Complemented Ideals in $A(\mathbb{R}^d)$ of Algebraic Curves

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Dedicated to Professor Heinz König on the occasion of his 80th birthday

Abstract. It is shown that for an algebraic curve $X \subset \mathbb{R}^d$ the ideal of real analytic functions vanishing on $X$ is complemented in $A(\mathbb{R}^d)$ if and only if in every $a \in X$ every irreducible component of the germ $X_a$ is either regular or a point.

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In [4, 10, 11] the problem is studied under which conditions on a real analytic subvariety $X \subset \mathbb{R}^d$ its ideal $J_X(\mathbb{R}^d)$ in the topological algebra $A(\mathbb{R}^d)$ of real analytic functions on $\mathbb{R}^d$ is complemented or, which turns out to be equivalent, under which conditions there exists a continuous linear extension operator from $A(X)$ to $A(\mathbb{R}^d)$. It is shown there that a necessary condition is that $X$ is of type PL, which is a condition on the singularities, and this condition is sufficient if $X$ is either compact or homogeneous. In the present paper we show that PL is also sufficient in the case, when $X$ is an algebraic curve. In this case, by a result of Braun, Meise and Taylor, type PL means that the only singularities are isolated points or self-intersections. So we obtain a complete solution in terms of a purely geometrical condition. In particular it yields a complete solution for all algebraic subvarieties of $\mathbb{R}^2$ by a purely geometrical condition.

1. Preliminaries

Let $X \subset \mathbb{R}^d$ be a real analytic variety and $J_X(\mathbb{R}^d) = \{f \in A(\mathbb{R}^d) : f|_X = 0\}$. $X$ is called $C$-analytic if it is the common zero set of finitely many functions in $A(\mathbb{R}^d)$. We have to recall some definitions. Let $V_a$ be the germ of a complex variety in the real point $a$. We identify $V_a$ with a bounded representative and and set $X_a = V_a \cap \mathbb{R}^d$. For any function $f$ on a set $M$ we set $\|f\|_M = \sup_{x \in M} |f(x)|$. 
Definition 1.1. $V_a$ satisfies $PL_{loc}$ if there is a constant $A > 0$ and a neighborhood $W_a$ of $a$ in $V_a$ such that for all holomorphic functions $f$ on $V_a$ and all $z \in W_a$

$$|f(z)| \leq \|f\|_{V_a}^{A|\text{Im}z|} \|f\|_{X_a}^{1-A|\text{Im}z|}.$$ 

This is equivalent to the following condition, due to Hörmander [7]: there is a constant $A > 0$ and a neighborhood $W_a$ of $a$ in $V_a$ such that for all plurisubharmonic functions $u$ on $V_a$ with $u \leq 1$ on $V_a$ and $u \leq 0$ on $X_a$ we have $u(z) \leq A|\text{Im}z|$ for all $z \in W_a$.

One direction of the equivalence is seen by applying the second condition to

$$u(z) = \frac{\log |f(z)| - \log \|f\|_{V_a}}{\log \|f\|_{V_a} - \log \|f\|_{X_a}}.$$ 

For the other direction one observes that the first condition implies the second one for functions $u(z) = \log |f(z)|$, $f$ holomorphic on $V_a$. The transfer to general plurisubharmonic functions is then standard (see the references below).

Let $X_a$ be the germ of a real analytic variety in the real point $a$. We call $J_{X_a}$ the ideal generated by $X_a$ in $O_a$. The germ $\tilde{X}_a := \{z : f(z) = 0\}$ for all $f \in J_{X_a}$ is called the (local) complexification of $X_a$. For its existence remember that $J_{X_a}$ is finitely generated.

A complex variety $V$ satisfies $PL_{loc}$ in the real point $a$ if its germ in $a$ satisfies $PL_{loc}$. The germ $X_a$ of a real analytic variety in the real point $a$ is of type $PL$ if its complexification satisfies $PL_{loc}$. $X$ is of type $PL$ if it is of type $PL$ in every point.

We will use the following result of Braun, Meise and Taylor:

Lemma 1.2 ([2], Proposition 3.16). A purely 1-dimensional irreducible germ is of type $PL$ if and only if it is regular.

For more information concerning varieties of type $PL$ see [2, 4, 7, 10, 11], for the topology of $A(\mathbb{R}^d)$ and $A(X)$ see [8] and also the previously mentioned papers. For real analytic and algebraic varieties see [1, 3, 9, 12].

2. Real analytic covers

Let $p < d$ and $U \subset \mathbb{R}^p$ be open and $\pi : X \ni x = (x_1, \ldots, x_d) \longrightarrow x' = (x_1, \ldots, x_p)$ the canonical projection. We call $X$ a locally bianalytic cover of $U$, if $\pi : X \cap \pi^{-1}U \longrightarrow U$ is surjective, proper, locally bianalytic and $\pi^{-1}x$ is finite for every $x \in U$. For $x \in \mathbb{R}^d$ we will always set $x = (x', x'')$, $x' = (x_1, \ldots, x_p)$, $x'' = (x_{p+1}, \ldots, x_d)$.

Lemma 2.1. If $X$ is a locally bianalytic cover of $U$, then the interpolation formula

$$\varphi^* f(x) := \sum_{\xi \in \pi^{-1}x'} f(\xi) \prod_{\eta \in \pi^{-1}x', \eta \neq \xi} \frac{|x'' - \eta|^2}{|\xi - \eta|^2}$$

defines a continuous linear extension operator $\varphi^* : A(X) \rightarrow A(U \times \mathbb{R}^{d-p})$. 
We want to study varieties which are real analytic covers outside a compact set. For this we need some preparation. For $0 < r < \rho \leq \infty$ we put $D^r = \{ x \in \mathbb{R}^p : r \leq \| x \| \leq \rho \}$ and $D^{r,s} = \{ x \in \mathbb{R}^d : r \leq \| x' \| \leq \rho, \| x'' \| \leq s \}$. Here $\| \|$ denotes the euclidian norm.

For compact $K \subset \mathbb{R}^d$ we denote by $H(K)$ the space of germs of holomorphic functions on $K$ with its natural (LB)-topology. It is the inductive limit of the spaces $H^\infty(U_n)$ where the $U_n$ runs through an open complex neighborhood basis of $K$. The space $\Lambda(\mathbb{R}^d)$ is then the projective limit of the spaces $H(K_n)$ where the $K_n$ are a compact exhaustion of $\mathbb{R}^d$. We use [11, Lemma 6.2] which we first quote:

**Lemma 2.2.** For any $0 < r < \rho$ there are $\sigma_1$, $\sigma_2$ with $0 < \sigma_1 < r < \rho < \sigma_2$ and continuous linear maps $\psi_0 : H(D^{r,r}_0) \to H(D^{r,r}_0)$ and $\psi_\infty : H(D^{r,r}_\infty) \to H(D^{r,r}_\infty)$ so that $\psi_0 f + \psi_\infty f = f$ on $D^{r,r}$.

**Corollary 2.3.** For every $\sigma_1 > 0$ there are $0 < \sigma_1 < r < \rho < \sigma_2$ such that the assertion of Lemma 2.2 holds.

**Proof:** If for $0 < \sigma_1 < r < \rho < \sigma_2$ and $\alpha > 0$ the assertion of Lemma 2.2 holds then it holds also for $0 < \alpha \sigma_1 < \alpha r < \alpha \rho < \alpha \sigma_2$, which implies our result. □

By a simple tensor argument we obtain now:

**Lemma 2.4.** For every $R > 0$ there are $0 < R < r < s < S$ and continuous linear maps $\Psi_0 : H(D^{S,\sigma}_R) \to H(D^{S,\sigma}_0)$ and $\Psi_\infty : H(D^{S,\sigma}_\infty) \to H(D^{S,\sigma}_\infty)$ so that $\Psi_0 f + \Psi_\infty f = f$ on $D^{S,\sigma}$.

For the following lemma we assume that $X \subset \{ x \in \mathbb{R}^d : |x'| < \sigma \}$.

**Lemma 2.5.** For any $0 < R < S$ and $0 < \sigma$ there are $\sigma_1$, $\sigma_2$ and $\tau$ with $0 < \sigma_1 < R < S < \sigma_2$, $\sigma < \tau$ and there exist finitely many functions $f_1, \ldots, f_m \in J_X(\mathbb{R}^d)$ and a continuous linear map $\chi : J_X(D^{S,\tau}_R) \to H(D^{S,\sigma}_R)^m$ such that with $\chi(f) = \sum_{j=1}^m g_j f_j = f$ on $D^{S,\sigma}_R$. $\sigma_1$ can be chosen large for large $R < S$.

**Proof:** We proceed like in the proof of [11, Lemma 6.2]. The diagram which we have to handle is in our situation

$$
\begin{array}{cccc}
0 & \longrightarrow & \Gamma(D^r, \mathcal{F}) & \longrightarrow & H(D^r) \times \longrightarrow & J_X(D^r) & \longrightarrow & 0 \\
& & & \uparrow & & \\
& & & J_X(D^S) & & \\
\end{array}
$$

where $\mathcal{F}$ denotes the ‘kernel sheaf’ of the sheaf-map $\sum_{j=1}^m g_j f_j$. Since this is a coherent sheaf on a holomorphically convex neighborhood of $\mathbb{R}^d$ the sections on $D_r^r$ are again finitely generated and we can represent $\Gamma(D^r, \mathcal{F})$ as a quotient of $H(D^r)^M$ with suitable $M$. Therefore we are led to the situation in the proof of [11, Lemma 6.2]. □

From [11, Theorem 8.1] we obtain the following lemma:
Lemma 2.6. If \( X \) is of type PL then there exists for every \( S > 0 \) and \( \sigma > 0 \) a continuous linear extension operator \( \varphi_* : A(X) \to H(D^{S,\sigma}_X) \).

**Definition** We call \( X \) regular over \( \mathbb{R}^p \) near infinity if

1. \( \pi : X \to \mathbb{R}^p \) is proper,
2. there is \( r_0 \) such that for \( U = \mathbb{R}^p \setminus B_{r_0} \) (for any component \( U \) of \( \mathbb{R}^p \setminus B_{r_0} \) if \( p = 1 \)) \( X \cap (U \times \mathbb{R}^{d-p}) \) is either empty or a locally bianalytic cover of \( U \).

**Proposition 2.7.** Let \( X \) be regular over \( \mathbb{R}^p \) near infinity and \( X \subset \{ x \in \mathbb{R}^d : |x^0| < \sigma \} \). If \( X \) is of type PL then there is a continuous linear extension operator \( A(X) \to H(D^{\infty,\sigma}_X) \).

**Proof:** The case of compact \( X \) is shown in [10] therefore we may assume that \( X \) is not compact. We first assume \( p > 1 \), then \( U = \mathbb{R}^p \setminus B_{r_0} \) is connected and \( X \cap (U \times \mathbb{R}^{d-p}) \) is an analytic cover of \( U \).

We choose \( 0 < r < s \) with \( r \) so large that for \( 0 < R < r < s < S \) chosen according to Lemma 2.4 and \( 0 < \sigma_1 < R < S < \sigma_2 \) chosen according to Lemma 2.5 we have \( 0 < r_0 < \sigma_1 < R < r < s < S < \sigma_2 \). We choose \( \tau > \sigma \) according to Lemma 2.5.

By Lemma 2.1 there is a continuous linear extension operator \( \varphi_* : A(X) \to A(U \times \mathbb{R}^{d-p}) \). Since \( X \) is of type PL there is, by Lemma 2.6, a continuous linear extension operator \( \varphi_* : A(X) \to H(D^{S,\sigma}_X) \). We set \( \Phi = \varphi^* - \varphi_* : A(X) \to J_X(D^{S,\sigma}_X) \).

By Lemma 2.5 we find finitely many function \( f_1, \ldots, f_m \in J_X(\mathbb{R}^d) \) and a continuous linear map \( \chi : J_X(D^{S,\sigma}_X) \to H(D^{\infty,\sigma}_X) \) such that with \( \chi(f) = (g_1, \ldots, g_m) \) we have \( \sum_{j=1}^m g_j f_j = f \) on \( D^{S,\sigma}_X \). From Lemma 2.4 we get continuous linear maps \( \Psi_0 : H(D^{S,\sigma}_X) \to H(D^{\infty,\sigma}_X) \) and \( \Psi_\infty : H(D^{\infty,\sigma}_X) \to H(D^{\infty,\sigma}_X) \) so that \( \Psi_0 f + \Psi_\infty f = f \) on \( D^{S,\sigma}_X \).

We put for \( f \in A(X) \) with \( \chi \circ \Phi(f) =: (g_1, \ldots, g_m) \)

\[
\varphi f = \varphi_* f + \sum_{j=1}^m f_j \Psi_0(g_j) \text{ on } D^{S,\sigma}_X \text{ and } \varphi f = \varphi^* f - \sum_{j=1}^m f_j \Psi_\infty(g_j) \text{ on } D^{\infty,\sigma}_X.
\]

On \( D^{S,\sigma}_X \) we have

\[
\sum_{j=1}^m f_j \Psi_0(g_j) + \sum_{j=1}^m f_j \Psi_\infty(g_j) = \sum_{j=1}^m f_j g_j = \Phi(f) = \varphi^* f - \varphi_* f.
\]

Therefore \( \varphi \) is well defined and the assertion is proved.

For \( p = 1 \) we have either that \( X \) is an analytic cover of \( \mathbb{R} \setminus B_R \) for some \( R > 0 \) then the proof as before applies, or \( X \) is an analytic cover of one of the components of \( \mathbb{R} \setminus B_R \), that means one of the half-lines. In any case it remains true that the interpolation formula (or also Langrange interpolation) on the nontrivial component and the zero-map on the trivial one gives a continuous linear extension operator \( \varphi_* : A(X) \to A(\mathbb{R}^d) \) and we may continue as in the proof for \( p > 1 \). \( \square \)

**Theorem 2.8.** Let \( X \) be regular over \( \mathbb{R}^p \) near infinity. If \( X \) is of type PL then there is a continuous linear extension operator \( A(X) \to A(\mathbb{R}^d) \).
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Proof: Since $\pi$ is proper the proof is by use of the transformation $x \mapsto (x', \arctan x_{p+1}, \ldots, \arctan x_d)$ reduced to Proposition 2.7 with $\sigma = \frac{\sqrt{d-p}}{2}$. □

3. Algebraic curves

Let now $V$ be an algebraic subvariety of $\mathbb{C}^d$. This means that $V$ is the common zero set of finitely many polynomials in $\mathbb{C}[z_1, \ldots, z_d]$. We call it an algebraic curve, if it is purely one-dimensional, and then we call $X = \mathbb{R}^d \cap V$ a real algebraic curve. The following Lemma is probably well known to specialists. We give a proof for the convenience of the reader. As previously we set $\pi(z_1, \ldots, z_d) = (z_1, \ldots, z_p)$.

Lemma 3.1. Let $V$ be a purely $p$-dimensional algebraic subvariety of $\mathbb{C}^d$, then after a real coordinate transformation $\pi : V \to \mathbb{C}^p$ is an analytic cover.

Proof: This follows from an analysis of the proof of the Normalization Theorem [1, A.1.1]. We first remark the for any finite set of polynomials one can find a common real non-characteristic vector. Let $p$ be the ideal of $V$ in $\mathbb{C}[z_1, \ldots, z_d]$ and $p = p_1 \cdots p_m$ its decomposition into prime ideals which corresponds to the decomposition of $V$ into irreducible components $V_j, j = 1, \ldots, m$. Then the subsequent choices of vectors in the proof of part (2) of [1, A.1.1] on p. 118 can be done real and simultaneously for all $p_j$ and, since $V$ is purely $p$-dimensional, the procedure stops at the same step for all $p_j$. Therefore the relevant coordinate transformation in the proof of Theorem [1, A.1.1] can be chosen real and simultaneously for all $p_j$, which implies that after some real coordinate transformation $\pi : V_j \to \mathbb{C}^p$ is an analytic cover for all $j$. This proves the assertion. □

The real algebraic variety is then, over the complement of the critical set intersected with $\mathbb{R}^p$, the collection of all real sheets. The number of such sheets can, of course, be smaller than the number of all sheets and it can vary between the components of the complement of the critical set intersected with $\mathbb{R}^p$.

Lemma 3.2. If $X$ is a real algebraic curve then after some real coordinate transformation it is regular over $\mathbb{R}$ at infinity.

Proof: We choose a real coordinate transformation according to Lemma 3.1. Since the critical set is finite the result follows. □

We arrive at our main theorem:

Theorem 3.3. If $X$ is an real algebraic curve in $\mathbb{R}^d$ then the following are equivalent:

1. $J_X(\mathbb{R}^d)$ is complemented.
2. There exists a continuous linear extension operator $A(X) \to A(\mathbb{R}^d)$.
3. $X$ is of type PL.
4. In every $a \in X$ every irreducible component of the germ $X_a$ is either regular or a point.
**Proof:** Since, by definition, \( X \) is \( C \)-analytic. \( \Leftrightarrow \) 2. \( \Rightarrow \) 3. follows from [4]. 3. \( \Rightarrow \) 2. is Theorem 2.8 together with Lemma 3.2 and the equivalence 3. \( \Leftrightarrow \) 4. follows from Lemma 1.2. Notice that isolated points of \( X \) obviously are of type \( PL \). □

As an immediate consequence we obtain:

**Corollary 3.4.** If \( X \) is an algebraic subvariety of \( \mathbb{R}^2 \) then the equivalences of Theorem 3.3 hold.

**Examples for \( d=2 \):**

1. \( y^2 = x^2(x + \lambda) \). For \( \lambda > 0 \) this is Newton’s knot, for \( \lambda = 0 \) it is Neill’s parabola and for \( \lambda < 0 \) it has two components, a smooth curve in the \( \{ x > 0 \} \)-halfplane and \( (0, 0) \) as a singular point. They all are of type \( PL \), except Neill’s parabola.

2. \( P(x, y) = y^3 - x^3(1 + x^2) \) this is a smooth curve with \( (0, 0) \) as a hidden singular point, it is of type \( PL \).

A lot more curves for which one may check what happens can be found e. g. in [5, 6].

**References**


