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Continuous linear right inverses for partial differential operators of order 2 and fundamental solutions in half spaces

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Summary. Let P be a complex polynomial in n variables of degree 2 and $P(D)$ the corresponding partial differential operator with constant coefficients. It is shown that $P(D): C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ admits a continuous linear right inverse if and only if after a separation of variables and up to a complex factor for some $c \in \mathbb{C}$ the polynomial P has the form

$$P(x_1, ..., x_n) = Q(x_1, ..., x_r) + L(x_{r+1}, ..., x_n) + c$$

where either $r = 1$ and $L \equiv 0$ or $r > 1$, Q and L are real and Q is indefinite. The proof of this characterization is based on the general solution of the right inverse problem for such operators and the fact that for each operator $P(D)$ of the given form and each characteristic vector N there exists a fundamental solution for $P(D)$ supported by $\{x \in \mathbb{R}^n : \langle x, N\rangle \geq 0\}$, which can be constructed explicitely using partial Fourier transform. The existence of sufficiently many fundamental solutions with support in closed half spaces implies that some right inverse can be given by a concrete formula. An example shows that the present characterization is restricted to operators of order 2.

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0. Introduction

In the early fifties L. Schwartz posed the problem to characterize those linear partial differential operators $P(D)$ that admit a (continuous linear) right inverse on the Fréchet space $\mathcal{E}(\Omega)$ of all infinitely differentiable functions on an open set Ω in $\mathbb{R}^n$ respectively on the space $\mathcal{D}'(\Omega)$ of all distributions on Ω. This problem was solved in [6], [7]. Its solution was extended to differential complexes over convex sets Ω by Palamodov [12] and to nonquasianalytic classes and ultradistributions in [8] and [9]. The evaluation of the general solution leads essentially to two cases that are handled by different methods. In the case of convex open sets Ω, including $\Omega = \mathbb{R}^n$, the existence of a right inverse for $P(D)$ on $\mathcal{E}(\Omega)$ or $\mathcal{D}'(\Omega)$ is equivalent to the fact that a Phragmén-Lindelöf condition depending on Ω, holds for the plurisubharmonic functions on the variety $V(P) := \{z \in \mathbb{C}^n : P(z) = 0\}$. For a comprehensive study of these Phragmén-Lindelöf conditions we refer to [11]. In the case of open sets Ω with a non-empty C^1-boundary it turns out that $P(D)$ admits a right inverse on $\mathcal{E}(\Omega)$ or $\mathcal{D}'(\Omega)$ only if P is hyperbolic with respect to each non-characteristic vector that is normal to $\partial\Omega$ at some point. However, the case of a characteristic half space Ω remained open.

In the present paper we give a more detailed characterization of the differential operators $P(D)$ of order 2 that admit a right inverse on $\mathcal{E}(\mathbb{R}^n)$. For such operators, the property is equivalent to the existence of a basis $\{N_1, \ldots, N_n\}$ of $\mathbb{R}^n$ such that $P(D)$ admits fundamental solutions $E_j^{\pm}$ in $\mathcal{D}'(\mathbb{R}^n)$ that are supported in the closed half spaces $H_{\pm}(N_j)$ determined by $N_j, 1 \leq j \leq n$. Moreover, it is equivalent to the existence of some bounded open convex set Ω in $\mathbb{R}^n$ for which $P(D)$ admits a right inverse on $\mathcal{E}(\Omega)$. An example shows that these equivalences fail for operators of order 3. The existence of sufficiently many fundamental solutions supported by half spaces implies that the existing right inverse on $\mathcal{E}(\mathbb{R}^n)$ can be given by a formula, involving only a finite partition of unity and convolutions with appropriate fundamental solutions that are constructed explicitely.

To prove our characterization, we reduce the study of general quadratic polynomials to certain normal forms. For these we derive necessary conditions using results from [11] and Holmgren's uniqueness theorem. To show that these necessary conditions are also sufficient, we prove that each polynomial which is of degree 2 and real up to a possibly complex additive constant admits fundamental solutions supported by any characteristic closed half space. This result can be easily proved by applying Hörmander's sophisticated characterization of the operators admitting fundamental solutions with support in a characteristic half space [4], Thm. 12.8.1. The intent is to give a simple, explicit formula for these fundamental solutions.

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1. Preliminaries

In this section we fix the notation and recall some results which will be used subsequently.

Definition 1. *Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{E}(\Omega)$ denotes the complex vector space of all infinitely differentiable functions on Ω, endowed with the Fréchet-space topology of uniform convergence of all derivatives on all compact subsets of Ω. Also, $\mathcal{D}(\Omega)$ denotes the space of all functions in $\mathcal{E}(\Omega)$ which have compact support in Ω. It is endowed with the standard* (LF)*-space topology. Its dual space $\mathcal{D}'(\Omega)$ is the space of all distributions on Ω.*

By $\mathbb{C}[z_1, \ldots, z_n]$ we denote the ring of all complex polynomials in n variables, which will be also regarded as functions on $\mathbb{C}^n$. For $P \in \mathbb{C}[z_1, \ldots, z_n]$,

$$P(z) = \sum_{|\alpha| \le m} a_\alpha z^\alpha,$$

with $\sum_{|\alpha|=m} |a_\alpha| \ne 0$, we call

$$P_m : z \mapsto \sum_{|\alpha|=m} a_\alpha z^\alpha$$

the principal part of P. Note that P_m is a homogeneous polynomial of degree m.

For $P \in \mathbb{C}[z_1, \ldots, z_n]$ and an open set Ω in $\mathbb{R}^n$ we define the linear partial differential operator

$$P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), \quad P(D)f := \sum_{|\alpha| \le m} a_\alpha i^{-|\alpha|} f^{(\alpha)}.$$

Then $P(D)$ is a continuous endomorphism of $\mathcal{D}'(\Omega)$ and its restriction to $\mathcal{E}(\Omega)$ is a continuous endomorphism of $\mathcal{E}(\Omega)$.

A distribution E in $\mathcal{D}'(\mathbb{R}^n)$ is called a *fundamental solution* for $P(D)$ if $P(D)E = \delta$, where δ denotes the point evaluation at zero.

A vector $N \in \mathbb{R}^n \setminus \{0\}$ is called non-characteristic for $P \in \mathbb{C}[z_1, \ldots, z_n]$ if $P_m(N) \ne 0$. $P(D)$ or P is called hyperbolic with respect to $N \in \mathbb{R}^n$ if N is non-characteristic for P and if $P(D)$ admits a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ satisfying Supp $E \subset \overline{H_+(N)}$, where

$$H_\pm(N) := \{x \in \mathbb{R}^n : \pm\langle x, N\rangle > 0\}.$$

$P(D)$ or P is called hyperbolic if it is hyperbolic with respect to some vector $N \in \mathbb{R}^n$.

We will say that $P(D)$ admits a right inverse on $\mathcal{E}(\Omega)$ (resp. on $\mathcal{D}'(\Omega)$) if there exists a continuous linear map $R : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ (resp. $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$) so that $P(D) \circ R = \mathrm{id}_{\mathcal{E}(\Omega)}$ (resp. $= \mathrm{id}_{\mathcal{D}'(\Omega)}$). By [8], 2.10, $P(D)$ admits a right inverse on $\mathcal{D}'(\Omega)$ if and only if $P(D)$ admits a right inverse on $\mathcal{E}(\Omega)$. If this is the case the open set Ω is said to be P-convex with bounds. Note that [8], 2.10, also shows that many other properties are equivalent for Ω to be P-convex with bounds.

To state several conditions which are equivalent to $\mathbb{R}^n$ being P-convex with bounds and which are needed subsequently, we recall the following definition.

Definition 2. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant. Then the zero variety of P is defined as*

$$V(P) := \{z \in \mathbb{C}^n : P(z) = 0\}.$$

A function $u : V(P) \to [-\infty, \infty[$ is called plurisubharmonic (psh), if it is locally bounded from above and plurisubharmonic in the usual sense at all regular points $V_{\mathrm{reg}} \subset V$. We assume in the sequel that at the singular points $V_{\mathrm{sing}} \subset V$ we have

$$u(z) = \limsup_{\xi \in V_{\mathrm{reg}}, \xi \to z} u(\xi) \quad \textit{for all} \quad z \in V_{\mathrm{sing}}.$$

By $PSH(V(P))$ we denote the set of all psh functions on $V(P)$ which satisfy this condition.

By [7], 4.6, and 2.10 and [6] we have:

Theorem 1. *For each non-constant polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ the following assertions are equivalent:*

1. $\mathbb{R}^n$ *is P-convex with bounds.*
2. *For each $r > 0$ there exists $R > 0$ such that for each $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$ there exists $E \in \mathcal{D}'(\mathbb{R}^n)$ satisfying $P(D)E = \delta$ and* $\mathrm{Supp} E \subset \{x \in \mathbb{R}^n : |x - \xi| \geq r\}$.
3. *$V(P)$ satisfies the following Phragmén-Lindelöf condition* PL(log)*: There exists $A > 0$ such that for each $\rho > 0$ there exists $B_\rho > 0$ such that each $u \in PSH(V(P))$ satisfying (α) and (β) also satisfies (γ), where*
   (α) $u(z) \leq |\Im z| + O(\log(2 + |z|^2)), \quad z \in V(P)$
   (β) $u(z) \leq \rho|\Im z|, \quad z \in V(P)$
   (δ) $u(z) \leq A|\Im z| + B_\rho \log(2 + |z|^2), \quad z \in V(P)$.

The following proposition shows that under appropriate hypotheses a right inverse for $P(D)$ on $\mathcal{E}(\mathbb{R}^n)$ can be obtained by an explicit construction.

Proposition 1. *Assume that $P \in \mathbb{C}[z_1, \ldots, z_n]$ admits fundamental solutions $E_1, \ldots, E_k \in \mathcal{D}'(\mathbb{R}^n)$ so that Supp $E_j \subset \mathbb{R}^n \setminus \Gamma_j$, where Γ_j is an open convex cone with vertex at zero for $1 \leq j \leq k$. If $\bigcup_{j=1}^k \Gamma_j$ covers the unit sphere of $\mathbb{R}^n$ then $\mathbb{R}^n$ is P-convex with bounds.*

Proof. Fix any fundamental solution E_0 for $P(D)$ and denote by Γ_0 the open unit ball in $\mathbb{R}^n$. Then choose a C^∞-partition of unity $(\varphi_j)_{j=0}^k$ subordinate to the cover $(-\Gamma_j)_{j=0}^k$ of $\mathbb{R}^n$ and define

$$R(f) := \sum_{j=0}^{k} E_j * (\varphi_j f).$$

Then it is easy to check that R is a continuous linear right inverse for $P(D)$ on $\mathcal{E}(\mathbb{R}^n)$ and also on $\mathcal{D}'(\mathbb{R}^n)$.

The following lemma is obvious, since $\mathcal{E}(\mathbb{R}^{n+k})$ can be identified in a natural way with the space of all C^∞-functions on $\mathbb{R}^k$ with values in $\mathcal{E}(\mathbb{R}^n)$.

Lemma 1. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant and define $Q(z_1, \ldots, z_{n+k}) := P(z_1, \ldots, z_n)$. Then $\mathbb{R}^n$ is P-convex with bounds if and only if $\mathbb{R}^{n+k}$ is Q-convex with bounds.*

2. Necessary conditions for differential operators of order two

In this section we derive necessary conditions on a polynomial P of degree two, that $\mathbb{R}^n$ is P-convex with bounds. To do this we show that it suffices to consider operators having a certain normal form. Then we discuss these normal forms using the Phragmén-Lindelöf condition stated in Theorem 1 and Holmgren's uniqueness theorem to get the desired necessary conditions. They are also sufficient, as we will show in the next section.

First we note that each polynomial P of degree two for which $\mathbb{R}^n$ is P-convex with bounds has a specific principal part.

Lemma 2. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be of degree $m > 0$ and denote by P_m its principal part. If $\mathbb{R}^n$ is P-convex with bounds then there exists $\lambda \in \mathbb{C}$ so that λP_m is real.*

Proof. If $\mathbb{R}^n$ is P-convex with bounds, then it follows from Theorem 1 and [11], 4.1, that $\mathbb{R}^n$ is also P_m-convex with bounds. By [11], 2.6, each irreducible factor S of P_m also has this property. Hence $V(S)$ satisfies the Phragmén-Lindelöf condition PL(log) stated in Theorem 1 Since S is homogeneous, [11], 3.3, implies that $V(S)$ even satisfies the following condition (PL): There exists $A \geq 1$ such that $u \in PSH(V(S))$ which satisfies

$$u(z) \leq |\Im z| + o(|z|), \quad z \in V(S) \text{ and}$$

$$u(z) \leq 0, \quad z \in V(S) \cap \mathbb{R}^n$$

also satisfies

$$u(z) \leq A|\Im z|, \quad z \in V(S).$$

Now fix $Q \in \mathbb{C}[z_1, \ldots, z_n]$ and assume that $Q|_{V(S)\cap\mathbb{R}^n} = 0$. We claim that Q vanishes on $V(S)$. Arguing by contradiction, assume there exists $z_0 \in V(S)$ such that $Q(z_0) \neq 0$. This implies $z_0 \in V(S) \setminus \mathbb{R}^n$, hence there exists $\mu > 0$ such that $\log|\mu Q(z_0)| > A|\Im z_0|$. Now note that $v(z) := \log|\mu Q(z)|$ satisfies the two hypotheses of (PL) and hence also the conclusion, which contradicts our choice of μ. Consequently, $Q|_{V(S)} = 0$. Since S is irreducible, we get $Q = R \cdot S$ for some $R \in \mathbb{C}[z_1, \ldots, z_n]$. Applying this to $Q := \mathrm{Re}(S)$, we obtain $\mathrm{Re}(S) = R_0 \cdot S$, where R_0 has to be a constant. Since S was any irreducible factor of P_m, the proof of the lemma is complete.

Because of Lemma 2 it suffices to consider only polynomials with real principal part when we want to determine all polynomials P of degree 2 for which $\mathbb{R}^n$ is P-convex with bounds. The following lemma shows that we can reduce the general case to the consideration of certain normal forms.

Lemma 3. *Let $Q \in \mathbb{C}[z_1, ..., z_n]$ have degree 2 and real principal part. Then there exist a real matrix $A \in GL(\mathbb{R}^n)$ and $a \in \mathbb{C}^n$ so that $P(z) := Q(Az + a)$ has one of the following normal forms:*

$$(I) \quad P(z) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{s} z_j^2 + C, \quad 1 \leq r \leq s \leq n,\ C \in \mathbb{C}$$

$$(II) \quad P(z) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{s} z_j^2 + \lambda z_{s+1} \quad 1 \leq r \leq s \leq n-1,\ \lambda \in \mathbb{C}\setminus\{0\}$$

$$(III) \quad P(z) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{s} z_j^2 + iz_{s+1} + z_{s+2}, \quad 1 \leq r \leq s \leq n-2.$$

Further, we have:

(*i*) *An open set Ω in $\mathbb{R}^n$ is Q-convex with bounds if and only if $A^t\Omega = \{A^t x : x \in \Omega\}$ is P-convex with bounds*

(*ii*) *$Q(D)$ has a fundamental solution supported in a (characteristic) half space H if and only if $P(D)$ has a fundamental solution supported in $A^t H = \{A^t x : x \in H\}$.*

Proof. Since the principal part of Q is of degree 2 and real it is the quadratic form of a real symmetric matrix. Hence there exists a real change of variables so that in these variables we have

$$Q(z)=\sum_{j=1}^{r}z_j^2-\sum_{j=r+1}^{s}z_j^2+L(z_1,\dots,z_n)+C$$

where $0\le r\le s\le n$, L is a $\mathbb{C}$-linear form and $C\in\mathbb{C}$. Replacing Q by $-Q$ if necessary, we can assume $1\le r\le s\le n$.

If $L\equiv 0$ then Q already is of type (I). Therefore we can assume that

$$L(z)=\sum_{j=1}^{n}b_jz_j \text{ for some } b\in\mathbb{C}^n,\ b\neq 0.$$

Next fix $a\in\mathbb{C}^n$ and note that

$$(*)\quad Q(z+a) := \sum_{j=1}^{r}z_j^2-\sum_{j=r+1}^{s}z_j^2+\sum_{j=1}^{r}(b_j+2a_j)z_j+\sum_{j=r+1}^{s}(b_j-2a_j)z_j + \sum_{j=s+1}^{n}b_jz_j+C'$$

$$\text{where } C':=C+\sum_{j=1}^{r}a_j^2-\sum_{j=r+1}^{s}a_j^2+\sum_{j=1}^{n}a_jb_j.$$

Then we consider the following two cases:

case 1: $s=n$ or $s<n$ and $b_j=0$ for $s+1\le j\le n$.

Then let $a_j:=-\frac{1}{2}b_j$ for $1\le j\le r$, $a_j:=\frac{1}{2}b_j$ for $r+1\le j\le s$ and $a_j=0$ for $s+1\le j\le n$. By $(*)$, this choice implies that $P(z):=Q(z+a)$ is of type (I).

case 2: $s<n$ and there exists k with $s+1\le k\le n$ and $b_k\neq 0$.
Then define a_j for $j\neq k$ as in case 1 and let

$$a_k:=-\frac{1}{b_k}\left(C+\sum_{j=1}^{r}a_j^2-\sum_{j=r+1}^{s}a_j^2+\sum_{j=1}^{s}a_jb_j\right).$$

With this choice we get from $(*)$ that

$$Q(z+a)=\sum_{j=1}^{r}z_j^2-\sum_{j=r+1}^{s}z_j^2+l(z_{s+1},\dots,z_n)$$

where l is a non-trivial $\mathbb{C}$-linear form in $n-s$ variables. If the range of the $\mathbb{R}$-linear map $l:\ \mathbb{R}^{n-s}\to\mathbb{C}$ has real dimension one we can find a real linear change A' of the variables $z_{s+1},\dots,z_n$ so that in the new variables $\rho_{s+1},\dots,\rho_n$ we have $l(\rho)=\lambda\rho_{s+1}$ for some $\lambda\in\mathbb{C}\backslash\{0\}$. Hence $P(z):=Q(A'(z+a))$ is of type (II) in this case.

If the range of $l:\ \mathbb{R}^{n-s}\to\mathbb{C}$ has real dimension two then we can find a real linear change A' of the variables $z_{s+1},\dots,z_n$ so that $P(z):=Q(A'(z+a))$ is of type (III).

The assertions (i) and (ii) are simple properties of how the symbols $Q(z) := e^{-i\langle x,z\rangle}Q(D)e^{i\langle x,z\rangle}$ behave under linear transformations and multiplication by exponentials. For, if $A \in GL(\mathbb{R}^n)$ then $A^t : \mathbb{R}^n \to \mathbb{R}^n$ induces the pull-back map $(A^t)^*$ on functions by $(A^t)^* : \mathcal{E}(A^t\Omega) \to \mathcal{E}(\Omega)$ that defines the constant coefficient operator $P(D) = (A^{-t})^* \circ Q(D) \circ (A^t)^*$ acting on $\mathcal{E}(A^t_-\Omega)$ whose symbol is $P(z) = Q(Az)$. Similarly, $P(z) = Q(z+a)$ is the symbol of the operator $P(D) = M_{-a} \circ Q(D) \circ M_a$, where $M_a(f) : x \mapsto e^{iax}f(x)$ is multiplication by e^{iax}. Both these maps clearly transform right inverses of $Q(D)$ to ones for $P(D)$ and vice-versa, which implies (i). Similarly, supports of distributions are transformed as indicated in (ii).

Lemma 4. *For $n \in \mathbb{N}$ let $P \in \mathbb{C}[s,t,w_1,\ldots,w_n]$ have the form*

$$P(s,t,w) = \lambda s + \mu t + \sum_{j=1}^{n} \varepsilon_j w_j^2$$

where $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $\mu \in \mathbb{R}$ and $\varepsilon_j = \pm 1$ for $1 \le j \le n$. Then $\mathbb{R}^{2+n}$ is not P-convex with bounds.

Proof. To argue by contradiction, assume that $\mathbb{R}^{2+n}$ is P-convex with bounds. Then, by Theorem 1, the variety $V(P)$ satisfies the condition PL(log) for certain constants $A > 0$ and $B_\rho > 0$ for $\rho > 0$. Let $t_0 := 2(A+2)$ and note that without restriction we can assume $\varepsilon_1 = -1, |\mu| \le 1$ and $\lambda = e^{i\varphi}$, where $\delta \le \varphi \le \pi - \delta$ for some δ satisfying $0 < \delta < \frac{\pi}{2}$. Then for $R > 0$ we let

$$z_R := (R^2, 0, e^{i\varphi/2}R, 0, \ldots, 0).$$

Note that $\Im z_R = R\sin\frac{\varphi}{2}$ and that $P(z_R) = 0$, i.e. $z_R \in V(P)$. We claim the following:

(1) There exist $R_0 > 0$ and $M > 0$ so that for $R \ge R_0$ each $z \in V(P)$ satisfying $|z - z_R| \le t_0|\Im z_R|$ already satisfies $|\Im z_R| \le M|\Im z|$.

If (1) holds then [11], 4.7, (note that $c = \frac{1}{2}$, by [10], 3.5) for $\omega(t) = \log(2 + |t|)$ implies the existence of $B' > 0$ such that

(2) $R\sin\frac{\delta}{2} \le |\Im z_R| \le B_{(A+2)M+1}\omega(z_R) \le B'\omega(R^2) = B'\log(2+R^2)$ for all $R \ge R_0$.

Since this is a contradiction, our assumption was wrong. Hence $\mathbb{R}^{2+n}$ is not P-convex with bounds.

To prove (1) let $\tau := t_0\sin\frac{\delta}{2}, M := \dfrac{8n(1+\tau)\sin\frac{\varphi}{2}}{\sin\frac{\delta}{2}}$ and choose $R_0 \ge \dfrac{8\tau}{\sin\frac{\delta}{2}}$ so large that each $\rho \in \mathbb{C}$ satisfying $|\rho - 1| \le \frac{\tau}{R_0}$ has the form $\rho = |\rho|e^{i\alpha}$ for some $\alpha \in \mathbb{R}$ with $|\alpha| \le \frac{\delta}{2}$. Next fix $R \ge R_0$ and $z = (s,t,w_1,\ldots,w_n)$ in $V(P)$ satisfying

$$|z - z_R| \le \tau R = t_0|\Im z_R|. \tag{3}$$

From the definition of z_R and (3) we get

$$|s - R^2| \le \tau R \text{ and hence } \left|\frac{s}{R^2} - 1\right| \le \frac{\tau}{R} \le \frac{\tau}{R_0}.$$

By our choice of R_0 this implies $s = |s|e^{i\alpha}$ for some α with $|\alpha| \leq \frac{\delta}{2}$, and it also implies

(4) $$|s| \geq R^2 - \tau R = R^2\left(1 - \frac{\tau}{R}\right) \geq \frac{R^2}{2}.$$

From (3) and the definition of z_R we derive

(5) $$|t| \leq \tau R \quad\text{and}\quad |w_j| \leq (1+\tau)R,\ 1 \leq j \leq n.$$

Now assume that $\max_{j=1,\ldots,n} |\Im w_j| < \frac{\sin\frac{\delta}{2}}{8n(1+\tau)}R$. Since z is in $V(P)$ we have

(6) $$|s|\sin(\varphi+\alpha) = \Im\lambda s = -\left(\mu\Im t + \sum_{j=1}^{n}\varepsilon_j\Im(w_j^2)\right).$$

If $w_j = |w_j|e^{i\varphi_j}$ for $1 \leq j \leq n$ then

(7) $$|\Im(w_j^2)| = |w_j|^2|\sin 2\varphi_j| = 2|w_j|^2|\sin\varphi_j\cos\varphi_j| = 2|\Im w_j||w_j|.$$

Since $|\alpha| \leq \frac{\delta}{2}$, we get from (4) - (7) and our assumption

$$\frac{R^2}{2}\sin\frac{\delta}{2} \leq |\Im\lambda s| \leq |\mu t| + \frac{\sin\frac{\delta}{2}}{4}R^2 \leq \left(\frac{\tau}{R\sin\frac{\delta}{2}} + \frac{1}{4}\right)R^2\sin\frac{\delta}{2} \leq \frac{3}{8}R^2\sin\frac{\delta}{2}.$$

From this contradiction we conclude that our assumption was wrong. Hence there exists k with $1 \leq k \leq n$ satisfying

$$|\Im w_k| \geq \frac{\sin\frac{\delta}{2}}{8n(1+\tau)}R = \frac{1}{8n(1+\tau)}\frac{\sin\frac{\delta}{2}}{\sin\frac{\varphi}{2}}|\Im z_R| = \frac{1}{M}|\Im z_R|.$$

Obviously, this implies that (1) holds.

Remark 1. Note that the proof of Lemma 4 also shows the following: Whenever ω is a weight function as defined in [8], 1.1, which satisfies $\log(t) = o(\omega(t))$ as t tends to ∞ and if P is as in Lemma 4 then $P(D)$ does not admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ and also not on $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$. Here $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ denotes the Fréchet-space of (ω)-ultradifferentiable functions and $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ denotes the (ω)-ultradistributions on $\mathbb{R}^n$ (see [8], 1.3, for the corresponding definitions).

The following lemma is an easy consequence of Holmgren's uniqueness Theorem as it is stated in Hörmander [4], Thm. 8.6.8. For its formulation we let $M^\perp := \{x \in \mathbb{R}^n : \langle x, m\rangle = 0 \text{ for all } m \in M\}$, if $M \neq \emptyset$ is a subset of $\mathbb{R}^n$.

Lemma 5. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant, let P_m denote its principal part and assume that*

$$C := \{N \in \mathbb{R}^n : P_m(N) = 0\}^\perp \neq \{0\}.$$

Then for any pair ω, Ω of open subsets of $\mathbb{R}^n$ with $\emptyset \neq \omega \subset \Omega$ the set

$$U := \{x \in \Omega : \text{ there exists } \xi \in \omega \text{ such that } [\xi, x] \subset \Omega,\ x - \xi \in C\}$$

is open and each $T \in \mathcal{D}'(\Omega)$ satisfying $P(D)T = 0$ and $T|_\omega = 0$ also satisfies $T|_U = 0$.

Lemma 6. *Let P and C be as in Lemma 5. If* $\dim_{\mathbb{R}} C \geq 2$ *then no open set Ω in $\mathbb{R}^n$ is P-convex with bounds.*

Proof. Arguing by contradiction, we assume that there exists an open set $\Omega \neq \emptyset$ in $\mathbb{R}^n$ which is P-convex with bounds. Let $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ denote a continuous linear right inverse for $P(D)$ on $\mathcal{D}'(\Omega)$. Then fix a relatively compact open subset $\omega \neq \emptyset$ of Ω and note that by the continuity estimates for R there exists a relatively compact set ω' in Ω, $\omega' \supset \omega$ such that for $x_0 \in \Omega \setminus \omega'$ the distribution $E_{x_0} := R(\delta_{x_0})$ satisfies $E_{x_0}|_\omega = 0$.

Next fix $\xi \in \omega$ and a one-dimensional linear subspace g of C, let I denote the connected component of ξ in $(\xi + g) \cap \Omega$ and fix $x_0 \in I \setminus \omega'$. Then apply Lemma 5 with ω and $\Omega \setminus \{x_0\}$ to E_{x_0} and get an open neighborhood V of x_0 such that $V \subset \Omega$ and

$$V \cap \operatorname{Supp} E_{x_0} \subset \{\xi + t(x_0 - \xi) : t \geq 1\} \cap V =: M \cap V.$$

Next choose $\xi_1 \in V \setminus M$ sufficiently close to x_0 and another one-dimensional linear subspace g_1 of C satisfying $g_1 \neq g$. Then apply Lemma 5 with $V \setminus M$ and $V \setminus \{x_0\}$ to E_{x_0} to get an open neighborhood $W \subset V$ of x_0 such that $\operatorname{Supp}(E_{x_0}|_W) = \{x_0\}$. Choose $\varphi \in \mathcal{D}(W)$ satisfying $\varphi \equiv 1$ on some neighborhood of x_0 and let $T := (\varphi E_{x_0}) * \delta_{-x_0}$. Then $P(D)T = \delta$ and $\operatorname{Supp} T = \{0\}$. Since such a distribution T does not exist, we derived a contradiction.

Lemma 7. *For $n \geq 2$ and $k \geq 1$ let $P \in \mathbb{C}[w_1, \ldots, w_{n+k}]$ be of the form*

$$P(w) = \sum_{j=1}^{n} \varepsilon_j w_j^2 + \sum_{j=n+1}^{n+k} a_j w_j$$

where $\varepsilon_j = \pm 1$ for $1 \leq j \leq n$ and $a_j \in \mathbb{C}$ for $n+1 \leq j \leq n+k$. If there exists an open set $\Omega \neq \emptyset$ in $\mathbb{R}^n$ which is P-convex with bounds then there exists $1 < l \leq n$ with $\varepsilon_l \neq \varepsilon_1$.

Proof. If all ε_j have the same sign then

$$\{N \in \mathbb{R}^{n+k} : P_2(N) = 0\} = \{w \in \mathbb{R}^{n+k} : w_1 = \cdots = w_n = 0\}.$$

Hence the real dimension of $C = \{N \in \mathbb{R}^{n+k} : P_2(N) = 0\}^\perp$ is at least 2. Therefore the result follows from Lemma 6 by contraposition.

Proposition 2. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be of one of three types stated in Lemma 3 then $\mathbb{R}^n$ is P-convex with bounds only if the following conditions are satisfied:*

If P is of type (I): $r = s = 1$ or $s > r$.
If P is of type (II): $\lambda \in \mathbb{R}$ and $s > r$.

Proof. (I): By Lemma 1 we can assume $s = n$. If $s = r \geq 2$ then $\mathbb{R}^n$ is not P-convex with bounds by Lemma 6.

(II): By Lemma 1 we can assume $s + 1 = n$. If $\mathbb{R}^n$ is P-convex with bounds it follows from Theorem 1 and Lemma 4 that λ has to be real.

If $r = 1$ and $s = r$ then $P(z_1, z_2) = z_1^2 + \lambda z_2$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. By Hörmander [4], Thm. 12.4.6, P is not hyperbolic. Hence $\mathbb{R}^2$ is not P-convex with bounds by [7], Thm. 4.11.

If $r \geq 2$ and $s = r$, we consider P as a polynomial in $s+2$ variables. By Lemma 7, $\mathbb{R}^{s+2}$ is not P-convex with (ω)-bounds. Hence $\mathbb{R}^n$ is not P-convex with bounds by Lemma 1.

(III): From Theorem 1, Lemma 1 and Lemma 4 it follows that $\mathbb{R}^n$ is not P-convex with bounds in this case.

In the next section we show that the necessary conditions of Proposition 2 are also sufficient.

3. Fundamental solutions supported by a characteristic half space

In this section we characterize the polynomials P of degree 2 for which $\mathbb{R}^n$ is P-convex with bounds by proving the converse of Proposition 2. The essential step in the proof is based on the following lemma.

Lemma 8. *For $\lambda \in \mathbb{C}$ define the polynomial $Q(\lambda, \cdot) \in \mathbb{C}[z_1, z_2]$ by $Q(\lambda, z_1, z_2) := z_1 z_2 + \lambda$. Then there exists a function E in $L_{\infty,\mathrm{loc}}(\mathbb{C} \times \mathbb{R}^2)$ satisfying*

$$(*) \qquad |E(\lambda, \xi_1, \xi_2)| \leq \exp\left(\sqrt{2\xi_1 |\Im\lambda|\,|\xi_2|}\right),\ \xi_1 > 0,\ \xi_2 \in \mathbb{R}$$

such that $E(\lambda, \cdot)$ is a fundamental solution for $Q(\lambda, D)$ which satisfies

$$\mathrm{Supp} E(\lambda, \cdot) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 0\}.$$

Proof. Let J_0 denote the Bessel function of order zero, i.e.

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} t^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\sin\zeta} d\zeta$$

(see e.g. Courant and Hilbert [1], Kap. VII, (10) and (21)). We define

$$\begin{aligned} P(\lambda, \xi_1, \xi_2) &:= J_0(i2\sqrt{\lambda} \cdot \sqrt{\xi_1 \xi_2}) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^2} \xi_1^n \xi_2^n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-2\sqrt{\lambda}\sqrt{\xi_1\xi_2} \sin\zeta) d\zeta \end{aligned}$$

and note that P is analytic in λ, ξ_1 and ξ_2.

To estimate the modulus of P, fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \leq 0$ and let $\sqrt{\lambda} = \alpha + i\beta$, where α, β are real. Then $\alpha^2 - \beta^2 = \mathrm{Re}\lambda \leq 0$ implies $|\alpha| \leq |\beta|$, hence $2|\alpha|^2 \leq 2|\alpha\beta| = |\Im\lambda|$ and consequently $|\mathrm{Re}\sqrt{\lambda}| = |\alpha| \leq \sqrt{|\Im\lambda|/2}$. Now use this to estimate the integral representation for P and to get

$$|P(\lambda, \xi_1, \xi_2)| \leq \exp\left(\sqrt{2\xi_1 |\Im\lambda|\,|\xi_2|}\right) \quad \text{if } \mathrm{Re}\lambda \leq 0, \xi_1 \geq 0,\ \xi_2 \geq 0.$$

Since $P(\lambda, \xi_1, \xi_2) = P(-\lambda, \xi_1, -\xi_2)$, the same estimate holds if $\mathrm{Re}\lambda \geq 0, \xi_1 \geq 0$ and $\xi_2 \leq 0$.

Next let

$$E(\lambda,\xi_1,\xi_2) := \begin{cases} -P(\lambda,\xi_1,\xi_2) & \text{if } \xi_1 > 0, \xi_2 > 0 \text{ and } \mathrm{Re}\lambda \le 0 \\ P(\lambda,\xi_1,\xi_2) & \text{if } \xi_1 > 0, \xi_2 < 0 \text{ and } \mathrm{Re}\lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then E is in $L_{\infty,\mathrm{loc}}(\mathbb{C}\times\mathbb{R}^2)$, satisfies $(*)$ and vanishes whenever $\xi_1 < 0$. Hence the proof of the lemma is complete if we show $Q(\lambda, D)E(\lambda,\cdot) = \delta$. To do this fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \le 0$ and note that

$$P(\lambda,0,\xi_2) = 1,\ \frac{\partial P}{\partial \xi_1}(\lambda,\xi_1,0) = 0,\ \frac{\partial^2 P}{\partial\xi_1\partial\xi_2}(\lambda,\xi_1,\xi_2) = \lambda P(\lambda,\xi_1,\xi_2).$$

Consequently, we have for each $\varphi \in \mathcal{D}(\mathbb{R}^2)$:

$$\begin{aligned} (Q(\lambda,D)E(\lambda,\cdot))\varphi &= E(\lambda,\cdot)(Q(\lambda,-D)\varphi) \\ &= \int_0^\infty\int_0^\infty -P(\lambda,\cdot)\left(-\frac{\partial^2\varphi}{\partial\xi_1\partial\xi_2}+\lambda\varphi\right)d\xi_1 d\xi_2 \\ &= -\int_0^\infty \frac{\partial\varphi}{\partial\xi_2}(0,\xi_2)d\xi_2 - \int_0^\infty\int_0^\infty \frac{\partial P}{\partial\xi_1}(\lambda,\cdot)\frac{\partial\varphi}{\partial\xi_2}d\xi_1 d\xi_2 + \\ &\quad -\lambda\int_0^\infty\int_0^\infty P(\lambda,\cdot)\varphi d\xi_1 d\xi_2 \\ &= \varphi(0) = \delta(\varphi). \end{aligned}$$

The case $\mathrm{Re}\lambda > 0$ follows by the same arguments.

Proposition 3. *For $n \ge 2$ assume that $P \in \mathbb{C}[z_1,\ldots,z_n]$ has degree 2 and that $P = P_0 + ic$, where $P_0 \in \mathbb{R}[z_1,\ldots,z_n]$ and $c \in \mathbb{R}$. Then for each characteristic vector $N \neq 0$ there exists a fundamental solution E for $P(D)$ satisfying* $\mathrm{Supp}E \subset \overline{H_+(N)}$.

Proof. If P admits a characteristic vector N we may assume without restriction that $N = e_1$ and hence

$$H_+(N) = \{x \in \mathbb{R}^n : x_1 > 0\} =: H.$$

Moreover, P has the form

$$P(\xi) = \xi_1 A(\xi') + P_1(\xi') + ic,$$

where A is a real affine form and P_1 a real polynomial, both depending only on $\xi' = (\xi_2,\ldots,\xi_n)$. Now distinguish the following three cases:

case 1: $A = 0$
case 2: A is a real constant
case 3: A is not constant,

in which different arguments are applied. **case 1:** Choose a fundamental solution $E_1 \in \mathcal{D}'(\mathbb{R}^{n-1})$ for $P_1(D') + ic$ and note that $E := \delta_{x_1} \otimes E_1(x')$ has the required properties in the present case.

case 2: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ denote by $\tilde{\varphi}$ its partial Fourier transform with respect to x', i.e.

$$\tilde{\varphi}(x_1,\xi') := \left(\frac{1}{2\pi}\right)^{n-1}\int_{\mathbb{R}^{n-1}} e^{-i\langle x',\xi'\rangle}\varphi(x_1,x')dx',$$

and define

$$E(\varphi) := (2\pi)^{n-1} \int_{\mathbb{R}^{n-1}} \left\{ \int_0^\infty \exp(-i(P_1(-\xi') + ic)\frac{x_1}{A})\widetilde{\varphi}(x_1, \xi')dx_1 \right\} d\xi.$$

To see that the integral exists, note that $\mathrm{Supp}\widetilde{\varphi} \subset [-\alpha, \alpha] \times \mathbb{R}^{n-1}$ for some $\alpha > 0$ and that

$$\left| \exp\left(-i(P_1(-\xi') + ic)\frac{x_1}{A}\right) \right| \le \exp\left(\frac{cx_1}{A}\right),$$

since P_1 is real. From this it follows easily that E is in $\mathcal{D}'(\mathbb{R}^n)$ and that $\mathrm{Supp}E \subset \overline{H}$. Using integration by parts and Fourier's inversion formula, one obtains that E is a fundamental solution for $P(D)$.

case 3: After a real linear change in the ξ' variables, we may assume that $A(\xi') = \xi_2 + a$ for some $a \in \mathbb{R}$. Hence P has the form

$$\begin{aligned} P(\xi) &= \xi_1(\xi_2 + a) + b\xi_2^2 + \xi_2 L(\xi'') + P_2(\xi'') + ic \\ &= (\xi_1 + L(\xi'') - 2ab)(\xi_2 + a) + b(\xi_2 + a)^2 + P_3(\xi'') + ic, \end{aligned}$$

where b is a real number, L is a real affine form and P_2, P_3 are real polynomials in $\xi'' = (\xi_3, \ldots, \xi_n)$. Next fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and let $\widetilde{\varphi}$ denote its partial Fourier transform with respect to x''. Then define

$$\begin{aligned} F\varphi := (2\pi)^{n-2} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{+\infty} \int_0^\infty & \exp\left(-i\left(L(-\xi'') - 2ab\right)x_1 + a(bx_1 + x_2)\right) \cdot \\ & \cdot E\left(P_3(-\xi'') + ic, x_1, x_2\right) \widetilde{\varphi}\left(x_1, bx_1 + x_2, \xi''\right) dx_1 dx_2 d\xi'', \end{aligned}$$

where $E(\lambda, x_1, x_2)$ denotes the function from Lemma 8. Using this lemma, it is easy to check that F is in $\mathcal{D}'(\mathbb{R}^n)$ and $\mathrm{Supp}F \subset \overline{H}$. To show that F is a fundamental solution for $P(D)$, fix $\chi \in C^2(\mathbb{R}^2)$ and note that for any real number R

$$\begin{aligned} & D_1 D_2 \left(\exp(-i(Rx_1 + a(bx_1 + x_2)) \chi\left(x_1, bx_1 + x_2\right)\right) \\ = \; & \exp(-i(Rx_1 + a(bx_1 + x_2)) \\ & \cdot \left\{ \left((-D_1 + R)(-D_2 + a) + b(-D_2 + a)^2\right) \chi \right\} (x_1, bx_1 + x_2). \end{aligned}$$

Note further that for $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi := P(-D)\psi$ and $R := L(-\xi'') - 2ab$ we have

$$\begin{aligned} \widetilde{\varphi}(x_1, x_2, \xi'') &= \left(\left\{(-D_1 + R)(-D_2 + a) + b(-D_2 + a)^2\right\} \widetilde{\psi}\right)(x_1, x_2, \xi'') \\ &\quad + (P_3(-\xi'') + ic)\,\widetilde{\psi}(x_1, x_2, \xi''). \end{aligned}$$

Hence

$$\begin{aligned} & \exp\left(-i(Rx_1 + a(bx_1 + x_2)\right) \widetilde{\varphi}\left(x_1, bx_1 + x_2, \xi''\right) E(P_3(-\xi'') + ic, x_1, x_2) \\ = \; & \Big\{ D_1 D_2 \left(\exp(-i(Rx_1 + a(bx_1 + x_2))\widetilde{\psi}(x_1, bx_1 + x_2, \xi'')\right) \\ & + \exp(-(Rx_1 + a(bx_1 + x_2))(P_3(-\xi'') + ic)\widetilde{\psi}(x_1, x_2, \xi'') \Big\} \\ & \cdot E(P_3(-\xi'') + ic, x_1, x_2). \end{aligned}$$

By our choice of $E(\lambda, \cdot)$, this implies

$$(P(D)F)(\psi) = F(\varphi) = (2\pi)^{n-2} \int_{\mathbb{R}^{n-2}} \widetilde{\psi}(0,0,\xi'')d\xi'' = \psi(0),$$

by Fourier's inversion formula. Hence F has all the required properties.

Remark 2. It is easy to check that in all the cases which we discuss in the proof of Proposition 3, condition 12.8.1 (iii) of Hörmander [4] is satisfied. Hence Proposition 3 is a consequence of [4], Thm. 12.8.1. We give the above proof since it is elementary and constructive.

Lemma 9. *Let $P = P_0 + ic_0$, where $P_0 \in \mathbb{R}[z_1, \ldots, z_n]$ and $c_0 \in \mathbb{R}$. If the principal part P_2 of P is an indefinite quadratic form (hence has at least rank 2) then $\mathbb{R}^n$ is P-convex with bounds.*

Proof. After a real linear change of variables we may assume that

$$P(x_1, \ldots, x_n) = \sum_{j=1}^{r} x_j^2 - \sum_{j=r+1}^{s} x_j^2 + L(x) + C,$$

where L is a real linear form, $C \in \mathbb{C}$ and $r \geq 1$, $s \geq r+1$. If $N \in \mathbb{R}^n \setminus \{0\}$ is characteristic for P then Proposition 3 implies that $P(D)$ admits a fundamental solution E_N satisfying $\operatorname{Supp} E_N \subset \overline{H_+(N)}$. Hence the result follows from Proposition 1 if we show that for each $x \in \mathbb{R}^n \setminus \{0\}$ there exists a characteristic vector $N \in \mathbb{R}^n \setminus \{0\}$ such that $x \in H_+(N)$. To prove this, fix $x = (x_1, \ldots, x_n)$ and consider the following cases:

case 1: there exists $s < j \leq n$ such that $x_j \neq 0$. Then $N := \operatorname{sign}(x_j)e_j$ does the job.

case 2: there exists $1 \leq j \leq r$ such that $x_j \neq 0$. Then let $N := \operatorname{sign}(x_j)e_j + \lambda e_s$, where $\lambda = 1$ if $x_s \geq 0$ and $\lambda = -1$ if $x_s < 0$.

case 3: there exists $r+1 \leq j \leq s$ such that $x_j \neq 0$. Then let $N := -\operatorname{sign}(x_j)e_j + \mu e_1$, where $\mu = 1$ if $x_1 \geq 0$ and $\mu = -1$ if $x_1 < 0$.

Theorem 2. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial of degree 2 in normal form (see Lemma 3). Then $\mathbb{R}^n$ is P-convex with bounds if and only if P satisfies the necessary conditions stated in Proposition 2.*

Proof. Assume that P is of type (I). If $r = s = 1$ then P is an ordinary differential operator in x_1. Hence its kernel in one variable admits a topological complement. Therefore, the result follows from Lemma 1 in this case.

If $s > r$ then the principal part of P is indefinite. Hence the result follows from Lemma 9.

If P is of type (II) and satisfies $\lambda \in \mathbb{R}$ as well as $s > r$ then its principal part is indefinite and P is real. Hence the result follows from Lemma 9.

If P is a real polynomial of degree 2 in $n \geq 2$ variables and if its principal part has at least rank 2 then the P-convexity with bounds of $\mathbb{R}^n$ can be characterized by a condition on P and by other interesting properties, as we will show next. To do so we need the following proposition which is interesting on its own.

Proposition 4. *Let $N \in \mathbb{R}^n$ be a characteristic vector for $P \in \mathbb{C}[z, \ldots, z_n]$ and assume that the following two conditions are satisfied:*

1. $\mathbb{R}^n$ is P-convex with bounds

2. *there exists a fundamental solution E in $\mathcal{D}'(\mathbb{R}^n)$ for $P(D)$ with* $\operatorname{Supp}(E) \subset \overline{H_-(N)}$.

Then $H_+(N)$ is P-convex with bounds.

Proof. Without restriction we can assume that $N = (0, \ldots, 0, 1)$, so that $H_+(N) = \{x \in \mathbb{R}^n : x_n > 0\}$. Then we choose a function $\psi \in \mathcal{E}(\mathbb{R}^n)$ with $\operatorname{Supp}(\psi) \subset \{(x', x_n) \in \mathbb{R}^n : x_n \leq 2\exp(-|x'|^2)\}$ and $\psi(x) = 1$ for all $x = (x', x_n) \in \mathbb{R}^n$ with $x_n \leq \exp(-|x'|^2)$. Next we fix a fundamental solution E for $P(D)$ with $\operatorname{Supp}(E) \subset \overline{H_-(N)}$, according to (2). Then it is easy to check that for each $f \in \mathcal{E}(H_+(N))$, the convolution $E * (\psi f)$ can be defined as an element of $\mathcal{E}(H_+(N))$ and that the map $f \mapsto E * (\psi f)$ is a continuous endomorphism of $\mathcal{E}(H_+(N))$. According to (1), there exists a continuous linear right inverse R for $P(D)$ on $\mathcal{E}(\mathbb{R}^n)$. Therefore it is easy to check that

$$S : f \mapsto E * (\psi f) + R((1 - \psi)f)|_{H_+(N)} \quad , \quad f \in \mathcal{E}(H_+),$$

defines a right inverse for $P(D)$ on $\mathcal{E}(H_+(N))$. Hence $H_+(N)$ is P-convex with bounds.

Remark 3. Note that Proposition 4 extends the sufficient results for the P-convexity with bounds of open half spaces that were known so far, namely, [7], 3.2 (non-characteristic) and 3.6 (characteristic).

Theorem 3. *Assume that $P \in \mathbb{R}[x_1, \ldots, x_n]$ is of degree 2 and that its principal part P_2 has at least rank 2. Then the following assertions are equivalent:*

1. *$\mathbb{R}^n$ is P-convex with bounds*
2. *P is not elliptic and each characteristic half space $H_+(N)$ is P-convex with bounds*
3. *there exists an open half space H which is P-convex with bounds*
4. *there exists an open set $\Omega \neq \emptyset$ in $\mathbb{R}^n$ which is P-convex with bounds*
5. *P_2 is indefinite.*

Proof. (1) $\Rightarrow$ (2): By a classical result of Grothendieck (see [7], 2.11 for its extension), P is not elliptic since (1) holds. If $N \in \mathbb{R}^n \setminus \{0\}$ is characteristic for P then Proposition 3 implies that $P(D)$ admits a fundamental solution that has support in $\overline{H_-(N)}$. This and (1) show that the hypotheses of Proposition 4 are satisfied. Hence $H_+(N)$ is P-convex with bounds.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): These implications hold by obvious reasons.

(4) $\Rightarrow$ (5): We transform P into its normal form according to Lemma 3 and note that condition (4) also holds for the transformed operator. By Lemma 7 its principal part is an indefinite form. Since P and the transformed operator have equivalent principal parts, (5) holds.

(5) $\Rightarrow$ (1): This holds by Lemma 9.

Corollary 1. *For each non-constant polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ of degree 2 the following assertions are equivalent:*

1. *$\mathbb{R}^n$ is P-convex with bounds*
2. *there exists a basis $\{N_1, \ldots, N_n\}$ of $\mathbb{R}^n$ and fundamental solutions $E_1^\pm, \ldots, E_n^\pm$ for $P(D)$ satisfying* $\operatorname{Supp} E_j^\pm \subset \overline{H_\pm(N_j)}$ *for $1 \leq j \leq n$*

3. *there exists a basis $\{N_1, \ldots, N_n\}$ of $\mathbb{R}^n$ for which $H_\pm(N_j)$ is P-convex with bounds for all j*
4. *there exists an open half space H which is P-convex with bounds*
5. *there exists a bounded convex open set Ω in $\mathbb{R}^n$ which is P-convex with bounds.*

Proof. (1) $\Rightarrow$ (2): By Lemma 2 and 3 we can assume that up to a constant factor and a real orthogonal change of variables, P_2 is a quadratic form induced by a real diagonal matrix and that the affine part of P is as in Lemma 3. By Proposition 2 there are two cases: Either P_2 is indefinite or $P(x_1, \ldots, x_n) = ax_1^2 + c$ where $a \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{C}$. If P_2 is indefinite then the proof of Lemma 9 shows that $N_1, \ldots, N_n$ exist and can be chosen to be characteristic for P. In the remaining case P admits fundamental solutions $E_\pm$ having support in $\{x \in \mathbb{R}^n : \pm x_1 \geq 0, x_j = 0 \text{ for } 2 \leq j \leq n\}$. Hence (2) holds also in this case.

(2) $\Rightarrow$ (3): By Proposition 1, the present hypothesis implies (1). Hence (3) follows from Proposition 4.

(3) $\Rightarrow$ (4) and (5): This follows from the fact that P-convexity with bounds is invariant under translations and finite intersections (see [7], 2.10).

(4) or (5) $\Rightarrow$ (1): This is an immediate consequence of [7], 4.5, and [11], 2.11.

Remark 4. In Franken and Meise [2], 3.6, it is shown that the conditions (2) and (5) of Corollary 1 are equivalent for each homogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$.

The following example shows that Corollary 1 does not hold for arbitrary polynomials without additional hypotheses.

Example 1. Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be given by $P(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^3$. Then $\mathbb{R}^3$ is P-convex with bounds while $P(D)$ does not admit any fundamental solution E supported by a closed half space.

To prove this note that $\mathbb{R}^3$ is P-convex with bounds by [7], 4.9. If $N \in \mathbb{R}^3$ is non-characteristic for P and if we assume that $P(D)$ admits a fundamental solution E satisfying $\operatorname{Supp} E \subset \overline{H_+(N)}$ then P is hyperbolic with respect to N, which is easily seen not to hold. Hence there is no such fundamental solution.

If $N \in S^2$ is characteristic for P then we use Hörmander [5], Thm. 4.2, to show that there is no fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ for $P(D)$ satisfying $\operatorname{Supp} E \subset \overline{H_+(N)}$. By this theorem it suffices to show

(1) there exists $w \in \mathbb{R}^3$ so that the polynomial $\tau \mapsto P(w + \tau N)$ does not vanish identically and has some non-real zero.

To prove (1) fix a characteristic vector $N = (a, b, c) \in S^2$ and let $w = (1, 1, 1)$. Then it is easy to check that

$$P(w + \tau N) = 3(\tau^2 + (a + b + c)\tau + 1), \quad \tau \in \mathbb{C}. \tag{2}$$

Because of

$$\sup \left\{ \sum_{j=1}^{3} \xi_j : \xi \in S^2 \right\} = \sqrt{3},$$

we have

$$\left(\frac{a+b+c}{2}\right)^2 \leq \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} < 1.$$

Hence the equation (2) has non-real zeros for each $N \in S^2$, and consequently (1) holds.

Remark 5. Example 1 in connection with Franken and Meise [2], Thm. 3.5, implies that no convex open bounded set in $\mathbb{R}^3$ is P-convex with bounds for the differential operator

$$\left(\frac{\partial}{\partial x}\right)^3 + \left(\frac{\partial}{\partial y}\right)^3 + \left(\frac{\partial}{\partial z}\right)^3.$$

Next let $Q(D)$ denote this operator, acting on $\mathcal{E}(\mathbb{R}^4)$. Then $\mathbb{R}^4$ is Q-convex with bounds and obviously $Q(D)$ admits a fundamental solution supported by the hyperplane $x_4 = 0$. Hence the open half spaces $\{x \in \mathbb{R}^4 : \pm x_4 > 0\}$ are Q-convex with bounds by Proposition 4. However, the arguments which we used above imply that no bounded open set Ω in $\mathbb{R}^4$ is Q-convex with bounds.

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