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Continuous Linear Right Inverses for Partial Differential Operators on Non-Quasianalytic Classes and on Ultradistributions

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Dedicated to PROFESSOR DR. H.-G. TILLMANN *on the occasion of his 70 birthday*

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Abstract. Characterizations are given of those linear partial differential operators with constant coefficients which admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\Omega)$ $\left(\text{resp. } \mathcal{E}_{\{\omega\}}(\Omega)\right)$ and/or $\mathcal{D}'_{(\omega)}(\Omega)$ $\left(\text{resp. } \mathcal{D}'_{\{\omega\}}(\Omega)\right)$, where Ω is an open set in $\mathbb{R}^n$. The characterizations are in the same spirit as in the previous results of the authors on the existence of right inverses on $C^\infty(\Omega)$ and/or $\mathcal{D}'(\Omega)$.

0. Introduction

In the early fifties L. SCHWARTZ posed the problem of determining when a linear differential operator $P(D)$ with constant coefficients admits a (continuous linear) right inverse on $\mathcal{E}(\Omega)$ or $\mathcal{D}'(\Omega)$, Ω an open subset of $\mathbb{R}^n$. This problem was solved by the present authors in [17] (see also [15], [16]) and for systems over convex open sets by PALAMODOV [23].

In the present article we consider the same problem for the non-quasianalytic classes of Beurling type $\mathcal{E}_{(\omega)}(\Omega)$, of Roumieu type $\mathcal{E}_{\{\omega\}}(\Omega)$ and for the corresponding classes of ultradistributions $\mathcal{D}'_{(\omega)}(\Omega)$ and $\mathcal{D}'_{\{\omega\}}(\Omega)$, where ω is a weight function in the sense of BRAUN, MEISE and TAYLOR [6]. Extending our results in [17], we characterize by various conditions when a given operator $P(D)$ admits a right inverse on any of these classes. In particular, we show that $P(D)$ admits a right inverse on $\mathcal{E}_{(\omega)}(\Omega)$ $\left(\text{resp. } \mathcal{E}_{\{\omega\}}(\Omega)\right)$ if and only if it admits a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ $\left(\text{resp. } \mathcal{D}'_{\{\omega\}}(\Omega)\right)$. A consequence of our characterization is that $P(D)$ admits a right inverse on $\mathcal{D}'_{\{\omega\}}(\Omega)$ if

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and only if there exists a weight function κ satisfying $\kappa = O(\omega)$ such that $P(D)$ admits a right inverse on $\mathcal{D}'_{(\kappa)}(\Omega)$. Hence the Roumieu case can be reduced to the Beurling case. The proof of these results – to a certain extent – can be given by variations of the arguments that we used in [17]. However, as a new ingredient we need a recent result of BRAUN [3] on the local structure of ultradistributions, which extends an earlier one of KOMATSU [13]. Also we make extensive use of the results in [6].

To evaluate our characterization further, two cases are distinguished that are treated by different methods. For a bounded open set Ω with C^1 - boundary, an application of Holmgren's uniqueness theorem shows that $P(D)$ has a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ only if $P(D)$ is (ω)-hyperbolic with respect to each non-characteristic vector N in the sense that there exists a fundamental solution $E_N \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ for $P(D)$ which satisfies $\operatorname{Supp} E_N \subset \{x \in \mathbb{R}^n : \langle x, N\rangle \geq 0\}$. Using results on (ω)-hyperbolic operators from [21], we show that an operator $P(D)$ admits a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ for some bounded open set Ω with C^1 - boundary if and only if $P(D)$ admits a right inverse on $\mathcal{D}'_{(\omega)}(G)$ for each convex open set G.

For convex open sets Ω an application of Fourier analysis gives that an operator $P(D)$ admits a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ if and only if the zero variety of P in $\mathbb{C}^n$ satisfies a condition $\mathrm{PL}(\Omega,(\omega))$ of Phragmén - Lindelöf type, which is related to a similar but different condition which HÖRMANDER [11] introduced to characterize when $P(D)$ acts surjectively on all real-analytic functions on Ω. For a comprehensive study of the condition $\mathrm{PL}(\Omega,(\omega))$ we refer to our article [20].

The paper is organized as follows: In the preliminary Section 1 we introduce ultra-differentiable functions and ultradistributions. The existence of a right inverse for $P(D)$ in the Beurling case is characterized in Section 2. The same is done for the Roumieu case in Section 3. The connection between right inverses and ω-hyperbolicity is investigated in Section 4 and in Section 5 we discuss the characterization of the existence of right inverses in terms of Phragmén - Lindelöf conditions.

The main results of the present paper were announced in our survey article [19].

1. Preliminaries

In this preliminary section we introduce the non-quasianalytic classes, the spaces of ultradistributions and most of the notation that will be used in the sequel.

Definition 1.1. A continuous increasing function $\omega : [0,\infty[\;\to\; [0,\infty[$ is called a *weight function* if it satisfies the following conditions:

(α) there exists $K \geq 1$ with $\omega(2t) \leq K(1+\omega(t))$ for all $t \geq 0$,

(β) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}\,dt < \infty$,

(γ) $\log t = o(\omega(t))$ as $t \to \infty$,

(δ) $\varphi : t \mapsto \omega(e^t)$ is convex.

For a weight function ω we define $\tilde{\omega} : \mathbb{C}^n \to [0,\infty[$ by $\tilde{\omega}(z) = \omega(|z|)$ and again call this function ω, by abuse of notation.

The Young conjugate $\varphi^* : [0,\infty[\to \mathbb{R}$ of φ is defined by

$$\varphi^*(y) := \sup\{xy - \varphi(x) : x \geq 0\}.$$

Remark 1.2. (a) Each weight function ω satisfies $\lim_{t\to\infty} \frac{\omega(t)}{t} = 0$ by the remark following 1.3 of [14].

(b) For each weight function ω there exists a weight function σ satisfying $\sigma(t) = \omega(t)$ for all large $t > 0$ and $\sigma|[0,1] \equiv 0$. This implies $\varphi_\sigma(y) = \varphi_\omega(y)$ for all large y, $\varphi_\sigma^*([0,\infty[) \subset [0,\infty[$ and $\varphi_\sigma^{**} = \varphi_\sigma$. From this it follows that all subsequent definitions do not change if ω is replaced by σ. On the other hand they also do not change if ω is replaced by $\omega + c$, c some positive number. Therefore we can and will assume that $\omega(0) \geq 1$.

Definition 1.3. Let ω be a weight function.

(a) For a compact set $K \subset \mathbb{R}^n$ and $\lambda > 0$ let

$$\mathcal{E}_\omega(K,\lambda) := \left\{ f \in C^\infty(K) : \|f\|_{K,\lambda} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^n} |f^{(\alpha)}(x)| \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right) < \infty \right\}.$$

(b) For an open set $\Omega \subset \mathbb{R}^n$ define

$$\begin{aligned} \mathcal{E}_{(\omega)}(\Omega) &:= \operatorname*{proj}_{\substack{\longleftarrow \\ K\subset\subset\Omega}} \operatorname*{proj}_{\substack{\longleftarrow \\ m\in\mathbb{N}}} \mathcal{E}_\omega(K,m) \\ &= \left\{ f \in C^\infty(\Omega) : \|f\|_{K,m} < \infty \text{ for each } K \subset\subset \Omega \text{ and each } m \in \mathbb{N} \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{\{\omega\}}(\Omega) &:= \operatorname*{proj}_{\substack{\longleftarrow \\ K\subset\subset\Omega}} \operatorname*{ind}_{\substack{\longrightarrow \\ m\in\mathbb{N}}} \mathcal{E}_\omega\left(K,\frac{1}{m}\right) \\ &= \left\{ f \in C^\infty(\Omega) : \text{ for each } K \subset\subset \Omega \text{ there is } m \in \mathbb{N} \text{ with } \|f\|_{K,\frac{1}{m}} < \infty \right\}. \end{aligned}$$

The elements of $\mathcal{E}_{(\omega)}(\Omega)$ $\left(\text{resp. } \mathcal{E}_{\{\omega\}}(\Omega)\right)$ are called ω-ultradifferentiable functions of Beurling (resp. Roumieu) type on Ω. We write $\mathcal{E}_*(\Omega)$, where $*$ can be either (ω) or $\{\omega\}$ at all occuring places.

(c) For a compact set K in $\mathbb{R}^n$ we let

$$\mathcal{D}_*(K) := \left\{ f \in \mathcal{E}_*(\mathbb{R}^n) : \operatorname{Supp}(f) \subset K \right\},$$

endowed with the induced topology. For an open set $\Omega \subset \mathbb{R}^n$ and a fundamental sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of Ω we let

$$\mathcal{D}_*(\Omega) := \operatorname*{ind}_{j\to} \mathcal{D}_*(K_j).$$

The dual $\mathcal{D}'_*(\Omega)$ of $\mathcal{D}_*(\Omega)$ is endowed with its strong topology. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ $\left(\text{resp. } \mathcal{D}'_{\{\omega\}}(\Omega)\right)$ are called ω-ultradistributions of Beurling (resp. Roumieu) type on Ω.

(d) For an open set $\Omega \subset \mathbb{R}^n$, an open subset U of Ω and $X(\Omega)$, being one of the spaces introduced in (b) or (c), we let

$$X(\Omega, U) := \{f \in X(\Omega) : f|_U \equiv 0\}.$$

Remark 1.4. Definitions 1.1 and 1.3 are taken from BRAUN, MEISE and TAYLOR [6]. They are variations of the classical ones, introduced by BEURLING [1] (see also PETZSCHE and VOGT [24]). Though [6] is based on BEURLING's ideas, we shall mainly refer to it, since it is well adapted to our applications. If ω is a subadditive function on $[0, \infty[$ satisfying 1.1 $(\beta) - (\delta)$, then the classes $\mathcal{E}_{(\omega)}$ and $\mathcal{D}_{(\omega)}$ coincide with those of BEURLING [1]. In [6] it is shown that for each weight function ω the spaces $\mathcal{D}_{(\omega)}(\Omega)$ and $\mathcal{D}_{\{\omega\}}(\Omega)$ are non-trivial.

The classical case $\mathcal{E}_{(\omega)} = C^\infty$ is formally not a subcase of what we present here, since $\omega := \log^+$ is not a weight function in the sense of Definition 1.1. However, it can be regarded as such if one interpretes φ^* appropriately or if one uses an equivalent definition of $\mathcal{E}_{(\omega)}(\Omega)$ (see BRAUN, MEISE and TAYLOR [6], 4.5).

Example 1.5. The following functions $\omega : [0, \infty[\to [0, \infty[$ are examples of weight functions:

1. $\omega(t) = t^\alpha$, $0 < \alpha < 1$,
2. $\omega(t) = (\log(1 + t))^\beta$, $\beta > 1$,
3. $\omega(t) = t(\log(e + t))^{-\beta}$, $\beta > 1$.

Note that for $\omega(t) = t^\alpha$, the classes $\mathcal{E}_{(\omega)}$ resp. $\mathcal{E}_{\{\omega\}}$ coincide with the Gevrey classes $\Gamma^{(d)}$ resp. $\Gamma^{\{d\}}$ for $d := 1/\alpha$.

Polynomials and partial differential operators. By $\mathbb{C}[z_1, \dots, z_n]$ we denote the ring of all complex polynomials in n variables, which are also regarded as functions on $\mathbb{C}^n$. For $P \in \mathbb{C}[z_1, \dots, z_n]$, $P(z) = \sum_{|\alpha| \le m} a_\alpha z^\alpha$, with $\sum_{|\alpha| = m} |a_\alpha| \neq 0$ we call

$$P_m : z \mapsto \sum_{|\alpha| = m} a_\alpha z^\alpha$$

the principal part of P. Note that P_m is a homogeneous polynomial of degree m.

For P as above and an open set Ω in $\mathbb{R}^n$ we define the linear partial differential operator

$$P(D) : \mathcal{D}'_*(\Omega) \longrightarrow \mathcal{D}'_*(\Omega), \quad P(D)f := \sum_{|\alpha| \le m} a_\alpha i^{-|\alpha|} f^{(\alpha)}.$$

Then $P(D)$ is a continuous endomorphism of $\mathcal{D}'_*(\Omega)$ and its restriction to $\mathcal{E}_*(\Omega)$ is a continuous endomorphism of $\mathcal{E}_*(\Omega)$.

Definition 1.6. For P as above and an open set Ω in $\mathbb{R}^n$ we let

$$\mathcal{N}(\Omega) := \{f \in \mathcal{D}'_*(\Omega) : P(D)f = 0\} \quad \text{and} \quad N(\Omega) := \mathcal{N}(\Omega) \cap \mathcal{E}_*(\Omega).$$

Right inverses. For locally convex spaces E and F we let

$$L(E,F) := \{A : E \to F : A \text{ is continuous and linear}\}.$$

A map $A \in L(E,F)$ is said to admit a right inverse, if there exists $R \in L(F,E)$ so that $A \circ R = \mathrm{id}_F$.

2. Right inverses on $\mathcal{D}'_{(\omega)}(\Omega)$ and $\mathcal{E}_{(\omega)}(\Omega)$

For a given weight function ω and for an open set Ω in $\mathbb{R}^n$ we characterize in this section the partial differential operators $P(D)$ that admit a continuous linear right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ (resp. on $\mathcal{E}_{(\omega)}(\Omega)$). In particular, we show that $P(D)$ has a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ if and only if $P(D)$ has a right inverse on $\mathcal{E}_{(\omega)}(\Omega)$. Up to Lemma 2.6 this could be done as in Section 2 of [17]. However, we prefer a somewhat different line of argument. The results of the present section will be evaluated further in the subsequent sections.

Throughout this section, ω denotes a given weight function. For an open set Ω in $\mathbb{R}^n$ and $\varepsilon > 0$ let

$$\Omega_\varepsilon := \left\{x \in \Omega : |x| < \frac{1}{\varepsilon} \text{ and } \operatorname{dist}(x, \partial\Omega) > \varepsilon\right\},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) for the following assertions:*

1. $P(D) : \mathcal{D}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ *admits a right inverse;*
2. *for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $f \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_\varepsilon)$ with $P(D)g = f$;*
3. *for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $\mu \in \mathcal{N}(\Omega_\delta)$ there exists $\nu \in \mathcal{N}(\Omega)$ with $\nu|_{\Omega_\varepsilon} = \mu|_{\Omega_\varepsilon}$;*
4. *for each $\varepsilon > 0$ there exists $0 < \delta_0 < \varepsilon$ so that for all $0 < \zeta < \sigma < \eta < \delta < \delta_0$ and each $\xi \in \overline{\Omega}_\eta \setminus \Omega_\delta$ there exists $E_\xi \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ so that*
   (i) $\operatorname{Supp} E_\xi \subset (\mathbb{R}^n \setminus \Omega_\varepsilon) - \xi$,
   (ii) $P(D)E_\xi = \delta + T_\xi$ *where* $\operatorname{Supp} T_\xi \subset (\Omega_\zeta \setminus \Omega_\sigma) - \xi$.

Proof. Mutatis mutandis this can be shown as in Lemma 2.1 of [17]. □

Lemma 2.2. *Let P be a complex polynomial in n variables, Ω an open set in $\mathbb{R}^n$ and $(\Omega_k)_{k\in\mathbb{N}}$ an exhaustion of Ω by relatively compact subsets. Let $\Omega_0 := \Omega_{-1} := \Omega_{-2} := \emptyset$ and assume that for each $k \in \mathbb{N}_0$ there exists a continuous linear map $A_k : \mathcal{D}'_{(\omega)}(\Omega, \Omega_k) \to \mathcal{D}'_{(\omega)}(\Omega, \Omega_{k-2})$ which satisfies*

$$(*) \qquad P(D)A_k(f)|_{\Omega_{k+1}} = f|_{\Omega_{k+1}} \quad \text{for all} \quad f \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_k).$$

Then $P(D)$ admits a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$.

Proof. By the following induction argument we define a sequence $(R_k)_{k\in\mathbb{N}_0}$ in $L\Big(\mathcal{D}'_{(\omega)}(\Omega)\Big)$ which satisfies

$$(2.1)\qquad P(D)R_k(f)|_{\Omega_{k+1}} = f|_{\Omega_{k+1}} \quad \text{for all} \quad f\in\mathcal{D}'_{(\omega)}(\Omega),\ k\in\mathbb{N}_0 .$$

If we let $R_0 := A_0$ then $(*)$ and $\Omega_0 = \emptyset$ imply that (2.1) holds for $k=0$. Assume that R_k is already defined so that (2.1) is satisfied. Then note that because of (2.1) we have

$$f - P(D)R_k(f) \in \mathcal{D}'_{(\omega)}(\Omega,\Omega_{k+1}) \quad \text{for all} \quad f\in\mathcal{D}'_{(\omega)}(\Omega).$$

Therefore we can define

$$(2.2)\qquad R_{k+1}(f) := A_{k+1}\big(f - P(D)R_k(f)\big) + R_k(f), \qquad f\in\mathcal{D}'_{(\omega)}(\Omega).$$

Obviously, R_{k+1} is in $L\Big(\mathcal{D}'_{(\omega)}(\Omega)\Big)$. Moreover, $(*)$ implies

$$P(D)R_{k+1}(f)|_{\Omega_{k+2}} = (f - P(D)R_k(f))|_{\Omega_{k+2}} + P(D)R_k(f)|_{\Omega_{k+2}} = f|_{\Omega_{k+2}}.$$

Hence condition (2.1) is satisfied by R_{k+1}.

To see that R_k converges to some $R\in L\Big(\mathcal{D}'_{(\omega)}(\Omega)\Big)$, note that from (2.2) and $A_{k+1}\Big(\mathcal{D}'_{(\omega)}(\Omega,\Omega_{k+1})\Big)\subset\mathcal{D}'_{(\omega)}(\Omega,\Omega_{k-1})$ we get

$$R_{k+1}(f)|_{\Omega_{k-1}} = R_k(f)|_{\Omega_{k-1}} \quad \text{for all} \quad k\in\mathbb{N}.$$

Finally note that (1) implies $P(D)R = \mathrm{id}_{\mathcal{D}'_{(\omega)}(\Omega)}$. □

Remark 2.3. It is easily seen that Lemma 2.2 remains true if the symbol $\mathcal{D}'_{(\omega)}$ is replaced everywhere by one of the symbols $\mathcal{E}_{(\omega)}$, $\mathcal{D}_{\{\omega\}}$ or $\mathcal{E}_{\{\omega\}}$.

Lemma 2.4. *Assume that for $P\in\mathbb{C}[z_1,\dots,z_n]$ and an open set $\Omega\subset\mathbb{R}^n$ condition* 2.1 (4) *holds. Then $P(D)$ admits a right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$ and on $\mathcal{E}_{(\omega)}(\Omega)$.*

Proof. For $\varepsilon>0$ let $0<\delta_0(\varepsilon)<\varepsilon$ denote the number which exists by condition 2.1 (4). Using this condition recursively, choose a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $]0,\infty[$ which decreases strictly to zero and satisfies $\varepsilon_1<\frac{1}{2}\operatorname{diam}\Omega$ and $\varepsilon_{k+1}<\delta_0(\varepsilon_k)$ for all $k\in\mathbb{N}$. Then define $\Omega_k=\emptyset$ for $k=-2,-1,0$ and $\Omega_k:=\Omega_{\varepsilon_k}$ for $k\in\mathbb{N}$. In order to be able to apply Lemma 2.2 we are going to define continuous linear operators $A_k:\mathcal{D}'_{(\omega)}(\Omega,\Omega_k)\to\mathcal{D}'_{(\omega)}(\Omega,\Omega_{k-2})$ for $k\in\mathbb{N}_0$, which satisfy condition 2.2 $(*)$. To do this, we distinguish two cases.

Case 1: $k=0,1,2$. In this case choose $\psi\in\mathcal{D}_{(\omega)}(\Omega_4)$ with $\psi|_{\Omega_3}\equiv 1$ and $E\in\mathcal{D}'(\mathbb{R}^n)$ with $P(D)E=\delta$ and define

$$A_0 : \mathcal{D}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega), \quad A_0(f) := E*(\psi f).$$

Furthermore, let $A_k := A_0|_{\mathcal{D}'_{(\omega)}(\Omega,\Omega_k)}$ for $k = 1,2$. Then it is easy to see that condition 2.2 (*) holds for $k = 0,1,2$ by the choice of ψ.

Case 2: $k \in \mathbb{N}$, $k \geq 3$. In this case choose a number t which satisfies

$$0 < t < \min\left(\operatorname{dist}(\Omega_{k-2}, \mathbb{R}^n \setminus \Omega_{k-1}), \operatorname{dist}(\Omega_{k+1}, \mathbb{R}^n \setminus \Omega_{k+2})\right). \tag{2.3}$$

Since $\overline{\Omega}_{k+1} \setminus \Omega_k$ is compact, we can choose $m \in \mathbb{N}$ and $\xi_j \in \overline{\Omega}_{k+1} \setminus \Omega_k$ so that for $B_t(\xi) := \{x \in \mathbb{R}^n : |x - \xi| < t\} = \xi + B_t(0)$ we have

$$\overline{\Omega}_{k+1} \setminus \Omega_k \subset \bigcup_{j=1}^{m} B_t(\xi_j).$$

Next use condition 2.1 (4) with $\varepsilon = \varepsilon_{k-1}$, $\delta = \varepsilon_k < \delta_0(\varepsilon_{k-1})$, $\eta = \varepsilon_{k+1}$, $\sigma = \varepsilon_{k+2}$ and $\zeta = \varepsilon_{k+3}$ to find $E_{\xi_j} \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ so that for $1 \leq j \leq m$,

$$\operatorname{Supp} E_{\xi_j} \subset (\mathbb{R}^n \setminus \Omega_{k-1}) - \xi_j\,, \tag{2.4}$$

$$P(D)E_{\xi_j} = \delta + T_{\xi_j} \quad \text{where} \quad \operatorname{Supp} T_{\xi_j} \subset (\Omega_{k+3} \setminus \Omega_{k+2}) - \xi_j\,. \tag{2.5}$$

Further choose functions $\varphi_j \in \mathcal{D}_{(\omega)}(B_t(\xi_j))$, $1 \leq j \leq m$, so that $\sum_{j=1}^{m} \varphi_j(x) = 1$ for all $x \in \overline{\Omega}_{k+1} \setminus \Omega_k$. This implies

$$\left.\left(\sum_{j=1}^{m} \varphi_j f\right)\right|_{\Omega_{k+1}} = f|_{\Omega_{k+1}} \quad \text{for each} \quad f \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_k)\,. \tag{2.6}$$

Next define

$$A_k(f) := \sum_{j=1}^{m} E_{\xi_j} * (\varphi_j f)\,, \quad f \in \mathcal{D}'_{(\omega)}(\Omega)\,.$$

Obviously, A_k is a continuous linear map from $\mathcal{D}'_{(\omega)}(\Omega)$ into $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$. To show $A_k\left(\mathcal{D}'_{(\omega)}(\Omega)\right) \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^n, \Omega_k)$, note that for $f \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_k)$ we get from (2.1) and (2.2) the inclusion

$$\operatorname{Supp}(A_k(f)) \subset \bigcup_{j=1}^{m} (\mathbb{R}^n \setminus \Omega_{k-1}) - \xi_j + B_t(\xi_j) \subset \mathbb{R}^n \setminus \Omega_{k-2}\,.$$

Hence $A_k : \mathcal{D}'_{(\omega)}(\Omega, \Omega_k) \to \mathcal{D}'_{(\omega)}(\Omega, \Omega_{k-2})$ is well-defined. To show that A_k satisfies 2.2 (*) note that (2.3) implies

$$\begin{aligned} P(D)A_k(f) &= \sum_{j=1}^{m} (\delta + T_{\xi_j}) * (\varphi_j f) \\ &= \sum_{j=1}^{m} \varphi_j f + \sum_{j=1}^{m} T_{\xi_j} * (\varphi_j f) \quad \text{for all} \quad f \in \mathcal{D}'_{(\omega)}(\Omega)\,. \end{aligned} \tag{2.7}$$

From (2.1) and (2.3) it follows that

$$\operatorname{Supp} T_{\xi_j} * (\varphi_j f) \subset (\Omega_{k+3} \setminus \Omega_{k+2}) - \xi_j + B_t(\xi_j) \subset \Omega_{k+3} \setminus \Omega_{k+1} .$$

Hence (2.5) and (2.4) imply

$$P(D)A_k(f)|_{\Omega_{k+1}} = f|_{\Omega_{k+1}} \quad \text{for all} \quad f \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_k).$$

Consequently, Lemma 2.2 shows that $P(D)$ admits a continuous linear right inverse on $\mathcal{D}'_{(\omega)}(\Omega)$.

It is easy to see that the arguments above also apply to $\mathcal{E}_{(\omega)}(\Omega, \Omega_k)$. Because of this we get from Remark 2.3 that $P(D)$ also admits a right inverse on $\mathcal{E}_{(\omega)}(\Omega)$. □

Combining Lemma 2.1 with Lemma 2.4 we see that the existence of a right inverse for $P(D)$ on $\mathcal{E}_{(\omega)}(\Omega)$ is necessary for the existence of a right inverse for $P(D)$ on $\mathcal{D}'_{(\omega)}(\Omega)$. To prove the converse, we introduce the following notation.

Notation. For an open set $\Omega \subset \mathbb{R}^n, \lambda > 0$ and $\varepsilon > 0$ let

$$\mathcal{E}^\lambda_\omega(\Omega) := \{f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < \infty \text{ for all } K \subset\subset \Omega\}$$

and

$$\mathcal{E}^\lambda_\omega(\Omega, \Omega_\varepsilon) := \{f \in \mathcal{E}^\lambda_\omega(\Omega) : f|_{\Omega_\varepsilon} \equiv 0\}.$$

Lemma 2.5. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. If $P(D) : \mathcal{E}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ admits a right inverse then the following condition holds:*

(∗) *For each $\varepsilon > 0$ there is $0 < \delta < \varepsilon$ so that for each $0 < \eta < \delta$ there exists $l \in \mathbb{N}$ so that for each $f \in \mathcal{E}^l_\omega(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{D}'_{(\omega)}(\Omega_\eta, \Omega_\varepsilon)$ so that $P(D)g = f|_{\Omega_\eta}$ in $\mathcal{D}'_{(\omega)}(\Omega_\eta)$.*

Proof. Let R denote a right inverse for $P(D)$. Since R is continuous, we get

(2.8) for each $K \subset\subset \Omega$ and each $m \in \mathbb{N}$ there exist $Q \subset\subset \Omega$, $j \in \mathbb{N}$, $C > 0$ so that $\|R(f)\|_{K,m} \le C\,\|f\|_{Q,j}$ for all $f \in \mathcal{E}_{(\omega)}(\Omega)$.

To derive (∗) from this, let $\varepsilon > 0$ be given. Then (2.8) implies the existence of $Q \subset\subset \Omega$, $j \in \mathbb{N}$ and $C > 0$ so that

$$\|R(f)\|_{\Omega_\varepsilon,1} \le C\,\|f\|_{Q,j} \quad \text{for all} \quad f \in \mathcal{E}_{(\omega)}(\Omega). \tag{2.9}$$

Choose $0 < \delta < \varepsilon$ so that $\overline{Q} \subset \Omega_\delta$. Next fix $0 < \eta < \delta$ and use (2.8) to find $L \subset\subset \Omega, m \in \mathbb{N}$, $m > j$ and $M > 0$ so that

$$\|R(f)\|_{\Omega_\eta,1} \le M\|f\|_{L,m} \quad \text{for all} \quad f \in \mathcal{E}_{(\omega)}(\Omega). \tag{2.10}$$

Now note that there exist $l \in \mathbb{N}$ and $D > 0$ so that

$$\exp\left(-m\varphi^*\left(\frac{p}{m}\right)\right) \le D\exp\left(-l\varphi^*\left(\frac{p+1}{l}\right)\right) \quad \text{for all} \quad p \in \mathbb{N}_0. \tag{2.11}$$

This is a consequence of the following facts. On the (DFN)-space A_ω of all entire functions h which for some $A, B > 0$ satisfy the estimate $|h(z)| \le A\exp(B\omega(z))$ for all $z \in \mathbb{C}$, the operator M, defined by $M(h) : z \mapsto zh(z)$ is linear and continuous. Furthermore, an entire function h belongs to A_ω if and only if its Taylor coefficients h_j, $j \in \mathbb{N}_0$, satisfy for some $m \in \mathbb{N}$ and $D > 0$ the estimate $|h_j| \le D\exp\left(-m\varphi^*\left(\frac{j}{m}\right)\right)$ for all $j \in \mathbb{N}_0$. Now (2.11) follows from the observation that M on this sequence space is the forward shift operator.

Next fix $f \in \mathcal{E}^l_\omega(\Omega, \Omega_\delta)$ and choose $\psi \in \mathcal{D}_{(\omega)}(\Omega)$ so that $\psi \equiv 1$ in a neighbourhood of L. Then fix $\rho \in \mathcal{D}_{(\omega)}(B_1(0))$ with $\int \rho(x)\,dx = 1$, let $\rho_t : x \mapsto \frac{1}{t^n}\rho(\frac{x}{t})$ for $t > 0$ and define $f_t := \psi(f * \rho_t)$. For sufficiently small $t > 0$ the function f_t is in $\mathcal{E}_{(\omega)}(\Omega, Q)$, since f is in $\mathcal{E}^l_\omega(\Omega, \Omega_\delta)$. Moreover, a direct estimate, using (2.10) shows that $\|f_t - f\|_{L,m} \to 0$ as $t \to 0$. Hence (2.10) implies that $g := \lim_{t\to 0} R(f_t)$ exists in $C(\Omega_\eta)$. From (2.9) we conclude that $g|_{\Omega_\varepsilon} \equiv 0$. Next observe that for each $\varphi \in \mathcal{D}_{(\omega)}(\Omega_\eta)$ we have

$$\begin{aligned} \langle P(D)g, \varphi\rangle &= \langle g, P(-D)\varphi\rangle = \int_\Omega gP(-D)\varphi\,d\lambda = \lim_{t\to 0}\int_\Omega R(f_t)P(-D)\varphi\,d\lambda \\ &= \lim_{t\to 0}\int f_t\varphi\,d\lambda = \langle f, \varphi\rangle, \end{aligned}$$

which completes the proof. □

Lemma 2.6. *For each open set $\Omega \subset \mathbb{R}^n$ and each $P \in \mathbb{C}[z_1, \dots, z_n]$ condition* 2.5 (∗) *implies condition* 2.1 (4).

Proof. For a given number $\varepsilon > 0$ choose $0 < \delta_0 < \varepsilon$ according to 2.5 (∗) and note that the conclusion of 2.5 (∗) then holds for all $0 < \delta < \delta_0$. Next fix $0 < \zeta < \sigma < \eta < \delta < \delta_0$ and $\xi \in \overline{\Omega}_\eta \setminus \Omega_\delta$ and choose $l \in \mathbb{N}$ according to 2.5 (∗) with ε, δ_0 and ζ. Then the proof of Braun, Meise and Taylor [6], 4.4, implies the existence of $m \in \mathbb{N}$ so that for each $f \in \mathcal{E}^{l+m}_\omega(\Omega)$ and each $\varphi \in \mathcal{E}_{(\omega)}(\Omega)$ we have $\varphi f \in \mathcal{E}^l_\omega(\Omega)$. Further, by Braun [3], Thm.8, there exists an elliptic ultradifferential operator $Q(D)$ acting on $\mathcal{E}_{(\omega)}$ so that the equation $Q(D)F_\xi = \delta_\xi$ has a solution in $\mathcal{E}^{l+m}_\omega(\mathbb{R}^n)$. Now choose $\varphi_\xi \in \mathcal{D}_{(\omega)}\left(\Omega_\sigma \setminus \overline{\Omega}_{\delta_0}\right)$ so that $\varphi_\xi \equiv 1$ in a neighbourhood of ξ. Then $f_\xi := \varphi_\xi F_\xi$ belongs to $\mathcal{E}^l_\omega(\Omega, \Omega_{\delta_0})$ and satisfies

$$Q(D)f_\xi = \delta_\xi + h_\xi, \quad \text{where} \quad h_\xi \in \mathcal{E}_{(\omega)}(\Omega, \Omega_{\delta_0}),$$

since $Q(D)$ is elliptic. Hence we can apply condition 2.5 (∗) with η replaced by ζ to get $g_\xi, H_\xi \in \mathcal{D}'_{(\omega)}(\Omega_\zeta, \Omega_\varepsilon)$ satisfying

$$P(D)g_\xi = f_\xi|_{\Omega_\zeta} \quad \text{and} \quad P(D)H_\xi = h_\xi|_{\Omega_\zeta}.$$

Next choose $\psi \in \mathcal{D}_{(\omega)}(\Omega_\zeta)$ with $\psi|_{\Omega_\sigma} \equiv 1$ and let

$$G_\xi := \psi(Q(D)g_\xi - H_\xi) \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n).$$

Then we have

$$\operatorname{Supp} G_\xi \subset \Omega_\zeta \setminus \Omega_\varepsilon \subset \mathbb{R}^n \setminus \Omega_\varepsilon,$$

and since $P(Q)$ commutes with $Q(D)$

$$P(D)G_\xi|_{\Omega_\sigma} = (P(D)Q(D)g_\xi - P(D)H_\xi)|_{\Omega_\sigma} = (Q(D)f_\xi - h_\xi)|_{\Omega_\sigma} = \delta_\xi|_{\Omega_\sigma}.$$

This implies

$$P(D)G_\xi = \delta_\xi + T_\xi, \quad \text{where} \quad \operatorname{Supp} T_\xi \subset \Omega_\zeta \setminus \Omega_\sigma .$$

It is easy to check that

$$E_\xi : \varphi \mapsto \langle G_\xi, \varphi(\cdot - \xi)\rangle, \quad \varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^n)$$

satisfies the conditions (i) and (ii) in 2.1 (4). □

To derive further conditions that are equivalent to the existence of a right inverse for $P(D)$ on $\mathcal{E}_{(\omega)}(\Omega)$, we introduce the following notation.

Notation. For an open set Ω in $\mathbb{R}^n$, $\varepsilon > 0$, and $m \in \mathbb{N}$ let

$$B_{\varepsilon,m} := \left\{\mu \in \mathcal{E}'_{(\omega)}(\Omega) : \operatorname{Supp}\mu \subset \Omega_\varepsilon,\ |\mu(f)| \le \|f\|_{\varepsilon,m} \text{ for all } f \in \mathcal{E}_{(\omega)}(\Omega)\right\}.$$

Obviously, $B_{\varepsilon,m}$ is a relatively compact subset of $\mathcal{E}'_{(\omega)}(\Omega)$. Moreover, for each compact set $M \subset \mathcal{E}'_{(\omega)}(\Omega)$ there exist $m \in \mathbb{N}$ and $\varepsilon > 0$ with $M \subset m B_{\varepsilon,m}$.

Remark 2.7. Recall that for $P \in \mathbb{C}[z_1, \dots, z_n]$ and an open set $\Omega \subset \mathbb{R}^n$ the P-convexity of Ω (see HÖRMANDER [10], Def. 3.5.1) is equivalent to the surjectivity of $P(D)$ on $\mathcal{E}(\Omega)$. Variations of the proof of this result show that Ω is P-convex if and only if $P(D) : \mathcal{E}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ is surjective (see e.g. BJÖRCK [2], 3.3.2 and 3.3.4).

Using the preceding remark, a smoothing argument, and the notation introduced above, the proofs of [17], 2.4 and 2.5 can be modified easily to prove also the following two lemmas.

Lemma 2.8. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. If $P(D) : \mathcal{E}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ admits a right inverse then the following condition $(*)$ holds:*

$(*)$ *For each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $0 < \eta < \delta$ and each $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ and $C > 0$ so that for each $\mu \in \mathcal{E}'_{(\omega)}(\Omega_\varepsilon)$ with $(\mu + \operatorname{im} P(D)^t) \bigcap B_{\eta,m} \neq \emptyset$ there exists $\lambda \in \mathcal{E}'_{(\omega)}(\Omega_\delta)$ so that $\mu + P(D)^t\lambda \in CB_{\delta,k}$.*

Lemma 2.9. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. If $(*)$ is satisfied*

$(*)$ *For each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $0 < \eta < \delta$ there exist $m, k \in \mathbb{N}$ and $C > 0$ so that for each $\mu \in \mathcal{E}'_{(\omega)}(\Omega_\varepsilon)$ with $(\mu + \operatorname{im} P(D)^t) \bigcap B_{\eta,m} \neq \emptyset$ there exists $\lambda \in \mathcal{E}'_{(\omega)}(\Omega_\delta)$ so that $\mu + P(D)^t\lambda \in CB_{\delta,k}$*

then Ω is P-convex and condition 2.5 $(*)$ holds.

Theorem 2.10. *For an open set Ω in $\mathbb{R}^n$ and for a complex polynomial P in n variables the following assertions are equivalent:*

1. $P(D) : \mathcal{D}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ *admits a right inverse;*
2. $P(D) : \mathcal{E}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ *admits a right inverse;*
3. *One of the conditions* 2.1 (2), 2.1 (3), 2.1 (4), 2.5 (∗), 2.8 (∗) *or* 2.9 (∗) *holds.*

Proof. Because of 2.1, 2.4, 2.5 and 2.6 the following implications hold:

$$(1) \Rightarrow 2.1(2) \Rightarrow 2.1(3) \Rightarrow 2.1(4) \Rightarrow (2) \Rightarrow 2.5(*) \Rightarrow 2.1(4) \Rightarrow (1).$$

Because of 2.8, 2.9, 2.6 and 2.4 we also have

$$(2) \Rightarrow 2.8(*) \Rightarrow 2.9(*) \Rightarrow 2.5(*) \Rightarrow 2.1(4) \Rightarrow (2).$$

□

Remark 2.11. Theorem 2.10 extends [17], Thm. 2.7 from $\mathcal{E}(\Omega)$ and $\mathcal{D}'(\Omega)$ to $\mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{D}'_{(\omega)}(\Omega)$. In [17] all the equivalent properties for an open set Ω were called P-convexity with bounds. Since now also the weight function ω matters, we introduce the following definition.

Definition 2.12. Let ω be a weight function, Ω an open set in $\mathbb{R}^n$ and P a complex polynomial in n variables. Ω is called P-convex with (ω)-bounds if one of the equivalent conditions in Theorem 2.10 holds.

Because of Theorem 2.10, the arguments of the proof of [17], Cor. 2.10, also prove the following corollary.

Corollary 2.13. *Let P be a complex polynomial in n variables and let $(\Omega_i)_{i\in I}$ be a family of open sets in $\mathbb{R}^n$ for which $\Omega := \cap_{i\in I}\Omega_i \neq \emptyset$ is open. If Ω_i is P-convex with (ω)-bounds for each $i \in I$ then Ω is P-convex with (ω)-bounds.*

Corollary 2.14. *Let Ω be an open set in $\mathbb{R}^n$, let $P \in \mathbb{C}[z_1, \ldots, z_n]$ and let ω be a weight function. If Ω is P-convex with (ω)-bounds, then Ω is P-convex with (κ)-bounds for each weight function κ satisfying $\omega = O(\kappa)$.*

Proof. By Theorem 2.10 condition 2.1(4) holds for Ω and ω. Since $\omega = O(\kappa)$ implies $\mathcal{D}'_{(\omega)}(\mathbb{R}^n) \subset \mathcal{D}'_{(\kappa)}(\mathbb{R}^n)$ and since supports do not change under this inclusion by [6], 3.9, we see that condition 2.1 (4) holds for Ω and κ. By Theorem 2.10 this completes the proof. □

To indicate that there are quite a number of polynomials P for which no open set is P-convex with (ω)-bounds, we recall the following definition.

Definition 2.15. A polynomial P in n variables is called (ω)-*hypoelliptic* if the operator $P(D)$ admits a fundamental solution $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ that satisfies $E|_{\mathbb{R}^n\setminus\{0\}} \in \mathcal{E}_{(\omega)}(\mathbb{R}^n \setminus \{0\})$.

It is easy to check (see BJÖRCK [2], Thm. 4.1.1) that for each (ω)-hypoelliptic polynomial P and each open set Ω in $\mathbb{R}^n$ we have $\mathcal{N}(\Omega) = N(\Omega) \subset \mathcal{E}_{(\omega)}(\Omega)$. Hence $\mathcal{N}(\Omega)$ is a nuclear Fréchet space. From this and 2.1 (3) we get by the proof of [17], Cor. 2.11, the following result:

Corollary 2.16. *For $n \geq 2$ let P be an (ω)-hypoelliptic polynomial in n variables. Then each open set Ω in $\mathbb{R}^n$ is not P-convex with (ω)-bounds.*

3. Right inverses for $\mathcal{D}'_{\{\omega\}}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$

In this section we characterize when a partial differential operator $P(D)$ admits a right inverse on $\mathcal{D}'_{\{\omega\}}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$, Ω an open subset of $\mathbb{R}^n$. Since the topology of $\mathcal{E}_{\{\omega\}}(\Omega)$ is more complicated than the one of $\mathcal{E}_{(\omega)}(\Omega)$, we first describe it in a way which is suitable for our purposes.

Throughout this section ω will denote a fixed weight function. Further we let

$$S_\omega := \{\sigma : \sigma \text{ is a weight function satisfying } \sigma = o(\omega)\}.$$

Remark 3.1. For each $\sigma \in S_\omega$ the following is easy to show:

For each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that

$$\varepsilon\varphi^*_\omega\left(\frac{x}{\varepsilon}\right) \leq \varphi^*_\sigma(x) + C_\varepsilon \quad \text{for all} \quad x > 0.$$

This implies that for each open set $\Omega \subset \mathbb{R}^n$ and each $K \subset\subset \Omega$

$$\|f\|_{K,\sigma} := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^n} \left|f^{(\alpha)}(x)\right| \exp\left(-\varphi^*_\sigma(|\alpha|)\right), \quad f \in \mathcal{E}_{\{\omega\}}(\Omega)$$

defines a continuous semi-norm on $\mathcal{E}_{\{\omega\}}(\Omega)$.

Notation. For an open set $\Omega \subset \mathbb{R}^n$, $\varepsilon > 0$, and $\sigma \in S_\omega$ define

$$B_{\varepsilon,\sigma} := \left\{\mu \in \mathcal{E}'_{\{\omega\}}(\Omega) : \operatorname{Supp}\mu \subset \Omega_\varepsilon,\ |\mu(f)| \leq \|f\|_{\Omega_\varepsilon,\sigma} \text{ for all } f \in \mathcal{E}_{\{\omega\}}(\Omega)\right\}.$$

By the preceding remark, each set $B_{\varepsilon,\sigma}$ is equicontinuous, hence bounded. The following lemma shows that even more holds.

Lemma 3.2. *The sets $(B_{\varepsilon,\sigma})_{\varepsilon>0,\sigma\in S_\omega}$ form a fundamental system of the bounded subsets of $\mathcal{E}'_{\{\omega\}}(\Omega)$.*

Proof. Let B be a bounded subset of $\mathcal{E}'_{\{\omega\}}(\Omega)$. Since $\mathcal{E}_{\{\omega\}}(\Omega)$ is reflexive by BRAUN, MEISE and TAYLOR [6], 4.9, B is equicontinuous. Hence there exists a compact subset K of Ω so that $\bigcup_{\mu\in B}\operatorname{Supp}\mu \subset \mathring{K}$. Choose $0 < t < \frac{1}{2}\operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$

and find $m \in \mathbb{N}$ and $\xi_1, \dots, \xi_m \in K$ so that $K \subset \bigcup_{j=1}^m B_t(\xi_j)$. Further choose $\varphi_j \in \mathcal{D}_{\{\omega\}}(B_t(\xi_j)), 1 \le j \le m$, so that $\sum_{j=1}^m \varphi_j(x) = 1$ for all $x \in K$. Then

$$(3.1) \qquad \mu(f) = \sum_{j=1}^{m} \mu(\varphi_j f) \quad \text{for all} \quad f \in \mathcal{D}_{\{\omega\}}(\Omega) \quad \text{and all} \quad \mu \in B.$$

Since B is bounded, we get from BRAUN, MEISE and TAYLOR [6], 4.4, that for $1 \le j \le m$ the sets

$$B_j := \{\varphi_j \mu : \mu \in B\} \subset \mathcal{E}'_{\{\omega\}}(B_t(\xi_j))$$

are bounded. Consequently, [6], 4.7, implies that for each k there exists $M_k > 0$ so that for $1 \le j \le m$, all $\nu \in B_j$,

$$(3.2) \ |\widehat{\nu}(z)| = |\langle \nu_x, e^{-i\langle x,z\rangle}\rangle| \le M_k \exp\left(h_j(\operatorname{Im} z) + \frac{1}{k}\omega(z)\right) \quad \text{for all} \quad z \in \mathbb{C}^n,$$

where h_j denotes the support functional of the convex set $B_t(\xi_j)$. From (3.2) we get for each $k \in \mathbb{N}, r > 0$

$$g(r) := \max_{1 \le j \le n} \sup_{|z|=r} \log^+ \left(|\widehat{\nu}(z)| \exp\left(-h_j(\operatorname{Im} z)\right)\right) \le \frac{1}{k}\omega(r) + \log M_k,$$

i.e., $g = o(\omega)$. By BRAUN, MEISE and TAYLOR [6], 1.7, there exists $\sigma \in S_\omega$ so that $g = o(\sigma)$. This implies

(3.3) For each $\delta > 0$ there exists $C_\delta > 0$ so that for $1 \le j \le m$ and each $\nu \in B_j$
$|\widehat{\nu}(z)| \le C_\delta \exp\left(h_j(\operatorname{Im} z) + \delta\sigma(z)\right)$ for all $z \in \mathbb{C}^n$.

Now choose $\varepsilon > 0$ so that $\bigcup_{j=1}^m B_t(\xi_j) \subset \Omega_\varepsilon$ and note that by [6], 3.3, there exist $L > 0$ and $M > 0$ so that for each $f \in \mathcal{D}_{\{\omega\}}(\Omega_\varepsilon)$ we have

$$(3.4) \quad |\widehat{f}(x)| = \left|\int_{\mathbb{R}^n} f(t)\, e^{-i\langle x,t\rangle}\, dt\right| \le M\, \|f\|_{\Omega_\varepsilon,\sigma}\, e^{-\frac{1}{L}\sigma(x)} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Next choose $\delta = \frac{1}{2L}$ in (3.3). Then we get from (3.3) and (3.4) that for each $\nu \in B_j$, $1 \le j \le m$, and each $f \in \mathcal{D}_{\{\omega\}}(\Omega_\varepsilon)$,

$$\begin{aligned} |\nu(f)| &= \left| \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \widehat{\nu}(-x)\, \widehat{f}(x)\, dx \right| \\ &\le \left(\frac{1}{2\pi}\right)^n M C_{\frac{1}{2L}}\, \|f\|_{\Omega_\varepsilon,\sigma} \int \exp\left(\left(\frac{1}{2L} - \frac{1}{L}\right)\sigma(x)\right) dx. \end{aligned}$$

Since the weight function σ satisfies $\log^+ t = o(\sigma(t))$, this together with (3.1) implies that for a suitable number $\lambda > 0$ we have $B \subset \lambda B_{\varepsilon,\sigma}$. □

Corollary 3.3. *For each weight function ω and each open set Ω in $\mathbb{R}^n$ the family $(\|\cdot\|_{K,\sigma})_{K \subset\subset \Omega, \sigma \in S_\omega}$ is a fundamental system of continuous semi-norms on $\mathcal{E}_{\{\omega\}}(\Omega)$.*

Proof. We have already remarked that for $K \subset\subset \Omega$ and $\sigma \in S_\omega$ the semi-norm $\|\cdot\|_{K,\sigma}$ is continuous on $\mathcal{E}_{\{\omega\}}(\Omega)$. If q is a given continuous semi-norm on $\mathcal{E}_{\{\omega\}}(\Omega)$, we denote by U the closed unit ball with respect to q. Then U° (the polar of U) is an equicontinuous subset of $\mathcal{E}'_{\{\omega\}}(\Omega)$. Hence Lemma 3.2 implies the existence of $\varepsilon > 0, \sigma \in S_\omega$ and $\lambda \geq 1$ with $U^\circ \subset \lambda B_{\varepsilon,\sigma}$. Note that for the closed unit ball $V_{\varepsilon,\sigma}$ of the semi-norm $\|\cdot\|_{\Omega_\varepsilon,\sigma}$ we have $B_{\varepsilon,\sigma} \subset V^\circ_{\varepsilon,\sigma}$ and hence $U^\circ \subset \lambda V^\circ_{\varepsilon,\sigma}$. From this we get by the theorem of bipolars

$$U = U^{\circ\circ} \supset \left(\lambda V^\circ_{\varepsilon,\sigma}\right)^\circ = \frac{1}{\lambda} V_{\varepsilon,\sigma},$$

which implies $q \leq \lambda \|\cdot\|_{\Omega_\varepsilon,\sigma}$. □

Lemma 3.4. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) for the following assertions:*

1. *$P(D) : \mathcal{D}'_{\{\omega\}}(\Omega) \to \mathcal{D}'_{\{\omega\}}(\Omega)$ admits a right inverse;*
2. *for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $f \in \mathcal{D}'_{\{\omega\}}(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{D}'_{\{\omega\}}(\Omega, \Omega_\varepsilon)$ with $P(D)g = f$;*
3. *for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $\mu \in \mathcal{N}(\Omega_\delta)$ there exists $\nu \in \mathcal{N}(\Omega)$ with $\nu|_{\Omega_\varepsilon} = \mu|_{\Omega_\varepsilon}$;*
4. *for each $\varepsilon > 0$ there exists $0 < \delta_0 < \varepsilon$ so that for all $0 < \zeta < \sigma < \eta < \delta < \delta_0$ and each $\xi \in \overline{\Omega}_\eta \setminus \Omega_\delta$ there exists $E_\xi \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$ so that*
   - (i) $\operatorname{Supp} E_\xi \subset (\mathbb{R}^n \setminus \Omega_\varepsilon) - \xi$
   - (ii) $P(D)E_\xi = \delta + T_\xi$ *where* $\operatorname{Supp} T_\xi \subset (\Omega_\zeta \setminus \Omega_\sigma) - \xi$.

Proof. (1) $\Rightarrow$ (2): Let $R : \mathcal{D}'_{\{\omega\}}(\Omega) \to \mathcal{D}'_{\{\omega\}}(\Omega)$ denote a right inverse for $P(D)$ and let $\varepsilon > 0$ be given. Then the set

$$B := \{\varphi \in \mathcal{D}_{\{\omega\}}(\overline{\Omega}_\varepsilon) : \|\varphi\|_{\Omega_\varepsilon,1} \leq 1\}$$

is bounded in $\mathcal{D}_{\{\omega\}}(\overline{\Omega}_\varepsilon)$, hence bounded in $\mathcal{D}_{\{\omega\}}(\Omega)$. Therefore,

$$q_B : \mathcal{D}'_{\{\omega\}}(\Omega) \longrightarrow \mathbb{R}, \quad q_B(\mu) := \sup_{\varphi \in B} |\mu(\varphi)|$$

is a continuous semi-norm on $\mathcal{D}'_{\{\omega\}}(\Omega)$. By the continuity of R there exist a bounded set C in $\mathcal{D}_{\{\omega\}}(\Omega)$ and $M > 0$ so that

$$q_B(R\mu) \leq M q_C(\mu) \quad \text{for all} \quad \mu \in \mathcal{D}'_{\{\omega\}}(\Omega).$$

Since $\mathcal{D}_{\{\omega\}}(Q)$ is a (DFS)-space for each compact set Q in Ω, we may assume that there exist a compact set L in Ω and $m \in \mathbb{N}$ so that

$$C = \left\{\varphi \in \mathcal{D}_{\{\omega\}}(L) : \|\varphi\|_{L,\frac{1}{m}} \leq 1\right\}.$$

Choose $0 < \delta < \varepsilon$ so that $L \subset \Omega_\delta$ and let $f \in \mathcal{D}'_{\{\omega\}}(\Omega, \Omega_\delta)$ be given. Then $g := R(f)$ is in $\mathcal{D}'_{\{\omega\}}(\Omega)$ and satisfies $P(D)g = P(D)Rf = f$. Moreover, g satisfies

$$q_B(g) = q_B(R(f)) \leq M q_C(f) = 0.$$

Hence we have

$$(*) \qquad g(\varphi) = 0 \quad \text{for all} \quad \varphi \in \operatorname{span} B.$$

Choose $\rho \in \mathcal{D}_{\{\omega\}}(B_1(0))$ with $\int \rho(x)\,dx = 1$ and define $\rho_t : x \mapsto \frac{1}{t^n}\rho(\frac{x}{t})$ for $t > 0$. Then it is easy to check that for each $\psi \in \mathcal{D}_{\{\omega\}}(\Omega_\varepsilon)$ we have $\lim_{t\to\infty} \psi * \rho_t = \psi$ in $\mathcal{D}_{\{\omega\}}(\Omega)$ and $\psi * \rho_t \in \operatorname{span} B$ for all sufficiently small $t > 0$. Hence $(*)$ implies

$$g(\psi) = \lim_{t\to 0} g(\psi * \rho_t) = 0 \quad \text{for each} \quad \psi \in \mathcal{D}_{\{\omega\}}(\Omega_\varepsilon).$$

Consequently, g belongs to $\mathcal{D}'_{\{\omega\}}(\Omega, \Omega_\varepsilon)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : This can be shown as in the proof of [17], Lemma 2.1. □

Lemma 3.5. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. If $P(D) : \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ admits a right inverse then the following condition $(*)$ holds:*

$(*)$ *For each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $0 < \eta < \delta$ and each $\sigma \in S_\omega$ there exist $\kappa \in S_\omega$ and $C > 0$ so that for each $\mu \in \mathcal{E}'_{\{\omega\}}(\Omega_\varepsilon)$ with $(\mu + \operatorname{im} P(D)^t) \cap B_{\eta,\sigma} \neq \emptyset$ there exists $\lambda \in \mathcal{E}'_{\{\omega\}}(\Omega_\delta)$ so that $\mu + P(D)^t\lambda \in CB_{\delta,\kappa}$.*

Proof. Choose a right inverse R for $P(D)$ and note that

$$R^t \circ P(-D) = R^t \circ P(D)^t = (P(D) \circ R)^t = \operatorname{id}_{\mathcal{E}'_{\{\omega\}}(\Omega)}.$$

To show that this implies the P-convexity of Ω, we fix a compact set K in Ω and choose $\varepsilon > 0$ so that $K \subset \Omega_\varepsilon$. Then fix $\sigma \in S_\omega$ and use the continuity of R^t to find $0 < \delta < \varepsilon$, $\kappa \in S_\omega$ and $\alpha > 0$ so that $R^t(B_{\varepsilon,\sigma}) \subset \alpha B_{\delta,\kappa}$. Since R^t is linear, this implies

$$R^t(\operatorname{span} B_{\varepsilon,\sigma}) \subset \operatorname{span} B_{\delta,\kappa}.$$

Fix $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{Supp} P(-D)\varphi \subset K$. Then it is easy to check that

$$P(-D)\varphi \in \operatorname{span} B_{\varepsilon,\sigma}, \quad \text{which implies} \quad \varphi = R^t \circ P(-D)\varphi \in \operatorname{span} B_{\delta,\kappa},$$

and hence $\operatorname{Supp}\varphi \subset \Omega_\delta$. Thus Ω is P-convex.

Next note that

$$\pi := (R \circ P(D))^t = P(D)^t \circ R^t$$

is a projection on $\mathcal{E}'_{\{\omega\}}(\Omega)$ with $\operatorname{im}\pi = \operatorname{im} P(D)^t$. Hence $Q := \operatorname{id}_{\mathcal{E}'_{\{\omega\}}(\Omega)} - \pi$ is a projection on $\mathcal{E}'_{\{\omega\}}(\Omega)$ and satisfies

$$(3.6) \qquad \ker Q = \operatorname{im}\pi = \operatorname{im} P(D)^t = \operatorname{im} P(-D).$$

Let $\varepsilon > 0$ be given. Note that $\mathcal{E}_{\{\omega\}}(\Omega)/\mathcal{E}_{\{\omega\}}(\Omega, \Omega_\varepsilon)$ is a (DFN)-space since it is equal to $\mathcal{D}_{\{\omega\}}(\Omega)/\mathcal{D}_{\{\omega\}}(\Omega, \Omega_\varepsilon)$. Moreover, $\left(\mathcal{E}_{\{\omega\}}(\Omega)/\mathcal{E}_{\{\omega\}}(\Omega, \Omega_\varepsilon)\right)'$ can be canonically identified with $\mathcal{E}_{\{\omega\}}(\Omega, \Omega_\varepsilon)^\perp$. Obviously, the set $\Delta = \{\delta_x : x \in \Omega_\varepsilon\}$ is weakly total in

$\mathcal{E}_{\{\omega\}}(\Omega,\Omega_\varepsilon)^\perp$, hence total in $\mathcal{E}_{\{\omega\}}(\Omega,\Omega_\varepsilon)^\perp$. Since Δ is relatively compact in $\mathcal{E}_{\{\omega\}}(\Omega)'$, the set $Q(\Delta)$ is relatively compact in $\mathcal{E}'_{\{\omega\}}(\Omega)$. By 3.2 there exist $0<\delta_0<\varepsilon$, $\theta\in S_\omega$ and $\alpha>0$ with $Q(\Delta)\subset\alpha B_{\delta_0,\theta}\subset\mathcal{E}_{\{\omega\}}(\Omega,\Omega_{\delta_0})^\perp$. Since $\mathcal{E}_{\{\omega\}}(\Omega,\Omega_{\delta_0})^\perp$ is closed in $\mathcal{E}'_{\{\omega\}}(\Omega)$, this implies

$$Q\Big(\mathcal{E}_{\{\omega\}}(\Omega,\Omega_\varepsilon)^\perp\Big)\subset\mathcal{E}_{\{\omega\}}(\Omega,\Omega_{\delta_0})^\perp. \tag{3.7}$$

Fix $0<\delta<\delta_0$ and let $0<\eta<\delta$ and $\sigma\in S_\omega$ be given. Since Q is continuous, we get from Lemma 3.2 the existence of $0<\zeta<\eta$, $\kappa\in S_\omega$ and $C_1>0$ so that

$$Q(B_{\eta,\sigma})\subset C_1B_{\zeta,\kappa}. \tag{3.8}$$

Fix $\mu\in\mathcal{E}'_{\{\omega\}}(\Omega_\varepsilon)$ and assume that for some $\nu\in\mathcal{E}'_{\{\omega\}}(\Omega)$ we have $\mu+P(D)^t\nu\in B_{\eta,\sigma}$. Then (1) implies

$$Q\big(\mu+P(D)^t\nu\big)=Q\mu+Q\big(P(D)^t\nu\big)=Q\mu$$

and hence $Q\mu\in C_1B_{\zeta,\kappa}$, because of (3.7). Moreover, $\operatorname{Supp}\mu\subset\Omega_\varepsilon$ and (3.6) imply $Q\mu\in\mathcal{E}(\Omega,\Omega_{\delta_0})^\perp$. This gives

$$\operatorname{Supp}Q\mu\subset\overline{\Omega}_{\delta_0}\subset\Omega_\delta. \tag{3.9}$$

Choose $\psi\in\mathcal{D}_{\{\omega\}}(\Omega_\delta)$ so that $\psi\equiv1$ in a neighbourhood of $\overline{\Omega}_{\delta_0}$. Then we get from the inclusion (3.8)

$$|Q\mu(f)|=|Q\mu(\psi f)|\le C_1\,\|\psi f\|_{\Omega_\delta,\kappa}\quad\text{for all}\quad f\in\mathcal{E}_{\{\omega\}}(\Omega).$$

Note that by the proof of Braun, Meise and Taylor [6], 4.4, there exist $L\ge1$ and C_2, depending only on κ, so that

$$\|\psi f\|_{\Omega_\delta,\kappa}\le C_2\,\|\psi\|_{\Omega_\delta,L\kappa}\,\|f\|_{\Omega_\delta,L\kappa}\quad\text{for all}\quad f\in\mathcal{E}_{\{\omega\}}(\Omega).$$

Hence there exists $C>0$ depending only on δ_0, δ, κ, C_1, C_2 so that $Q\mu\in CB_{\delta,L\kappa}$. Define $\lambda:=-R^t(\mu)$ and note that

$$\mu+P(D)^t\lambda=\mu-P(D)^tR^t(\mu)=\mu-\pi(\mu)=Q\mu\in CB_{\delta,L\kappa}.$$

From this and (3.8) we get

$$\operatorname{Supp}P(-D)\lambda=\operatorname{Supp}P(D)^t\lambda=\operatorname{Supp}(Q\mu-\mu)\subset\Omega_\delta. \tag{3.10}$$

Note that we have already shown that Ω is P-convex. Therefore a standard smoothing argument together with (3.9) and Hörmander [10], Thm. 3.5.2, imply $\operatorname{Supp}\lambda\subset\Omega_\delta$. Since $L\kappa$ is in S_ω, this completes the proof. □

Lemma 3.6. *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. If $(*)$ is satisfied*

$(*)$ *For each $\varepsilon>0$ there exists $0<\delta<\varepsilon$ so that for each $0<\eta<\delta$ there exist $\sigma,\kappa\in S_\omega$ and $C>0$ so that for each $\mu\in\mathcal{E}'_{\{\omega\}}(\Omega_\varepsilon)$ with $(\mu+\operatorname{im}P(D)^t)\cap B_{\eta,\sigma}\neq\emptyset$ there exists $\lambda\in\mathcal{E}'_{\{\omega\}}(\Omega_\delta)$ so that $\mu+P(D)^t\lambda\in CB_{\delta,\kappa}$*

then the following assertions hold:

1. *Ω is P-convex.*

2. *For each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that for each $0 < \eta < \delta$ there exists $\theta \in S_\omega$ so that for each $f \in \mathcal{E}_{(\theta)}(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{D}'_{\{\omega\}}(\Omega_\eta, \Omega_\varepsilon)$ so that $P(D)g = f|_{\Omega_\eta}$ holds in $\mathcal{D}'_{\{\omega\}}(\Omega_\eta)$.*

Proof. (1): To show that Ω is P-convex, let K be a given compact subset of Ω. Then there exists $\varepsilon > 0$ with $K \subset \Omega_\varepsilon$. Choose $0 < \delta < \varepsilon$ according to $(*)$, fix $0 < \eta < \delta$ and choose $\sigma, \kappa \in S_\omega$ and $C > 0$ according to $(*)$. Fix $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{Supp} P(-D)\varphi \subset K$ and let $\mu := -P(-D)\varphi = -P(D)^t\varphi \in \mathcal{E}'_{\{\omega\}}(\Omega)$. Then $\mu + P(D)^t\varphi = 0 \in B_{\eta,\sigma}$ which implies that for each $s \in]0,1]$ we have

$$\frac{1}{s}\left(\mu + P(D)^t\varphi\right) \in B_{\eta,\sigma}.$$

Hence $(*)$ implies the existence of $\lambda_s \in \mathcal{E}'_{\{\omega\}}(\Omega_\delta)$ so that

$$P(D)^t\left(-\frac{1}{s}\varphi + \lambda_s\right) = \frac{1}{s}\mu + P(D)^t\lambda_s \in CB_{\delta,\kappa}.$$

Now note that by Remark 2.7, $P(D) : \mathcal{E}_{(\kappa)}(\mathbb{R}^n) \to \mathcal{E}_{(\kappa)}(\mathbb{R}^n)$ is surjective. Hence $P(D)^t : \mathcal{E}'_{(\kappa)}(\mathbb{R}^n) \to \mathcal{E}'_{(\kappa)}(\mathbb{R}^n)$ is an injective topological homomorphism. Regarding $B_{\delta,\kappa}$ as an equicontinuous subset of $\mathcal{E}'_{(\kappa)}(\mathbb{R}^n)$, we can therefore find a bounded open set G in $\mathbb{R}^n$ and $l \in \mathbb{N}$, $D > 0$ so that

$$\begin{aligned}&\left(P(D)^t\right)^{-1}(CB_{\delta,\kappa}) \subset B_{G,l\kappa}\\ :=\ &\left\{\nu \in \mathcal{E}'_{(\kappa)}(\mathbb{R}^n) : \operatorname{Supp}\nu \subset G,\ |\nu(f)| \le D\,\|f\|_{G,l\kappa} \text{ for all } f \in \mathcal{E}_{(\kappa)}(\mathbb{R}^n)\right\}.\end{aligned}$$

This implies

$$-\frac{1}{s}\varphi + \lambda_s \in B_{G,l\kappa} \quad \text{for all} \quad s \in]0,1],$$

and consequently $\varphi = \lim_{s\to 0} s\lambda_s$ in $\mathcal{E}'_{(\kappa)}(\mathbb{R}^n) \subset \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$. Since $\operatorname{Supp}\lambda_s \subset \Omega_\delta$ for each $s \in]0,1]$, this proves $\operatorname{Supp}\varphi \subset \overline{\Omega}_\delta$.

(2): For a given number $\varepsilon > 0$ choose $0 < \delta < \varepsilon$ according to $(*)$, fix $0 < \eta < \delta$ and choose $\sigma, \kappa \in S_\omega$ and $C \ge 1$ according to $(*)$. Without restriction we may assume $\sigma = o(\kappa)$. Then note that Ω is P-convex by (1). Therefore, $P(D) : \mathcal{E}_{(\kappa)}(\Omega) \to \mathcal{E}_{(\kappa)}(\Omega)$ is surjective, hence $P(-D) = P(D)^t : \mathcal{E}'_{(\kappa)}(\Omega) \to \mathcal{E}'_{(\kappa)}(\Omega)$ is an injective topological homomorphism. Hence there exist $l \in \mathbb{N}$, $L \ge 1$ and $0 < \zeta < \eta$ so that

$$P(-D)^{-1}(B_{\eta,\kappa}) \subset LB_{\zeta,l\kappa}.$$

For each $\nu \in \mathcal{E}'_{\{\omega\}}(\Omega)$ with $P(-D)\nu \in B_{\eta,\kappa}$ we therefore have

$$\nu \in P(-D)^{-1}(P(-D)\nu) \in LB_{\zeta,l\kappa}.$$

Since Ω is P-convex, a smoothing argument together with HÖRMANDER [10], Thm. 3.5.2, and Lemma 3.4.3, implies $\operatorname{Supp}\nu \subset \Omega_\eta$. Hence we get

$$(3.11)\quad \frac{1}{L}P(-D)^{-1}(B_{\eta,\kappa}) \subset \widetilde{B} := B_{\zeta,l\kappa} \cap \left\{\mu \in \mathcal{E}'_{\{\omega\}}(\Omega) : \operatorname{Supp}\mu \subset \Omega_\eta\right\}.$$

Fix $f \in \mathcal{E}_{(\kappa)}(\Omega, \Omega_\delta)$ and let

$$X := \operatorname{span}\left\{(P(-D)\mathcal{E}'_{\{\omega\}}(\Omega)) \cap B_{\eta,\kappa}\,,\ \mathcal{E}'_{(\kappa)}(\Omega_\varepsilon)\right\} \subset \mathcal{E}'_{\{\omega\}}(\Omega).$$

Note that for $\nu \in \mathcal{E}'_{\{\omega\}}(\Omega)$ satisfying $P(-D)\nu \in B_{\eta,\kappa}$, we have $\nu \in L\widetilde{B}$ because of (3.11). Hence $\langle\nu, f\rangle$ is defined. Now define $F : X \to \mathbb{C}$ by

$$F\ :\ P(-D)\nu + \mu \mapsto \begin{cases} 0 & \text{if } \operatorname{Supp}(P(-D)\nu + \mu) \subset \Omega_\delta\,, \\ \langle\nu, f\rangle & \text{otherwise}, \end{cases}$$

for $P(-D)\nu \in \operatorname{span}\left(P(-D)\mathcal{E}'_{\{\omega\}}(\Omega) \cap B_{\eta,\kappa}\right)$ and $\mu \in \mathcal{E}'_{(\kappa)}(\Omega_\varepsilon)$.

To show that F is well-defined, assume that for $\mu_1, \mu_2 \in \mathcal{E}'_{(\kappa)}(\Omega_\varepsilon)$ and $P(-D)\nu_1$, $P(-D)\nu_2 \in \operatorname{span} B_{\eta,\kappa}$ we have

$$P(-D)\nu_1 + \mu_1 \ = \ P(-D)\nu_2 + \mu_2\,.$$

Then we get

$$\operatorname{Supp}(P(-D)(\nu_1 - \nu_2)) \ = \ \operatorname{Supp}(\mu_2 - \mu_1) \subset \Omega_\varepsilon\,.$$

Since Ω is P-convex by (1), this and a smoothing argument imply

$$\operatorname{Supp}(\nu_1 - \nu_2) \subset \Omega_\varepsilon \quad \text{hence} \quad \langle\nu_1 - \nu_2, f\rangle \ = \ 0\,.$$

Obviously, F is 1-homogeneous. The additivity of F follows easily from the observation that for $P(-D)\nu + \mu \in X$ with $\operatorname{Supp}(P(-D)\nu + \mu) \subset \Omega_\delta$ and $\operatorname{Supp}\mu \subset \Omega_\varepsilon$ the P-convexity of Ω_δ implies $\operatorname{Supp}\nu \subset \Omega_\delta$ and hence $\langle\nu, f\rangle = 0$.

Next, denote by E_0 the normed space which is generated by the bounded, absolutely convex set $B_{\eta,\sigma} \subset \mathcal{E}'_{\{\omega\}}(\Omega)$. We claim that $F|_{X\cap E_0}$ is continuous. To show this, fix $P(-D)\nu + \mu \in X \cap B_{\eta,\sigma}$. Then $(*)$ implies the existence of $\lambda \in \mathcal{E}'_{\{\omega\}}(\Omega_\delta)$ with

$$(3.12)\qquad \mu + P(D)^t\lambda \in CB_{\delta,\kappa}\,.$$

From this we get

$$(3.13)\qquad \begin{aligned} &P(-D)(\nu - \lambda) \\ &= (P(-D)\nu + \mu) - (P(-D)\lambda + \mu) \in B_{\eta,\sigma} + CB_{\delta,\kappa} \subset MB_{\eta,\kappa}\,, \end{aligned}$$

where M depends only on C, κ and σ. By (3.11), this shows

$$(3.14)\qquad \nu - \lambda \in ML\widetilde{B}\,.$$

Moreover, (3.13) and $P(-D)\nu \in \operatorname{span} B_{\eta,\kappa}$ imply $P(-D)\lambda \in \operatorname{span} B_{\eta,\kappa}$ and hence $P(-D)\lambda + \mu \in X$. Therefore we get from (3.12) and the definition of F that

$$\begin{aligned} F(P(-D)\nu + \mu) &= F(P(-D)\lambda + \mu) + F(P(-D)(\nu - \lambda)) \\ &= F(P(-D)(\nu - \lambda)) = \langle \nu - \lambda, f \rangle \end{aligned}$$

and hence by (3.14)

$$|F(P(-D)\nu + \mu)| \le ML \|f\|_{\zeta, l\kappa}.$$

Since $P(-D)\nu + \mu$ was an arbitrary element of $X \cap B_{\eta,\sigma}$, this shows that $F|_{X\cap E_0}$ is bounded on the unit ball of $X \cap E_0$, hence it is continuous. Therefore, the theorem of Hahn-Banach implies the existence of $\widetilde{F} \in E_0'$ satisfying $\widetilde{F}|_{X\cap E_0} = F$. Let $\Phi : \mathcal{D}_{\{\omega\}}(\Omega_\eta) \to \mathcal{E}'_{(\kappa)}(\Omega_\eta)$ denote the canonical injection, defined by

$$\Phi(\varphi) : h \mapsto \int_{\Omega_\eta} \varphi(x) h(x)\, dx, \quad \varphi \in \mathcal{D}_{\{\omega\}}(\Omega_\eta), \quad h \in \mathcal{E}_{(\kappa)}(\Omega_\eta).$$

It is easily seen that Φ maps $\mathcal{D}_{\{\omega\}}(\Omega_\eta)$ continuously into E_0. Therefore, $g := \widetilde{F} \circ \Phi = \Phi^t(\widetilde{F})$ is in $\mathcal{D}'_{\{\omega\}}(\Omega_\eta)$. Since Ω_δ is P-convex and since f vanishes on Ω_δ, the definition of $\widetilde{F}$ and F gives for each $\varphi \in \mathcal{D}_{\{\omega\}}(\Omega_\eta)$

$$\begin{aligned} \langle P(D)g, \varphi \rangle &= \langle g, P(-D)\varphi \rangle = \widetilde{F}(\Phi(P(-D)\varphi) = F(P(-D)\Phi(\varphi)) \\ &= \langle \Phi(\varphi), f \rangle = \int_{\Omega_\eta} \varphi(x) f(x)\, dx = \langle f, \varphi \rangle. \end{aligned}$$

Further, $\varphi \in \mathcal{D}_{\{\omega\}}(\Omega_\varepsilon)$ implies $\Phi(\varphi) \in \mathcal{E}'_{\{\omega\}}(\Omega_\varepsilon)$ and hence

$$\langle g, \varphi \rangle = \widetilde{F}(\Phi(\varphi)) = F(\Phi(\varphi)) = 0.$$

Hence, g is in $\mathcal{D}'_{\{\omega\}}(\Omega_\eta, \Omega_\varepsilon)$ and satisfies $P(D)g = f|_{\Omega_\eta}$. □

Lemma 3.7. *For each open set $\Omega \subset \mathbb{R}^n$ and each $P \in \mathbb{C}[z_1, \dots, z_n]$ condition* 3.6 (2) *implies condition* 3.4 (4).

Proof. For a given number $\varepsilon > 0$ choose $0 < \delta_0 < \varepsilon$ according to 3.6 (2) and note that the conclusion of 3.6 (2) then holds for all $0 < \delta < \delta_0$. Next fix $0 < \zeta < \sigma < \eta < \delta < \delta_0$ and $\xi \in \overline{\Omega}_\eta \setminus \Omega_\delta$ and choose $\theta \in S_\omega$ according to 3.6 (2) with η replaced by ζ. By Braun, Meise and Taylor [6], 1.7, we can choose $\kappa \in S_\omega$ so that $\theta = o(\kappa)$. Furthermore, note that by the proof of [6], 4.4, there exists $m \in \mathbb{N}$ so that for each $f \in \mathcal{E}_\kappa^{m+1}(\Omega)$ and each $\varphi \in \mathcal{E}_{(\kappa)}(\Omega)$ we have $\varphi f \in \mathcal{E}_\kappa^1(\Omega)$. Then, by Braun [3], Thm. 8, we can choose an elliptic ultradifferential operator $Q(D)$ on $\mathcal{E}_{(\kappa)}$ so that the equation $Q(D)F_\xi = \delta_\xi$ has a solution in $\mathcal{E}_\kappa^{m+1}(\mathbb{R}^n)$. Choose $\varphi_\xi \in \mathcal{D}_{(\kappa)}(\Omega_\sigma \setminus \overline{\Omega}_{\delta_0})$ so that $\varphi_\xi \equiv 1$ in a neighbourhood of ξ. Then $f_\xi := \varphi_\xi F_\xi$ belongs to $\mathcal{E}_\kappa^1(\Omega, \Omega_{\delta_0})$ and satisfies

$$Q(D)f_\xi = \delta_\xi + h_\xi, \quad \text{where} \quad h_\xi \in \mathcal{E}_{(\kappa)}(\Omega, \Omega_{\delta_0}),$$

since $Q(D)$ is elliptic. Now note that $\mathcal{E}^1_\kappa(\Omega, \Omega_{\delta_0})$ is contained in $\mathcal{E}_{(\theta)}(\Omega, \Omega_{\delta_0})$ because of $\theta = o(\kappa)$. Hence we get from 3.6 (2) with η replaced by ζ the existence of $g_\xi, H_\xi \in \mathcal{D}'_{\{\omega\}}(\Omega_\zeta, \Omega_\varepsilon)$ satisfying

$$P(D)g_\xi = f_\xi|_{\Omega_\zeta} \quad \text{and} \quad P(D)H_\xi = h_\xi|_{\Omega_\zeta}.$$

Next, choose $\psi \in \mathcal{D}_{\{\omega\}}(\Omega_\zeta)$ with $\psi|_{\Omega_\sigma} \equiv 1$. Since $Q(D)$ acts continuously on $\mathcal{D}'_{\{\omega\}}(\Omega)$,

$$G_\xi := \psi(Q(D)g_\xi - H_\xi)$$

is in $\mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$ and satisfies

$$\operatorname{Supp} G_\xi \subset \Omega_\zeta \backslash \Omega_\varepsilon \subset \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Since $P(D)$ commutes with $Q(D)$, we get

$$P(D)G_\xi|_{\Omega_\sigma} = (P(D)Q(D)g_\xi - P(D)H_\xi)|_{\Omega_\sigma} = (Q(D)f_\xi - h_\xi)|_{\Omega_\sigma} = \delta_\xi|_{\Omega_\sigma},$$

which implies

$$P(D)G_\xi = \delta_\xi + T_\xi \quad \text{where} \quad \operatorname{Supp} T_\xi \subset (\Omega_\zeta \backslash \Omega_\sigma).$$

Now it is easy to check that

$$E_\xi : \varphi \mapsto \langle G_\xi, \varphi(\cdot - \xi)\rangle, \quad \varphi \in \mathcal{D}_{\{\omega\}}(\mathbb{R}^N)$$

satisfies the conditions (i) and (ii) in 3.4 (4). □

Because of Remark 2.3, the arguments in the proof of Lemma 2.4 also give the following lemma.

Lemma 3.8. *Assume that for $P \in \mathbb{C}[z_1, \dots, z_n]$ and an open set $\Omega \subset \mathbb{R}^n$ condition* 3.4 (4) *holds. Then $P(D)$ admits a right inverse on $\mathcal{D}'_{\{\omega\}}(\Omega)$ and on $\mathcal{E}_{\{\omega\}}(\Omega)$.*

Theorem 3.9. *For an open set Ω in $\mathbb{R}^n$ and for a complex polynomial P in n variables the following assertions are equivalent:*

1. $P(D) : \mathcal{D}'_{\{\omega\}}(\Omega) \to \mathcal{D}'_{\{\omega\}}(\Omega)$ *admits a right inverse;*
2. $P(D) : \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ *admits a right inverse;.*
3. *One of the conditions* 3.4 (2), 3.4 (3), 3.4 (4), 3.5 (∗), 3.6 (∗) *or* 3.6 (2) *holds.*

Proof. Because of 3.4, 3.8 and 3.5 – 3.7 the following implications hold:

$$(1) \Rightarrow 3.4\,(2) \Rightarrow 3.4\,(3) \Rightarrow 3.4\,(4) \Rightarrow (2) \Rightarrow 3.5(*) \Rightarrow 3.6(*) \Rightarrow 3.6\,(2) \Rightarrow 3.4\,(4) \Rightarrow (1).$$

□

Definition 3.10. Let ω be a weight function, Ω an open set in $\mathbb{R}^n$ and P a complex polynomial in n variables. Ω is called P-convex with $\{\omega\}$-bounds if one of the equivalent conditions in Theorem 3.9 holds.

As in Section 2 we get the following corollary:

Corollary 3.11. *Let P be a polynomial in n variables and let $(\Omega_i)_{i\in I}$ be a family of open sets in $\mathbb{R}^n$ for which $\Omega := \cap_{i\in I}\Omega_i \neq \emptyset$ is open. If Ω_i is P-convex with $\{\omega\}$-bounds for each $i \in I$ then Ω is P-convex with $\{\omega\}$-bounds.*

Corollary 3.12. *For each open set Ω in $\mathbb{R}^n$ and each $P \in \mathbb{C}[z_1, \dots, z_n]$ the following assertions are equivalent:*

1. *Ω is P-convex with $\{\omega\}$-bounds;*
2. *Ω is P-convex with (κ)-bounds for some $\kappa \in S_\omega$.*

Proof. (1) $\Rightarrow$ (2): By Theorem 3.9 we know that condition 3.4 (4) holds. Now an inspection of the proof of Lemma 3.8 (or of Lemma 2.4) shows that in the construction of each map A_k only finitely many ultradistributions enter. Hence only countably many $F_j \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$, $j \in \mathbb{N}$, are envolved in the whole construction. By BRAUN, MEISE and TAYLOR [6], 7.6, there exist weight functions $\sigma_j \in S_\omega$ so that $F_j \in \mathcal{D}'_{(\sigma_j)}(\mathbb{R}^n)$ for each $j \in \mathbb{N}$. By [6], 1.9, there exists $\kappa \in S_\omega$ such that $\sigma_j = o(\kappa)$ for each $j \in \mathbb{N}$. This implies that the operators A_k, constructed in the proof of Lemma 3.8 are in fact continuous linear operators from $\mathcal{D}'_{(\kappa)}$ into $\mathcal{D}'_{(\kappa)}(\Omega, \Omega_{k-2})$ which satisfy the corresponding condition 2.2 $(*)$. Hence Lemma 2.2 implies that (2) holds.

(2) $\Rightarrow$ (1): From (2) and Theorem 2.10 we get that condition 2.1 (4) holds with ω replaced by κ. Now $\kappa \in S_\omega$ implies $\mathcal{D}'_{(\kappa)}(\mathbb{R}^n) \subset \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$, by BRAUN, MEISE and TAYLOR [6], 3.9. Therefore condition 3.4 (4) holds for ω. By Theorem 3.9 this implies (1). □

4. Right inverses and ω-hyperbolicity

In this section we show that an open set Ω in $\mathbb{R}^n$ with C^1-boundary is P-convex with $*$-bounds only if P satisfies certain hyperbolicity conditions. In particular, we obtain a characterization of the polynomials P for which the Euclidean unit ball is P-convex with $*$-bounds. To introduce the hyperbolicity conditions that are used we first recall some notation.

Notation. For a vector $N \in \mathbb{R}^n \setminus \{0\}$ we let

$$H_+(N) := \{x \in \mathbb{R}^n : \langle x, N\rangle > 0\}, \quad H_-(N) := H_+(-N).$$

Recall the definition of ω-hyperbolicity from [21].

Definition 4.1. Let $P \in \mathbb{C}[z_1, \dots, z_n]$ and let $N \in \mathbb{R}^n$ be non-characteristic for P. Then the operator $P(D)$ is called $*$-hyperbolic with respect to N if there exists $E \in \mathcal{D}'_*(\mathbb{R}^n)$ satisfying $P(D)E = \delta$ and $\operatorname{Supp} E \subset \overline{H_+(N)}$. $P(D)$ is called $*$-hyperbolic, if it is $*$-hyperbolic with respect to some direction.

Remark 4.2. (a) If $P(D)$ is $*$-hyperbolic with respect to N then Holmgren's uniqueness theorem implies that the fundamental solution E which exists by 4.1 actually satisfies $\operatorname{Supp} E \subset \Gamma$, where Γ is a closed convex cone with vertex at the origin satisfying $\Gamma \setminus \{0\} \subset H_+(N)$.

(b) If $P(D)$ is $*$-hyperbolic with respect to N, then $P(D)$ is also $*$-hyperbolic with respect to $-N$ by [21], 2.9 (a) and 2.13 (b).

(c) If $\omega(t) = \log(2+t)$ then (ω)-hyperbolicity coincides with ordinary hyperbolicity by HÖRMANDER [10], Thm. 5.6.1 and Thm. 5.6.2.

(d) For a homogeneous polynomial P the operator $P(D)$ is $*$-hyperbolic with respect to N if and only if $P(D)$ is hyperbolic with respect to N, by [21], 2.9 (b) and 2.13 (b).

The significance of $*$-hyperbolicity for our considerations depends on the following lemma. In view of Remark 4.2 it can be proved by an easy modification of the arguments that we applied in the proof of [17], 3.1. For an exposition of the underlying idea in the special case of an open half space, see [19], 5.1.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be P-convex with $*$-bounds and let $N \in \mathbb{R}^n$ be non-characteristic for the polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$. If there exists $x_0 \in \partial\Omega$ so that $\partial\Omega$ is continuously differentiable in some neighborhood of x_0 and if N is normal to $\partial\Omega$ at x_0 then $P(D)$ is $*$-hyperbolic with respect to N.*

From Lemma 4.3 in connection with Remark 4.2) and (b), we deduce the following proposition (see also [19], 5.4).

Proposition 4.4. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ and assume that $N \in \mathbb{R}^n$ is non-characteristic for P. Then the following conditions are equivalent:*

1. *$H_+(N)$ and/or $H_-(N)$ is P-convex with $*$-bounds.*
2. *$P(D)$ is $*$-hyperbolic with respect to N.*

In the case of characteristic half spaces, the following proposition can be derived as in [22].

Proposition 4.5. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be non-constant and let $N \in \mathbb{R}^n \setminus \{0\}$ be non-characteristic for P. If the following two conditions are satisfied,*

1. *$\mathbb{R}^n$ is P-convex with $*$-bounds*

and

2. *$P(D)$ admits a fundamental solution $E \in \mathcal{D}'_*(\mathbb{R}^n)$ satisfying $\operatorname{Supp} E \subset \overline{H_-(N)}$,*

then $H_+(N)$ is P-convex with $*$-bounds.

To formulate the main result of this section, we will use the following notation: For an open subset Ω of $\mathbb{R}^n$ with (non-empty) C^1-boundary, the Gauss map

$$G : \partial\Omega \longrightarrow S^{n-1} \quad \text{is defined by} \quad G(x) := N_x ,$$

where N_x denotes the outer unit normal to $\partial\Omega$ at x. For $P \in \mathbb{C}[z_1, \dots, z_n]$ we define following HÖRMANDER [12], (10.4.2) –

$$\widetilde{P}(\xi, t) := \left(\sum_{\alpha \in \mathbb{N}_0^n} |P^{(\alpha)}(\xi)|^2 \, t^{2|\alpha|} \right)^{1/2} \quad \text{for} \quad \xi \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

Theorem 4.6. *For each non-constant polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ the following assertions are equivalent:*

1. *There exists a bounded open set $\Omega \neq \emptyset$ in $\mathbb{R}^n$ with C^1-boundary which is P-convex with $*$-bounds;*

2. *There exists an open subset $\Omega \neq \emptyset$ of $\mathbb{R}^n$ with C^1-boundary and surjective Gauss map which is P-convex with $*$-bounds;*

3. *$P(D)$ is $*$-hyperbolic with respect to each $N \in \mathbb{R}^n$ which is non-characteristic for P;*

4. *Each open convex subset of $\mathbb{R}^n$ is P-convex with $*$-bounds;*

5. *The principal part P_m of P is proportional to a product of linear forms with real coefficients and $Q := P - P_m$ has the following property;:*

If $* = (\omega)$ then there exists $C > 0$ such that $|Q(\xi)| \leq C\widetilde{P}_m(\xi, \omega(\xi))$ for all $\xi \in \mathbb{R}^n$.

If $* = \{\omega\}$ then there exist $\kappa \in S_\omega$ and $C > 0$ such that $|Q(\xi)| \leq C\widetilde{P}_m(\xi, \kappa(\xi))$ for all $\xi \in \mathbb{R}^n$.

Proof. (1) $\Rightarrow$ (2): This is easy to check.

(2) $\Rightarrow$ (3): This is a consequence of Lemma 4.3.

(3) $\Rightarrow$ (4): Let Ω be a given convex open subset of $\mathbb{R}^n$. Assume first that $* = (\omega)$. Since the non-characteristic vectors of P are dense in S^{n-1} we can find a sequence $(\Omega_j)_{j \in \mathbb{N}}$ of open convex polyhedra such that each Ω_j has only faces which have non-characteristic normals. Moreover, we can assume $\overline{\Omega}_j \subset \Omega_{j+1}$ for each $j \in \mathbb{N}$ and $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$. To show that Ω satisfies the condition 2.5 $(*)$, let $\varepsilon > 0$ be given. Choose $j \in \mathbb{N}$ with $\overline{\Omega}_\varepsilon \subset \Omega_j$ and $0 < \delta < \varepsilon$ with $\overline{\Omega}_j \subset \Omega_\delta$. Next fix $0 < \eta < \delta$ and find $m \in \mathbb{N}$ with $\overline{\Omega}_\eta \subset \Omega_m$. Then fix $f \in \mathcal{D}'_{(\omega)}(\Omega, \Omega_\delta)$, choose $\varphi \in \mathcal{D}_{(\omega)}(\Omega_m)$ with $\varphi|_{\overline{\Omega}_\eta} \equiv 1$, and note that φf is in $\mathcal{E}'_{(\omega)}(\mathbb{R}^n, \Omega_\delta)$. By our choice of Ω_j we can find $\mu \in \mathbb{N}$ and $b_\nu, N_\nu \in \mathbb{R}^n$ for $1 \leq \nu \leq \mu$ so that

$$\mathbb{R}^n \setminus \Omega_j = \bigcup_{\nu=1}^{\mu} \left(\overline{H_+(N_\nu)} + b_\nu \right),$$

where the N_ν are non-characteristic for P. Since the sets Ω_δ and $H_+(N_\nu) + b_\nu$, $1 \leq \nu \leq \mu$, form an open cover of $\mathbb{R}^n$, we can use a partition of unity in $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ subordinate to this cover, to obtain $\varphi f = \sum_{\nu=1}^{\mu} f_\nu$, where $\operatorname{Supp} f_\nu \subset H_+(N_\nu) + b_\nu$ for $1 \leq \nu \leq \mu$. Since N_ν is non-characteristic for P the hypothesis together with Remark 4.2 (a) implies the existence of a fundamental solution $E_\nu \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ satisfying

$\operatorname{Supp} E_\nu \subset \Gamma_\nu$, where Γ_ν is a closed convex cone with $\Gamma_\nu \setminus \{0\} \subset H_+(N_\nu)$. Consequently $g_\nu := E_\nu * f\nu$ is in $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ and $\operatorname{Supp} g_\nu \subset H_+(N_\nu) + b_\nu$. Hence $g := \sum_{\nu=1}^{\mu} g_\nu$ is in $\mathcal{D}'_{(\omega)}(\mathbb{R}^n, \Omega_j)$ which is a subset of $\mathcal{D}'_{(\omega)}(\mathbb{R}^n, \Omega_\varepsilon)$ and satisfies

$$P(D)g = \sum_{\nu=1}^{\mu} P(D)g_\nu = \sum_{\nu=1}^{\mu} f_\nu = \varphi f .$$

By our choice of φ, this implies $P(D)g|_{\Omega_\eta} = f|_{\Omega_\eta}$. Hence Ω satisfies condition 2.5 $(*)$. By Theorem 2.10 this implies (4).

If $* = \{\omega\}$ then, by [21], 2.13, there exists a weight function $\sigma \in S_\omega$ such that $P(D)$ is (σ) – hyperbolic with respect to each non – characteristic direction. Consequently, Ω is P – convex with (σ) – bounds, hence P – convex with $\{\omega\}$ – bounds by Corollary 3.12.

(4) $\Rightarrow$ (1): This is obviously true.

(3) $\Rightarrow$ (5): By [21], 2.9 (b) and 2.13 (b), $P_m(D)$ is hyperbolic with respect to each non – characteristic direction. Hence DE CRISTOFORIS [8], Thm. 1, implies that P_m is a complex multiple of a product of linear forms with real coefficients. Moreover, the perturbation theorem [21], 3.1, together with [21], 2.12, shows that Q satisfies the conditions stated in (5).

(5) $\Rightarrow$ (3): Obviously, P_m is hyperbolic and hence $*$ – hyperbolic with respect to each non – characteristic direction. Since Q satisfies the conditions in (5), the perturbation theorem [21], 3.1 and [21], 2.12, imply (3). □

Remark 4.7. Note that there are several other conditions equivalent to the condition on Q in Theorem 4.6 (5). They are explained in Section 3 of our article [21].

From Theorem 4.6 (resp. [21], 2.10) the following examples are easily derived.

Example 4.8. Assume that the weight function ω satisfies $t^{1/2} = O(\omega(t))$ as $t \to \infty$.

(a) Each convex open set Ω in $\mathbb{R}^2$ is P – convex with (ω) – bounds for the polynomials $P(z_1, z_2) := z_1^2 + az_2,\ a \in \mathbb{C}$.

(b) Each convex open set Ω in $\mathbb{R}^3$ is P – convex with (ω) – bounds for the polynomials $P(z_1, z_2, z_3) := z_1^2 - z_2^2 + az_3,\ a \in \mathbb{C}$.

5. Right inverses and Phragmén – Lindelöf conditions

In this section we explain why open convex sets Ω in $\mathbb{R}^n$ are P – convex with $*$ – bounds if and only if the zero variety $V(P)$ of P satisfies a condition of Phragmén Lindelöf type. To introduce these conditions several definitions are needed.

Definition 5.1. Let V be an analytic variety in $\mathbb{C}^n$. A function $u : V \to [-\infty, \infty[$ is called *plurisubharmonic* (*psh*) *on* V if it is locally bounded from above and psh at the regular points V_{reg} of V. The values of u at the singular points V_{sing} of V are not important in our considerations. However, for the formulation of our results it is convenient in the sequel to assume that

$$u(z) = \limsup_{\zeta \in V_{\text{reg}},\, \zeta \to z} u(\zeta) \quad \text{for all} \quad z \in V_{\text{sing}}.$$

By PSH(V) we denote the set of all psh functions on V which satisfy this condition.

Definition 5.2. Let Ω and K be convex subsets of $\mathbb{R}^n$, Ω being open and K being compact. Then we define the support functional h_K of K by

$$h_K(x) := \sup\{\langle x, y\rangle : y \in K\}$$

and we let

$$\mathcal{K}(\Omega) := \{L \subset \Omega : L \text{ is convex and compact}\}.$$

Definition 5.3. Let Ω be a convex open subset of $\mathbb{R}^n$, let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant and let ω be a weight function or $\omega(t) = \log(2+t)$. Also let

$$V(P) := \{z \in \mathbb{C}^n : P(-z) = 0\}.$$

(a) P or $V(P)$ satisfies the Phragmén-Lindelöf condition PL($\Omega, (\omega)$) if the following holds:

For each $K \in \mathcal{K}(\Omega)$ there exists $K' \in \mathcal{K}(\Omega)$ so that for each $K'' \in \mathcal{K}(\Omega)$ there exists $B > 0$ so that each $u \in$ PSH($V(P)$) satisfying (α) and (β) also satisfies (γ), where

(α) $u(z) \le h_K(\operatorname{Im} z) + O(\omega(z)), \quad z \in V(P)$,

(β) $u(z) \le h_{K''}(\operatorname{Im} z), \quad z \in V(P)$,

(γ) $u(z) \le h_{K'}(\operatorname{Im} z) + B\omega(z), \quad z \in V(P)$.

If the above condition holds only for all $u = \log|f|$, f an entire function on $\mathbb{C}^n$, then we say that P or $V(P)$ satisfies the condition APL($\Omega, (\omega)$).

(b) P or $V(P)$ satisfies the Phragmén-Lindelöf condition PL($\Omega, \{\omega\}$) if the following holds:

For each $K \in \mathcal{K}(\Omega)$ there exists $K' \in \mathcal{K}(\Omega)$ so that for each $K'' \in \mathcal{K}(\Omega)$ there exists $\sigma \in S_\omega$ so that each $u \in$ PSH($V(P)$) satisfying (α) and (β) also satisfies (γ), where

(α) $u(z) \le h_K(\operatorname{Im} z) + o(\omega(z)), \quad z \in V(P)$,

(β) $u(z) \le h_{K''}(\operatorname{Im} z), \quad z \in V(P)$,

(γ) $u(z) \le h_{K'}(\operatorname{Im} z) + \sigma(z), \quad z \in V(P)$.

If the above holds only for all $u = \log|f|$, f an entire function on $\mathbb{C}^n$, then we say that P or $V(P)$ satisfies the condition APL($\Omega, \{\omega\}$).

(c) P or $V(P)$ satisfies the Phragmén-Lindelöf condition HPL(Ω) if the following holds for the principal part P_m of P:

For each $K \in \mathcal{K}(\Omega)$ there exists $K' \in \mathcal{K}(\Omega)$ and $\delta > 0$ so that each $u \in$ PSH($V(P_m)$) satisfying (α) and (β) also satisfies (γ), where

(α) $u(z) \le h_K(\operatorname{Im} z) + \delta|z|, \quad z \in V(P_m)$,

(β) $u(z) \le 0, \quad z \in V(P_m) \cap \mathbb{R}^n$,

(γ) $u(z) \le h_{K'}(\operatorname{Im} z), \quad z \in V(P_m)$.

Remark 5.4. HÖRMANDER [11] showed that P satisfies HPL(Ω) if and only if $P(D)$ is surjective on the space $\mathcal{A}(\Omega)$ of all real-analytic functions on a convex open set Ω in $\mathbb{R}^n$. He was the first to prove that conditions of Phragmén-Lindelöf type for algebraic varieties are important in studying certain properties of linear partial differential operators with constant coefficients. Also, he introduced the name "Phragmén-Lindelöf principles" for conditions of this type.

Related but different Phragmén-Lindelöf conditions were used by ZAMPIERI [25], BRAUN, MEISE and VOGT [7] and BRAUN [4] to investigate and characterize when $P(D)$ is surjective on $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ or $\mathcal{E}_{\{\omega\}}(\Omega)$, Ω a convex open subset of $\mathbb{R}^n$.

The significance of the conditions APL($\Omega, *$) and PL($\Omega, *$) for the questions studied in this article is shown by the following theorem.

Theorem 5.5. *Let Ω be a convex open subset of $\mathbb{R}^n$ and let $P \in \mathbb{C}[z_1, \dots, z_n]$ be non-constant. Then the following assertions are equivalent:*

1. *Ω is P-convex with $*$-bounds;*
2. *P satisfies the condition* APL($\Omega, *$);
3. *P satisfies the condition* PL($\Omega, *$).

Proof. Let us first consider the case $* = (\omega)$. Then the proof can be given by the same arguments that we applied in [17], Sect. 4, using Fourier analysis. More precisely, as in [17], 4.4, one shows that condition 2.8 ($*$) implies APL($\Omega, (\omega)$) and as in [17], 4.5, one proves that APL($\Omega, (\omega)$) implies 2.9 ($*$). Since 2.8 ($*$) and 2.9 ($*$) are equivalent to (1) by Theorem 2.10, we see that (1) and (2) are equivalent. The equivalence of (2) and (3) was proved in [18], Thm. 6.2. Note that, based on Theorem 2.10, a detailed (different) proof of the present theorem for $* = (\omega)$ is given in [19], Sect. 3.

If $* = \{\omega\}$ then, by Corollary 3.12 and the preceding case, (1) is equivalent to the existence of a weight function $\kappa \in S_\omega$ such that P satisfies PL($\Omega, (\kappa)$). By [20], 6.2, this is equivalent to PL($\Omega, \{\omega\}$). Hence (1) and (3) are equivalent. Obviously (3) implies (2). To see that (2) implies (1), note that the arguments used in the proof of the implication (1) $\Rightarrow$ (3) in [20], 6.2, also give that (2) implies the existence of $\kappa \in S_\omega$ so that $V(P)$ satisfies APL($\Omega, (\kappa)$). By the preceding case and Corollary 3.12 this implies (1). □

For a comprehensive study of the Phragmén-Lindelöf conditions PL($\Omega, *$) we refer to our article [20]. In Theorem 3.3 and Theorem 6.3 of [20] we show that for a homogeneous polynomial P the conditions PL($\Omega, *$) do not depend on $*$. This characterization together with Theorem 5.4 and [17], Thm. 4.5 implies the following result.

Theorem 5.6. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be non-constant and homogeneous. Then for each open convex set Ω in $\mathbb{R}^n$ the following assertions are equivalent:*

1. *Ω is P-convex with (ω)-bounds for some/all weight functions ω;*

2. *Ω is P - convex with $\{\omega\}$ - bounds for some/all weight functions ω;*

3. *Ω is P - convex with bounds.*

Remark 5.7. Note that for homogeneous polynomials P and $\Omega = \mathbb{R}^n$ condition 5.6 (2) was characterized in [17], Thm. 4.7 in terms of a dimension condition for $V(P) \cap \mathbb{R}^n$ and of a local Phragmén-Lindelöf condition at points in $V(P) \cap S^{n-1}$, and that a quite different characterization was obtained in BRAUN [4], Thm. 5.2.

To formulate a sufficient condition for $\mathbb{R}^n$ to be P-convex with $*$-bounds, we introduce the following definition.

Definition 5.8. Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be non-constant. The variety $V(P)$ satisfies the strong dimension condition if $V(P) \cap \mathbb{R}^n \neq \emptyset$ and if for each $\xi \in V(P) \cap \mathbb{R}^n$ each local irreducible component W_ξ of $V(P)$ at ξ satisfies $\dim_{\mathbb{R}} W_\xi \cap \mathbb{R}^n = n - 1$.

The next corollary follows from [20], 3.17.

Corollary 5.9. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be homogeneous and non-constant. If $(V(P) \cap \mathbb{R}^n) \setminus \{0\} \subset V(P)_{\text{reg}}$ and if $V(P)$ satisfies the strong dimension condition then $\mathbb{R}^n$ is P - convex with bounds and hence P - convex with $*$ - bounds.*

From [20], Thm. 4.1, we get the following theorem.

Theorem 5.10. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be non-constant and denote by P_m its principal part. If a convex open subset of Ω of $\mathbb{R}^n$ is P - convex with $*$ - bounds then Ω is P_m - convex with bounds.*

Corollary 5.11. *Let Ω be a convex open subset of $\mathbb{R}^n$ and let $P \in \mathbb{C}[z_1, \dots, z_n]$ be non-constant. If Ω is P - convex with $*$ - bounds then $V(P)$ satisfies* HPL(Ω), *i. e., $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective.*

Proof. Let P_m denote the principal part of P. By [20], Thm. 4.1, the hypothesis implies that $V(P_m)$ satisfies PL($\Omega, \log(2+t)$). Hence $V(P_m)$ satisfies HPL(Ω) by [17], Thm. 4.12. By HÖRMANDER [11] this implies that $P(D)$ is surjective on $\mathcal{A}(\Omega)$. □

Theorem 5.10 suggests to treat P as a perturbation of its principle part P_m. While a general perturbation theorem for the property PL($\Omega, *$) is missing, a partial result in this direction are [20], Thm. 5.6 and Thm. 6.5. From these and Corollary 5.9 we get the next theorem.

Theorem 5.12. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be irreducible and denote by P_m its principal part. Assume that $V(P_m) \setminus \{0\}$ is a manifold. Then the following conditions are equivalent:*

1. *$\mathbb{R}^n$ is P - convex with (ω) - bounds (resp. $\{\omega\}$ - bounds).*

2. *$V(P_m)$ satisfies the strong dimension condition and* $\operatorname{dist}(z, V(P_m)) = O(\omega(z))$ *(resp.* $\operatorname{dist}(z, V(P_m)) = o(\omega(z))$*), $z \in V$, $|z| \to \infty$.*

For a further evaluation of Theorem 5.12 we refer to [20], 5.8 and 5.9. As a consequence of these we get the following examples which are recalled from [20], 5.10.

Example 5.13. For $n \geq 3$ and $m \geq 2$ let $P_m \in \mathbb{C}[z_1, \dots, z_n]$ be of the form

$$P_m(z_1, \dots, z_n) = \sum_{j=1}^{n} a_j z_j^m,$$

where $a_j \in \mathbb{R} \setminus \{0\}$ for $1 \leq j \leq n$. Then $\mathbb{R}^n$ is $(P_m + Q)$-convex with bounds for all $Q \in \mathbb{C}[z_1, \dots, z_n]$ with $\deg Q < m$, whenever either m is odd or m is even and there are j, k such that sign $a_j \neq$ sign a_k.

From 5.13 we conclude in particular that $\mathbb{R}^3$ is P-convex with bounds for $P(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^3$. Since P is homogeneous and not hyperbolic it is not $*$-hyperbolic by Remark 4.2 (d).

The next result shows that the situation is different in the case of two variables.

Theorem 5.14. *For each non-constant polynomial $P \in \mathbb{C}[z_1, z_2]$ the following assertions are equivalent:*

1. *$P(D)$ is $*$-hyperbolic;*
2. *$\mathbb{R}^2$ is P-convex with $*$-bounds;*
3. *$P(D)$ is $*$-hyperbolic with respect to each non-characteristic direction;*
4. *Each convex open set in $\mathbb{R}^2$ is P-convex with $*$-bounds.*

Proof. (1) $\Rightarrow$ (2): This is an easy consequence of the existence of fundamental solutions E_+ and E_- in $\mathcal{D}'_*(\mathbb{R}^2)$ for $P(D)$ having support in closed cones as described in Remark 4.2 (a).

(2) $\Rightarrow$ (3): Using Theorem 5.12 and [21], 2.7, this can be shown for $* = (\omega)$ by the same arguments that were used to prove the implication (2) $\Rightarrow$ (3) of Theorem 4.11 in [17]. The case $* = \{\omega\}$ is then reduced to the previous case by Corollary 3.12 and [21], 2.14.

(3) $\Rightarrow$ (4) $\Rightarrow$ (1): This holds by Theorem 4.6. □

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