ an exact sequence

$$0 \rightarrow F_n \rightarrow G_n \rightarrow^m E_n \rightarrow 0$$

of Hilbert spaces which splits. Again we may assume $F_n = \ker q_n$. Let $r_n$ be a right inverse for $q_n$ then $a_n := j_{n+1}^n \circ q_{n+1} - q_n$ defines a map in $L(E, F_n)$. By Lemma 3.2 we get a sequence $\varphi_n \in L(E, F_n)$, $n \in \mathbb{N}$, with $a_n = j_{n+1}^n \circ \varphi_{n+1} - \varphi_n$ for every $n$. Then $R_n = r_n - \varphi_n$ satisfies $j_{n+1}^n \circ R_{n+1} = R_n$ for every $n$, hence defines a map $R \in L(E, G)$ which is easily seen to be a right inverse for $q$. □

### 4 Splitting theorem, complete version

To obtain the complete characterization of Fréchet-Hilbert splitting pairs we need analogous to (WP) the weaker version of (S).

**Definition.** $(E, F)$ satisfies condition (WS) if the following holds:

$$\forall \mu \exists k \forall K \exists \nu \forall m \exists N, S \forall x \in E, y \in F_n^*: $$

$$||x||_m ||y||_{K}^* \leq S(||x||_\nu ||y||_{\mu}^* + ||x||_N ||y||_{K}^*).$$

We first state the well known fact that in most relevant cases (S) and (WS) are equivalent.
Proposition 4.1 If $E$ is a proper (not normable) Fréchet space then (S) is equivalent to (WS).

We consider exact sequences of Fréchet-Hilbert spaces

$$0 \longrightarrow F \longrightarrow G \xrightarrow{q} H \longrightarrow 0$$

and assume that $E$ is an infinite dimensional Fréchet-Hilbert space. Then we ask when the induced sequence

$$0 \longrightarrow L(E, F) \longrightarrow L(E, G) \xrightarrow{q_*} L(E, H) \longrightarrow 0$$

is exact which means, when is $q_*$ is surjective.

The equivalence $1 \iff 2.$ in the following Theorem is Theorem 4.2 in [3].

Theorem 4.2 The following are equivalent:

1. $(E, F)$ satisfies (S).
2. Every exact sequence (1) of Fréchet-Hilbert spaces splits.
3. For any exact sequence (3) the exact sequence (4) is exact.
4. for any exact sequence (3) every equicontinuous set in $L(E, H)$ can be lifted in (4) to an equicontinuous set in $L(E, G)$.

Proof. $1. \Rightarrow 4.$: By Lemma 3.1 we get condition (P) for $X$, defined as in Section 3. The argument leading to Lemma 3.2 also shows that $\sigma$ lifts products of equicontinuous sets in $L(E, F_n)$ into products of equicontinuous sets. In the proof of Theorem 3.3 an equicontinuous set in $L(E, F)$ leads to a product of equicontinuous sets in $L(E, F_n)$.

$4. \Rightarrow 3.$ and $3. \Rightarrow 2.$ are obvious. $2. \Rightarrow 3.$ is well known.

3. $\Rightarrow 1.$ 3. implies that $X$ is weakly acyclic and therefore, by Proposition 2.5, satisfies (WP) which implies (WS). If $E$ is a proper Fréchet space then Proposition 4.1 gives the result. If $E$ is a Hilbert space then, due to Lemma 2.4, we may assume $E$ and $F$ to be separable. As a sample sequence (3) we use the canonical resolution

$$0 \longrightarrow F \longrightarrow \prod_n F_n \xrightarrow{\sigma} \prod_n F_n \longrightarrow 0.$$ 

Any bounded set in $\prod_n F_n$ is contained in a Hilbert ball $B$ which generates a separable Hilbert space $H_B$. There is an isomorphism from $E$ onto $H_B$ which can be lifted under $\sigma$, hence $B$ can be lifted under $\sigma$ to a bounded set. This means that the spectrum $F_1^* \subset F_2^* \subset \ldots$ is acyclic and the necessity part of Theorem 2.3 yields the result. Notice that in (S) we have $\|x\|_m = \|x\|_\nu = \|x\|_N$. □
References


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