LECTURES ON RESTRICTION SPACES OF A^∞

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1 Preliminaries

With $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| < 1\}$ we set

$$A^\infty := C^\infty(\overline{\mathbb{D}}) \cap H(\mathbb{D})$$

where $H(\mathbb{D})$ denotes the holomorphic functions on $\mathbb{D}$. Let E be a proper compact subset of the unit circle. We will study the space

$$A_\infty(E) := A^\infty|_E = \{f|_E \,:\, f \in A^\infty\}$$

equipped with the quotient topology of the restriction map.

An equivalent representation of A^∞ is

$$A^\infty = \{f \in C^\infty_{2\pi}(\mathbb{R}) \,:\, \text{ all negative Fourier coefficients vanish}\}.$$

Here $C^\infty_{2\pi}(\mathbb{R})$ denotes the 2π-periodic C^∞-functions on $\mathbb{R}$.

So A^∞ can be considered as the space of all functions on $\mathbb{R}$ which have a Fourier expansion $f(t) = \sum_{k=0}^\infty a_k e^{ikt}$ with rapidly decreasing Fourier coefficients $(a_0, a_1, \dots)$, that is $\sum_{k=0}^\infty |a_k|(k+1)^p < \infty$ for all $p \in \mathbb{N}_0$.

The map $F \mapsto (a_0, a_1, \dots)$ defines an isomorphism $A^\infty \cong s$, where s denotes the space of rapidly decreasing sequences.

We will adopt throughout this draft the representation of A^∞ as periodic C^∞-functions. Then $A_\infty(E)$ is given in the following way: $E \subset [0, 2\pi[$ compact and $A_\infty(E)$ is defined as above.

We set $I_A(E) := \{f \in A^\infty \,:\, f|_E = 0\}$, that is, $I_A(E)$ is the ideal of E in the algebra A^∞. Then

$$A_\infty(E) \cong A^\infty / I_A(E)$$

and it is a nuclear Fréchet space.

Example: If E has not Lebesgue measure 0, then $I_A(E) = \{0\}$ hence $A_\infty(E) = A^\infty$.

Definition 1.1 *E is called a Carleson set if*

$$\int_0^{2\pi} \log \frac{1}{d(x,E)}\,dx < +\infty.$$

Theorem 1.2 (Taylor and Williams, Novinger) *The following are equivalent:*

1. *$I_A(E) \neq \{0\}$.*
2. *There is $f \in A^\infty$ such that $\{t \in [0, 2\pi[: f(t) = 0\} = E$.*
3. *E is a Carleson set.*

Lemma 1.3 *If $\varepsilon_1, \varepsilon_2, \ldots$ denote the lengths of the disjoint intervals of which $[0, 2\pi[\setminus E$ consists, then E is a Carleson set if, and only if,*

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} < +\infty.$$

Proof: For $0 \leq a < b$ we have

$$\int_a^b \log \frac{1}{d(x,\{a,b\})} dx = (b-a)\log\frac{1}{b-a} + (1+\log 2)(b-a). \tag{1}$$

If $]a,b[$ is one of the disjoint intervals in $[0,2\pi[\setminus E$, then $d(x,E) = d(x,\{a,b\})$, from where easily follows the equivalence. □

Examples: 1. $E = \{x_1, x_2, \ldots\} \cup \{0\}$, $x_n \searrow 0$. Then $\varepsilon_n = x_n - x_{n+1}$.

Assumption: There are $q \in \mathbb{N}$, $C > 0$ such that $x_n^q \leq C\,\varepsilon_n$ for all $n \in \mathbb{N}$.

Then

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} \leq 2\pi \log C + q \sum_n \varepsilon_n \log \frac{1}{x_n} \leq 2\pi \log C + q \int_0^{2\pi} \log \frac{1}{x} dx < +\infty.$$

2. $E =$ the classical Cantor set. Then, obtaining the ε_n from the stepwise construction,

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \sum_{k=1}^{\infty} 2^{k-1} 3^{-k} \log 3^k = \frac{\log 3}{3} \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^{k-1} = 3 \log 3 < +\infty.$$

In both cases the set E is a Carleson set.

3. An example of a set of the type in case 1. failing our assumption and not being a Carleson set would be $x_n = \frac{1}{\log n}$ for $n = 2, 3, \ldots$.

In this case $\frac{1}{n \log(n+1) \log n} \geq \varepsilon_n \geq \frac{1}{(n+1) \log(n+1) \log n}$, $\log \frac{1}{\varepsilon_n} \geq \log n$ for large n, from which the claim is easily derived.

2 The problem

Claim (Patel 2011): For every Carleson set E the space $A_\infty(E)$ does not have a basis.

Basis: $e_1, e_2, \ldots$ is a basis of the topological vector space X if every $x \in X$ has a unique expansion $x = \sum_n x_n e_n$.

It was an important Problem of Grothendieck whether every nuclear Fréchet space has a basis. It was solved in the negative by Mityagin and Zobin . Many counterexamples have been given since then. If the claim was true it would have been quite interesting as being appearing in a not ad hoc constructed natural environment as a consequence of structural properties.

Our main result will be

Theorem 2.1 *For $E = \{2^{-n} : n \in \mathbb{N}\}$ and the for E being the classical Cantor set the space $A_\infty(E)$ has a basis.*

so disproving the above mentioned claim.

We will proceed in two steps:

1. Study the space $C_\infty(E) := \{f|_E : f \in C^\infty(\mathbb{R})\}$.
2. Compare the spaces $C_\infty(E)$ and $A_\infty(E)$. Show that in our cases they coincide.

3 Structure of $C_\infty(E)$

A complete characterization of the function in $C_\infty(E)$ has been given by Whitney 1934 . We will not use this and give a more suitable description in our special cases.

Most of the following applies to arbitrary compact subsets E of $\mathbb{R}$. But, due to our intended application we will assume $E \subset [0, 2\pi[$. It is quite obvious that

$$E_\infty(\mathbb{R}) := \{f|_E \,:\, f \in C^\infty(\mathbb{R})\} = \{f|_E \,:\, f \in C^\infty_{2\pi}(\mathbb{R})\} = \{f|_E \,:\, f \in C^\infty([0, \pi])\}$$

and the quotient topologies of the restriction in all three descriptions coincide.

Therefore the topology of $E_\infty(E)$ can be given by the quotient norms

$$|||\varphi|||_k := \inf\{\|f\|_k \,:\, f \in C^\infty(\mathbb{R}),\ f|_E = \varphi\}$$

where $\|f\|_k = \sup\{|f^{(p)}(x)| \,:\, p = 0, .., k,\ x \in [0, 2\pi]\}$.

We set $J(E) := \{f \in C^\infty(\mathbb{R}) \,:\, f|_E = 0\}$, that is, $J(E)$ is the ideal of E in $C^\infty(\mathbb{R})$. Then

$$C_\infty(E) = C^\infty(\mathbb{R})/J(E)$$

and $C_\infty(E)$ is a nuclear Fréchet space.

Lemma 3.1 *Let 0 be an accumulation point of E, $\varphi \in C_\infty(E)$ and $\varphi = f|_E$ where $f \in C^\infty(\mathbb{R})$. Then $f^{(p)}(0)$ is uniquely determined by φ for all $p \in \mathbb{N}_0$.*

Proof: We proceed by induction: $f^{(0)}(0) = f(0) = \varphi(0)$

Assume $f^{(0)}(0), \ldots, f^{(p)}(0)$ to be determined. For $x \in E,\ x \neq 0$ there is ξ between x and 0 such that

$$f^{(p+1)}(\xi) = \frac{(p+1)!}{x^{p+1}} \left(f(x) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j!}\, x^j \right).$$

For $x \to 0$ we have $\xi \to 0$ and therefore

$$f^{(p+1)}(0) = \lim_{x \in E, x \to 0} \frac{(p+1)!}{x^{p+1}} \left(\varphi(x) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j!}\, x^j \right) \tag{2}$$

and this determines explicitly $f^{(p+1)}(0)$. □

Definition 3.2 $\varphi^{(p)}(0) := f^{(p)}(0)$ *where $f \in C^\infty(\mathbb{R})$ with $f|_E = \varphi$.*

Corollary 3.3 *If 0 is an accumulation point of E and $f \in J(E)$ then $f^{(p)}(0) = 0$ for all $p \in \mathbb{N}_0$.*

Proof: f is an extension of $\varphi \equiv 0$. $\square$

Of course the previous applies, mutatis mutandis, to any accumulation point of E.

We set $J^\infty(E) := \{f \in C^\infty(\mathbb{R}) : f^{(p)}(x) = 0 \text{ for all } p \in \mathbb{N}_0 \text{ and } x \in E\}$. Then $\mathscr{E}(E) = C^\infty(\mathbb{R})/J^\infty(E)$ is the space of Whitney-jets on E.

Corollary 3.4 *If E is perfect, then $J(E) = J^\infty(E)$.*

and this implies

Proposition 3.5 *If E is perfect, then $C_\infty(E) = \mathscr{E}(E)$.*

This applies, in particular, to the Cantor set.

We will use the following two theorems.

Theorem 3.6 (Tidten) *If E is the Cantor set, then $\mathscr{E}(E)$ is isomorphic to a complemented subspace of s.*

Theorem 3.7 (Aytuna, Krone, Terzioğlu) *IF X is a complemented subspace of s and $X \oplus X \cong X$, then X has a basis, more precisely: then $X \cong \Lambda_\infty(\alpha)$ for some α.*

Here we define for any sequence $\alpha_1 \leq \alpha_2, \leq \cdots \nearrow +\infty$

$$\Lambda_\infty(\alpha) = \{\xi = (\xi_1, \xi_2, \dots) : |\xi|_p := \sup_n |\xi_n| e^{p\alpha_n} < +\infty \text{ for all } p \in \mathbb{N}_0\}.$$

Equipped with the norms $|\cdot|_p$ this is a Fréchet space.

We obtain:

Proposition 3.8 *If E is the Cantor set, then $C_\infty(E)$ has a basis.*

4 Structure of $C^\infty(E)$ if E has one accumulation point

We assume that $E = \{x_n : n \in \mathbb{N}\} \cup \{0\}$ where $x_n \searrow 0$. In this section we set $\varepsilon_n = \min(x_n - x_{n-1}, x_{n+1} - x_n)$ and as before we make the

Assumption: There are $q \in \mathbb{N}$, $C > 0$ such that $x_n^q \leq C\,\varepsilon_n$ for all $n \in \mathbb{N}$.

From our previous example we know that E is a Carleson set.

We introduce the following notation:

$$\begin{aligned} J^\infty(0) &= \{f \in C^\infty(\mathbb{R}) \,:\, f^{(p)}(0) = 0 \text{ for all } p \in \mathbb{N}_0\} \\ J_\infty(0) &= \{\varphi \in C_\infty(E) \,:\, \varphi^{(p)}(0) = 0 \text{ for all } p \in \mathbb{N}_0\} = \{f|_E \,:\, f \in J^\infty(0)\} \end{aligned}$$

and we first study the space $J_\infty(0)$.

We choose an even $\chi \in \mathscr{D}[-\frac{1}{2}, +\frac{1}{2}]$ with $\chi \equiv 1$ in a neighborhood of 0 and we set $\chi_\varepsilon(x) := \chi(\frac{x}{\varepsilon})$.

For any scalar sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$ we set $f(x) := \sum_{n=1}^\infty \xi_n \chi_{\varepsilon_n}(x - x_n)$. Then $f \in C^\infty(\mathbb{R} \setminus \{0\})$ and $f(x_n) = \xi_n$ for all n.

Lemma 4.1 *$f \in J^\infty(0)$ if, and only if, $\lim_{n\to\infty} \frac{\xi_n}{\varepsilon_n^p} = 0$ for all $p \in \mathbb{N}_0$.*

Proof: For every p and N we have

$$\sup_{0<|x|\le|x_N|} |f^{(p)}(x)| = \sup_{n\ge N} |\xi_n| \, \|\chi_{\varepsilon_n}^{(p)}\| = \|\chi^{(p)}\| \sup_{n\ge N} \frac{|\xi_n|}{\varepsilon_n^p}$$

which proves the assertion. □

The following holds without our general assumption on the sequence $(x_n)_{n\in\mathbb{N}}$.

Lemma 4.2 *1. If $\varphi \in J_\infty(E)$ then $\lim_{n\to\infty} \frac{1}{x_n^p}|\varphi(x_n)| = 0$ for all $p \in \mathbb{N}_0$.*
2. If $\varphi \in C(E)$ and $\lim_{n\to\infty} \frac{1}{\varepsilon_n^p}|\varphi(x_n)| = 0$ for all $p \in \mathbb{N}_0$ then $\varphi \in J_\infty(0)$.

Proof: 1. Let $\varphi = f|_E \in J_\infty(0)$, $f \in J^\infty(0)$. Then we have

$$\frac{1}{x_n^p}|\varphi(x_n)| = \frac{1}{x_n^p}|f(x_n)| \le \frac{1}{p!}\|f^{(p)}\|_{[0,x_n]} \tag{3}$$

and the right hand side converges to zero.

2. Follows from the previous Lemma, because

$$f(x) = \sum_{n=1}^\infty \varphi(x_n)\chi_{\varepsilon_n}(x) \in J^\infty(0)$$

and $f|_E = \varphi$. □

Using our assumption on the sequence we obtain:

Proposition 4.3 *Let $\varphi \in C(E)$. Then $\varphi \in J_\infty(E)$ if, and only if, $\lim_{n\to\infty} \frac{1}{x_n^p}|\varphi(x_n)| = 0$ for all $p \in \mathbb{N}_0$.*

We set $\alpha_n = -\log x_n$ and set, as defined before:

$$\Lambda_\infty(\alpha) = \{\xi = (\xi_1, \xi_2, \dots) \,:\, |\xi|_p := \sup_n |\xi_n| e^{p\alpha_n} < +\infty \text{ for all } p \in \mathbb{N}_0\}.$$

Equipped with the norms $|\cdot|_p$ this is a Fréchet space.

Theorem 4.4 *$\Phi : \varphi \mapsto (\varphi(x_n))_{n\in\mathbb{N}}$ maps $J_\infty(0)$ isomorphically onto $\Lambda_\infty(\alpha)$.*

Proof: That Φ is an algebraic isomorphism follows from the previous proposition. From equation 3 we see that

$$|\Phi(\varphi)|_p \le \frac{1}{p!} \inf\{\|f\|_p \,:\, f \in C^\infty(\mathbb{R}) \text{ with } f|_E = \varphi\} = |||f|||_p.$$

Therefore Φ is continuous and, due to the open mapping theorem, an isomorphism. □

Theorem 4.5 *Let $\varphi \in C(E)$. Then $\varphi \in C_\infty(E)$ if, and only if, there are numbers A_p, $p \in \mathbb{N}_0$, such that $A_0 = \varphi(0)$ and for all $p \in \mathbb{N}_0$ we have*

$$A_{p+1} = \lim_{n\to\infty} \frac{(p+1)!}{x_n^{p+1}} \left(\varphi(x_n) - \sum_{n=1}^{p} \frac{A_j}{j!}\, x_n^j \right).$$

In this case $A_p = \varphi^{(p)}(0)$ for all p.

Proof: Necessity follows from formula 2, from there also that $A_p = \varphi^{(p)}(0)$ for all p.

To show sufficiency we us the E. Borel Theorem to find $g \in C^\infty(\mathbb{R})$ with $g^{(p)}(0) = A_p$ for all $n \in \mathbb{N}_0$.

We set $h = \varphi - g|_E$ and estimate:

$$\begin{aligned} (p+1)! \frac{h(x_n)}{x_n^{(p+1)}} &= \frac{(p+1)!}{x_n^{p+1}} (\varphi(x_n) - g(x_n)) \\ &= \frac{(p+1)!}{x_n^{p+1}} \left(\varphi(x_n) - \sum_{j=0}^{p} \frac{A_j}{j!} x_n^j \right) - \frac{(p+1)!}{x_n^{p+1}} \left(g(x_n) - \sum_{j=0}^{p} \frac{g^{(j)}(0)}{j!} x_n^j \right). \end{aligned}$$

The second line converges to $A_{p+1} - g^{(p+1)}(0) = 0$. So there exists $H \in J^\infty(0)$ with $H|_E = h$. We put $f = H + g$. Then $f \in C^\infty(\mathbb{R})$ and $f|_E = \varphi - g|_E + g|_E = \varphi$. □

Theorem 4.6 *The norms*

$$\|\varphi\|_k := \max_{p=0,..,k} \left\{ |\varphi^{(p)}(0)| + \sup_{n\in\mathbb{N}} \frac{1}{x_n^p} \left(\varphi(x_n) - \sum_{j=0}^{p} \frac{\varphi^{(j)}(0)}{j!} x_n^j \right) \right\}$$

are a fundamental system of seminorms in $C^\infty(E)$.

Proof: If $f \in \mathbb{C}^\infty(\mathbb{R})$ with $f|_E = \varphi$ then we have, by elementary estimates, $\|\varphi\|_k \leq 3\|f\|_k$. Since that holds for any such f we obtain $\|\varphi\|_k \leq 3|||\varphi|||_k$ for all k. So the topology generated by the norms $\|\cdot\|_k$ is weaker than the topology of $C_\infty(E)$.

By standard arguments one shows that $C^\infty(E)$ is complete in this topology hence, by the open mapping theorem, the topologies coincide. □

We have already remarked that $C^\infty(E)$ can be considered as well as a restriction space of $C^\infty(\mathbb{R})$ as also of $C^\infty_{2\pi}(\mathbb{R})$. We collect some information about the latter space.

Lemma 4.7 *The following norms are a fundamental system of seminorms for* $C^\infty_{2\pi}(\mathbb{R})$

$$|f|_n := \sum_{k=-\infty}^{\infty} |a_k|(|k|+1)^n, \qquad n \in \mathbb{N}_0,\ a_k \textit{ Fourier coefficients.}$$

The dual norms are

$$|\mu|_n^* = \sup_{k=-\infty}^{\infty} |b_k|(|k|+1)^{-n}, \qquad n \in \mathbb{N}_0,\ b_k = \mu(e^{ikt}).$$

They satisfy $|\mu|_n^{*\,2} \leq |\mu|_{n-1}^* \, |\mu|_{n+1}^*$ *for all* $n \in \mathbb{N}$.

Here for any seminorm $\|\cdot\|$ the extended real valued dual norm is defined by $\|\mu\|^* = \sup_{\|x\|\leq 1} |\mu(x)|$.

Theorem 4.8 *If there is* $C > 0$ *such that* $x_n \leq C x_{n+1}$ *for all* $n \in \mathbb{N}$, *then* $\|\cdot\|_k^2 \leq C_k \|\cdot\|_{k-1} \|\cdot\|_{k+1}$ *for all* k *with suitable* C_k.

Proof: not given here, see the original paper [12].

REMAK: The rather restrictive assumption on the sequence $(x_n)_{n\in\mathbb{N}}$ cannot be relaxed. To show this we define

$$\varphi(x_n) = \frac{x_n^m}{m!} \text{ for } n > N, \quad \varphi(x_n) = 0 \text{ otherwise.}$$

We may assume that $x_1 \leq 1$. We obtain

$$\frac{1}{x_n^p}\left(\varphi(x_n) - \sum_{j=0}^{p} \frac{\varphi(j)}{j!} x_n^j\right) = \frac{1}{x_n^p}\varphi(x_n) \qquad \text{for } p < m$$

$$= \frac{1}{x_n^p}\big(\varphi(x_n) - \frac{x_n^m}{m!}\big) \text{ for } p \geq m$$

and therefore

$$\begin{aligned} &= 0 && \text{for } p < m, \quad n \leq N \\ &= \frac{1}{m!} x_n^{m-p} && \text{for } p < m, \quad n > N \\ &= -\frac{1}{m!} x_n^{m-p} && \text{for } p \geq m, \quad n \leq N \\ &= 0 && \text{for } p \geq m, \quad n > N. \end{aligned}$$

This yields:
for $k < m$

$$\|\varphi\|_k = \frac{1}{m!} x_{N+1}^{m-k}$$

for $k = m$

$$\|\varphi\|_k = \max(\frac{1}{m!} x_{N+1}; 1) = 1$$

and for $k > m$

$$\|\varphi\|_k = \max(\frac{1}{m!} x_{N+1}; 1; \frac{1}{m!} x_N^{m-k}).$$

For given k we choose $m = k$ and obtain:

$$\|\varphi\|_k = 1, \ \|\varphi\|_{k-1} = \frac{1}{k!} x_{N+1}, \ \|\varphi\|_{k+1} = \frac{1}{k!} x_N^{-1}.$$

The norm inequality in Theorem 4.8 then gives $(k!)^2 \leq C_k \frac{x_{N+1}}{x_N}$

$$x_N \leq C_k (k!)^{-2} x_{N+1}$$

for all N, hence the assumption in Theorem 4.6.

We use the following results:

Theorem 4.9 (V.) *Let E and F be nuclear Fréchet spaces. $A \in L(F, E)$ surjective and there are constants $C_k > 0$ and $p \in \mathbb{N}_0$ such that*

1. $\|Ax\|_k \leq C_k \|x\|_{k+p}$ *for all k and $x \in F$.*

2. $\|x\|_k^2 \leq C_k \|x\|_{k-1} \, \|x\|_{k+1}$ *for all* k *and* $x \in E$.

3. $\|y\|_k^{*2} \leq C_k \|y\|_{k-1}^* \, \|y\|_{k+1}^*$ *for all* k *and* $y \in F'$.

then E *has a basis, more precisely: there is* β *such that* $E \cong \Lambda_\infty(\beta)$.

Theorem 4.10 (V.) *If* $\limsup \frac{\alpha_{n+1}}{\alpha_n} < +\infty$, $\Lambda_\infty(\alpha)$ *nuclear and*

$$0 \longrightarrow \Lambda_\infty \alpha) \longrightarrow \Lambda_\infty(\beta) \longrightarrow \omega \longrightarrow 0$$

exact, then $\Lambda_\infty(\alpha) \cong \Lambda_\infty(\beta)$.

Here $\omega := \mathbb{C}^{\mathbb{N}}$ with the product topology.

Finally we obtain:

Theorem 4.11 *If there are constants* $C > 0$ *and* $q \in \mathbb{N}$ *such that* $x_n^q \leq C\varepsilon_n$ *and* $x_n \leq C x_{n+1}$ *for all* n, *then* $C_\infty(E) \cong \Lambda_\infty(\alpha)$ *where* $\alpha_n = -\log x_n$. *In particular,* $C^\infty(E)$ *has a basis.*

Proof: We apply Theorem 4.9 to $F = C_{2\pi}^\infty(\mathbb{R})$, $E = C_\infty(E)$ and $A : C_{2\pi}^\infty \to C_\infty(E)$ the restriction map. The assumptions are fulfilled by Lemma 4.7, Theorem 4.8 and the fact that $\|Af\|_k \leq 3\|f\|_k \leq 3|f|_k$ for every $f \in C_{2\pi}^\infty(\mathbb{R})$ (cf. proof of Theorem 4.6).

This shows that $C_\infty(E)$ is isomorphic to some space $\Lambda_\infty(\beta))$. By use of the E. Borel theorem we have an exact sequence

$$0 \longrightarrow J_\infty(0) \longrightarrow C_\infty(E) \stackrel{\rho}{\longrightarrow} \omega \longrightarrow 0$$

where $\rho(\varphi) = (\varphi^{(p)}(0))_{p \in \mathbb{N}_0}$. Moreover $x_n \leq C x_{n+1}$ implies that for $\alpha_n = -\log x_n$ we have $\limsup_n \frac{\alpha_{n+1}}{\alpha_n} = 1$. Since in the above exact sequence $J_\infty(0) \cong \Lambda_\infty(\alpha)$ and $C_\infty(E) = \Lambda_\infty(\beta)$ we obtain from Theorem 4.10 that $C_\infty(E) \cong \Lambda_\infty(\beta) \cong \Lambda_\infty(\alpha)$. $\square$

EXAMPLE. $x_n = 2^{-n}$, $\varepsilon_n = 2^{-n+1}$ hence $x_n \leq 2\varepsilon_n$, $x_n \leq 2x_{n+1}$. Therefore $C_\infty(E) \cong \Lambda_\infty(n) \cong H(\mathbb{C})$. Here $H(\mathbb{C})$ denotes the space of entire functions which is isomorphic to $\Lambda_\infty(n)$ by $f \mapsto (a_n)_{n \in \mathbb{N}_0}$ where $f(z) = \sum_{k=0}^\infty a_k z^k$.

5 Comparison of $A_\infty(E)$ and $C_\infty(E)$

We will use the following result:

Theorem 5.1 (Alexander, Taylor, Williams) *If there are constants C_1, C_2 such that*

$$\frac{1}{b-a}\int_a^b \log\frac{1}{d(x,E)}\,dx \le C_1 \log\frac{1}{b-a} + C_2 \tag{4}$$

for all $0 \le a < b \le 2\pi$, then $A_\infty(E) = C_\infty(E)$.

We first study the case of $E = \{x_1, x_2, \dots\} \cup \{0\}$, $x_n \searrow 0$. We set $\varepsilon_n = x_n - x_{n+1}$. To study formula (4) in the special case of $a = x_M$, $b = x_n$ we use formula (1) to obtain:

$$\sum_{n=m}^{M-1} \varepsilon_n \log\frac{1}{\varepsilon_n} = \int_{x_M}^{x_m} \log\frac{1}{d(x,E)}dx - (1+\log 2)(x_m - x_M).$$

Hence in this case the condition in Theorem 5.1 means the existence of C_1, C_2 such that

$$\frac{1}{x_n - x_M}\sum_{n=m}^{M-1} \varepsilon_n \log\frac{1}{\varepsilon_n} \le C_1 \log\frac{1}{x_n - x_M} + C_2. \tag{5}$$

We use this to give an example of a Carleson set which does not fulfill the condition in Theorem 5.1.

Example: We set $E = \{x_{m,k} = 2^{-m+1} - k2^{-m-m^2} \,:\, m \in \mathbb{N}_0,\ k = 0,..,2^{m^2}\}$.

We fix $m \in \mathbb{N}$ and consider inequality (5) for the points $2^{-m} < 2^{-m+1}$ in E. We obtain:

$$2^m 2^{m^2}(2^{-m-m^2}(m+m^2)\log 2) = (m+m^2)\log 2 \le C_2 m \log 2 + C_2.$$

This should hold for all $m \in \mathbb{N}$ which is impossible. So E does not fulfill the condition in Theorem 5.1.

On the other hand

$$\sum_n \varepsilon_n \log\frac{1}{\varepsilon_n} = \sum_{m=1}^{\infty} 2^{m^2} 2^{-m-m^2}(m+m^2)\log 2 = \log 2 \sum_{m=1}^{\infty} 2^{-m}(m+m^2) < +\infty$$

hence E is a Carleson set.

We will need the following elementary inequality. For that let $0 < a \le b$. From the mean value theorem we obtain $a < \xi < a + b$ such that

$$(a+b)\log(a+b) - a\log a = b(\log\xi + 1) \le b(\log(a+b)+1) \le b\log b + (1+\log 2)b$$

and this implies

$$a\log\frac{1}{a} + b\log\frac{1}{b} \le (a+b)\log\frac{1}{a+b} + 2\log b. \tag{6}$$

Assume now that $0 < a_1 \le a_2 \le \cdots \le a_m$ with $\sum_{j=1}^{k} a_j \le a_{k+1}$ for $k = 1,\ldots,m-1$. Set $a = \sum_{j=1}^{m} a_j$, then we obtain, using estimate (6) inductively,

$$\sum_{j=1}^{m} a_j \log\frac{1}{a_j} \le a\log\frac{1}{a} + 2a. \tag{7}$$

Assumption: $x_{n+1} \le \varepsilon_n$ for all $n \in \mathbb{N}$.

We analyze estimate (4) under this assumption. Let $0 < a < b$. We assume

$$x_{M+1} \le a \le x_M, \quad x_{m+1} \le b \le x_m$$

and set $]\alpha_{j+1}, \alpha_j[\; := \;]x_{j+1}, x_j[\, \cap\,]a, b[$, $\varepsilon'_j = \alpha_j - \alpha_{j+1}$ and assume first $m < M$

$$\sum_{j=k+1}^{M} \varepsilon'_j = x_{k+1} - a \le x_{k+1} \le \varepsilon_k \text{ for } m \le k < M.$$

By use of formulae (4) and (7) we obtain

$$\begin{aligned}\int_a^{x_{m+1}} \log\frac{1}{d(x,E)}\,dx &\le \sum_{n=m+1}^{M} \varepsilon'_n \log\frac{1}{\varepsilon'_n} + 4(x_{m+1} - a)\\ &\le (x_{m+1} - a)\log\frac{1}{x_{m+1} - a} + 6(x_{m+1} - a).\end{aligned}$$

To handle the case $m = M$ or the interval $]x_{m+1}, b[$ we have to estimate the following situation $0 < A \le \alpha < \beta \le B$. We set $\xi' = \min(\xi, \frac{A+B}{2})$, $\xi'' = \max(\xi, \frac{A+B}{2})$ and obtain

$$\begin{aligned}\int_\alpha^\beta \log\frac{1}{d(x,\{A,B\})}\,dx &= \int_{\alpha'}^{\beta'} \log\frac{1}{x-A}\,dx + \int_{\alpha''}^{\beta''} \log\frac{1}{B-x}\,dx\\ &\le \int_{\alpha'}^{\beta'} \log\frac{1}{x-\alpha'}\,dx + \int_{\alpha''}^{\beta''} \log\frac{1}{\beta''-x}\,dx\\ &= (\beta'-\alpha')\log\frac{1}{\beta'-\alpha'} + (\beta'-\alpha') +\\ &\qquad + (\beta''-\alpha'')\log\frac{1}{\beta''-\alpha''} + (\beta''-\alpha'').\end{aligned}$$

Considering the cases $\alpha \le \beta \le \frac{A+B}{2}$, $\frac{A+B}{2} \le \alpha \le \beta$, $\alpha < \frac{A+B}{2} < \beta$ separately we arrive in every case at

$$\int_\alpha^\beta \log \frac{1}{d(x, \{A, B\})}\, dx \le (\beta - \alpha) \log \frac{1}{\beta - \alpha} + 3(\beta - \alpha).$$

The factor three appears only in the last case and comes from the application of (6).

With $A = x_{m+1}$, $B = x_m$ and $\alpha = a$, $\beta = b$ this takes care of the case $m = M$. With $A = \alpha = x_{m+1}$, $\beta = b$ we obtain, using (6) for the last estimate,

$$\begin{aligned} \int_a^b \log \frac{1}{d(x, E)}\, dx &= \int_a^{x_{m+1}} \log \frac{1}{d(x, E)}\, dx + \int_{x_{m+1}}^b \log \frac{1}{d(x, E)}\, dx \\ &\le (x_{m+1} - a) \log \frac{1}{x_{m+1} - a} + 6(x_{m+1} - a) + \\ &\qquad + (b - x_{m+1}) \log \frac{1}{b - x_{m+1}} + 3(b - x_{m+1}) \\ &\le (b - a) \log \frac{1}{b - a} + 8(b - a). \end{aligned}$$

We have shown, rephrasing the assumption $x_{n+1} \le \varepsilon_n$,

Theorem 5.2 *If $x_n \le 2\varepsilon_n$ for all $n \in \mathbb{N}$ then $C_\infty(E) = A_\infty(E)$.*

If $\varepsilon_{n-1} \ge \varepsilon_n$ for $n = 2, 3, \ldots$ or, equivalently $x_n \le \frac{1}{2}(x_{n-1} + x_{n+1})$ then the above assumption is special case of the assumption in Section 4. It can also be written as $2x_{n+1} \le x_n$. Hence we have the following final result for the case of sets with only one accumulation point.

Theorem 5.3 *If $(\varepsilon_n)_{n\in\mathbb{N}}$ is decreasing and there is $C > 0$ such that $2x_{n+1} \le x_n \le Cx_{n+1}$ for all $n \in \mathbb{N}$ then $A_\infty(E) = C_\infty(E) \cong \Lambda_\infty(\alpha)$ with $\alpha_n = -\log x_n$. In particular, the space $A_\infty(E)$ has a basis.*

Example: For $E = \{2^{-n} : n \in \mathbb{N}\}$ the space $A_\infty(E)$ has a basis, more precisely $A_\infty(E) = C^\infty(E) \cong \Lambda_\infty(n) \cong H(\mathbb{C})$.

Next we study the $A_\infty(E)$ for the classical Cantor set. So E will denote in the sequel this set. We first make the

Remark: $(3^k E) \cap [0, 1] = E$ for all $k \in \mathbb{N}_0$.

This is the reason to state the following elementary formula: For $M \subset [0,1]$ and $a > 0$ we have

$$\int_0^a \log \frac{1}{d(x, aM)}\, dx = a \log \frac{1}{a} + a \int_0^1 \log \frac{1}{d(x, M)}\, dx. \tag{8}$$

Let now $0 \le a < b < 1$ be given. We set $b - a := \gamma = 0,\gamma_1\gamma_2 \ldots \gamma_m$ where the last term denotes the triadic expansion of γ which we assume to be finite.

We set $a_0 = a$ and $a_k := a + 0,\gamma_1 \ldots \gamma_k$ for $k = 1, \ldots, m$. That means $a_{k+1} = a_k + \gamma_{k+1} 3^{-k-1}$. Therefore

$$\int_a^b \log \frac{1}{d(x, E)}\, dx = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx.$$

We study the $k - th$ term. It is an integral over an interval of length $\gamma_{k+1} 3^{-k-1}$ and γ_{k+1} can take on the values $0, 1, 2$. If $\gamma_{k+1} = 0$ the term is 0, hence we assume $\gamma_{k+1} = 1$ or 2.

We consider the subdivision of $[0,1]$ into 3^k intervals of lengths 3^{-k}. We call them windows and we call a window:
white, if it has been already been excluded from the Cantor set,
black, if it waits for treatment.

We first assume that $\gamma_{k+1} = 1$ and we consider three cases.

1st Case: Our interval is contained in a white window, then

$$\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx \le \int_0^{3^{-k-1}} \log \frac{1}{x}\, dx = 3^{-k-1} \log \frac{1}{3^{-k-1}} + 3^{-k-1}.$$

2nd Case: Our interval is contained in a black window. We estimate the integral by the integral over the whole window. Shifting the lower end of the window into 0 and then multiplying by 3^k we get the again the Cantor set. By use of formula (8) we obtain

$$\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx \le 3^{-k} \log \frac{1}{3^{-k-1}} + D_0 3^{-k}$$

where $D_0 = \log 3 + \int_0^1 \log \frac{1}{d(x,E)}\, dx$ and this estimate is valid also in the 1st case.

3rd Case: Our interval hits a black and a white window. Then we estimate roughly by the sum of the estimates in the 1st and 2nd case and this estimate is, of course valid for all three cases.

If $\gamma_{k+1} = 2$ we have two intervals of length 3^{-k-1}. Therefore we may estimate by two times the estimate of the previous 3rd case and therefore four times the estimate in the 2nd case. So finally we obtain:

$$\begin{aligned}\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x,E)}\, dx &\leq 4 \cdot 3^{-k} \log \frac{1}{3^{-k-1}} + 4D_0 3^{-k} \\ &\leq 12\gamma_{k+1} 3^{-k-1} \log \frac{1}{3^{-k-1}} + 4D_0 \gamma_{k+1} 3^{-k-1} \\ &\leq 12\gamma_{k+1} 3^{-k-1} \log \frac{1}{\gamma_{k+1} 3^{-k-1}} + D\gamma_{k+1} 3^{-k-1}.\end{aligned}$$

where $D = 4D_0 + 12 \log 2$.

So we have

$$\int_a^b \log \frac{1}{d(x,E)}\, dx \leq 12 \sum_{k=0}^{m-1} \gamma_{k+1} 3^{-k-1} \log \frac{1}{\gamma_{k+1} 3^{-k-1}} + D(b-a).$$

We wish to apply inequality (7). Therefore we need

$$\sum_{k=n}^{m-1} \gamma_{k+1} 3^{-k-1} \leq 2 \sum_{k=n}^{\infty} 3^{-k-1} = 3^{-n} \leq 3^{-\nu}$$

where $\nu = \max\{k \leq n \,:\, \gamma_k \neq 0\}$. Now (7) delivers

$$\int_a^b \log \frac{1}{d(x,E)}\, dx \leq 12(b-a) \log \frac{1}{b-a} + (24 + D)(b-a)$$

for all triadic numbers $a < b$ in $[0, 1[$. Since E is Carleson, the function $\log \frac{1}{d(x,B)}$ is integrable, hence both sides of the above inequality depend continuously on a and b. Therefore the inequality is true for all $0 \leq a < b \leq 1$ and we have proved:

Proposition 5.4 *If E is the classical Cantor set then $A_\infty(E) = C_\infty(E)$.*

Together with our previous results we have shown:

Theorem 5.5 *If E is the classical Cantor set then $A_\infty(E)$ has a basis. It is isomorphic to some $\Lambda_\infty(\alpha)$.*

Theorems 5.3 and 5.5 show our main result Theorem 2.1.

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