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Extremal plurisubharmonic functions of linear growth on algebraic varieties

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1 Introduction

The classical Phragmén–Lindelöf Theorem shows that subharmonic functions $u(z)$ on the complex plane with a given asymptotic linear rate of growth, $u(z) \leqq |z| + o(|z|)$, and a uniform bound at real points, $u(z) \leqq 0$ for $z = x$ real, must also satisfy a uniform linear growth estimate, $u(z) \leqq |\operatorname{Im} z|$. In recent years, it has been shown that the validity of some estimates in a similar spirit for plurisubharmonic (psh) functions on an algebraic variety $V \subset \mathbb{C}^n$ characterizes whether or not the (system of) constant coefficient partial differential operator associated to V has a given property. Hörmander initiated the study of such conditions in [10], where he showed that the constant coefficient partial differential equation $P(D)f = g$ associated to a polynomial P has a real analytic solution f for every real analytic function g on $\mathbb{R}^n$ if and only if a certain estimate of Phragmén–Lindelöf type is satisfied by all the psh functions on $V(P) = \{z \in \mathbb{C}^n : P(z) = 0\}$. In [14] it is shown that the existence of a continuous linear right inverse for $P(D)$, as a linear transformation of $C^\infty(\mathbb{R}^n)$ to itself, is characterized by the validity of a Phragmén–Lindelöf-type estimate for psh functions on $V(P)$, and this was extended to the case of systems of equations by Palamodov [18] and to ultradifferentiable functions in [15]. Other Phragmén–Lindelöf conditions that characterize the surjectivity of $P(D)$ on spaces of ultradifferentiable functions are given in [6]. In many examples, use of the Phragmén–Lindelöf condition is the easiest, and possibly the only, way to verify that given examples of partial differential operators have the property in question.

In this paper, we focus on one aspect of the Phragmén–Lindelöf conditions, the existence of a uniform bound on the linear rate of growth of the psh function. For this purpose, we define in Section 2 a condition called RPL, the radial Phragmén–Lindelöf condition (Definition 2.2). This condition requires that psh functions u on V satisfying $u(z) \leqq |z| + o(|z|)$ and $u(z) \leqq \rho|\operatorname{Im} z|$ also satisfy a bound $u(z) \leqq A|z| + B_\rho$. The essential feature here is that the constant A depends only on the variety V, and is independent of u and ρ. The role of the constant B_ρ, which depends on ρ but is independent of u, is to account for the nonhomogeneity of the variety. When V is homogeneous, one can always take the constant B to be 0. Our main result, Theorem 5.1, shows that an algebraic variety of (pure) dimension k satisfies the condition RPL if and only if the homogeneous algebraic variety V_h tangent to V at infinity satisfies the *dimension condition* (Definition 2.6); i.e. for every irreducible component W of V_h, the real algebraic variety $W \cap \mathbb{R}^n$ also has (real) dimension k. The result is applied in [16], to characterize the homogeneous varieties which satisfy the Phragmén–Lindelöf condition studied there. For varieties of the form $V = \{P(z) = 0\}$ where P is a homogeneous polynomial on $\mathbb{C}^n$, this latter Phragmén–Lindelöf condition is equivalent to the existence of a continuous linear right inverse for the associated constant coefficient partial differential operator $P(D)$. Also, Theorem 5.1 is used to prove a perturbation result for this condition.

Thus, while the main application of most Phragmén–Lindelöf conditions has been to characterize properties of constant coefficient partial differential operators, the main application of the RPL condition is to characterize an important intermediate step in the study of such principles. We do not know a property of the partial differential operator $P(D)$ that is equivalent to the variety $V = \{P(z) = 0\}$ satisfying the condition RPL.

The precise definitions of what we mean by a psh function on V, the condition RPL, and the dimension condition are given in Section 2. In Section 3 we treat the case of homogeneous varieties where the relation between the estimate we are studying and the dimension of the real points in V_h is explained by a theorem of Sibony–Wong [19], as extended by Siciak [20]. Section 4 contains some technical results about continuity properties of extremal psh functions. The main result in this section, Theorem 4.4, gives a natural semicontinuity property of extremal functions as a function of the variety. This result, whose proof depends on an "extension with bounds" lemma for psh functions, Lemma 4.6, is then used in Section 5 to show that the condition RPL carries over from V to its tangent cone V_h. The same method and the essential results of Section 4 are applied in [16], Section 4 to prove the analogous result for a different Phragmén–Lindelöf condition. We also give in Section 5 a different version of the Sibony–Wong estimate which allows an exceptional set. This is a key point in proving the main result, Theorem 5.1.

2 Definitions and preliminaries

In this section, we introduce the notation and terminology that will be used throughout the paper. There are several possible definitions of what is meant by a psh function on a variety in $\mathbb{C}^n$, see e.g. Fornaess and Narasimhan [8]. For our purposes it is convenient to have the largest possible class, the so-called weakly psh functions.

Definition 2.1 *Let V be an analytic variety in an open subset D of $\mathbb{C}^n$. A function $u : V \to [-\infty, \infty[$ is called psh on V if it is locally bounded above and psh at the regular points $V_{\text{reg}} \subset V$. The values of u at the singular points $V_{\text{sing}} \subset V$ are not important in our considerations. However, it is convenient in the sequel to assume that*

$$u(z) = \limsup_{\xi \in V_{\text{reg}},\, \xi \to z} u(\xi) \quad \text{for all } z \in V_{\text{sing}} .$$

By PSH(V) *we denote the set of all psh functions on V.*

The type of estimate studied here is the following, which we call the radial Phragmén–Lindelöf condition.

Definition 2.2 *An algebraic variety V in $\mathbb{C}^n$ satisfies the condition RPL if and only if there exists $A > 0$ such that for each $\rho > 1$, a constant B_ρ exists such that each $u \in$ PSH(V) satisfying* (1) *and* (2) *also satisfies* (3), *where*:

(1) $u(z) \leqq |z| + o(|z|)\,,\ z \in V$

(2) $u(z) \leqq \rho|\operatorname{Im} z|\,,\ z \in V$

(3) $u(z) \leqq A|z| + B_\rho\,,\ z \in V.$

The main theorem of the paper, given in Section 5, characterizes when pure dimensional algebraic varieties satisfy the condition RPL.

It is also useful to study a local version of the RPL condition.

Definition 2.3 *Let V be an analytic variety in a neighborhood of a point $\xi \in V \cap \mathbb{R}^n$. We say that V satisfies the condition RPL_{loc} at ξ if and only if there are constants $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > 0$ and $A > 0$ such that each $u \in$* PSH $(V \cap \{|z - \xi| < \varepsilon_1\})$ *satisfying* (1) *and* (2) *also satisfies* (3), *where*:

(1) $u(z) \leqq 1,\ z \in V \cap \{|z - \xi| < \varepsilon_1\}$

(2) $u(z) \leqq 0,\ z \in V \cap \mathbb{R}^n \cap \{|z - \xi| \leqq \varepsilon_2\}$

(3) $u(z) \leqq A|z - \xi|\,,\ z \in V \cap \{|z - \xi| \leqq \varepsilon_3\}\,.$

The last definition can perhaps be better expressed in terms of relative extremal psh functions (Klimek [12], Chapter 4, Section 5).

Definition 2.4 *Let V be an analytic variety in a pseudoconvex domain $D \subset \mathbb{C}^n$ and let K be a compact subset of D. By the* **extremal function of K relative to D and V**, *we mean the function*

$$U_K(z) = U(z; K, D, V), \quad z \in V\ ;$$

that is the upper envelope of all the functions $u(z)$ that are psh and bounded above by 1 on V and that satisfy $u(z) \leqq 0$ whenever $z \in K \cap V$. The upper-semicontinuous regularization $U_K^*(z) = \limsup_{\zeta \to z,\, \zeta \in V_{\text{reg}}} U_K(\zeta)$ is psh on

V and equal to $U_K(z)$ outside a pluripolar subset of V (see e.g.[12], *Theorem* 4.7.6).

Clearly, if we take $D = \{|z - \xi| < \varepsilon_1\}$ and $K = V \cap \mathbb{R}^n \cap \{|z - \xi| \leqq \varepsilon_2\}$, then V satisfies $\mathrm{RPL_{loc}}$ at ξ if and only if

$$U_K^*(z) \leqq A|z - \xi|\,, \qquad z \in V \cap \{|z - \xi| \leqq \varepsilon_3\}\,.$$

The definition of the condition RPL can also be phrased in terms of the extremal psh function on V that is the upper envelope of all the psh functions on V satisfying conditions (1) and (2) of that definition.

In studying the conditions RPL, RPL_{loc}, the component structure of the variety is not important, as Hörmander has already pointed out in [10].

Proposition 2.5 (i) *An algebraic variety V satisfies the condition RPL if and only if each irreducible component of V satisfies RPL.*

(ii) *An analytic variety V in a neighborhood of $\xi \in \mathbb{C}^n$ satisfies RPL_{loc} at ξ if and only if each local irreducible component of V satisfies RPL_{loc} at ξ.*

Proof. Let $V_1, \ldots, V_k$ be the irreducible components of $V \subset \mathbb{C}^n$. If RPL holds for V_j, $1 \leqq j \leqq k$, then it clearly holds for V. Conversely, if V satisfies RPL and if V_m is given, then we can find a psh function v on $\mathbb{C}^n$ which is $-\infty$ on V_j for $j \neq m$, satisfies $v|_{V_m} \not\equiv -\infty$, and $v(z) \leqq \log(1+|z|)$ as $|z| \to \infty$ (see e.g. [12], Chapter 5, or [3]). Now let u be psh on V_m and satisfy the estimates (1) and (2) of the definition of the RPL condition. The function $u_\varepsilon = u + \varepsilon v$ can be extended to a psh function on V by defining it as $-\infty$ on $V \setminus V_m$. Then the function

$$w_\varepsilon = \frac{1}{1+2\varepsilon}(u_\varepsilon + 2\varepsilon \sum_{1 \leqq i \leqq n} \log|\sin z_i/z_i|)$$

satisfies the estimates (1) and (2) of the RPL condition. Hence, RPL for V implies that w_ε is bounded by $A|z| + B_\rho$ on V and consequently on V_m. Then letting $\varepsilon \to 0$, we see that u has the same bound on V_m. The proof for the condition $\mathrm{RPL_{loc}}$ goes exactly the same way.

It seems clear that for a variety to satisfy the condition RPL, it must have lots of "nearly real" points. One measure of this is the dimension of the set of real points in the variety.

Definition 2.6 *Let V be an analytic variety in a neighborhood of a point $\xi \in V \cap \mathbb{R}^n$. We say that V satisfies the* **dimension condition at ξ** *if, for every local irreducible component W of V at ξ, the dimension of $W \cap \mathbb{R}^n$ as a real analytic variety at ξ is equal to the dimension of V at ξ as a complex analytic variety. If V is a pure dimensional global variety in $\mathbb{C}^n$ then we say that V satisfies the* **dimension condition** *if, for every irreducible component W of V, the dimension of $W \cap \mathbb{R}^n$, as a real analytic variety, is equal to the dimension of V as a complex variety. Finally, a global variety $V \subset \mathbb{C}^n$ satisfies the* **strong dimension condition** *if and only if $V \cap \mathbb{R}^n \neq \emptyset$ and, for each*

$\xi \in V \cap \mathbb{R}^n$, V satisfies the dimension condition at ξ, when V is considered as a local variety in a neighborhood of ξ.

For the definition of real analytic sets and their dimensions we refer to Narasimhan [17].

The dimension condition and strong dimension condition are different requirements. For example, the algebraic variety $z_2^2 = (z_1 - 1)(z_1 + 1)^2$ has complex dimension 1. At the point $\xi = (1, 0)$ it also has dimension 1 as a real algebraic variety but the point $(-1, 0)$ is isolated in the set of real points in the variety. Hence, this variety satisfies the dimension condition but not the strong dimension condition. An example of a homogeneous variety with this property is given by making this example homogeneous, $z_2^2 z_3 = (z_1 - z_3)(z_1 + z_3)^2$.

The dimension condition measures if the set of real points in the variety V is pluripolar. Recall that a set E is pluripolar in V if and only if there is a plurisubharmonic function on V that is $-\infty$ on E but is not identically $-\infty$ on V. If W is any irreducible component of V, there are psh functions on V that are identically $-\infty$ on any other irreducible component of V but not identically $-\infty$ on W. Thus, a subset E of V is not pluripolar in V if and only if $E \cap W$ is not pluripolar in W for each irreducible component W of V.

Proposition 2.7 (i) *A pure dimensional variety V in $\mathbb{C}^n$ satisfies the dimension condition if and only if $V \cap \mathbb{R}^n$ is not pluripolar in V.*

(ii) *A local analytic variety V in a neighborhood of a point $\xi \in V \cap \mathbb{R}^n$ satisfies the dimension condition at ξ if and only if $V \cap \mathbb{R}^n \cap \{|z - \xi| < \varepsilon\}$ is not pluripolar in V for each $\varepsilon > 0$.*

Proof. By the above remark, it is no loss of generality to suppose that V is irreducible. If $V \cap \mathbb{R}^n$ has dimension less than the dimension of V, then in a neighborhood of each point of $V \cap \mathbb{R}^n$ there are finitely many analytic functions $f_1, \ldots, f_q$ that vanish identically on $V \cap \mathbb{R}^n$ but do not vanish all on $V \setminus \mathbb{R}^n$ in this neighborhood. That is, the set $V \cap \mathbb{R}^n$ is the set where the psh function $u = \log \sum |f_j|$ is $-\infty$. Thus, the set $V \cap \mathbb{R}^n$ is locally pluripolar in V and, consequently, is globally pluripolar in the Stein space V by a theorem of B. Josefson (as extended to Stein spaces by E. Bedford [1]). Consequently, if V fails the dimension condition, then $V \cap \mathbb{R}^n$ is pluripolar in V. Conversely, if $V \cap \mathbb{R}^n$ has dimension $k = \dim V$ then there are regular points $\xi \in V \cap \mathbb{R}^n$ such that $V \cap \mathbb{R}^n$ is a real analytic manifold of real dimension k in a neighborhood of ξ. Therefore, any psh function u on V that is $-\infty$ on $V \cap \mathbb{R}^n$ must be $-\infty$ on a neighborhood of ξ in V. Hence, u must be identically $-\infty$ on the regular points of V, since these form a connected complex manifold. That is, $u \equiv -\infty$ on V, so $V \cap \mathbb{R}^n$ is not pluripolar in V. This completes the proof of (i). The proof of (ii) is similar.

Lemma 2.8 *Let V be an analytic variety in a neighborhood of $\xi \in V \cap \mathbb{R}^n$ which satisfies RPL_{loc} at ξ. Then V satisfies the dimension condition at ξ.*

Proof. Let $K = V \cap \mathbb{R}^n \cap \{|z| \leqq \varepsilon\}$. The extremal function U_K satisfies $U_K(z) \leqq A|z-\xi|$ for $z \in V$ near ξ, since V satisfies $\mathrm{RPL}_{\mathrm{loc}}$ at ξ. Therefore, K is not pluripolar in V, so V satisfies the dimension condition by the previous proposition.

We conclude this section with a lemma that will be needed several times in the paper.

Lemma 2.9 *There exists a psh function H on $\{|z| < 1\} \subset \mathbb{C}^n$ that is continuous on the closed ball $\{|z| \leqq 1\}$ and has the following properties:*

(1) $H(z) \leqq |\operatorname{Im} z|$ *for* $|z| \leqq 1$
(2) $H(z) \leqq |\operatorname{Im} z| - c$ *for* $|z| = 1$ *and* $c = \frac{1}{2}$
(3) $H(x) \leqq 0$ *for* $x \in \mathbb{R}^n, |x| \leqq 1$
(4) $H(iy) \geqq 0$ *for* $y \in \mathbb{R}^n, |y| \leqq 1$
(5) $H(z) = O(|z|^2)$ *as* $z \to 0$.

Proof. It is easy to check that $H(z) = \frac{1}{2}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2)$ has all the given properties. (We thank the referee for suggesting this function to replace a more complicated one used previously.)

3 Homogeneous varieties

In this section we study the condition RPL on homogeneous algebraic varieties in $\mathbb{C}^n$; that is, on varieties V satisfying $z \in V$ if and only if $\zeta z \in V$ for all complex scalars ζ. For such varieties, we will show that the condition RPL holds if and only if the variety satisfies the dimension condition. This result is closely related to the Sibony–Wong estimate [19], as improved by Siciak [20], which implies that a psh function u on $\mathbb{C}^n$ satisfying the inequality $u(z) \leqq |z|$ on a nonpluripolar set of complex lines in $\mathbb{C}^n$ must in fact satisfy an estimate $u(z) \leqq A|z|$ for all $z \in \mathbb{C}^n$. The constant A depends only on the set of lines and is independent of the function u. In fact, when $V = \mathbb{C}^n$, our result is a special case of the Sibony–Wong estimate.. However, we need the result on homogeneous algebraic varieties.

Lemma 3.1 *Let V be a homogeneous algebraic variety in $\mathbb{C}^n$, and E a set of complex lines in V which is not pluripolar in V. Then there exists a constant $A > 0$, depending on E and V, such that if u is psh on V and satisfies $u(z) \leqq |z|$ for all z in a complex line belonging to E, then $u(z) \leqq A|z|$ for all $z \in V$.*

We will postpone the proof of this lemma until the remark following Lemma 5.5 in Section 5, where a version of it is given that allows u to be psh outside a small exceptional set. However, using the lemma we can prove the main result of this section.

Theorem 3.2 *Let V be a pure dimensional homogeneous algebraic variety in $\mathbb{C}^n$. Then the following are equivalent:*

(i) *V satisfies the condition RPL.*

(ii) *There exists a constant $A \geqq 1$ such that every psh function u on V that satisfies* (1) *and* (2) *also satisfies* (3), *where*

(1) $u(z) \leqq |z| + o(|z|),\ z \in V$

(2) $u(z) \leqq 0,\ z \in V \cap \mathbb{R}^n$

(3) $u(z) \leqq A|z|,\ z \in V.$

(iii) *V satisfies* $\mathrm{RPL}_{\mathrm{loc}}$ *at* 0.

(iv) *V satisfies the dimension condition at* 0.

(v) *V satisfies the dimension condition.*

(vi) *There exists a constant $A \geqq 1$ such that every psh function u on V satisfying $u(z) \leqq |z|$ for all $z \in V$ of the form $z = \zeta x$, where $x \in V \cap \mathbb{R}^n$, $\zeta \in \mathbb{C}$, also satisfies $u(z) \leqq A|z|$ for all $z \in V$.*

Proof. (i) $\Rightarrow$ (ii): Let u be a psh function on V satisfying (1) and (2) of (ii). Let H be the psh function of Lemma 2.9 and c the constant appearing in that lemma. For $R > 0$ large, define for $z \in V, |z| \leqq R$,

$$v(z) = v(z, R) = \frac{c}{2}\max\left\{u(z) - \frac{1}{R} + \frac{2R}{c}H\left(\frac{z}{R}\right),\ \frac{2|\mathrm{Im}\, z|}{c}\right\}.$$

When $|z| = R$ and R is sufficiently large, the first term in the maximum does not exceed

$$R + o(R) + (2R/c)(|\mathrm{Im}\, z/R| - c) \leqq (2/c)|\,\mathrm{Im}\, z|\,,$$

so if we extend the definition of $v(z, R)$ to all of V by making it equal to $|\,\mathrm{Im}\, z|$ at points of V outside the ball $\{|z| \leqq R\}$, then v is psh on V. Note that v satisfies the inequality (1) of the condition RPL, since it is equal to $|\,\mathrm{Im}\, z|$ outside a compact subset of V. We also claim that v satisfies the estimate (2) of that condition for some constant ρ. This is obvious for $|z| > R$. In the compact set $V \cap \{|z| \leqq R\}$, we have that the first term in the maximum defining v is negative on a neighborhood of the set of points where $|\,\mathrm{Im}\, z| = 0$, since $H(z) \leqq |\,\mathrm{Im}\, z|$ and u is uppersemicontinuous. Therefore, we can choose ρ sufficiently large, depending on R and u, such that $v(z) \leqq \rho|\,\mathrm{Im}\, z|$ holds on all of V. Since V is a homogeneous variety, for any $r > 0$ the function $v(rz)/r$ is also psh on V and satisfies (1) and (2) of the condition RPL with the same constant ρ. Therefore, we conclude that $v(rz)/r \leqq A|z| + B_\rho$ for all $z \in V$. Multiply this equation by r and replace rz by z to obtain that $v(z) \leqq A|z| + rB_\rho$ holds for all $z \in V$. Letting $r \to 0$, we get $v(z, R) \leqq A|z|$. Finally, let $R \to \infty$, and use the fact that $H(z) = O(|z|^2)$ as $z \to 0$, to see that $(c/2)u(z) \leqq \limsup_{R\to\infty} v(z, R) \leqq A|z|$ holds for all $z \in V$.

(ii) $\Rightarrow$ (iii): Let $\varepsilon_1 > 2\varepsilon_2 > 0$ and let u be psh on $V \cap \{|z| < \varepsilon_1\}$ with $u(z) \leqq 1$ and $u(z) \leqq 0$ for $z \in V \cap \mathbb{R}^n \cap \{|z| \leqq \varepsilon_2\}$. Fix a point $z' \in V \cap \{|z| \leqq \varepsilon_2\}$

and then define a psh function on V by setting

$$v(z) = c\varepsilon_2 \max\left\{u(z) + \frac{1}{c}H\left(\frac{z - \operatorname{Re} z'}{\varepsilon_2}\right), \frac{|\operatorname{Im} z|}{c\varepsilon_2}\right\}.$$

on $V \cap \{|z - \operatorname{Re} z'| \leqq \varepsilon_2\}$ and setting $v(z) = |\operatorname{Im} z|$ at points of V outside this ball. Exactly as in the proof of (i) ⇒ (ii), we have that v is psh on V. The function v also clearly satisfies (1) and (2) of (ii), so we conclude that it also satisfies (3). Applying this estimate at the point z' and using the fact that $H(iy) \geqq 0$, we conclude that $u(z') \leqq \frac{A}{c\varepsilon_2}|z'|$. Since z' is an arbitrary point of $V \cap \{|z| \leqq \varepsilon_2\}$, we have therefore proved that V satisfies $\mathrm{RPL}_{\mathrm{loc}}$ at 0 (with $\varepsilon_3 = \varepsilon_2$).

(iii) ⇒ (iv): If V fails to satisfy the dimension condition at 0, then by Proposition 2.7, part (ii), the set $K = V \cap \mathbb{R}^n \cap \{|z| \leqq \varepsilon\}$ is pluripolar in V for some $\varepsilon > 0$. Therefore, the extremal function of K relative to $V \cap \{|z| < \varepsilon_1\}$ satisfies $U_K^*(z) \equiv 1$ (see e.g. Klimek [12], Chapter 4). In particular, it cannot satisfy $U_K(z) \leqq A|z|$ in any neighborhood of 0, so as was already noted in Section 2, V cannot satisfy $\mathrm{RPL}_{\mathrm{loc}}$ at 0. This is a contradiction, so V must satisfy the dimension condition at 0.

(iv) ⇒ (v): This is obvious, since the dimension of a variety is the maximum over all points in the variety of the dimension of the variety at the point.

(v) ⇒ (vi): This is a direct consequence of Lemma 3.1. For, by Proposition 2.7, the set E of all complex lines $l(x) = \{\zeta x : \zeta \in \mathbb{C}\}$ where $x \in V \cap \mathbb{R}^n$ is not a pluripolar set of lines in V if and only if V satisfies the dimension condition. Therefore, the hypothesis on u in (vi) shows that the hypothesis of Lemma 3.1 is satisfied. The constant A in (vi) is the one associated to the set E of real lines in V.

(vi) ⇒ (i): We will show that (vi) implies (ii), which clearly implies (i). Let u be psh on V and satisfy the estimates (1) and (2) of (ii), and let $x \neq 0$ be a point of $V \cap \mathbb{R}^n$. Then we obtain from the ordinary Phragmén–Lindelöf theorem in the complex plane, applied to the function $\zeta \mapsto u(\zeta x)$, that $u(\zeta x) \leqq |\operatorname{Im} \zeta x| \leqq |\zeta x|$; that is, $u(z) \leqq |z|$ for all $z = \zeta x$ with $\zeta \in \mathbb{C}$ and $x \in V \cap \mathbb{R}^n$. By (vi), we therefore conclude that u satisfies the estimate (3) of (ii).

Example. It seems reasonable that the estimate $u(z) \leqq A|z|$ of (3) of the condition $\mathrm{RPL}_{\mathrm{loc}}$ or of (ii) of Theorem 3.2 might be improved to one of the form

$$u(z) \leqq A|\operatorname{Im} z|^{\varepsilon}|z|^{1-\varepsilon} \tag{1}$$

for some constant $\varepsilon > 0$. However, this is not the case, as is shown by the homogeneous variety $V = \{z = (z_1, z_2, z_3) | z_2 z_3^2 = -(z_1 - z_2)^2(z_1 + z_2)\}$ (which satisfies the dimension condition but not the strong dimension condition). This can be seen as follows. Let $\xi = (1, 1, 0)/\sqrt{2} \in V \cap \mathbb{R}^3$ and for small $r > 0$, let $U_r = \{|z - r\xi| < r/2\}$. For small $\delta > 0$, define a psh function in $V \cap U_r$ by

$$u(z) = u(z, r, \delta) = \max\left\{\frac{cr}{4} + \delta \log|z_1 - z_2| + \frac{r}{2}H\left(\frac{2(z - r\xi)}{r}\right), |\operatorname{Im} z|\right\}.$$

where H, c are the function and constant of Lemma 2.9. Then u is psh on $V \cap U_r$ and $-\infty = u(z) \leqq 0$ at all real points of V inside this ball, since $z_1 = z_2$ at all such points. At points in the boundary of U_r, we have that the first term in the maximum does not exceed $cr/4 + (r/2)(2|\operatorname{Im} z|/r - c) \leqq |\operatorname{Im} z|$ from the estimate (2) of Lemma 2.9. Therefore, we can extend u to a psh function on all of V by defining it to be equal to $|\operatorname{Im} z|$ outside this ball. For small r, δ, this function clearly satisfies $u \leqq 1$ and $u \leqq 0$ on $V \cap \mathbb{R}^n$. On the other hand, $\lim_{\delta \to 0} u(z, r, \delta) = cr/4 + (r/2)H(2(z - r\xi)/r) > cr/8$ at nonreal points of U_r near $r\xi$. Since there are such points with $|\operatorname{Im} z|$ arbitrarily small, no estimate of the form (1) can hold.

It seems likely that such an estimate may hold if V satisfies the strong dimension condition.

4 Local extremal functions

We pointed out in Section 2 that the definition of the property RPL_{loc} could be phrased in terms of local extremal functions U_K on a variety. In this section two continuity properties of these local extremal functions are given, one in terms of K and the other in V. These properties will be used in the next section to show that if the condition RPL holds on V then it also holds on the homogeneous variety that is tangent to V at infinity. The results are given in a slightly more general form than is needed here, since other applications, such as the one in [16] require the stronger version. For this reason, we give the following generalization of the local extremal function defined in Section 2. Similar extremal functions have been studied previously by several authors; e.g. Siciak [21], Zeriahi [23].

Definition 4.1 *Let D be a domain in $\mathbb{C}^n$, h a psh function on D, E a subset of D, and V an analytic variety in D. The* **extremal psh function of E relative to h, V, and D** *is the function $U_E(z) = U_E(z; h, V, D)$ defined on V by*

$$\sup\{u(z) : u \text{ is psh on } V,\ u(z) \leqq h(z),\ z \in V;\ u(z) \leqq 0,\ z \in E\}.$$

When some or all of the parameters h, V, D are fixed, we will often drop them from the notation. As usual, we will let U_E^* denote the uppersemicontinuous regularization of U_E through regular points of V.

Recall that a pseudoconvex domain $D \subset \mathbb{C}^n$ is called *hyperconvex* if it has a bounded continuous psh exhaustion function. That is, if there exists a negative continuous psh function ρ on D such that $\rho(z) \to 0$ as $z \to \partial D$ and $\{z \in D : \rho(z) \leqq -\delta\}$ is a compact subset of D for each $\delta > 0$.

Proposition 4.2 *Let D be a hyperconvex domain in $\mathbb{C}^n$, h a continuous psh function on D, V an analytic variety in D, and K_j a sequence of compact subsets of D.*

If $K_1 \supset K_2 \supset \cdots$ with $K = \cap_j K_j$, then

$$\lim_{j\to\infty} U_{K_j}(z;\, h,\, V,\, D) = \sup_j U_{K_j}(z;\, h,\, V,\, D) = U_K(z;\, h,\, V,\, D)\,.$$

If $K_1 \subset K_2 \subset \cdots$ with $E = \cup_j K_j$, then

$$\lim_{j\to\infty} U^*_{K_j}(z;\, h,\, V,\, D) = \inf_j U^*_{K_j}(z;\, h,\, V,\, D) = U^*_E(z;\, h,\, V,\, D)\,.$$

Proof. This proposition is well-known. The proof of the first assertion is the same as that of Proposition 4.5.10 of [12]. The second assertion is a consequence of the following three general facts about extremal functions: (i) $U_{K_j} = 0$ at all points of K_j; (ii) $U^*_{K_j} = U_{K_j}$ except on a pluripolar set ([2], Theorem 7.1); (iii) the countable union of pluripolar sets is pluripolar. We omit the details.

We also need to discuss the limits of the extremal functions under convergent sequences of varieties $V_j \subset D$. There are several ways this can be done; for example, convergence in the Hausdorff metric on sets or as currents on D. However, for our purposes it seems the simplest and most convenient way is in terms of the following definition.

Definition 4.3 *Let D be a domain in $\mathbb{C}^n$ and let V and V_j, $j \in \mathbb{N}$, be subsets of D. We say that $(V_j)_{j\in\mathbb{N}}$ converges to V if the following two conditions are satisfied:*

(1) *Each $z_0 \in V$ is the limit of a sequence $(z_j)_{j\in\mathbb{N}}$ satisfying $z_j \in V_j$ for all $j \in \mathbb{N}$.*
(2) *Each sequence $(z_k)_{k\in\mathbb{N}}$ satisfying $z_k \in V_{j_k}$ for all $k \in \mathbb{N}$ and some subsequence $(j_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$, which is convergent in D has its limit in V.*

Theorem 4.4 *Let D be a hyperconvex domain in $\mathbb{C}^n$. Let K be a compact subset of D, h a continuous psh function on D and V_j a sequence of pure k-dimensional analytic varieties that converge to an analytic variety V in D, in the sense of Definition 4.3. Then*

$$U_K(z; h, V, D) \leqq \liminf_{j\to\infty} U_K(z_j; h,\, V_j,\, D)$$

for every sequence of points $z_j \in V_j$ such that $z_j \to z \in V_{\mathrm{reg}}$.

For the proof, we need several lemmas.

Lemma 4.5 *Let D be a hyperconvex domain in $\mathbb{C}^n$, V an analytic variety in D, $\mathcal{O}$ an open neighborhood of V in D, and $\varepsilon > 0$. Then there exists a psh* *function* ψ *on* D *such that*

(i) $V = \{z \in D | \psi(z) = -\infty\}$
(ii) $\psi(z) < \varepsilon$, $z \in D$
(iii) $\psi(z) \geqq 0$, $z \in D \backslash \mathcal{O}$.

Proof. This is very much like Theorem 7.2 of [3]. Since V is an analytic variety in a pseudoconvex domain, there is a psh function Ψ on D such that $V = \{z \in D | \Psi(z) = -\infty\}$. This function will be modified to obtain the function ψ of the lemma. The lemma holds with V replaced by any complete pluripolar set in D.

Let ρ be a continuous psh function on D that is an exhaustion function for D and vanishes on ∂D. Let $(\delta_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers that decreases to zero so fast that $\sum \delta_j / j < \varepsilon/2$ and $(1 + \log j)\delta_{j-1} < \varepsilon/4$. Exhaust D by the compact sets $K_j = \{z \in D | \rho(z) \leqq -\delta_j\}$ and set $M_j = \sup\{\Psi(z) | z \in K_j\}$. It is no loss of generality to suppose that $M_j \geqq 0$. Choose numbers $\varepsilon_j > 0$ so small that

$$\sum_{j=1}^{\infty} \varepsilon_j (M_j + 1) < \infty$$

$$0 \geqq \varepsilon_j (\Psi(z) - M_j) \geqq -\frac{\varepsilon}{2^{j+2}}, \quad z \in K_j \backslash \mathcal{O}.$$

Define for $z \in K_j, \psi_j(z) = \max\{\varepsilon_j(\Psi(z) - M_j), (1/j)(\rho(z) + \delta_j)\}$. On the boundary of K_j, we have $\psi_j(z) = 0 = (1/j)(\rho + \delta_j)$, so ψ_j can be extended to a psh function on all of D by setting it equal to $(1/j)(\rho + \delta_j)$ in $D \backslash K_j$. Then let

$$\psi(z) = \sum_{j=1}^{\infty} \psi_j(z).$$

Since $\psi_j \leqq 0$ on K_j, the partial sums of the infinite series are eventually decreasing on each compact set. If $z \in V$, then $\psi_j(z) = (1/j)(\rho(z) + \delta_j)$, so $\psi(z) = -\infty$ because $\sum (1/j) = +\infty$ and $\sum \delta_j / j < +\infty$. For any $z \in D$, there is a unique integer $k = k(z)$ such that $z \in K_k \backslash K_{k-1}$ (where $K_0 = \emptyset$). Then

$$\psi(z) = \sum_{j=1}^{k-1} (1/j)(\rho(z) + \delta_j) + \sum_{j=k}^{\infty} \max\{\varepsilon_j(\Psi(z) - M_j), (1/j)(\rho(z) + \delta_j)\}.$$

From this we see that $\psi(z) > -\infty$ for $z \notin V$, so ψ is psh on D and (i) of the lemma holds. It is also clear from this equation that $\psi(z) < \sum \delta_j / j < \varepsilon/2$. And, when $z \notin \mathcal{O}$, we have (with $\delta_0 := 0$)

$$\psi(z) \geqq \sum_{j=1}^{k-1} \frac{\rho(z)}{j} + \sum_{j=k}^{\infty} \varepsilon_j (\Psi(z) - M_j) > -(1 + \log k)\delta_{k-1} - \varepsilon/4 > -\varepsilon/2$$

by the choice of the δ_j, ε_j. Thus, the function $\psi + \varepsilon/2$ has all the properties asserted in the lemma.

Using this lemma, the following "extension with bounds" result can be proved for psh functions.

Lemma 4.6 *Let D be a hyperconvex domain, h a continuous psh function on D and V an analytic variety in D. If $\varepsilon > 0$ and U is any psh function on D such that $U \leqq h$ on V, then there is a psh function U_1 on D such that $U_1 = U$ on V and $U_1 \leqq h + \varepsilon$ for all $z \in D$.*

Proof. Let $\mathcal{O} = \{z \in D | U(z) < h(z)+\varepsilon/4\}$. From the lemma just proved, there is a psh function ψ on D that is $-\infty$ on V, nonnegative on $D\backslash\mathcal{O}$, and bounded above by $\varepsilon/4$ on D. If we define U_1 as equal to $\max\{U, (h+\varepsilon/4)+(\psi+\varepsilon/4)\}$ on $\mathcal{O}$, then U_1 is equal to $h+\psi+\varepsilon/2$ near $\partial\mathcal{O} \cap D$. Hence, we can extend U_1 to a psh function on all of D by setting it equal to $h+\psi+\varepsilon/2$ on $D\backslash\mathcal{O}$. We clearly have $U_1 = U$ on $V \subset \mathcal{O}$ since $\psi = -\infty$ on V. Also, $U_1 < h+\varepsilon$ since $\psi < \varepsilon/2$.

Lemma 4.7 *Let $(V_j)_{j\in\mathbb{N}}$ be a sequence of pure k-dimensional analytic varieties in an open set $D \subset \mathbb{C}^n$ that converge to an analytic variety V in the sense of Definition 4.3. If K is a compact subset of D and $\mathcal{O}$ is an open neighborhood of $K \cap V$ in D, then $K \cap V_j \subset \mathcal{O}$ for sufficiently large j.*

Proof. Assume that $K \cap V_j$ is not contained in $\mathcal{O}$ for infinitely many $j \in \mathbb{N}$. By passing to a subsequence we can then assume that there exists a sequence $(z_j)_{j\in\mathbb{N}}$ converging to $z_0 \in D$ and satisfying $z_j \in (K \cap V_j)\backslash\mathcal{O}$ for all $j \in \mathbb{N}$. Since $(V_j)_{j\in\mathbb{N}}$ converges to V, and since K is compact, we conclude $z_0 \in K \cap V \subset \mathcal{O}$, so it must be the case that $z_j \in \mathcal{O}$ for all large $j \in \mathbb{N}$, contradicting our choice of z_j.

Lemma 4.8 *Let D be a hyperconvex domain in $\mathbb{C}^n$, h a continuous psh function on D, and U a psh function on D such that $U \leqq h-\delta$ on D for some $\delta > 0$. Then there is a sequence of continuous psh functions U_j on D such that $U_1 \geqq U_2 \geqq \cdots \to U$ on D and $U_1 < h$ on D.*

Proof. Let $\chi(z) = \chi(|z|)$ be a smoothing kernel; i.e. an infinitely differentiable, nonnegative function with support in $\{|z| \leqq 1\}$ and integral equal to 1. With $\chi_\varepsilon(z) = \varepsilon^{-2n}\chi(z/\varepsilon)$ the usual approximation of the δ-function, the functions $U_\varepsilon = U * \chi_\varepsilon$ are psh and smooth on the set D_ε of points in D whose distance to the boundary of D is greater than ε, and $U_\varepsilon \searrow U$ as $\varepsilon \searrow 0$. Let ρ be a continuous psh defining function for D with $\rho = 0$ on ∂D. Let $K_j = \{z \in D | \rho(z) \leqq -\delta/2j\}$. Since K_j is a compact subset of D and h is continuous, it follows from Dini's theorem that there is a number $\varepsilon = \varepsilon_j > 0$ so small that $D_\varepsilon \supset K_j$ and $U_\varepsilon < h - 3\delta/4$ for $z \in K_j$. Then for $z \in K_j$ near ∂K_j, we have $U_\varepsilon < h-3\delta/4 < h+j\rho$. Consequently, the function U_j defined on D by setting $U_j = h+j\rho$ on $D\backslash K_j$ and $U_j = \max\{U_\varepsilon, h+j\rho\}$ on K_j is psh and continuous on D. We clearly have $U_j < h$ on D. Moreover, since for $\varepsilon = \varepsilon_{j+1} < \varepsilon_j$ we have $U_\varepsilon < h - 3\delta/4 < h + j\rho$ on $K_{j+1}\backslash K_j$, the sequence U_j is decreasing in j. This completes the proof.

We can now give the proof of Theorem 4.4.

Proof. Let z_0 be a regular point of the pure k-dimensional variety V and $\eta > 0$. Let u be a psh function on V with $u \leqq 0$ on $K \cap V$, $u \leqq h$ on V, and $u(z_0) > U_K(z_0; h, V, D) - \eta$. Since h, D are fixed throughout the argument, we will drop the h and D from the notation and write the extremal functions as $U_K(z; V)$. By Lemma 4.5, there exists a negative psh function v on D that is equal to $-\infty$ exactly on the singular set of V. The function $u + \eta v$ is therefore psh at all the regular points of V and tends to $-\infty$ on the singular set of V. Therefore, by a Theorem of Fornaess and Narasimhan [8], it has an extension to a psh function on all of D (for the case of the theorem needed here, there is a simpler proof). By Lemma 4.6, there is a psh function U_η on D such that $U_\eta = u + \eta v$ on V and $U_\eta < h + \eta$ on all of D. Since $u \leqq 0$ on $K \cap V$, we have that $U_\eta \leqq 0$ on $K \cap V$. By Lemma 4.8, there exists a larger continuous psh function on D, again denoted U_η, such that $U_\eta < h + 2\eta$ and $U_\eta < \eta$ on an open set $\mathcal{O}$ containing $K \cap V$. By Lemma 4.7, the sets $K \cap V_j$ are contained in $\mathcal{O}$ for sufficiently large j. Therefore, the functions $U_\eta - 2\eta$ are competitors in the supremum defining the extremal functions $U_K(z; V_j)$ for large j. Consequently, if $z_j \in V_j$ converges to $z_0 \in V$ then because U_η is continuous

$$\begin{aligned}\liminf_{j\to\infty} U_K(z_j; V_j) &\geqq \liminf_{j\to\infty} U_\eta(z_j) - 2\eta \geqq U_\eta(z_0) - 2\eta \\ &\geqq u(z_0) + \eta v(z_0) - 2\eta \geqq U_K(z_0; V) + \eta v(z_0) - 3\eta\,.\end{aligned}$$

Since $\eta > 0$ is arbitrary and $v(z_0) > -\infty$, the assertion of the Theorem follows.

5 Nonhomogeneous varieties

In this section, we show that the dimension condition holds for the variety tangent to V at infinity if and only if V satisfies the condition RPL. If V is an algebraic variety, recall that V_h, the homogeneous variety tangent to V at infinity is defined to be the set of all points of the form ζz where $\zeta \in \mathbb{C}$ and there exist points $z_j \in V$ such that $|z_j| \to \infty$ and $z = \lim_{j\to\infty} z_j/|z_j|$. Equivalently, if $\mathscr{I}(V)$ is the ideal of all polynomials that vanish on V, then V_h is the variety of common zeros of the highest degree homogeneous terms of these polynomials. We refer to the books of Chirka [7] or Whitney [22] for more details about tangent varieties.

Theorem 5.1 *Let V be a pure dimensional algebraic variety in $\mathbb{C}^n$. Then the following are equivalent*:

(i) *V satisfies the condition RPL.*
(ii) *V_h satisfies the condition RPL.*
(iii) *V_h satisfies the dimension condition.*

For the proof of Theorem 5.1, we need some lemmas.

Lemma 5.2 *Suppose V is an algebraic variety in $\mathbb{C}^n$. Let $V_j = \{z/j : z \in V\}$. Then the sequence of algebraic varieties V_j converges to V_h in $\mathbb{C}^n$, in the sense of Definition* 4.3.

Proof. It is easy to verify the second part of Definition 4.3. The first part is less clear, but we omit the proof since it is well-known. It can be easily proved using the good coordinates for V and V_h introduced later in this section. A result stronger than the lemma is given in [22], Chapter 7, Section 3, which is a local version of the lemma. An analogous proof can also be used for the global case considered here.

Lemma 5.3 *Suppose V is an algebraic variety that satisfies the condition RPL, that $D = \{z \in \mathbb{C}^n : |z| < 2\}$, and that $V_j = \{z \in D : jz \in V\}$. For $\varepsilon > 0$, let*

$$K_\varepsilon = \{|z| \leqq 1 : |\operatorname{Im} z| \leqq \varepsilon|z| \text{ or } |z| \leqq \varepsilon\}.$$

Then the local extremal function $U_{K_\varepsilon}(z; V_j) = U_{K_\varepsilon}(z; 1, V_j)$ of K_ε relative to V_j, D, and the psh function 1 on D satisfies the estimate

$$U_{K_\varepsilon}(z; V_j) \leqq A|z| + \frac{B_\varepsilon}{j}, \quad z \in V_j, \quad |z| \leqq \frac{1}{4}, \tag{2}$$

where A is independent of j and ε and B_ε is independent of j.

Proof. Let u be psh on V_j, satisfy $u \leqq 1$ on V_j, and $u \leqq 0$ on K_ε. Let H be the psh function and c the constant from Lemma 2.9. Fix $z_0 \in V_j$ with $|z_0| \leqq 1/4$, and let $r = r(j) = j/2$. Define a psh function U on $V \cap \{|z - \operatorname{Re} jz_0| < r\}$ by

$$U(z) = c \max\left\{ r u(z/j) + \frac{r}{c} H\left(\frac{z - \operatorname{Re} jz_0}{r}\right), \frac{1}{c}|\operatorname{Im} z|\right\}.$$

On the set of $z \in V$ with $|z - \operatorname{Re} jz_0| = r$, we have from $u \leqq 1$ and the estimate (2) for H from Lemma 2.9 that the first term in the maximum does not exceed $r + (r/c)(|\operatorname{Im} z|/r - c) \leqq (1/c)|\operatorname{Im} z|$, so U can be extended to a psh function on all of V by defining it to be equal to $|\operatorname{Im} z|$ outside of this ball. The function U clearly satisfies the estimate (1) of the condition RPL, $U(z) \leqq |z| + o(|z|)$. The estimate (2) of the condition is obvious outside the ball $\{|z - \operatorname{Re} jz_0| < r\}$ with $\rho = 1$. Inside this ball, $U(z) \leqq |\operatorname{Im} z|$ if $z/j \in K_\varepsilon$ since $H(z) \leqq |\operatorname{Im} z|$. Otherwise, $z/j \notin K_\varepsilon$, so

$$U(z) \leqq \frac{cj}{2} + |\operatorname{Im} z| \leqq \frac{c|z|}{2\varepsilon} + |\operatorname{Im} z| \leqq \left(1 + \frac{c}{2\varepsilon^2}\right)|\operatorname{Im} z|,$$

so (2) of the condition RPL holds for U with $\rho = 1 + c(2\varepsilon^2)^{-1}$. Since V satisfies RPL, we conclude that $U(z) \leqq A|z| + B_\varepsilon$ holds for all $z \in V$. Let $z = jz_0$ and use the fact that $H(iy) \geqq 0$ to see that $u(z_0) \leqq (A/c)(j|z_0|/r) + B_\varepsilon/cr$. Consequently, equation (2) holds with constant A equal to $2A/c$, where A is the constant from the condition RPL. This completes the proof of the lemma.

Proof of Theorem 5.1 (i) $\Rightarrow$ (ii) This part is a consequence of the convergence theorems in Section 4. Let D denote the ball $\{|z| < 2\} \subset \mathbb{C}^n$ and let $K = \mathbb{R}^n \cap \{|z| \leqq 1\}$. By Theorem 3.2 it is enough to prove that V_h satisfies $\mathrm{RPL_{loc}}$ at 0; that is, the extremal function $U_K(z) = U_K(z; 1, V, D)$ satisfies

$$(3) \qquad U_K(z) \leqq A|z| \quad \text{for } z \in V_h, \quad |z| \leqq \frac{1}{4}.$$

Define $K_\varepsilon = \{|z| \leqq 1 : |\mathrm{Im}\, z| \leqq \varepsilon|z| \text{ or } |z| \leqq \varepsilon\}$ so that $K = \cap_{\varepsilon>0} K_\varepsilon$. By Proposition 4.2, it suffices to prove that (3) holds with K replaced by K_ε, where the constant A is independent of $\varepsilon > 0$.

To prove this, let V_j denote the variety in $\mathbb{C}^n$ given by $V_j = \{z/j : z \in V\}$. By Lemma 5.2, we have that $V_j \cap D \to V_h \cap D$, in the sense of Definition 4.3. We also have the estimate (2) from Lemma 5.3. Letting $j \to \infty$ in this inequality and applying the convergence theorem, Theorem 4.4, then shows that (3) holds with K replaced by K_ε, as asserted.

Proof of Theorem 5.1 (ii) $\Rightarrow$ (iii) This is a part of Theorem 3.2.

To prove the remaining implication of the theorem, we shall use some special coordinates that are good for the study of V and V_h. The assumption that V is an algebraic variety of pure dimension k in $\mathbb{C}^n$ is continued. Then the variety V_h also has dimension k. Therefore, after a suitable (real linear) change of variables we can choose coordinates $z = (s, w)$ on $\mathbb{C}^n, s \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k$ so that the projection map $\pi : (s, w) \mapsto w$ is a proper map of V and V_h onto $\mathbb{C}^k$. Then there are positive integers m, m' with $m \geqq m'$ such that

$$(4) \qquad \begin{aligned} V &= \{(\alpha_j(w), w) : 1 \leqq j \leqq m\} \\ V_h &= \{(\beta_j(w), w) : 1 \leqq j \leqq m'\} \end{aligned}$$

where the α_j and β_j are locally multiple-valued analytic functions. Moreover we can assume that the coordinates are such that there exists $C > 0$ such that

$$(5) \qquad \begin{aligned} |\alpha_j(w)| &\leqq C(1 + |w|) \\ |\beta_j(w)| &\leqq C|w| \end{aligned}$$

for all $w \in \mathbb{C}^k$. (See, for example, [7], Theorem 2, p. 77). As $|w| \to \infty$, the varieties V and V_h are closer together than $|z|^{1-\varepsilon}$; that is, constants $\varepsilon > 0, C \geqq 1$ exist, depending only on V and the choice of coordinates, such that for each $w \in \mathbb{C}^k$ we have

$$(6) \qquad \begin{aligned} \max_{1 \leqq j \leqq m'} \{ \min_{1 \leqq l \leqq m} |\beta_j(w) - \alpha_l(w)| \} &\leqq C(1 + |w|)^{1-\varepsilon} \\ \max_{1 \leqq l \leqq m} \{ \min_{1 \leqq j \leqq m'} |\beta_j(w) - \alpha_l(w)| \} &\leqq C(1 + |w|)^{1-\varepsilon}. \end{aligned}$$

Also, there is a homogeneous polynomial $D(w)$ of degree d on $\mathbb{C}^k$ such that the branched cover $\pi : V_h \to \mathbb{C}^k$ is an analytic cover over $\{w \in \mathbb{C}^k : D(w) \neq 0\}$ and such that each fiber over $w \in \mathbb{C}^k$ has exactly m' distinct points. Since V_h

is a homogeneous variety, it therefore follows that when $D(w) \neq 0$, the distinct points $\beta_j(\zeta w) - \beta_l(\zeta w) = \zeta(\beta_j(w) - \beta_l(w))$ are further apart than a small constant times $|\zeta w|$. In particular, for each $\eta > 0$ there exists $R_\eta > 0$ such that for each $w \in \mathbb{C}^k$ with $|w| = 1$ and $|D(w)| \geqq \eta$ there exist positive integers $\mu_j, 1 \leqq j \leqq m'$ so that for all $R > R_\eta$ and each $1 \leqq j \leqq m'$ there are exactly μ_j of the $\alpha_l(w)$ satisfying

$$(7) \qquad |R\beta_j(w) - \alpha_l(Rw)| \leqq C(1 + R|w|)^{1-\varepsilon}.$$

Lemma 5.4 *There exists $\eta > 0$ small enough, such that, given any $w_0 \in \mathbb{C}^k$, there exist $\lambda \in \mathbb{C}^k$ with $|\lambda| = 1$ and a number $r, 0 < r < \max\{|w_0|, 1\}$, such that the analytic disk*

$$\Delta = \Delta(w_0, \lambda, r) = \{w_0 + \zeta\lambda : \zeta \in \mathbb{C}, |\zeta| \leqq r\}$$

has its boundary $\partial\Delta$ contained in the set $|D(w)| \geqq \eta|w|^d$.

Proof. This is a well known consequence of a minimum modulus theorem for polynomials (e.g. [4], Lemma 3.4.1), and the homogeneity of $|D(w)|$.

The basic estimate for the proof of the last part of the theorem is given in the next lemma. It is like Lemma 3.1, except that an exceptional set is allowed and a stronger upper bound is required. The idea is that psh functions must have the same rate of growth in all but a small exceptional set of directions. The proof is similar to that of Theorems 1.41 and 1.44 in Chapter 1 of [9]. See also Section 11 of [20], which contains the proof of Lemma 3.1 due to Siciak. After the proof, we will also point out how the same argument proves Lemma 3.1, the Sibony–Wong estimate. A different proof can be given using Kiselman's theorem on envelopes of psh functions [11].

Lemma 5.5 *Let $v : V_h \to [0, \infty[$ be usc. Assume that for some $\eta_0 > 0$ and $\rho > \eta_0^{-1/d}$ the following conditions are satisfied:*

(i) *v is psh on the set*

$$G := \{z = (s, w) \in V_h : |D(w)| > |w/\rho|^d\}.$$

(ii) *$v(z) \leqq |z|$ for all z in a set E of complex lines in V_h passing through the origin such that the union of all these lines is not a (locally) pluripolar set in V_h.*

(iii) *$v(z) \leqq \rho|z|$ for all $z \in V_h$ and some $\rho > 1$.*

Then there exists a constant $A \geqq 1$, depending only on V_h, η_0 and the set E of complex lines in (ii) *but independent of ρ and v, such that*

$$v(z) \leqq A|z| \quad \text{for all } z = (s, w) \in V_h \text{ with } |D(w)| \geqq \eta_0|w|^d.$$

Proof. In proving the lemma, it is no loss of generality to assume that $v(\zeta z) = v(|\zeta|z)$ for all complex numbers ζ, that $t \to v(tz)$ is nondecreasing for $t \geqq 0$, and that $v(z) \geqq |z|$; otherwise, replace v by $\max_{|\tau| \leqq |1|}\{v(\tau z), |z|\}$. Then for $m > v(0) = 0$ and $z \in V_h$, define the auxillary functions

$$r(z) = r(z,m) = \sup\{t \geqq 0 : v(tz) < m\}$$

$$\psi(z) = \psi(z,m) = \log \frac{1}{r(z,m)} .$$

The function $z \mapsto \psi(z)$ has the following properties:

(a) ψ is usc and $\psi(\zeta z) = \psi(z) + \log|\zeta|$;
(b) $\psi(z) = \log|z| + \log \frac{1}{m}, z \in l$, l a complex line in E;
(c) $\frac{m}{\rho} \leqq r(z)|z|, \quad z \in V_h$;
(d) $\psi(z) \leqq \log|z| + \log \frac{\rho}{m}$, $z \in V_h$;
(e) if $z_0 = (s_0, w_0)$ is a point of G, then $z \mapsto \psi(z,m)$ is psh on a neighborhood of z_0.

These properties are all direct consequences of the definitions of $r(z)$ and $\psi(z)$. For example, it is easy to check that ψ is usc, since v is usc, and that it has the homogeneity property of (a). The estimates of (b) and (c) are direct consequences of the estimates of hypotheses (ii) and (iii) of the lemma, together with the lower bound $v(z) \geqq |z|$. Part (d) is just a reformulation of (c). To see that (e) holds, let $(\beta_j(w), w)$ be the branch of V_h satisfying $z_0 = (\beta_j(w_0), w_0)$ and consider the function

$$\varphi(w,\zeta) := v(\zeta(\beta_j(w), w)), \quad |w - w_0| < \delta, \quad \zeta \in \mathbb{C} .$$

For δ sufficiently small, φ is psh on $\{(w,\zeta) : |w - w_0| < \delta, \zeta \neq 0\}$. Since φ is locally bounded from above, it has a psh extension to $\{(w,\zeta) : |w - w_0| < \delta, \zeta \in \mathbb{C}\}$ by Hörmander [10], 4.4. Hence, an application of Thm. I. 28 in Gruman and Lelong [9] shows that $\psi((\beta_j(w), w), m) = -\log \delta(w,m)$ is plurisubharmonic in a neighborhood of w_0. Obviously, this implies (e).

Next, let E^* denote the set of points $z \in V_h$ which satisfy $|z| \leqq 1$ and which belong to some complex line in E. Then E^* is not a pluripolar subset of V_h. Let $L(z)$ denote the extremal psh function of minimal growth for the set E^*,

$$L(z) = \sup\{u(z) : u \text{ psh on } V_h, u \leqq 0 \text{ on } E^*, \quad u(z) \leqq \log(1 + |z|) + O(1)\} ,$$

and let L^* be the usc regularization of L. By a theorem of Zeriahi [23] Thm. 4.2, extending a result of Siciak for the case when $V_h = \mathbb{C}^n$, the function L^* is also a psh function of minimal growth on V_h; that is, there exists a constant $\gamma > 0$ such that

$$L^*(z) \leqq \log^+|z| + \gamma, \quad z \in V_h$$

where $\log^+|z| = \max\{\log|z|, 0\}$. There is one technical point to note in the application of Zeriahi's theorem. He uses a different definition of psh function on V than the one in Definition 2.1. However, using the fact that there is a psh function v on $\mathbb{C}^n$ of logarithmic growth that is equal to $-\infty$ exactly on the singular subset of V_h, it is easy to check that the extremal function L^* is independent of the class of psh functions used in its definition.

Finally, define

$$U(z) = U(z,m) = \max\left\{\frac{1}{2}\psi(z,m) + \frac{1}{2d}\log|D(w)|,\ \log|z| + \frac{1}{2}\log\frac{1}{m}\right\}.$$

For $z = (s,w) \notin G$ we get from (d) and $|w| \leqq |z|$,

$$\tfrac{1}{2}\psi(z) + \tfrac{1}{2d}\log|D(w)| \leqq \tfrac{1}{2}\left[\log|z| + \log\tfrac{\rho}{m} + \tfrac{1}{d}\log|D(w)|\right]$$

$$\leqq \tfrac{1}{2}\left[\log|z| + \log\tfrac{\rho}{m} + \log\tfrac{|z|}{\rho}\right] = \log|z| + \tfrac{1}{2}\log\tfrac{1}{m}.$$

Therefore, the first term in the maximum defining U is dominated by the second one whenever $|D(w)| \leqq (|w|/\rho)^d$. Hence, we obtain from (e) that U is psh on V_h. Moreover, we get from (d) that U is bounded by $\log|z| + O(1)$ on V_h. Since we may assume that $|D(w)| \leqq |w|^d$ holds for all $w \in \mathbb{C}^k$, we get from (b):

$$\frac{1}{2}\psi(z) + \frac{1}{2d}\log|D(w)| \leqq \log|z| + \frac{1}{2}\log\frac{1}{m} \quad \text{for all } z \in E^*.$$

Hence $U - \frac{1}{2}\log\frac{1}{m}$ is a competitor in the definition of the extremal function L. Consequently, we have

$$U(z) \leqq L(z) + \frac{1}{2}\log\frac{1}{m} \leqq \log^+|z| + \gamma + \frac{1}{2}\log\frac{1}{m} \quad \text{for all } z \in V_h.$$

Evaluating this at $|z| = 1$ we get

$$\psi(z) \leqq 2\gamma + \log\frac{1}{m} - \frac{1}{d}\log|D(w)|, \quad z \in V_h, \quad |z| = 1.$$

Now, from (5) there is a constant $C_0 > 0$ such that $|w| \geqq 1/C_0$ whenever $z = (s,w) \in V_h$ with $|z| = 1$. Therefore, the last inequality can be rewritten as

$$\psi(z) \leqq 2\gamma + \log\frac{1}{m} + \log C_0 + \frac{1}{d}\log\frac{|w|^d}{|D(w)|}.$$

The homogeneity property (a) then implies

$$\psi(z) = \psi\left(|z|\frac{z}{|z|}\right) \leqq \log|z| + 2\gamma + \log\frac{C_0}{m} + \frac{1}{d}\log\frac{|w|^d}{|D(w)|}, \quad z = (s,w) \in V_h.$$

Exponentiate the last inequality, write ψ in terms of $r(z)$, and multiply by $mr(z)$ to obtain

$$m \leqq C_0 e^{2\gamma}(r(z)|z|)\left(\frac{|w|}{|D(w)|^{1/d}}\right) \quad \text{for all } z = (s,w) \in V_h.$$

Next fix $z = (s, w) \in V_h$ with $v(z) > 0 = v(0)$ and fix $0 < \varepsilon < v(z)$ arbitrarily. If we let $m := v(z) - \varepsilon$, the previous estimate gives

$$v(z) - \varepsilon \leqq C_0 e^{2\gamma} |z| \left(\frac{|w|}{|D(w)|^{1/d}} \right) .$$

Since $\varepsilon > 0$ can be chosen to be arbitrarily small, this implies

$$v(z) \leqq \frac{C_0 e^{2\gamma}}{\eta_0^{1/d}} |z|$$

for all $z = (s, w) \in V_h$ with $|D(w)| \geqq \eta_0 |w|^d$, which completes the proof.

Remark. The same argument also proves Lemma 3.1. For, in this case, if $m > m_0 = \limsup_{z \to 0} v(z)$, then the function $\psi(z)$ is always psh on V_h and satisfies $\psi(z) \leqq \log|z| + O(1)$. Hence, without using the hypothesis (iii), we conclude, again from the theorem of Zeriahi cited before, that

$$\psi(z) \leqq L^*(z) + \log\frac{1}{m} \leqq \log^+|z| + \gamma + \log\frac{1}{m} .$$

Consequently, we have $\psi(z) \leqq \gamma + \log\frac{1}{m}$ when $|z| = 1$, so by the homogeneity property of ψ in (a),

$$\psi(z) \leqq \log|z| + \gamma + \log\frac{1}{m} .$$

Exactly as in the proof of the lemma, this implies

$$v(z) \leqq \max\{m_0, e^{\gamma}|z|\}, \quad z \in V_h .$$

If v is replaced by

$$v_\varepsilon(z) = \max\left\{\frac{1}{1+\varepsilon}(v(z) + \varepsilon \log|z|), 0\right\}$$

then the conclusion also applies to v_ε which has $m_0 = 0$. Thus, $v_\varepsilon(z) \leqq e^{\gamma}|z|$ and, letting $\varepsilon \to 0$, we conclude that $v(z) \leqq e^{\gamma}|z|$.

Returning now to the special coordinates $z = (s, w)$, we can then associate to a psh function u on V a function v on V_h which is psh at most points of V_h, by

(8) $$v(\beta_j(w), w) = \max\{u(\alpha_l(w), w) : |\alpha_l(w) - \beta_j(w)| \leqq C(1 + |w|)^{1-\varepsilon}\}$$

where C, ε are the constants in (6). This function was used by Hörmander [10] in his proof that the Phragmén–Lindelöf condition he studied carries over to nearby varieties.

Lemma 5.6 *Let u be a psh function on V and let v be the function on V_h just defined. Then*

(i) *If $u(z) \leqq |z| + o(|z|)$, $z \in V$, then $v(z) \leqq |z| + o(|z|)$, $z \in V_h$;*
(ii) *If $u(z) \leqq \rho|\mathrm{Im}\, z|$, $z \in V$, then $v(z) \leqq \rho(|\mathrm{Im}\, z| + C(1 + |z|)^{1-\varepsilon})$, $z \in V_h$;*

(iii) *For each* $\eta > 0$, *there is a constant* $R = R_\eta > 0$ *such that* v *is psh on* $V_h \cap \Gamma(\eta, R)$, *where* $\Gamma(\eta, R)$ *is the cone with truncated tip,*

$$\Gamma(\eta, R) = \{z = (s, w) : |D(w)| > \eta |w|^d, |z| > R\} .$$

Proof. The estimates of (i) and (ii) follow directly from the first inequality of (6). Part (iii) is a consequence of (7), since then $v(\beta_j(w), w)$ is locally the maximum of the μ_j psh functions $u(\alpha_l(w), w)$.

Lemma 5.7 *Let* $\delta > 0$. *Then there exists a function* φ *psh on* $\mathbb{C}^n$ *and a constant* $C_1 \geqq 0$ *such that* $\varphi(0) = 0$ *and*

$$-C_1|z|^{1-\varepsilon} \leqq \varphi(z) \leqq \delta|\operatorname{Im} z| + C_1 - |z|^{1-\varepsilon} .$$

Proof. This is well-known, even in a more general form; see e.g. [5]. For the case at hand and one variable, one can take the function φ given by the formula

$$h(z) = \operatorname{Re}(-iz)^a = r^a \cos(a(\theta - \frac{\pi}{2})), \quad z = re^{i\theta}$$
$$\varphi(z) = \varphi(x + iy) = \delta|y| - h(x + i(\eta + |y|)) .$$

When $a = 1 - \varepsilon < 1$, the function $\varphi(z) - \varphi(0)$ satisfies the estimates of the lemma. It is harmonic off the real axis and easily checked that the jump in the normal derivative across the x-axis is positive provided $\eta > 0$ is large enough. Consequently, it is subharmonic in $\mathbb{C}$. For n variables, one can take a function of the form $\varphi(z_1) + \ldots + \varphi(z_n)$.

Proof of Theorem 5.1, (iii) $\Rightarrow$ *(i)* Let u be psh on V and satisfy the estimates (1) and (2) of the condition RPL with $\rho > 1$. To prove the existence of constants A, depending only on V and B_ρ, depending only on V and ρ but not on u, such that u satisfies (3) of RPL, we first fix a number $\eta_1 > 0$ so that the conclusion of Lemma 5.4 holds for all $0 < \eta \leqq \eta_1$. Next we let E denote the set of all complex lines of the form

$$l(x) = \{\zeta x : \zeta \in \mathbb{C}, x \in V_h \cap \mathbb{R}^n\} .$$

E is not a pluripolar set of lines, since $V_h \cap \mathbb{R}^n$ is not pluripolar in V_h by Proposition 2.7 and the present hypothesis. Consequently, the set E_η of these complex lines that also lie inside the set $|D(w)| \geqq \eta|w|^d$ is also not a pluripolar set provided $\eta > 0$ is sufficiently small, because the union of the sets $V_h \cap \{|D(w)| \geqq \eta|w|^d\}$ is equal to V_h, except for a pluripolar subset of V_h. We can therefore choose $0 < \eta_0 < \eta_1$, depending on V_h so small that E_{η_0} is not pluripolar.

Since it suffices to prove the desired conclusion for all sufficiently large $\rho > 0$, we can assume that $\rho > \rho_0 = \eta_0^{-1/d}$. To do this, consider the function v defined on V_h in terms of u in (8). Let $\varphi(z)$ be a psh function on $\mathbb{C}^n$ obtained by choosing C as in (ii) of Lemma 5.6 and then $\delta = 1/C\rho$ in Lemma 5.7. By the estimates of Lemma 5.6, there is a constant $C(\rho) > 0$, depending on ρ but independent of u and v, such that the function v_2 defined by

$$v_1(z) = \tfrac{1}{2}\{v(z) + \rho C\varphi(z) - C(\rho)\}$$

$$v_2(z) = \max\{v_1(z), 0\}$$

then satisfies the estimates

$$\text{(9)} \qquad \begin{aligned} v_2(z) &\leqq |z| + o(|z|), \quad z \in V_h \\ v_2(z) &\leqq \rho|\mathrm{Im}\, z|, \quad z \in V_h. \end{aligned}$$

If we choose $C(\rho)$, the constant in the definition of v_1, sufficiently large, then we can assume that

$$v_1(z) \leqq 0 \quad \text{for } |z| \leqq R_{\rho-d} .$$

Hence 5.6 (iii) implies that v_2 is psh on the set

$$V_h \cap \{z = (s, w) : |D(w)| > |w/\rho|^d\} .$$

Since $\rho^{-d} < \eta_0$, this implies that for each $z \in E_{\eta_0}$ the function $\zeta \to v_2(\zeta z)$ is a subharmonic function of $\zeta \in \mathbb{C}$. Therefore, from the classical Phragmén–Lindelöf principle and the estimates (9),

$$v_2(z) \leqq |z|, \quad z \in l(x) \in E_{\eta_0} .$$

In consequence, Lemma 5.5 implies that there is a constant A depending only on V_h and η_0 such that

$$v_2(z) = A|z|, \quad z = (s, w) \in V_h, \quad |D(w)| > \eta_0|w|^d .$$

From the definition of v_2 in terms of v and φ, and the lower bound for φ from Lemma 5.7, we deduce

$$\text{(10)} \qquad v(z) \leqq A|z| + B, \quad z = (s, w) \in V_h, \quad |D(w)| > \eta_0|w|^d$$

for a possibly larger constant A and a constant B depending also on ρ.

Next we consider the psh function U, defined on $\mathbb{C}^k$ by

$$U(w) = \max\{u(\alpha_j(w), w) : 1 \leqq j \leqq m\} .$$

From (5) and (10) we get a constant $C_\rho = C_\rho(B, A)$ such that

$$U(w) \leqq AC|w| + C_\rho, \quad w \in \mathbb{C}^k, \quad |D(w)| > \eta_0|w|^d .$$

To derive a similar estimate for U at all points in $\mathbb{C}^k$, we apply Lemma 5.4. It implies that for each $w \in \mathbb{C}^k, |w| \geqq 1$, there is a one-dimensional disk of radius not exceeding $|w|$ and whose boundary lies in the set where $|D(w)| \geqq \eta_1 |w|^d > \eta_0 |w|^d$. Hence the maximum principle implies that the same estimate holds for all $w \in \mathbb{C}^k$, if we replace AC by $2AC$ and C_ρ by some larger constant B_ρ. That is, the estimate (3) of the condition RPL holds for all $z \in V$.

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