Hadamard type operators on temperate distributions

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Abstract

We study Hadamard operators on $\mathscr{S}'(\mathbb{R}^d)$ and give a complete characterization. They have the form $L(S) = S \star T$ where $\star$ means the multiplicative convolution and $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$, the space of distributions which are θ-rapidly decreasing in infinity and at the coordinate hyperplanes. To show this we study and characterize convolution operators on the space $Y(\mathbb{R}^d)$ of exponentially decreasing C^∞-functions on $\mathbb{R}^d$. We use this and the exponential transformation to characterize the Hadamard operators on $\mathscr{S}'(Q)$, Q the positive quadrant, and this result we use as a building block for our main result.

In the present note we study Hadamard operators on $\mathscr{S}'(\mathbb{R}^d)$, that is, continuous linear operators on $\mathscr{S}'(\mathbb{R}^d)$ which admit all monomials as eigenvectors and we give a complete characterization. Operators of Hadamard type have attracted some attention in recent times. Such operators on $C^\infty(\mathbb{R}^d)$ have been studied and characterized in [10, 13], on $\mathscr{A}(\mathbb{R})$ in [1, 2, 3] and on $\mathscr{A}(\mathbb{R}^d)$ in [5]. There you find also references to the long history of such problems. Their surjectivity on $C^\infty(\mathbb{R}^d)$ has been characterized in [4]. Since it can be shown that Hadamard operators commute with dilations our problem is, by duality, closely related to the study of continuous linear operators in $\mathscr{S}(\mathbb{R}^d)$ which commute with dilations. In a first step we study such operators on $\mathscr{S}(Q)$, $Q =]0, +\infty[^d$. By means of the exponential transformation this can be transferred to the study of convolution operators on the space $Y(\mathbb{R}^d)$ of C^∞-functions on $\mathbb{R}^d$ with exponential decay.

In a first part of the paper we study such operators and give a complete characterization in terms of the class $\mathscr{O}'_Y(\mathbb{R}^d)$ of exponentially decreasing distributions, which is similar to the class $\mathscr{O}'_C$ of L. Schwartz of rapidly decreasing distributions,

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which are the convolution-multipliers in $\mathscr{S}(\mathbb{R}^d)$. We study the class $\mathscr{O}'_Y(\mathbb{R}^d)$ and these results are of independent interest.

By means of the exponential transformation we obtain a description of the operators on $\mathscr{S}'(Q)$ which commute with dilations in Q. They have the form $\varphi \mapsto T_x\varphi(xy)$ where T is a distribution in $\mathscr{O}'_H(Q)$. These are the exponential transforms of $\mathscr{O}'_Y(\mathbb{R}^d)$, we call them θ-rapidly decreasing distributions on Q. The class $\mathscr{O}'_H(\mathbb{R}^d)$ first appeared in [11] where the Hadamard operators in $\mathscr{D}'(\mathbb{R}^d)$ were described. For a more detailed study of this class and examples see [11, §3].

From there we obtain our main result: The Hadamard operators on $\mathscr{S}'(\mathbb{R}^d)$ have the form $S \mapsto S \star T$ where $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ the class of distributions on $\mathbb{R}^d$ which are θ-rapidly decreasing in infinity and at the coordinate hyperplanes. It is a subclass of $\mathscr{O}'_H(\mathbb{R}^d)$, known from [11].

We use standard notation of Functional Analysis, in particular, of distribution theory. For unexplained notation we refer to [6], [8], [9], [7].

1 Preliminaries

We use the following notation $\partial_j = \partial/\partial x_j$, $\theta_j = x_j\,\partial_j$. For a multiindex $\alpha \in \mathbb{N}_0^d$ we set $\partial^\alpha = \partial_1^{\alpha_1}..\partial_d^{\alpha_d}$, likewise for θ^α. $\mathbf{1}$ denotes the vector $(1,\ldots,1)$. For vectors $x, y \in \mathbb{R}^d$ we will use the definition $xy = (x_1y_1, ...x_dy_d)$. This will hold except for obvious cases like in the formula for the Fourier transform.

For a polynomial $P(z) = \sum_\alpha c_\alpha z^\alpha$ we consider the *Euler operator* $P(\theta) = \sum_\alpha c_\alpha \theta^\alpha$ and also the operator $P(\partial)$, defined likewise. The dual operator of $P(\theta)$ is $P(\theta^*)$ where $\theta^* = -\theta - 1$, hence also an Euler operator.

For $a \in \mathbb{R}^d_*$ the *dilation operator* D_a is defined by $(D_aT)\varphi = |a_1 \cdots a_d|^{-1} T_\xi\varphi(\xi/a)$. For the distribution $x^\alpha \in \mathscr{S}'(\mathbb{R}^d)$ this yields $D_a x^\alpha = (ax)^\alpha$. For $e \in \{-1,+1\}^d$ this definition simplifies to $(D_eT)\varphi = T_\xi\varphi(e\xi)$. These operators are called *reflections* .

For basic properties of Hadamard operators see [11]. They are a closed commutative sub-algebra of $L(\mathscr{S}'(\mathbb{R}^d))$. Euler operators and dilations are of Hadamard type, Therefore they commute with all Hadamard operators. On the other hand we have:

Lemma 1.1 *If $L \in L(\mathscr{S}'(\mathbb{R}^d))$ commutes with θ_j for all j and with all reflections then it is a Hadamard operator.*

Proof: We set $T = L(x^\alpha)$ and have to show that $T \in \operatorname{span}\{x^\alpha\}$. Since L commutes with θ_j we obtain $\theta_j T = \alpha_j T$. By use of the exponential transformation we obtain for Q and likewise for all quadrants $Q_e = eQ$ that $T = c_e x^\alpha$ on Q_e, with constants c_e. Since L commutes with reflections all c_e must be equal and we have $T = cx^\alpha$ on $\mathbb{R}^d_*$.

We set $S = T - cx^\alpha$. Then $\operatorname{supp} S \subset Z_0 = \{\xi \,:\, \xi_1 \cdots \xi_d = 0\}$ and $\theta_j S = \alpha_j S$. Since S is of finite order there is $\beta \in \mathbb{N}^d$ such that $x^\beta S = 0$. We have

$$\partial_j(x^\beta S) = \beta_j x^{\beta'} S + x^{\beta'}\theta_j S = (\beta_j + \alpha_j)\, x^{\beta'} S$$

where $\beta' = (\beta_1, \ldots, \beta_j - 1, \ldots, \beta_d)$. Repeating this we obtain:

$$0 = \partial^\beta (x^\beta S) = b\, S$$

with $b \neq 0$. Therefore $S = 0$, that is, $L(x^\alpha) = cx^\alpha$. □

We set for $x \in \mathbb{R}^d$

$$\operatorname{Exp}(x) = (\exp(x_1), .., \exp(x_d)).$$

Exp is a diffeomorphism from $\mathbb{R}^d$ onto $Q := (0, +\infty)^d$. Therefore

$$C_{\operatorname{Exp}} : f \longrightarrow f \circ \operatorname{Exp}$$

is a linear topological isomorphism from $C^\infty(Q)$ onto $C^\infty(\mathbb{R}^d)$. For $f \in C^\infty(Q)$ we have $P(\partial)(f \circ \operatorname{Exp}) = (P(\theta)f) \circ \operatorname{Exp}$ that is $P(\partial) \circ C_{\operatorname{Exp}} = C_{\operatorname{Exp}} \circ P(\theta)$. In this way the study of Hadamard operators on Q can be reduced to the study of operators on $\mathbb{R}^d$. This has be done in [13] for $C^\infty(Q)$. We apply the same argument to the space $\mathscr{S}(Q)$ where $\mathscr{S}(Q) = \{f \in \mathscr{S}(\mathbb{R}^d) \,:\, \operatorname{supp} f \subset \overline{Q}\}$.

As usual $\mathscr{S}(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$, its dual $\mathscr{S}'(\mathbb{R}^d)$ the space of temperate distributions. We consider is subspace $\mathscr{S}(Q)$ and its dual $\mathscr{S}'(Q)$.

We recall the following definitions of [9, Chap. VI, §8]: $\mathscr{B}'$ denotes the dual of the space of C^∞-space which are bounded including all derivatives and $\mathscr{D}'_{L_1}$ the dual of the space of C^∞-space such that all derivatives are in $L_1(\mathbb{R}^d)$.

2 Convolution operators on C^∞-functions with exponential decay

We start with studying convolution operators on the space of C^∞-functions with exponential decay on $\mathbb{R}^d$ and its dual. We will transfer our results by the exponential diffeomorphism to results on Hadamard operators on $\mathscr{S}'(Q)$ and use this as building blocks to study Hadamard operators on $\mathscr{S}'(\mathbb{R}^d)$. We set

$$\begin{aligned} Y(\mathbb{R}^d) &:= \{f \in C^\infty(\mathbb{R}^d) \,:\, \sup_x |f^{(\alpha)}(x)|\, e^{k|x|} < \infty \text{ for all } \alpha \text{ and } k \in \mathbb{N}\} \\ &= \{f \in C^\infty(\mathbb{R}^d) \,:\, \sup_x |f^{(\alpha)}(x)|\, e^{x\eta} < \infty \text{ for all } \alpha \text{ and } \eta \in \mathbb{R}^d\} \end{aligned}$$

with its natural topology.

Then $Y(\mathbb{R}^d)$ is a Fréchet space, closed under convolution and $P(\partial)$ is a continuous linear operator in $Y(\mathbb{R}^d)$ for every polynomial P. $\mathscr{D}(\mathbb{R}^d) \subset Y(\mathbb{R}^d)$ as a dense subspace, hence $Y(\mathbb{R}^d)' \subset \mathscr{D}'(\mathbb{R}^d)$. We obtain (see [14, Lemma 2.1]):

Lemma 2.1 $C_{\mathrm{Exp}}(\mathscr{S}(Q)) = Y(\mathbb{R}^d)$.

We set $\omega(x) = \sum_{\eta \in \{-1,+1\}^d} e^{\eta x}$. We have $\omega \in C^\infty(\mathbb{R}^d)$ and $e^{|x|} \le \omega(x) \le 2^d e^{|x|}$.

In analogy to [9, Chap. VII, §5, p. 100] we define

Definition 1 *$T \in \mathscr{O}_Y'(\mathbb{R}^d)$ if $\omega(kx)T \in \mathscr{B}'$ for every k.*

It is obvious that we might equivalently write $\omega(kx)T \in \mathscr{D}'_{L_1}$ for every k.

For the following theorem compare [9, Chap. VII, §5, Théorème IX].

Theorem 2.2 *For $T \in \mathscr{D}'(\mathbb{R}^d)$ the following are equivalent:*

1. *$T \in \mathscr{O}_Y'(\mathbb{R}^d)$.*
2. *For any k there are finitely many functions t_β such that $e^{k|x|}t_\beta \in L_\infty(\mathbb{R}^d)$ and such that $T = \sum_\beta \partial^\beta t_\beta$.*
3. *$T \in Y(\mathbb{R}^d)'$ and $T_x\varphi(x+y) \in Y(\mathbb{R}^d)$ for all $\varphi \in Y(\mathbb{R}^d)$.*
4. *$f(y) = T_x\varphi(x+y)$ is a exponentially decreasing continuous function (that is $\sup |f(y)|e^{k|y|} < \infty$ for all k) for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$.*
5. *$(\omega(kx)T) * \varphi$ is a continuous bounded function for every k and $\varphi \in \mathscr{D}(\mathbb{R}^d)$.*

Proof: (1) ⇒ (2) If $\omega(kx)T \in \mathscr{D}'_{L_1}$ then, by a standard conclusion, there are finitely many functions $\tau_\beta \in L_\infty(\mathbb{R}^d)$ such that $\omega(kx)T = \sum_\beta \partial^\beta \tau_\beta$. This yields

$$\begin{aligned}
T\varphi &= (\omega(kx)T)\left(\frac{1}{\omega(kx)}\varphi\right) = \sum_\beta \partial^\beta \tau_\beta \left(\frac{1}{\omega(kx)}\varphi\right) \\
&= \sum_\beta (-1)^{|\beta|} \int \tau_\beta(x) \partial^\beta \left(\frac{1}{\omega(kx)}\varphi(x)\right) dx \\
&= \sum_\beta (-1)^{|\beta|} \int \tau_\beta(x) \sum_{\alpha \le \beta} c_{\alpha,\beta} \left(\partial^{\beta-\alpha}\frac{1}{\omega(kx)}\right) \varphi^{(\alpha)}(x) dx \\
&= \sum_\alpha \left(\sum_{\beta \ge \alpha} (-1)^{|\beta|} \tau_\beta(x) c_{\alpha,\beta} \partial^{\beta-\alpha}\frac{1}{\omega(kx)}\right) \varphi^{(\alpha)}(x) dx \\
&= \langle \sum_\alpha \partial^\alpha t_\alpha, \varphi \rangle
\end{aligned}$$

where $\omega(kx)t_\alpha \in L_\infty(\mathbb{R}^d)$ for all the, finitely many, α.

(2) $\Rightarrow$ (1) is straightforward, because we may assume that $T = \partial^\beta t_\beta$.

(2) $\Rightarrow$ (3) The first part is clear from (2). Assume $T = \partial^\beta t_\beta$, $e^{(k+1)|x|}|t_\beta(x)| \in L_\infty(\mathbb{R}^d)$. Then

$$T_x\varphi(x+y) = (-1)^{|\beta|}\int t_\beta(x)\varphi^{(\beta)}(x+y)dx \in C^\infty(\mathbb{R}^d)$$

and we have

$$e^{k|y|}|\partial_y^\alpha T_x\varphi(x+y)| \leq \int e^{k|x|}|t_\beta(x)|e^{k|x+y|}|\varphi^{(\alpha+\beta)}(x+y)|dx \leq \infty.$$

If $T \in \mathscr{O}'_Y(\mathbb{R}^d)$ this holds for all summands in the representation of T with given k and since we have for all k such a representation the claim is proved.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (5) For $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and $\eta \in \mathbb{R}^d$ we obtain

$$((e^{\eta x}T) * \varphi)(y) = T_x(e^{\eta x}\varphi(y-x)) = e^{\eta y}Te^{-\eta(y-x)}\varphi(y-x) = e^{\eta y}(T * (e^{-\eta x}\varphi))(y).$$

Since $e^{-\eta x}\varphi \in \mathscr{D}(\mathbb{R}^d)$ the right hand side is bounded, by (4). Adding over all $\eta \in \{+k, -k\}^d$ we obtain the result.

(5) $\Rightarrow$ (2) This follows from Lemma 2.3. □

The following Lemma is essentially an adaptation of [9, Chap. VI, §8, Théorème XXV].

Lemma 2.3 *Let ω be measurable, $\omega(x) > 0$ for all $x \in \mathbb{R}^d$. Let $S \in \mathscr{D}'(\mathbb{R}^d)$ be a distribution such that $\sup_x \omega(x)|S_y\varphi(x-y)| < \infty$ for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$ then there are finitely many measurable functions τ_β with $\sup_x \omega(x)\tau_\beta(x) < \infty$ such that $S = \sum_\beta \tau_\beta^{(\beta)}$.*

Proof: We consider the map $\Psi : \mathscr{D}(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d)$ given by

$$\Psi(\varphi) = \omega(x)S_y\varphi(x-y).$$

Because of the Closed Graph Theorem Ψ is continuous. Let B denote the unit ball in $\mathbb{R}^d$. Then there is $m \in \mathbb{N}$ such that Ψ restricted to $\mathscr{D}(B)$ extends to a continuous map $\mathscr{D}^m(B) \to L_\infty(\mathbb{R}^d)$, where $\mathscr{D}^m(B)$ denotes the Banach space of m-times continuously differentiable functions with support in B. We choose $\gamma \in \mathscr{D}(B)$, $\gamma(x) = 1$ in a neighborhood of 0 and set $g = \gamma E \in \mathscr{D}^m(B)$ where E is an elementary solution of Δ^k, k large enough. Then $\Psi(g) \in L_\infty(\mathbb{R}^d)$ that means $\tau := S * g$ is a measurable function with $\omega(x)|\tau(x)| \leq C$ for suitable C and we obtain $\Delta^k\tau = S + S*\psi$ where $\psi \in \mathscr{D}(B)$. We have $\omega(x)(S*\psi) = \Psi(\psi) \in L_\infty(\mathbb{R}^d)$. Therefore the equality $S = \Delta^k\tau - S * \psi$ shows the result. □

We have to fix our notation on the convolution of distributions. For distributions T, S and a function ψ we define $(S * T)\psi := S_y(T_x\psi(x+y))$ whenever this makes sense.

Lemma 2.4 *If $T \in \mathscr{O}'_Y(\mathbb{R}^d)$ and $\varphi \in Y(\mathbb{R}^d)$ then both $T*\varphi \in Y(\mathbb{R}^d)'$ and $\varphi*T \in Y(\mathbb{R}^d)'$ are defined and equal and we have $T*\varphi = \varphi * T = T_y\varphi(x-y) \in Y(\mathbb{R}^d)$. $\varphi \mapsto T*\varphi$ is a continuous linear operator in $Y(\mathbb{R}^d)$.*

Proof: The first claim follows from Theorem 2.2, (3), the fact that $Y(\mathbb{R}^d)$ is closed under convolution and, finally, from the representation in Theorem 2.2, (2). The second is then easily shown or follows from the Closed Graph Theorem. □

This shows part of the following theorem.

Theorem 2.5 *For an operator $L \in L(Y(\mathbb{R}^d))$ the following are equivalent:*

1. *L commutes with translations.*
2. *There is $T \in \mathscr{O}'_Y(\mathbb{R}^d)$ such that $L\varphi = T*\varphi$ for all $\varphi \in Y(\mathbb{R}^d)$.*

Proof: (2) $\Rightarrow$ (1) is clear, we have to show the converse. We define $T \in Y(\mathbb{R}^d)'$ by $T\varphi := (L\varphi)(0)$. Then by standard arguments we have $(L\varphi)(x) = L(\varphi(\cdot + x))(0) = T_y\varphi(y+x) = \check{T}\varphi(x-y) = (\check{T}*\varphi)(x)$. Due to Theorem 2.2, (3) we have $T \in \mathscr{O}'_Y(\mathbb{R}^d)$, hence also $\check{T} \in \mathscr{O}'_Y(\mathbb{R}^d)$. □

The dual situation is a bit more complicated, since existence of $T*S$ and commutivity is not a priori clear. We define:

$$\mathscr{O}_Y(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \,:\, \exists k \in \mathbb{N}_0 \,\forall \alpha \in \mathbb{N}_0^d : \sup_{x\in\mathbb{R}^d} |f^{(\alpha)}(x)|\, e^{-k|x|} < \infty\}.$$

Equipped with its natural locally convex topology $\mathscr{O}_Y(\mathbb{R}^d)$ is the inductive limit of a sequence of Fréchet spaces, that is, an (LF)-space and we have

Lemma 2.6 *$\mathscr{O}'_Y(\mathbb{R}^d)$ is the dual space of $\mathscr{O}_Y(\mathbb{R}^d)$. For $S \in Y(\mathbb{R}^d)'$ the map $\varphi \mapsto S_y\varphi(x+y)$ is a continuous linear map from $Y(\mathbb{R}^d)$ to $\mathscr{O}_Y(\mathbb{R}^d)$.*

Proof: The first part by use of a standard argument using Theorem 2.2, (2). For the second part we estimate

$$\begin{aligned} |\partial^\alpha S_y\varphi(x+y)| &= |S_y\varphi^{(\alpha)}(x+y)| \le C \sup_{y,|\beta|\le k} |\varphi^{(\alpha+\beta)}(x+y)e^{k|y|} \\ &\le C\, e^{k|y|} \sup_{\xi,|\gamma|\le m} |\varphi^{(\gamma)}(\xi)|e^{k|\xi|} \end{aligned}$$

with C and k depending on S and $m = k + |\alpha|$. □

We obtain an analogue to Lemma 2.4.

Lemma 2.7 *If $T \in \mathscr{O}'_Y(\mathbb{R}^d)$ and $S \in Y(\mathbb{R}^d)'$ then both $T * S \in Y(\mathbb{R}^d)'$ and $S * T \in Y(\mathbb{R}^d)'$ are defined and equal. $S \mapsto T * S$ is a continuous linear operator in $Y(\mathbb{R}^d)'$.*

Proof: The existence of $S * T$ follows from Theorem 2.2, (3), the existence of $T * S$ from Lemma 2.7. $(T * S)\varphi = (S * T)\varphi$ for $\varphi \in \mathscr{D}(\mathbb{R}^d)$ equality follows by direct calculation by use of Theorem 2.2, (2). The continuity of $S \mapsto S * T$ is obvious. □

Theorem 2.8 *For an operator $L \in L(Y(\mathbb{R}^d)')$ the following are equivalent:*

1. *L commutes with translations.*
2. *There is $T \in \mathscr{O}'_Y(\mathbb{R}^d)$ such that $L(S) = T * S$ for all $S \in Y(\mathbb{R}^d)'$.*

Proof: (2) $\Rightarrow$ (1) is clear, we have to show the converse. The transpose $L^* \in L(Y(\mathbb{R}^d))$ also commutes with translation. Note that $Y(\mathbb{R}^d)$ is Montel, hence reflexive. Because of Theorem 2.5, Proof, there is $T \in \mathscr{O}'_Y(\mathbb{R}^d)$ such that $(L^*(\varphi))(x) = T_y\varphi(x+y)$. So for $S \in Y(\mathbb{R}^d)'$ we obtain $\langle L(S), \varphi\rangle = \langle S, L^*(\varphi)\rangle = S_x(T_y\varphi(x+y)) = ((S * T)\varphi)(x)$. □

3 Hadamard operators on $\mathscr{S}'(Q)$

Let L be a Hadamard operator on $\mathscr{S}'(Q)$, that is an operator which admits all monomials as eigen-functions. We need some preparations, cf. Section 1 in [11]. For $a \in Q$ we define the dilation $D_a \in L(\mathscr{S}'(Q))$ by

$$(D_aT)\varphi := T_x\left(\frac{1}{a_1\dots a_d}\,\varphi\Big(\frac{x}{a}\Big)\right)$$

for $T \in \mathscr{S}'(Q)$ and $\varphi \in \mathscr{S}(Q)$. By direct verification we see that $D_a\xi^\alpha = a^\alpha\xi^\alpha$.

Like in [11, Lemma 1.1] we obtain that L commutes with dilations, that is, $D_a \circ L = L \circ D_a$ for all $a \in Q$.

We set $M = L^* \in L(\mathscr{S}(Q))$ and obtain like in [11, Lemma 1.3] that M commutes with with dilations, that is,

$$M_\xi(\varphi(\eta\xi))[x] = (M\varphi)(\eta x)$$

for all $\varphi \in \mathscr{S}(Q)$ and $\eta \in Q$.

For $\varphi \in \mathscr{S}(Q)$ we define now

$$T\varphi = (M\varphi)(\mathbf{1}) = (L\delta_{\mathbf{1}})(\varphi).$$

Then $T \in \mathscr{S}'(Q)$ and for all $\eta \in Q$ we have

$$(M\varphi)(\eta) = T_\xi \varphi(\eta\xi). \tag{1}$$

We have to determine the set of distributions in $T \in \mathscr{S}'(Q)$ such that

$$T_\xi \varphi(\cdot\, \xi) \in \mathscr{S}(Q) \text{ for all } \varphi \in \mathscr{S}(Q). \tag{2}$$

For $\widetilde{T} = C^*_{\mathrm{Log}}(T)$ the condition (2) is equivalent to

$$\widetilde{T}_\xi \psi(\cdot + \xi) \in Y(\mathbb{R}^d) \text{ for all } \psi \in Y(\mathbb{R}^d) \tag{3}$$

which, by Theorem 2.2, is equivalent to $\widetilde{T} \in \mathscr{O}'_Y(\mathbb{R}^d)$.

In analogy to [11, Definition 3] we define the space $\mathscr{O}'_H(Q)$ of *θ-rapidly decreasing distributions* on Q.

Definition 2 *$T \in \mathscr{O}'_H(Q)$ if for any k there are finitely many functions t_β such that $(|x|^{2k} + |x|^{-2k})t_\beta \in L_\infty(Q)$ and such that $T = \sum_\beta \theta^\beta t_\beta$.*

By use of the description in Theorem 2.2, (2), we obtain:

Lemma 3.1 *$C^*_{\mathrm{Exp}}(\mathscr{O}'_Y(\mathbb{R}^d)) = \mathscr{O}'_H(Q)$.*

Hence we obtain the following translation of Theorem 2.2:

Theorem 3.2 *For $T \in \mathscr{D}'(\mathbb{R}^d)$ the following are equivalent:*

1. *$T \in \mathscr{O}'_H(Q)$*
2. *For any k there are finitely many functions t_β such that $(|x|^{2k} + |x|^{-2k})t_\beta \in L_\infty(Q)$ and such that $T = \sum_\beta \theta^\beta t_\beta$*
3. *$T \in \mathscr{S}'(Q)$ and $T_x\varphi(xy) \in \mathscr{S}(Q)$ for all $\varphi \in \mathscr{S}(Q)$.*
4. *$f(y) = T_x\varphi(xy)$ is a rapidly decreasing continuous function (that is $\sup |f(y)|$ $(|y|^{2k} + |y|^{-2k}) < \infty$ for all k) for all $\varphi \in \mathscr{D}(Q)$.*
5. *$((|x|^{2k} + |x|^{-2k})T) \star \varphi$ is a continuous bounded function for every k and $\varphi \in \mathscr{D}(Q)$.*

We have obtained the following.

Theorem 3.3 *L Hadamard operator on $\mathscr{S}'(Q)$ if and only if there is $T \in \mathscr{O}'_H(Q)$ such that $L(S) = S \star T$ for all $T \in \mathscr{S}'(Q)$.*

Here $\langle S \star T, \varphi\rangle = S_x(T_y\varphi(xy)$ for all $\varphi \in \mathscr{S}(Q)$.

4 Hadamard operators on $\mathscr{S}'(\mathbb{R}^d)$

Let now L be a Hadamard operator on $\mathscr{S}'(\mathbb{R}^d)$ and $M = L^* \in L(\mathscr{S}(\mathbb{R}^d))$ and obtain like in [11, Lemma 1.3] that M commutes with with dilations, that is,

$$M_\xi(\varphi(\eta\xi))[x] = (M\varphi)(\eta x) \tag{4}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $\eta \in \mathbb{R}^d_*$.

For $\varphi \in \mathscr{S}(\mathbb{R}^d)$ we define now

$$T\varphi = (M\varphi)(\mathbf{1}) = (L\delta_{\mathbf{1}})(\varphi). \tag{5}$$

Then $T \in \mathscr{S}'(\mathbb{R}^d)$ and for all $\eta \in \mathbb{R}^d_*$ we have

$$(M\varphi)(\eta) = T_\xi\varphi(\eta\xi). \tag{6}$$

We have to determine the set of distributions in $T \in \mathscr{S}'(\mathbb{R}^d)$ such that $T_\xi\varphi(\cdot\,\xi)$, $\xi \in \mathbb{R}^d_*$, extends to a function in $\mathscr{S}(\mathbb{R}^d)$ for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

We want to use the results of Section 3. We denote by H_j, $j = 1, .., d$, the coordinate hyperplanes and set $Z_0 = \bigcup_j H_j$.

$$\mathscr{S}(\mathbb{R}^d_*) = \{\varphi \in \mathscr{S}(\mathbb{R}^d) \,:\, \varphi \text{ flat on } Z_0\}.$$

We will show that $M(\mathscr{S}(\mathbb{R}^d_*)) \subset \mathscr{S}(\mathbb{R}^d_*)$. For that it suffices to show that $L(\mathscr{S}'(Z_0)) \subset \mathscr{S}'(Z_0)$. Here $\mathscr{S}'(Z_0)$ denotes the temperate distributions with support in Z_0.

By $\mathcal{F}$ we denote the Fourier transform and remark that for all j

$$\theta_j \circ \mathcal{F} = \mathcal{F} \circ \theta_j^*, \quad \theta_j^* \circ \mathcal{F} = \mathcal{F} \circ \theta_j. \tag{7}$$

We set $\widetilde{L} = \mathcal{F} \circ L \circ \mathcal{F}^{-1}$ and since θ_j^* commutes with L we conclude by use of (7) that $\widetilde{L}$ commutes with θ_j for all j. By straightforward calculation we see that $\widetilde{L}$ commutes with all reflections. By Lemma 1.1 this implies that $\widetilde{L}$ is a Hadamard operator. We have $L = \mathcal{F}^{-1} \circ \widetilde{L} \circ \mathcal{F}$.

We have $\mathcal{F}(\delta^{(\alpha)}) = i^\alpha (2\pi)^{-d/2} x^\alpha$, hence $\widetilde{L}(\mathcal{F}\delta^{(\alpha)}) = \widetilde{m}_\alpha \mathcal{F}\delta^{(\alpha)}$. Finally we obtain

$$L(\delta^{(\alpha)}) = \widetilde{m}_\alpha \delta^{(\alpha)}, \tag{8}$$

where $\widetilde{L}(x^\alpha) = \widetilde{m}_\alpha x^\alpha$.

Example: $L = \theta$ then $\widetilde{L} = \theta^* = -\theta - 1$. Since $\widetilde{m}_k = -k - 1$ we obtain $\theta\delta^{(k)} = (-k-1)\delta^{(k)}$ which, of course, can be verified by direct calculation.

In fact, we will need this result only for $d = 1$. We set $x = (x_1, x')$, $x' = (x_2, .., x_d)$ and consider distributions of the form $T_\alpha = \delta^{(\alpha)}(x_1) \otimes S(x')$, $S \in \mathscr{S}'(\mathbb{R}^{d-1})$.

We fix $\alpha' = (\alpha_2, .., \alpha_d)$ and $\psi \in \mathscr{D}(\mathbb{R}^{d-1})$. For $T \in \mathscr{S}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{S}(\mathbb{R})$ we set $(R_\psi T)\varphi := T(\varphi(x_1)\psi(x')$. This defines a map $R_\psi : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R})$.

For $U \in \mathscr{S}'(\mathbb{R})$ we set $L_1(U) := (R_\psi \circ L)(U \otimes x^{\alpha'}) \in \mathscr{S}'(\mathbb{R})$. We obtain for $\alpha \in \mathbb{N}_0$ and $\hat{\alpha} = (\alpha, \alpha')$

$$L_1(x^\alpha) = R_\psi(Lx^{\hat{\alpha}}) = R_\psi(m_{\hat{\alpha}} x^{\hat{\alpha}}) = m_{\hat{\alpha}} \int \xi^{\alpha'} \psi(\xi)\, d\xi \; x^\alpha.$$

Hence L_1 is a Hadamard operator on $\mathscr{S}'(\mathbb{R})$ and, by (8), $\delta^{(\alpha)}$ is an eigenvector of L_1. This means $L_1(\delta^{(\alpha)}) = \mu_\alpha \delta^{(\alpha)}$, hence $(-1)^\alpha \mu_\alpha \varphi^{(\alpha)}(0) = T(\varphi(x_1), \psi(x'))$ for all $\varphi \in \mathscr{S}(\mathbb{R})$ where $T = L(\delta^{(\alpha)} \otimes x^{\alpha'})$.

We choose $\chi \in \mathscr{D}(\mathbb{R})$, $\chi = 0$ in a neighborhood of 0, and set $\varphi_\alpha(x) = \frac{x^\alpha}{\alpha!}\chi(x)$. Then $\mu_\alpha = (-1)^\alpha T(\varphi_\alpha(x_1), \psi(x'))$. Setting $\mu_\alpha(\psi) = T(\varphi_\alpha(x_1), \psi(x'))$ we obtain a distribution $\mu_\alpha \in \mathscr{S}'(\mathbb{R}^{d-1})$ such that

$$L(\delta^{(\alpha)} \otimes x^{\alpha'}) = \delta^{(\alpha)} \otimes \mu_\alpha.$$

We fix $\alpha \in \mathbb{N}_0$ and we have shown, that $x^{\alpha'} \in \{S \in \mathscr{S}'(\mathbb{R}^{d-1}) : L(T_\alpha) \in \delta^{(\alpha)} \otimes \mathscr{S}(\mathbb{R}^{d-1})\}$ for all $\alpha' \in \mathbb{N}_0^{d-1}$. Since this set is a closed linear subspace of $\mathscr{S}'(\mathbb{R}^{d-1})$ we have shown: $L(T_\alpha) \in \delta^{(\alpha)} \otimes \mathscr{S}(\mathbb{R}^{d-1})$ for all $S \in \mathscr{S}(\mathbb{R}^{d-1})$.

Distributions $T \in \mathscr{S}'(H_1)$ have the form

$$T = \sum_{\alpha=0}^{m} \delta^{(\alpha)}(x_1) \otimes S_\alpha(x')$$

(cf. [8, Chap III, Théorème XXXVI]). So we have shown $L(\mathscr{S}'(H_1)) \subset \mathscr{S}'(H_1)$. By an analogous argument this holds also for H_j, $j = 2, .., d$.

Since $\mathscr{S}'(Z_0) = \sum_{j=1}^d \mathscr{S}'(H_j)$ (see [14, Lemma 3.3]) we have shown:

Lemma 4.1 $L(\mathscr{S}'(Z_0)) \subset \mathscr{S}'(Z_0)$.

As an immediate consequence we obtain:

Proposition 4.2 $M(\mathscr{S}(\mathbb{R}^d_*)) \subset \mathscr{S}(\mathbb{R}^d_*)$.

We put $M_+(\varphi) = M(\varphi)|_Q$ for $\varphi \in \mathscr{S}(Q)$. Then $M_+ \in L(\mathscr{S}(Q))$ and $L_+ := M_+^* \in L(\mathscr{S}'(Q))$ is a Hadamard operator. From Theorem 3.2 we get $T_+ = L_+(\mathbf{1}) = T|_{\mathscr{S}(Q)} \in \mathscr{O}'_H(Q)$.

Clearly we can do this for all quadrants $Q_e = \{x : ex \in Q\}$. We set $M_e(\varphi) = M(\varphi)|_Q$ for $\varphi \in \mathscr{S}(Q_e)$ and $T_e(\varphi) = (M_e\varphi)(\mathbf{1})$ for $\varphi \in \mathscr{S}(Q_e)$. By the same arguments as before we obtain that $T_e \in \mathscr{O}'_H(Q_e)$ (defined in obvious analogy).

In analogy to Definition 2 we define the space of distributions on $\mathbb{R}^d$, which are *θ-rapidly decreasing* in infinity and at the coordinate hyperplanes.

Definition 3 *$T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ if for any k there are finitely many functions t_β such that $(|x|^{2k} + |x|^{-2k})t_\beta \in L_\infty(\mathbb{R}^d)$ and such that $T = \sum_\beta \theta^\beta t_\beta$.*

Then we have for T as defined in (5):

Lemma 4.3 *$\widetilde{T} := \sum_e T_e \in \mathscr{O}'_H(\mathbb{R}^d_*)$ and $T|_{\mathscr{S}(\mathbb{R}^d_*)} = \widetilde{T}$.*

For $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ and $\varphi \in \mathscr{S}(\mathbb{R}^d)$ we define $(M_T\varphi)(x) = T_\xi\varphi(\xi x)$ which is defined for all $x \in \mathbb{R}^d$.

Lemma 4.4 *M_T is a continuous linear operator in $\mathscr{S}(\mathbb{R}^d)$, $L_T := M_T^*$ is a Hadamard operator.*

Proof: We have to estimate $\Psi(x) := x^\gamma(M_T\varphi)^{(\alpha)}(x)$. We first recall that $\theta^*_\xi\varphi(\xi x) = (\theta^*\varphi)(\xi x)$ and $(\theta^*)^\beta\xi^\alpha\varphi(\xi) = \xi^\alpha\sum_{\nu\le\beta} p_\nu(\xi)\,\varphi^{(\nu)}(\xi) =: \xi^\alpha\psi(\xi)$, where the p_ν are polynomials.

We choose $k = k(\alpha - \gamma)$ large enough and obtain

$$\begin{aligned}\Psi(x) &= x^\gamma\int\theta^\beta\tau_\beta(\xi)\,\xi^\alpha\,\varphi^{(\alpha)}(\xi x)\,d\xi = x^{\gamma-\alpha}\int\theta^\beta\tau_\beta(\xi)(\xi x)^\alpha\varphi^{(\alpha)}(\xi x)\,d\xi\\ &= x^{\gamma-\alpha}\int\tau_\beta(\xi)\,(\xi x)^\alpha\,\psi(\xi x)\,d\xi = \int\tau_\beta(\xi)\,\xi^{\alpha-\gamma}\,(\xi x)^\gamma\psi(\xi x)d\xi\end{aligned}$$

and therefore

$$\|\Psi\|_{\gamma,\alpha} \le \Big(\int|\tau_\beta(\xi)\,\xi^{\alpha-\gamma}|\,d\xi\Big)\|\varphi\|.$$

Here $\|\varphi\| := \sup_x|x^\gamma\psi(x)|$ is a continuous semi-norm on $\mathscr{S}(\mathbb{R}^d)$. This shows the first part of the claim.

For the second part we have to study $\int x^\gamma(M_T\varphi)(x)dx$. We obtain

$$\begin{aligned}\int x^\gamma\Big(\int\theta^\beta\tau_\beta(\xi)\varphi(\xi x)\,d\xi\Big)\,dx &= \int\theta^\beta\tau_\beta(\xi)\xi^{-\gamma}\Big(\int(\xi x)^\gamma\varphi(\xi x)dx\Big)\,d\xi\\ &= \Big(\int\theta^\beta\tau_\beta(\xi)\xi^{-\gamma-\mathbf{1}}d\xi\Big)\int x^\gamma\varphi(x)dx\\ &= (-\gamma-\mathbf{1})^\beta\Big(\int\tau_\beta(\xi)\xi^{-\gamma-\mathbf{1}}d\xi\Big)\int x^\gamma\varphi(x)dx\\ &= \int(m_\gamma x^\gamma)\varphi(x)dx.\end{aligned}$$

We have shown that $L_T x^\gamma = m_\gamma x^\gamma$ and this completes the proof. □

For $S \in \mathscr{S}'(\mathbb{R}^d)$, $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ we define $S \star T \in L(\mathscr{S}'(\mathbb{R}^d))$ by

$$(S \star T)(\varphi) = S_x(T_\xi\varphi(\xi x)) \text{ for all } \varphi \in \mathscr{S}(\mathbb{R}^d).$$

The following is the main result of this paper.

Theorem 4.5 *1. For every $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ the map $S \mapsto S \star T$ is a Hadamard operator on $\mathscr{S}'(\mathbb{R}^d)$.*
2. For every Hadamard operator L on $\mathscr{S}'(\mathbb{R}^d)$ there is $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ such that $L(S) = S \star T$ for all $S \in \mathscr{S}'(\mathbb{R}^d)$.

Proof: The first part is Lemma 4.4. For the second part we choose $T = L(\delta_{\mathbf{1}})$, then $(M\varphi)(\eta) = T_\xi \varphi(\eta\xi)$ for all $\eta \in \mathbb{R}^d_*$ (see (6)). This equation is true for all $\eta \in \mathbb{R}^d$ if $\varphi \in \mathscr{S}(\mathbb{R}^d_*)$. By Lemma 4.3 there is $\widetilde{T} \in \mathscr{O}'_H(\mathbb{R}^d)$ such that $T\varphi = \widetilde{T}\varphi$ for $\varphi \in \mathscr{S}(\mathbb{R}^d_*)$. This means that $L_{\widetilde{T}}(S) = S \star \widetilde{T}$ defines a Hadamard operator and $L(S)\varphi = L_{\widetilde{T}}(\varphi)$ for all $\varphi \in \mathscr{S}(\mathbb{R}^d_*)$. Therefore $L - L_{\widetilde{T}}$ is a Hadamard operator such that $(L - L_{\widetilde{T}})S$ vanishes on $\mathscr{S}(\mathbb{R}^d_*)$ for all $S \in \mathscr{S}'(\mathbb{R}^d)$, hence $(L - L_{\widetilde{T}})x^\alpha = 0$ for all α and therefore $L - L_{\widetilde{T}} = 0$. Finally we have $T = L(\delta_{\mathbf{1}}) = L_{\widetilde{T}}(\delta_{\mathbf{1}}) = \widetilde{T}$. Therefore we have $L(S) = S \star T$ for all $S \in \mathscr{S}'(\mathbb{R}^d)$. □

5 Final remarks

In [11] the Hadamard operators in $\mathscr{D}'(\mathbb{R}^d)$ were characterized. We can express the Main Theorem of [11] in the following way:

Theorem 5.1 *The Hadamard operators on $\mathscr{D}'(\mathbb{R}^d)$ are the operators of the form $S \mapsto S \star T$ where $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ and $\operatorname{supp} T$ has positive distance to the coordinate hyperplanes.*

This follows from the fact that for a distribution $T \in \mathscr{D}'(\mathbb{R}^d)$, the support of which has positive distance of the coordinate hyperplanes, the conditions $T \in \mathscr{O}'_H(\mathbb{R}^d)$ and $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ coincide. For the definition of $\mathscr{O}'_H(\mathbb{R}^d)$ see [11, Definition 3].

This implies:

Corollary 5.2 *Every Hadamard operator on $\mathscr{D}'(\mathbb{R}^d)$ maps $\mathscr{S}'(\mathbb{R}^d)$ into $\mathscr{S}'(\mathbb{R}^d)$.*

By $\sigma(x) = \prod_j \frac{x_j}{|x_j|}$ we denote the signum of x. For $\alpha \in \mathbb{N}_0^d$ and $T = \theta^\beta \tau_\beta$ with $(|x|^{2k} + |x|^{-2k})t_\beta \in L_\infty(\mathbb{R}^d)$ and k large enough we define

$$T\Big(\frac{\sigma(x)}{x^{\alpha+\mathbf{1}}}\Big) = \int \tau_\beta(x)(\theta^*)^\beta \frac{\sigma(x)}{x^{\alpha+\mathbf{1}}} dx.$$

Therefore, using a proper representation, we can define $T\big(\frac{\sigma(x)}{x^{\alpha+\mathbf{1}}}\big)$ for any $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$. The definition does not depend on the representation, as the following result shows.

Theorem 5.3 *If $T \in \mathscr{O}'_H(\mathbb{R}^d_*)$ and $L(S) := S \star T$ the related Hadamard operator on $\mathscr{S}'(\mathbb{R}^d)$, then the eigenvalues of L with respect to x^α are $m_\alpha = T\big(\frac{\sigma(x)}{x^{\alpha+\mathbf{1}}}\big)$.*

The **Proof** is the same as the proof of Theorem 4.2 in [11]. In a remark after the proof there it is pointed out that it holds in a very general context.

We could also, in analogy to Section 2, define

$$\mathscr{O}_H(\mathbb{R}^d_*) := \{f \in C^\infty(\mathbb{R}^d_*) \,:\, \exists k \in \mathbb{N}_0\, \forall \alpha \in \mathbb{N}_0^d : \sup_{x\in\mathbb{R}^d_*} |\theta^{(\alpha)} f(x)|(|x|^{2k}+|x|^{-2k}) < +\infty\}.$$

Equipped with its natural locally convex topology $\mathscr{O}_H(\mathbb{R}^d_*)$ is the inductive limit of a sequence of Fréchet spaces, that is, an (LF)-space and we have

Lemma 5.4 *$\mathscr{O}'_H(\mathbb{R}^d_*)$ is the dual space of $\mathscr{O}_H(\mathbb{R}^d_*)$.*

This can be derived from Lemma 2.6 by use of the exponential transformation applied to all quadrants, or by direct verification. In this setting the term $T\left(\frac{\sigma(x)}{x^{\alpha+\mathbf{1}}}\right)$ is properly defined.

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