A DIVISION THEOREM FOR REAL ANALYTIC FUNCTIONS

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Abstract

We characterize those homogeneous polynomials \( P \in \mathbb{C}[z_1, \ldots, z_d] \) for which the principal ideal \( (P) = P \cdot A(\mathbb{R}^d) \) is complemented in \( A(\mathbb{R}^d) \) or, equivalently, which admit a continuous linear division operator. The condition is the same that characterizes among the homogeneous polynomials those which are non-elliptic and \( P(D) \) is surjective in \( A(\mathbb{R}^d) \) and those for which \( P(D) \) admits a continuous linear right inverse in \( C^\infty(\mathbb{R}^d) \). It depends only on the type of real singularities.

In this note we study the problem when for a polynomial \( P \in \mathbb{C}[z_1, \ldots, z_d] \) the principal ideal \( (P) = P \cdot A(\mathbb{R}^d) \) is complemented in the space \( A(\mathbb{R}^d) \) of real analytic functions or, equivalently, when there exists a continuous linear operator \( T \) in \( A(\mathbb{R}^d) \) so that \( T(P \cdot f) = f \) for every \( f \in A(\mathbb{R}^d) \). \( T \) and its dual operator which divides analytic functionals through \( P \) are called division operators, so we study, when \( P \) admits continuous linear division. We give a necessary condition which is also sufficient if either \( P \) is homogeneous or its real zero set is compact. So we get a complete characterization for the case of homogeneous polynomials.

In this case the principal ideal \( (P) \) is complemented if and only if the variety \( V = \{ z \in \mathbb{C}^d : P(z) = 0 \} \) satisfies the so called local Phragmén-Lindelöf condition (\( =: PL_{loc} \) condition) in every real point. This condition appeared first in Hörmander \[9\] and it characterizes those non-elliptic homogeneous polynomials for which \( P(D) \) is surjective in \( A(\mathbb{R}^d) \). It also characterizes those homogeneous polynomials for which \( P(D) \) admits a continuous linear right inverse in \( C^\infty(\mathbb{R}^d) \) (see Meise, Taylor and Vogt \[14\]). It has been intensely studied in the meantime, see e.g. \[3, 4, 5, 15, 19\].

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\end{flushleft}
It is also interesting to compare our result with the classical results in the $C^\infty$-case and in the case of entire functions. In the $C^\infty$-case this comparison will be done in Section 4. The connection to the ideal property (= propriété des zéros) and the analytic ideal property will also be discussed there. By a result of Zahariuta [21] and Djakov-Mitiagin [6] the principal ideal $P \cdot H(\mathbb{C}^d)$ is always complemented in the space $H(\mathbb{C}^d)$ of entire functions.

1 Preliminaries

Throughout the paper $A(\mathbb{R}^d)$ will denote the space of complex-valued real analytic functions with its natural locally convex topology (see Martineau [13]). For compact $K \subset \mathbb{C}^d$ we denote by $H(K)$ the space of germs of holomorphic functions on $K$ with its natural (LB)-topology, i.e. it is the inductive limit of the spaces $H^\infty(U_n)$ where the $U_n$ runs through an open neighborhood basis of $K$. This understood, $A(\mathbb{R}^d)$ is the projective limit of the spaces $H(K_n)$ where $K_n$ is any compact exhaustion of $\mathbb{R}^d$. For open $\omega \subset \mathbb{C}^d$ we denote by $H(\omega)$ the Fréchet space of holomorphic functions on $\omega$ with the compact open topology.

We use common notation for locally convex spaces. In particular for locally convex spaces $E$ and $F$ we denote by $L(E,F)$ the space of all continuous linear maps from $E$ to $F$ and we set $L(E) := L(E,E)$. The dual space $E'$ is always assumed to carry its strong topology. For all unexplained concepts of functional analysis we refer to [16]. For homological concepts we refer to [20], for real analytic spaces to [8] and for concepts of pluripotential theory to [11].

Let $V_a$ be the germ of a complex variety in the real point $a$. We assume always that it is given by a relatively compact open connected neighborhood $\Omega$ of $a$ in $\mathbb{C}^d$ and finitely many holomorphic functions $f_1, \ldots, f_m$ on $\Omega$ so that $V_a = \{z \in \Omega : f_1(z) = \cdots = f_m(z) = 0\}$ and $f_1, \ldots, f_m$ generate the ideal of $V_a$ in $\mathcal{O}_a$. We set $X_a = V_a \cap \mathbb{R}^d$.

If the ideal of $X_a$ in $\mathcal{O}_a$ is generated by $f_1, \ldots, f_m$ we call $V_a$ the complexification of $X_a$.

We define

$$\omega_{a,V}(z) = \limsup_{\zeta \to z} \sup \{u(\zeta) : u \text{ plurisubharmonic on } V_a, u \leq 1, u \leq 0 \text{ on } X_a\}.$$

For two germs $\omega_a$ and $\tilde{\omega}_a$ we set $\omega_a \prec \tilde{\omega}_a$ if there is a constant $C > 0$ so that $\omega_a \leq C\tilde{\omega}_a$ in a neighborhood of $a$ in $V_a \cap \bar{V}_a$. If $\omega_a \prec \tilde{\omega}_a$ and $\tilde{\omega}_a \prec \omega_a$ we write $\omega_a \sim \tilde{\omega}_a$ and call such germs equivalent. Then we obtain: Up to equivalence the germ of $\omega_{a,V}$ depends
only on the germ of $V_a$. If $V_a$ is the complexification of $X_a$ then, of course, it depends only on the germ of $X_a$. In this case we write also $\omega_{a,X}$.

**Definition 1.1** $V_a$ satisfies $PL_{\text{loc}}$ if $\omega_{a,V} \prec |\text{Im } z|$.

A complex variety $V$ satisfies $PL_{\text{loc}}$ in the real point $a$ if its germ in $a$ satisfies $PL_{\text{loc}}$. The germ $X_a$ of a real analytic variety in the real point $a$ is of type $PL$ if its complexification satisfies $PL_{\text{loc}}$. Notice that for a complex variety $V$ and $X = V \cap \mathbb{R}^d$ the germ $X_a$ can be of type PL, while $V_a$ does not satisfy $PL_{\text{loc}}$ in $a$.

**Example:** Let $V = \{ z : \sum_{j=1}^{d} z_j^2 = 0 \}$ and $d > 2$ then $X_0 = \{0\}$ and coincides with its complexification. Therefore $X_0$ is of type PL while $V_0$ does not satisfy $PL_{\text{loc}}$.

**Definition 1.2** For a germ $W_a$ of a subvariety of $V_a$ we set $W_a^{PL} \subset V_a$ if $\omega_{a,W} \prec \omega_{a,V}$ on $W_a$.

Notice that with this notation we have $V_a^{PL} \subset \mathbb{C}_a^d$ if and only if $V_a$ satisfies $PL_{\text{loc}}$.

**Lemma 1.3** We have the following simple facts:

1. If $W_a^{PL} \subset \tilde{W}_a^{PL} \subset V_a$ then $W_a^{PL} \subset V_a$.

2. If $V_a$ is in $\mathbb{C}^{d_1}$ and $W_b$ is in $\mathbb{C}^{d_2}$, $d_1 + d_2 = d$ then $V_a \times \{b\}^{PL} \subset V_a \times W_b$.

It should be pointed out that these relations remain unchanged, up to the base point, under biholomorphic maps defined in a neighborhood of $a$, mapping reals to reals. This is used, in particular, in connection with Lemma 1.3. (2).

We consider now a homogeneous complex variety $V$. We set

$$S = \{ z \in \mathbb{C}^d : \sum_{j=1}^{d} z_j^2 = 1 \}$$

and $V^0 = S \cap V$. Then $V^0$ is a complex variety and $V^0 \cap \mathbb{R}^d = X \cap S^{d-1} =: X^0$ is a compact real analytic variety.

In a neighborhood of $a \in X^0$ we identify by polar coordinates $V_a$ with $(1 + \varepsilon \mathbb{D}) \times V_a^0$ where $\mathbb{D}$ is the open unit disc. Therefore $V_a^{PL} \subset V_a$. If $V_a$ satisfies $PL_{\text{loc}}$ then also $V_a^0$. 

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Lemma 1.4 If $V$ is homogeneous and satisfies $PL_{loc}$ in any real point, the there exist a continuous linear extension operator $A(X^0) \to A(\mathbb{R}^d)$. In particular there exists a continuous linear extension operator $A(X^0) \to A(S^{d-1})$.

Proof: According to the previous $V_0$ satisfies $PL_{loc}$ in any point of $X^0$. That means that $X^0$ is a compact subvariety of $\mathbb{R}^d$ of type $PL$. By [19], Theorem 2.2 there exists a continuous linear extension operator $A(X^0) \to A(\mathbb{R}^d)$. If we compose it with the restriction map to $S^{d-1}$ we get the second assertion. □

From there we get the following theorem (see [19], Theorem 7.2). We set $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$, $X_* = \mathbb{R}^d_* \cap X$.

Theorem 1.5 If the homogeneous real analytic variety $V$ satisfies $PL_{loc}$ in any real point then there exists a continuous linear extension operator $A(X_*) \to A(\mathbb{R}^d_*)$.

Proof: The map $z \mapsto (\log ||z||, \frac{z_1}{||z||}, \ldots, \frac{z_d}{||z||})$ is a real analytic diffeomorphism from $\mathbb{R}^d_*$ to $\mathbb{R} \times S^{d-1}$, and also from $X_*$ to $\mathbb{R} \times X^0$. So we may identify $A(\mathbb{R}^d_*) \cong A(\mathbb{R}) \hat{\otimes}_\pi A(S^{d-1})$ and $A(X_*) \cong A(\mathbb{R}) \hat{\otimes}_\pi A(X^0)$. If $\varphi : A(X^0) \to A(S^{d-1})$ is the extension map of Lemma 1.4 then $id \hat{\otimes}_\pi \varphi$ leads to the extension map as claimed. □

2 Description of the problem

Let $P \in \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial. We denote by $(P) = P \cdot A(\mathbb{R}^d)$ the principal ideal of $P$ in the real analytic functions. The following Lemma is then quite obvious.

Lemma 2.1 The following are equivalent:

1. There is a continuous linear operator $T = T_P \in L(A(\mathbb{R}^d))$ ("division operator") so that $T(Pf) = f$ for all $f \in A(\mathbb{R}^d)$.

2. There is a continuous linear operator $S = S_P \in L(A'(\mathbb{R}^d))$ so that $P \cdot S(\mu) = \mu$ for all $\mu \in A'(\mathbb{R}^d)$.

3. $(P)$ is complemented in $A(\mathbb{R}^d)$.

Proof: Since (1) and (2) are just dual to each other it suffices to show the equivalence of (1) and (3). If (1) is given, then $f \mapsto P \cdot T(f)$ defines a continuous linear projection onto $(P)$. If (3) is given and $\pi$ a continuous linear projection onto $(P)$, then $(P)$ is a quotient of $A(\mathbb{R}^d)$, hence ultrabornological. By the Grothendieck-de Wilde Theorem the map $T_0 : Pf \mapsto f$ is continuous linear from $(P)$ onto $A(\mathbb{R}^d)$. We set $T := T_0 \circ \pi$. □
Definition: We say that $P$ admits continuous linear division if the equivalent conditions of Lemma 2.1 are fulfilled.

Since obviously $P$ admits continuous linear division if and only if each of its irreducible factors does it, we may assume for our investigation, that $P$ is irreducible. In this case we can describe $(P)$ by the zeros of its functions.

We use the following notation:

\[ V = \{ z \in \mathbb{C}^d : P(z) = 0 \} , \]
\[ X = V \cap \mathbb{R}^d = \{ x \in \mathbb{R}^d : P(x) = 0 \} , \]

and we obtain:

**Lemma 2.2** $(P)$ is the set of all $f \in A(\mathbb{R}^d)$, for which there exists an open neighborhood $\omega$ of $\mathbb{R}^d$ and a function $F \in H(\omega)$ so that $F|_{\mathbb{R}^d} = f$ and $F|_{V \cap \omega} = 0$.

**Proof:** If $f \in (P)$ then there is $g \in A(\mathbb{R}^d)$ so that $f = Pg$. There is an open neighborhood $\omega$ of $\mathbb{R}^d$ and $G \in H(\omega)$ so that $G|_{\mathbb{R}^d} = g$. We put $F = PG$.

To show the converse we assume $\omega$ and $F$ as described in the lemma. Then $P$ divides $F$, i.e. there is $G \in H(\omega)$ with $F = PG$.

We set

\[ H_V(X) = \{ (f, \Omega) : \Omega \text{ open neighborhood of } X \text{ in } V, f \text{ holomorphic on } \Omega \} \]

with $(f_1, \Omega_1) = (f_2, \Omega_2)$ if there exists an open set $\Omega \subset V$ with $X \subset \Omega \subset \Omega_1 \cap \Omega_2$ and $f_1|\Omega = f_2|\Omega$.

We obtain a natural restriction map $\rho : A(\mathbb{R}^d) \rightarrow H_V(X)$ by setting $\rho(f) = F|_{V}$, where $F$ is an extension of $f$ to a holomorphic function on an open neighborhood of $\mathbb{R}^d$.

**Lemma 2.3** The sequence

\[ 0 \rightarrow (P) \hookrightarrow A(\mathbb{R}^d) \xrightarrow{\rho} H_V(X) \rightarrow 0 \]

is exact.

**Proof:** Due to Lemma 2.2 we have to show only the surjectivity of $\rho$. For given $(f, \Omega)$ we find, by use of the Cartan-Grauert Theorem, an open pseudoconvex set $\omega \subset \mathbb{C}^d$ so that $\mathbb{R}^d \subset \omega$ and $\omega \cap V \subset \Omega$. By the Cartan-Oka theory there exists an $F \in H(\omega)$ so that $F|_{\omega \cap V} = f$. \qed
3 Necessity of $PL_{loc}$

Using $|z| = \max_j |z_j|$ we put $D_r = \{ x \in \mathbb{R}^d : |x| \leq r \}$. By $V_r$ we denote the pluricomplex Green function of $D_r$ (see [11, p. 207]) and set

$$D_{r,\alpha} = \{ z \in \mathbb{C}^d : V_r(z) < \alpha \}, \quad W_{r,\alpha} = D_{r,\alpha} \cap V.$$ 

By $| |_{r,\alpha}$ we denote the norm of $H^\infty(D_{r,\alpha})$ and by $\| |_{r,\alpha}$ the norm in $H^\infty(W_{r,\alpha})$.

In complete analogy to [19] we obtain from [22, 23]

**Lemma 3.1** For $0 < \alpha_1 < \alpha'_2 < \alpha_2 < \alpha_3$ we have $C > 0$ so that

$$|\eta|^{s_{\alpha_3-\alpha_1}}_{r,\alpha_2} \leq C |\eta|^{s_{\alpha_3-\alpha'_2}}_{r,\alpha_1} |\eta|^{s_{\alpha'_2-\alpha_1}}_{r,\alpha_3}$$

for all $\eta \in H^\infty(D_{r,\alpha_1})'$.

**Lemma 3.2** For $0 < \alpha_1 < \alpha_2 < \alpha_3$ and $f \in H^\infty(D_{r,\alpha_3})$ we have

$$|f|^{s_{\alpha_3-\alpha_1}}_{r,\alpha_2} \leq |f|^{s_{\alpha_3-\alpha_2}}_{r,\alpha_1} |f|^{s_{\alpha_2-\alpha_1}}_{r,\alpha_3}.$$

Since the restriction of $H^\infty(W_{r,\alpha})$ to $W_{r,\alpha'}$ is contained in the range of the restriction of $H^\infty(D_{r,\alpha})$ to $W_{r,\alpha'}$ we obtain from Lemma 3.1:

**Lemma 3.3** For $0 < \alpha_1 < \alpha'_2 < \alpha_2 < \alpha_3$ we have $C > 0$ so that

$$\| \eta \|^{s_{\alpha_3-\alpha_1}}_{r,\alpha_2} \leq C \| \eta \|^{s_{\alpha_3-\alpha'_2}}_{r,\alpha_1} \| \eta \|^{s_{\alpha'_2-\alpha_1}}_{r,\alpha_3}$$

for all $\eta \in H^\infty(W_{r,\alpha_1})'$.

We set

$$A(\mathbb{R}^d) = \text{proj.} \text{ind}_a H^\infty(D_{r,\alpha})$$

with the locally convex limit topologies, and likewise

$$H_V(X) = \text{proj.} \text{ind}_a H^\infty(W_{r,\alpha})$$

and it easy to show that $H_V(X)$, equipped with this topology carries the quotient topology of $A(\mathbb{R}^d)$ under $\rho$. Both space are (PLB) spaces i.e. countable projective limits of unions of Banach spaces. From the exact sequence (1) we see that (P) is complemented if and only if $\rho$ has a continuous linear right inverse.
Proposition 3.4 If $(P)$ is complemented in $A(\mathbb{R}^d)$, then $V$ satisfies $PL_{loc}$ in every $x \in X$.

Proof: Let $\varphi$ be a right inverse for $\rho$. From the theory of (PLB)-spaces (see e.g. [7, p. 63]) we know that for very $r$ there is $R$ so that we have a factorization expressed in the left square of the following commutative diagram

$$
\begin{array}{ccc}
A(\mathbb{R}^d) & \longrightarrow & A(D_r) \\
\uparrow \varphi & & \uparrow \tilde{\varphi} \\
H_V(X \cap D_r) & \longrightarrow & H_V(X \cap D_r) \\
\end{array}
$$

The unnamed maps are the natural restrictions.

Now, arguing precisely as in the proof of Lemma 5.3 of [19] we obtain $\epsilon > 0$ and $C_\alpha$ so that

$$
|\tilde{\varphi}f|_{r,\epsilon\alpha} \leq C_\alpha \|f\|_{R,\alpha}
$$

for all $f \in H^\infty(V \cap D_{R,\alpha})$.

We get the following chain of inequalities for $0 < \alpha_1 < \alpha_2 < \alpha_3$ and $f \in H^\infty(W_{r,\alpha_3})$

$$
\|f\|_{r,\epsilon\alpha_2} \leq |\tilde{\varphi}f|_{r,\epsilon\alpha_2} \\
\leq |\tilde{\varphi}f|_{r,\epsilon\alpha_1}\|\tilde{\varphi}f\|_{r,\epsilon\alpha_3} \\
\leq C_{\alpha_1}^{\alpha_3-\alpha_2}C_{\alpha_3}^{\alpha_2-\alpha_1}\|f\|_{R,\alpha_1}\|f\|_{R,\alpha_3}.
$$

By applying this to $f^n$, taking $n$-th roots and letting $n$ go to infinity we obtain:

$$
\|f\|_{r,\epsilon\alpha_2} \leq \|f\|_{R,\alpha_2}^{\alpha_3-\alpha_2}\|f\|_{R,\alpha_1}^{\alpha_2-\alpha_1}.
$$

We set $\|f\|_{R,0} = \sup\{|f(x)| : x \in X \cap D_R\}$. Letting $\alpha_1$ tend to 0 and replacing $\alpha_2$ by $\alpha$ and $\alpha_3$ by $\gamma$ we get for $0 < \alpha < \gamma$ and $f \in H^\infty(W_{R,\gamma})$:

$$
\|f\|_{r,\epsilon\alpha} \leq \|f\|_{R,0}\|f\|_{R,\gamma}^{\frac{\gamma-\alpha}{\gamma-\alpha_1}}.
$$

This means that for any function $u(z) = c \log |f(z)|$ where $f$ is holomorphic on $W_{R,\gamma}$, $c > 0$, so that $u(z) < 0$ on $\mathbb{R}^d \cap W_{R,\gamma}$ and $u(z) < 1$ on $W_{R,\gamma}$ we have

$$
u(z) \leq \frac{\alpha}{\gamma}, \quad z \in W_{r,\epsilon\alpha}.
$$
We fix now $0 < \rho < r$, $\gamma > 0$. Form the explicit form of the level sets of $V_{D_r}$ (see [11], p. 207) we see that there is $\delta > 0$ so that for any $z \in \partial D_{r,\varepsilon\alpha}$, $\alpha < \gamma$, with $|x| \leq \rho$ we have $|y| \geq \delta \alpha$.

Therefore there is $A$ so that for any $u(z) = c \log |f(z)|$ where $f$ is holomorphic on $W_{R,\gamma}$, $c > 0$, so that $u(z) < 0$ on $R^d \cap W_{R,\gamma}$ and $u(z) < 1$ on $W_{R,\gamma}$ we have

$$u(z) \leq A |\text{Im} z|, \quad z \in W_{r,\varepsilon\gamma} \cap \{z : |\text{Re} z| \leq \rho\}.$$

By standard arguments (cf. [19]) we conclude that $V$ has $PL_{\text{loc}}$ in any $x \in R^d \cap V \cap D_{\rho}$.

As $r$ and $\rho < r$ was arbitrary this proves the result.

4 Consequences of the necessary condition

Before we narrow our focus to homogeneous polynomials for which we are able to prove the converse of Proposition 3.4 and obtain a complete characterization, we show some consequences of the necessary condition proved so far.

If $V$ satisfies $PL_{\text{loc}}$ in some point $a \in X$, then for any $f \in \mathcal{O}_a$ with $f|_{X_a} = 0$ we have $f|_{V_a} = 0$, i.e. the ideal $\mathcal{I}_{X_a}$ of $X_a$ in $\mathcal{O}_a$ coincides with the ideal $\mathcal{I}_{V_a}$ of $V_a$ and, for irreducible $P$, is $P \cdot \mathcal{O}_a$. Since $\tilde{P}(z) := \overline{P(z)}$ vanishes on $X_a$ the polynomial $P$ must, in this case, divide $\tilde{P}$, i.e. $P$ has to be proportional to a real polynomial. For general $P$ satisfying $PL_{\text{loc}}$ in $a$ we know that every irreducible factor satisfies $PL_{\text{loc}}$ in $a$. Therefore every irreducible factor which vanishes in $a$ has to be proportional to a real polynomial. Finally we conclude from the equality of ideals that $\dim R X_a = \dim C V_a = d - 1$.

If $V$ satisfies $PL_{\text{loc}}$ in every $a \in X$ we get immediately:

**Corollary 4.1** If $(P)$ is complemented in $A(R^d)$ then $P = P_1 P_2$ where $P_1$ has no real zeros and $P_2$ is a real polynomial, $X$ is of pure dimension $d - 1$ and $H_V(X) = A(X)$.

For all this cf. [10], [14], [15], [19]. We may prove an even stronger local version. Let $a \in X$ and $P_a = f_1 \cdots f_p$ an irreducible decomposition of the germ $P_a$ of $P$ in $\mathcal{O}_a$. Put $V_{a,j}$ the germ of the complex zero variety of $f_j$ in $a$ and $X_{a,j} = V_{a,j} \cap R^d$. Then every $V_{a,j}$ satisfies $PL_{\text{loc}}$, hence $X_{a,j}$ is coherent, of pure dimension $d - 1$, and $f_j$ is up to a unit in $\mathcal{O}_a$ real on $R^d$.

**Proposition 4.2** If $P$ is a real polynomial and $(P)$ complemented in $A(R^d)$ then for every $a \in X$ the germ $X_a$ is coherent and the germ $P_a$ has an, in $\mathcal{O}_a$, irreducible decomposition $P_a = f_1 \cdots f_p$, where all $f_j \in \mathcal{O}_{a}^{\mathbb{R}}$, and their real zero varieties are coherent and have pure dimension $d - 1$.
We use the following notation:
We set \( J_X = \{ f \in A(\mathbb{R}^d) : f|_X = 0 \} \), \( J_V = \{ f \in H(\mathbb{C}^d) : f|_V = 0 \} \). Moreover we set \((P)^\infty = P \cdot C^\infty(\mathbb{R}^d)\) and \( J^\infty_X = \{ f \in C^\infty(\mathbb{R}^d) : f|_X = 0 \} \). We say that \( P \) has the ideal property (or propriété des zéros) if \((P)^\infty = J^\infty_X\) and that \( P \) has the analytic ideal property if \((P) = J_X\).

**Lemma 4.3** If \( P \) has the ideal property then also the analytic ideal property. If \( P \) has the analytic ideal property and \( X_a \) is coherent in every \( a \in X \) then \( P \) has the ideal property.

**Proof:** For the first part see [2, Proposition 3]), the second follows from [12, Theorem 3.10] and a partition of unity argument. \( \square \)

Since it is obvious from the previous that the \( PL_{\text{loc}} \) condition being fulfilled in every point of \( X \) implies the analytic ideal property, and it implies also coherence of \( X \) in every \( a \in X \), we obtain from Lemma 4.3:

**Theorem 4.4** If \((P)\) is complemented in \( A(\mathbb{R}^d) \) then \( P \) has the ideal property and the analytic ideal property.

Now, from results of Bierstone and Schwarz [1] and Langenbruch [10, Theorem 1.6] we know that \((P)^\infty\) is complemented in \( C^\infty(\mathbb{R}^d)\) if and only if \( P \) has the ideal property and we can conclude:

**Corollary 4.5** If \((P)\) is complemented in \( A(\mathbb{R}^d) \), then \((P)^\infty\) is complemented in \( C^\infty(\mathbb{R}^d)\).

**Example:** The real zero variety of the irreducible polynomial \( P(x, y) = y^3 - x^3(1 + x^2) \) is of pure dimension 1. In \( \mathcal{O}_0 \) it decomposes into one real and two complex factors, hence, by Proposition 4.2 \( P \) is not complemented. In \( A(\mathbb{R}^d) \) it decomposes into two real valued factors,

\[
\begin{align*}
  f_1(x, y) &= y - x^{3/2}(1 + x^2), \\
  f_2 &= \left(y + \frac{x}{2} \sqrt{1 + x^2}\right)^2 + \frac{3}{4}x^2(1 + x^2)^{3/2}.
\end{align*}
\]

\( f_1 \) describes \( X \), \( f_2 \) vanishes in 0 only. Hence \( P \) does even not have the analytic ideal property. On the other hand there is a continuous linear projection in \( A(\mathbb{R}^d) \) onto \( J \), namely \( f(x, y) \mapsto f(x, y) - f(x, x \sqrt{1 + x^2}) \).

Let us remark that the description in Proposition 4.2 corresponds to the characterization of the ideal property contained in Bochnak [2, Corollaire 2].

Let \( P \) have the ideal property and \( \pi \) be a continuous linear projection in \( C^\infty(\mathbb{R}^d) \) onto \((P)^\infty\). Assume now that \((P)\) is not complemented, then \( \pi \) necessarily sends some
\[ f \in A(\mathbb{R}^d) \] to \( \pi(f) \in C^\infty(\mathbb{R}^d) \setminus A(\mathbb{R}^d) \), because \( \pi A(\mathbb{R}^d) \subset A(\mathbb{R}^d) \) would, due to the closed graph theorem, imply that \( \pi|_{A(\mathbb{R}^d)} \) is a continuous linear projection in \( A(\mathbb{R}^d) \) onto \((P)\).

## 5 Sufficiency of \( PL_{loc} \), local case

From Proposition 3.4 and [19], Theorem 2.2 we get the following theorem. Notice that \( PL_{loc} \) in any real point of \( V \) implies that \( X \) is coherent (see [19]), hence \( A(X) = H_v(X) \).

**Theorem 5.1** If \( X \) is compact, then \((P)\) is complemented if and only if \( V \) satisfies \( PL_{loc} \) in any \( x \in X \).

We will now study the case of a homogeneous \( P \). As a first step we extend the proof of [19], Proposition 4.5, i. e. the sufficiency part of Theorem 2.2 there, from the case of compact \( X \) to a semiglobal result in the general case.

**Theorem 5.2** If \( V \) satisfies \( PL_{loc} \) in any real point, then for every compact \( K \subset \mathbb{R}^d \) there is a continuous linear map \( \varphi_K : A(X) \to H(K) \) so that \( \varphi_K f|_{K \cap X} = f|_{K \cap X} \) for all \( f \in A(X) \).

**Proof:** We choose \( r > 0 \) so that \( K \) is contained in the interior of \( D_r \). We use the following neighborhoods of \( D_r \)

\[ U_{r,\alpha} = \{ z : |x| < r + \alpha, |y| < \alpha \}. \]

They are analytic polyhedra.

By a compactness argument the \( PL_{loc} \)-condition gives us a neighborhood

\[ U = \{ z : |x| < R, |y| < \gamma \} \]

of \( D_r \) in \( \mathbb{C}^d \) and constants \( A \) and \( \gamma_0 > 0 \), so that for any plurisubharmonic function \( u \) on \( V \) with \( u(z) < 1 \) for \( z \in U \cap V \) and \( u(x) < 0 \) for \( x \in U \cap X \) we have \( u(z) \leq A|\text{Im } z| \) for \( z \in U_{r,\gamma_0} \).

Let \( \omega(z) = \omega(D_r \cap X, U_{r,\gamma} \cap V, z) \) the relative extremal function on \( U_{r,\gamma} \cap V \), i. e. the upper regularization of

\[ \sup\{ u(z) : u \text{ plurisubharmonic on } V, u < 1 \text{ on } U_{r,\gamma} \cap V, u < 0 \text{ on } D_r \cap X \}. \]

Then the above condition says that there is a constant \( A \) and \( \gamma_0 > 0 \) so that

\[ \omega(z) \leq A|\text{Im } z|, \quad z \in V \cap U_{r,\gamma_0}. \]
We set
\[ \tilde{V}_{r, \alpha} = V \cap U_{r, \alpha}, \quad V_\beta = \{ z \in V : \omega(z) < \beta \} \]
and obtain for \( 0 < \alpha < \gamma_0 \)
\[ \tilde{V}_{r, \alpha} \subset V_{A\alpha} \]
Changing our previous notation we denote now by \( || \cdot ||_{r, \alpha} \) the norm in \( H^\infty(U_{r, \alpha}) \) and by \( || \cdot ||_\beta \) the norm in \( H^\infty(V_\beta) \). In analogy to Lemmas 3.1 and 3.2 we obtain
For \( 0 < \alpha_1 < \alpha_2 < \alpha_3 \) and \( \eta \in H^\infty(U_{r, \alpha_1})' \) we have
\[ ||\eta||_{r, \alpha_2}^{\alpha_3 - \alpha_1} \leq C \|||\eta||_{r, \alpha_1}^{\alpha_3 - \alpha_2} \||\eta||_{r, \alpha_3}^{\alpha_2 - \alpha_1}. \]
For \( 0 < \alpha_1 < \alpha_2 < \alpha_3 \) and \( f \in H^\infty(V_{\alpha_3}) \) we have
\[ |f|_{r, \alpha_2}^{\alpha_3 - \alpha_1} \leq |f|_{r, \alpha_1}^{\alpha_3 - \alpha_2} |f|_{r, \alpha_3}^{\alpha_2 - \alpha_1}. \]
Moreover we have for any small \( \alpha \) the following diagram with exact row:
\[
\begin{array}{cccccc}
0 & \longrightarrow & H(U_{r, \alpha}) & \xrightarrow{M_{P}} & H(U_{r, \alpha}) & \longrightarrow & H(\tilde{V}_{r, \alpha}) & \longrightarrow & 0 \\
& & & & & & \uparrow & & \\
& & & & & & H(V_{A\alpha}) & & \\
\end{array}
\]
where \( M_{P} \) is the multiplication with \( P \) and the unnamed arrows are the restriction maps.
As in the proof of [19], Proposition 4.5, we conclude that there is a continuous linear map \( \varphi : A(X) \rightarrow H(D_r) \) with the desired property. \( \square \)
If \( V \) satisfies \( PL_{loc} \) in any real point then we have for every \( n \in \mathbb{N} \) a map \( \varphi_n \in L(A(X), H(D_n)) \) with \( \varphi_n(f)|_X = f|_X, \) for all \( F \in A(\mathbb{R}^d) \). If there exists a sequence \( \psi_n \in L(A(X), H(D_n)) \), \( n \in \mathbb{N} \), so that (omitting the restriction map) \( \psi_{n+1} - \psi_n = \varphi_n \) for all \( n \), then we obtain a right inverse \( \varphi \) for \( \rho \). Therefore for a proof that \( (P) \) is complemented it would suffice to show that \( \text{Proj}_{\rho}^1 L(A(X), H(D_n)) = 0 \). Unfortunately this is not known. However, in the homogeneous case we can use a simpler argument.

6 Sufficiency of \( PL_{loc} \), homogeneous case

Let \( P \) now be a homogeneous polynomial of degree \( m > 0 \) so that \( V \) satisfies the \( PL_{loc} \)-condition in every real point. In consequence \( P \) cannot be elliptic.
Therefore $X \neq \{0\}$. We set as previously $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$, $X_* = \mathbb{R}^d_* \cap X$.

From Theorem 1.5 we obtain:

**Lemma 6.1** There exists a continuous linear extension operator $A(X_*) \longrightarrow A(\mathbb{R}^d_*)$.

So we have an extension operator near 0 and one off 0. To patch them together we need the following lemma. For $0 \leq r < \rho \leq \infty$ we put $D^0_{r,\alpha} = \{ x : r < \| x \| < \rho, \| \text{Im} \, x \| < \alpha \}$.

**Lemma 6.2** For any $0 < r < \rho$ there are $\sigma_1, \sigma_2$ with $0 < \sigma_1 < r < \rho < \sigma_2 < \pi/2 < R$.

**Proof:** Due to the real analytic diffeomorphism $x \mapsto (\arctan x_1, \ldots, \arctan x_d)$ it suffices to replace $\psi_\infty$ by $\psi_R : H(D^0_{\sigma_1}) \longrightarrow H(D^R_r)$ for large $R > \sigma_2$ in the statement of the lemma. In fact, one would need only to consider the case of $0 < \sigma_1 < r < \rho < \sigma_2 < \pi/2 < R$.

We put $w(z) = \sqrt{\sum_{j=1}^d z_j^2}$ and set for small $\alpha > 0$

$$D^0_{r,\alpha} = \{ z \in \mathbb{C}^d : r - \alpha < w(z) < \rho + \alpha; \| \text{Im} \, z \| < \alpha \}.$$  

The $D^0_{r,\alpha}$ are analytic polyhedra and $\bigcap_\alpha D^0_{r,\alpha} = D^0_r$. From Zaharjuta [22, 23] we learn that for the spaces $H^\infty(D^0_{r,\alpha})$ we have inequalities like in Lemma 3.1.

Since $D^0_{0,\alpha} \cap D^R_{r,\alpha} = D^0_{r,\alpha}$ we obtain by the Cartan-Oka theory the exactness of the row in the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H(D^0_{0,\alpha}) & \longrightarrow & H(D^0_{0,\alpha}) \oplus H(D^R_{r,\alpha}) & \longrightarrow & H(D^r_{r,\alpha}) & \longrightarrow & 0 \\
& & & \uparrow & & & & & \\
& & & & & H(U_{\alpha}) & & & &
\end{array}
$$

Here $U_{\alpha} = \{ z : \omega(z) < \alpha \}$ where $\gamma < \sigma_1$ and $\omega(z) = \omega(D^0_{\sigma_1}, D^0_{\sigma_1}, \gamma, z)$ is the relative extremal plurisubharmonic function. Comparison with the pluricomplex Green function of neighborhoods of points in $D^\sigma_{\sigma_1}$ shows that for small $\alpha$ and suitable $A > 0$ we have $\omega(z) \leq A|\text{Im} \, z|$ on $D^\rho_{r,\alpha}$, hence $D^\rho_{r,\alpha} \subset U_{\alpha}$. The vertical arrow in the diagram now means the restriction map.

For the norms in $H^\infty(U_{\alpha})$ we obtain, due to the maximality of $\omega$ the inequalities like in Lemma 3.2. As previously by small changes of the $\alpha$ we can set up the scheme for the application of [18] and obtain the result. **□**
Proposition 6.3 If $V$ satisfies $PL_{\text{loc}}$ in any point of $X$, then there is a continuous linear extension operator $\varphi : A(X) \rightarrow A(\mathbb{R}^d)$.

Proof: By Theorem 5.2 there is a continuous linear operator $\varphi_0 : A(X) \rightarrow H(D_{\sigma_1}^{\sigma_2})$ so that $\varphi_0 f = f$ on $X \cap D_{\sigma_1}^{\sigma_2}$, and by Lemma 6.1 there is a continuous linear map $\varphi_\infty : A(X) \rightarrow A(\mathbb{R}^d)$ so that $\varphi_\infty f = f$ on $X^*$.

We choose $\chi : A(X) \rightarrow H(D_{\sigma_1}^{\sigma_2})$ so that $P \cdot \chi f = \varphi_0 f - \varphi_\infty f$ on $D_{\sigma_1}^{\sigma_2} \cap X$. With $\psi_0$ and $\psi_\infty$ of Lemma 6.2 we put for $f \in A(X)$ $\varphi f = \varphi_0 f - P \cdot \psi_0(\chi f)$ on $D_0^\rho$ and $\varphi f = \varphi_\infty f + P \cdot \psi_\infty(\chi f)$ on $D_\infty^\rho$.

On $D_0^\rho$ we have

$$(\varphi_0 f - P \cdot \psi_0(\chi f)) - (\varphi_\infty + P \cdot \psi_\infty(\chi f)) = P \cdot \chi f - P \cdot \chi f = 0.$$ 

Therefore $\varphi$ is well defined and the assertion is proved. \qed

7 Main theorems

From Propositions 3.4, 6.3 and Theorem 5.1 we obtain the following characterization:

Theorem 7.1 If $P$ is homogeneous or has a compact real zero set then $(P)$ is complemented if and only if $V$ satisfies $PL_{\text{loc}}$ in every real point.

If we restrict our attention to the homogeneous case and combine our result with the results in [15, Theorem 3.13], [9] and [14] then we have the following result:

Theorem 7.2 For homogeneous $P$ the following are equivalent:

1. The principal ideal of $P$ is complemented in $A(\mathbb{R}^d)$.
2. $P(D)$ is non-elliptic and $P(D) : A(\mathbb{R}^d) \rightarrow A(\mathbb{R}^d)$ is surjective.
3. $P(D) : C^\infty(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d)$ has a continuous linear right inverse.
4. $V$ satisfies $PL_{\text{loc}}$ in any real point of $V$. 

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8 Examples and further results

Concrete example of homogeneous polynomials to satisfy or not satisfy $PL_{loc}$ in every real point of its zero variety, we call them of type $PL$ from now on, can be found in [3], [4], [5], [9], [14], [15], [19]. We present some of them. We always assume, without restriction of generality, that $P \in \mathbb{R}[x_1, \ldots, x_n]$.

If $n = 2$ the $P$ is a product of linear forms, and $PL$ means that all of them must be real. General, not necessarily homogeneous, $P$ is of type $PL$ if and only if all real singularities of $X$ are intersections of smooth lines (see [5]). A good example is $P_1(x, y) = y^2 - x^2 + x^4$ (lemniscate), a bad one $P_2(x, y) = y^2 - x^3 + x^5$ (see [19]).

For $\deg P = 2$ the polynomial $P$ is of type $PL$ if and only if the underlying quadratic form is indefinite or the product of two real linear forms (see [9]).

Similarly, if $n \geq 3$ and $P$ has the form

$$P(x_1, \ldots, x_n) = \sum_{k=1}^{n} a_k x_k^m,$$

then $P$ is of type $PL$ if and only if either $m$ is odd or the $a_k$ have different signs or only one $a_k \neq 0$ (see [14]).

We recall that a real analytic variety $X \subset \mathbb{R}^d$ is called of type $PL$ if its complexification in every point satisfies $PL_{loc}$. Now with exactly the same modifications of the proof of [19], Proposition 4.5. as applied to show Theorem 5.2 we can prove

**Theorem 8.1** If $X$ is of type $PL$, then for every compact $K \subset \mathbb{R}^d$ there is a continuous linear map $\varphi_K : A(X) \to H(K)$ so that $\varphi_K f|_{K \cap X} = f|_{K \cap X}$ for all $f \in A(X)$.

This modification could serve to extend [19], Theorem 2.2 also to noncompact real analytic varieties. That would be immediate if $\text{Proj}^1_L(A(X), H(K_n)) = 0$ which, unfortunately, is not known.

References


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