TOPICS ON PROJECTIVE SPECTRA OF (LB)–SPACES

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Abstract. The paper first gives a new introduction to Palamodov's theory of the projective limit functor avoiding categorical and abstract homological concepts. Then Retakh's condition for $\operatorname{Proj}^1 \mathcal{X} = 0$ for a spectrum $\mathcal{X}$ of (LB)–spaces is discussed. Conditions are derived which are accessible for evaluation. In §3 these conditions are connected to certain topological properties of the projective limit and finally the case of sequence spaces is presented, where we have a complete characterization in terms of the defining matrices.

Introduction.

The present paper, based on the author's lectures during the conference and on [13], first gives a brief introduction into the theory of the projective limit functor as developed by Palamodov in [9] and [10]. We do not use tools of abstract homological algebra and category theory, however define directly and explicitly Proj^0 and Proj^1 and obtain in a very natural way Palamodov's exact sequence of six spaces ([9, p. 542]). Then we turn to projective limits of regular (LB)–spaces. We present Retakh's result ([11, Theorem 3]) and derive from it conditions (P_1^*) and (P_2^*), which are estimates in terms of the dual norms of a given setting and can be used (see [1], [2], [3]) for the solution of solvability problems in analysis.

The §3 essentially gives a survey on results of [13] which connect the conditions for $\operatorname{Proj}^1 \mathcal{X} = 0$ to topological properties of the projective limit as barrelledness, reflexivity, etc.. In §4 we present a result of [13], which extends results of [7], and gives in the case of sequence spaces a necessary and sufficient condition for $\operatorname{Proj}^1 \mathcal{X} = 0$ and for a lot of topological properties of the projective limit and its dual, which in this case are equivalent. This in particular contains results of Grothendieck on topological properties of (F)– and (DF)–spaces ([4, II, §4]).

Finally we discuss how far all this depends on the given spectrum and what exactly has to be shown in the typical application (as eg. [1]).

T. Terzioğlu (ed.), Advances in the Theory of Fréchet Spaces, 11–27.

1. Projective spectra of linear spaces

In the following projective or injective spectrum will always mean countable projective or injective spectrum of linear spaces.

A projective spectrum is a sequence $\mathcal{X} = (X_n, \iota^n_{n+1})$ of linear spaces X_n and linear maps $\iota^n_{n+1} : X_{n+1} \to X_n$. We put $\iota^m_m = \mathrm{id}_{X_m}$ and $\iota^n_m = \iota^n_{n+1} \circ \cdots \circ \iota^{m-1}_m$ for all $m, n < m$.

If $\mathcal{X} = (X_n, \iota^n_{n+1})$ and $\mathcal{Y} = (Y_n, \iota^n_{n+1})$ are projective spectra, then a map $\Phi : \mathcal{X} \to \mathcal{Y}$ is a sequence $\varphi^n_{k(n)} : X_{k(n)} \to Y_n$ of linear maps, where $k(n) \leq k(n+1)$ and $\varphi^n_{k(n)} \circ \iota^{k(n)}_{k(n+1)} = \iota^n_{n+1} \circ \varphi^{n+1}_{k(n+1)}$ for all n. We put $\varphi^n_m = \varphi^n_{k(n)} \circ \iota^{k(n)}_m$ for all $k(n) \leq m$ and obtain linear maps $X_m \to Y_n$ satisfying $\varphi^n_m \circ \iota^m_M = \iota^n_\mu \circ \varphi^\mu_M$ whenever all maps are defined.

Two maps $\Phi = \left(\varphi^n_{k(n)}\right)$ and $\tilde{\Phi} = \left(\tilde{\varphi}^n_{l(n)}\right)$ are called equivalent $\left(\Phi \sim \tilde{\Phi}\right)$ if for every n there is $m(n) \geq \max(k(n), l(n))$ such that $\varphi^n_{m(n)} = \tilde{\varphi}^n_{m(n)}$ for all n. This defines an equivalence relation.

For two maps $\Phi : \mathcal{X} \to \mathcal{Y}, \Psi : \mathcal{Y} \to \mathcal{Z}$ the composition $\Psi \circ \Phi : \mathcal{X} \to \mathcal{Z}$ is defined by $\left(\psi^n_{l(n)} \circ \varphi^{l(n)}_{k(l(n))}\right)_n$, where $\Phi = \left(\varphi^n_{k(n)}\right)_n$, $\Psi = \left(\psi^n_{l(n)}\right)_n$. Composition respects equivalence. A map $\Phi : \mathcal{X} \to \mathcal{Y}$ is called an equivalence map if there is a map $\Phi^{-1} : \mathcal{Y} \to \mathcal{X}$ such that $\Phi^{-1} \circ \Phi \sim \mathrm{id}_{\mathcal{X}}$, $\Phi \circ \Phi^{-1} \sim \mathrm{id}_{\mathcal{Y}}$. $\mathcal{X}$ and $\mathcal{Y}$ are called equivalent if there is an equivalence map $\mathcal{X} \to \mathcal{Y}$ (resp. $\mathcal{Y} \to \mathcal{X}$).

Example. Let $I = \left(\iota^n_{k(n)}\right)_n$, then $I \sim \mathrm{id}$. Therefore $\left(X_{k(n)}, \iota^{k(n)}_{k(n+1)}\right)_n \sim \mathcal{X}$. $\Phi \sim \tilde{\Phi}$ if and only if there exists $\tilde{I}$ such that $\Phi \circ I = \tilde{\Phi} \circ \tilde{I}$.

Definition. For $\mathcal{X} = (X_n, \iota^n_{n+1})_n$ we set

$$\begin{aligned} \mathrm{Proj}^0\, \mathcal{X} &= \left\{ (x_n)_n \in \prod_n X_n : \iota^n_{n+1} x_{n+1} = x_n \text{ for all } n \right\} \\ \mathrm{Proj}^1\, \mathcal{X} &= \left(\prod_n X_n \right) / \mathrm{B}(\mathcal{X}) \end{aligned}$$

where

$$\mathrm{B}(\mathcal{X}) = \left\{ (a_n)_n \in \prod_n X_n : \text{there exists } (b_n)_n \in \prod_n X_n, \right. \\ \left. \text{such that } a_n = \iota^n_{n+1} b_{n+1} - b_n \text{ for all } n \right\}.$$

There is a natural exact sequence

$$0 \to \mathrm{Proj}^0\, \mathcal{X} \hookrightarrow \prod_n X_n \xrightarrow{\sigma} \prod_n X_n \xrightarrow{q} \mathrm{Proj}^1\, \mathcal{X} \to 0$$

where $\sigma : (x_n)_n \mapsto \left(\iota^n_{n+1} x_{n+1} - x_n\right)_n$ and q is the quotient map.

For $\Phi : \mathcal{X} \to \mathcal{Y}$, $\Phi = \left(\varphi^n_{k(n)}\right)_n$ we define

$$\begin{aligned} {}^\pi\Phi\, x &= \left(\varphi^n_{k(n)} x_{k(n)}\right)_n \\ \Phi^\pi x &= \left(\sum_{m=k(n)}^{k(n+1)-1} \varphi^n_m x_m\right)_n \end{aligned}$$

These are linear maps from $\prod_n X_n$ to $\prod_n Y_n$, satisfying $\Phi^\pi \circ \sigma = \sigma \circ {}^\pi\Phi$. This means we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Proj}^0 \mathcal{X} & \to & \prod_n X_n & \xrightarrow{\sigma} & \prod_n X_n & \to & \mathrm{Proj}^1 \mathcal{X} & \to 0 \\ & \Phi^0 \downarrow & & {}^\pi\Phi \downarrow & & \downarrow \Phi^\pi & & \downarrow \Phi^1 & \\ 0 \to & \mathrm{Proj}^0 \mathcal{Y} & \to & \prod_n Y_n & \xrightarrow{\sigma} & \prod_n Y_n & \to & \mathrm{Proj}^1 \mathcal{Y} & \to 0 \end{array}$$

where Φ^0 and Φ^1 are the maps induced by ${}^\pi\Phi$ and Φ^π. We used the following definition:

Definition. (1) For $x = (x_n)_n \in \mathrm{Proj}^0 \mathcal{X}$ we set

$$\Phi^0 x = \left(\varphi^n_{k(n)} x_{k(n)}\right)_n$$

(2) For $a = (a_n)_n + \mathrm{B}(\mathcal{X}) \in \mathrm{Proj}^1 \mathcal{X}$ we set

$$\Phi^1 a = \left(\sum_{m=k(n)}^{k(n+1)-1} \varphi^n_m a_m\right)_n + \mathrm{B}(\mathcal{Y}).$$

We have the following:

1.1 Proposition. *Let Φ, $\tilde{\Phi} : \mathcal{X} \to \mathcal{Y}$, $\Psi : \mathcal{Y} \to \mathcal{Z}$ be maps, $I : \mathcal{X} \to \mathcal{X}$ as above. Then*
(1) $(\Psi \circ \Phi)^0 = \Psi^0 \circ \Phi^0$, $(\Psi \circ \Phi)^1 = \Psi^1 \circ \Phi^1$
(2) $I^0 = \mathrm{id}_{\mathrm{Proj}^0 \mathcal{X}}$, $I^1 = \mathrm{id}_{\mathrm{Proj}^1 \mathcal{X}}$
(3) *if* $\Phi \sim \tilde{\Phi}$ *then* $\Phi^0 = \tilde{\Phi}^0$, $\Phi^1 = \tilde{\Phi}^1$.

The Proof of (1) is obvious. (2) requires some calculation. (3) is an immediate consequence of (1) and (2) since $\Phi \sim \tilde{\Phi}$ means the existence of I and $\tilde{I}$ such that $\Phi \circ I = \tilde{\Phi} \circ \tilde{I}$.

1.2 Corollary. *If $\mathcal{X} \sim \mathcal{Y}$ and $\Phi : \mathcal{X} \to \mathcal{Y}$ is an equivalence map then*

$$\Phi^0 : \mathrm{Proj}^0 \mathcal{X} \to \mathrm{Proj}^0 \mathcal{Y},\ \Phi^1 : \mathrm{Proj}^1 \mathcal{X} \to \mathrm{Proj}^1 \mathcal{Y}$$

are isomorphisms.

Let $\Phi : \mathcal{X} \to \mathcal{Y}$, $\Psi : \mathcal{Y} \to \mathcal{Z}$ be maps, $\Phi = \left(\varphi^n_{k(n)}\right)_n$, $\Psi = \left(\psi^n_{l(n)}\right)_n$.

Definition. $\mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z}$ is called exact in $\mathcal{Y}$ if

(1) $\Psi \circ \Phi \sim 0$
(2) for every $n, N \geq k(n)$ there are $\mu, m \geq \max(n, l(\mu))$ such that $\mathrm{im}\, \varphi^n_N \supset \iota^n_m \ker \psi^\mu_m$.

It is not difficult to see that this is invariant under equivalence. More precise:

Remark. If

$$\begin{array}{ccccc} \mathcal{X} & \to & \mathcal{Y} & \to & \mathcal{Z} \\ \updownarrow\sim & & \updownarrow\sim & & \updownarrow\sim \\ \tilde{\mathcal{X}} & \to & \tilde{\mathcal{Y}} & \to & \tilde{\mathcal{Z}} \end{array}$$

is a diagram, commutative up to equivalence, and one row is exact, then also the other.

The definition of exactness of $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ means that by just taking subsequences we can find equivalent spectra such that (using the same notation after the change) $\Phi = (\varphi_n^n)_n$, $\Psi = (\psi_n^n)_n$ and

$$\iota_{n+1}^n \ker \psi_{n+1}^{n+1} \subset \operatorname{im} \varphi_n^n \subset \ker \psi_n^n$$

for all n.

1.3 Proposition. *(1) If* $0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ *is exact, then*

$$0 \to \operatorname{Proj}^0 \mathcal{X} \to \operatorname{Proj}^0 \mathcal{Y} \to \operatorname{Proj}^0 \mathcal{Z}$$

is exact.
(2) If $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$ *is exact, then*

$$\operatorname{Proj}^1 \mathcal{X} \to \operatorname{Proj}^1 \mathcal{Y} \to \operatorname{Proj}^1 \mathcal{Z} \to 0$$

is exact.

Let $0 \to \mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z} \to 0$ be a short exact sequence of spectra. Then by going to equivalent spectra by taking subsequences we first may assume the standardization described before Proposition 1.3. From there it is again easy to see, that the spectra $(\ker \psi_n^n, \iota_{n+1}^n)_n$, $(\operatorname{im} \psi_n^n, \iota_{n+1}^n)_n$ are equivalent to $\mathcal{X}$ (resp. $\mathcal{Z}$). Hence we obtained spectra $\tilde{\mathcal{X}}$, $\tilde{\mathcal{Y}}$, $\tilde{\mathcal{Z}}$ and maps $\tilde{\Phi}$, $\tilde{\Psi}$ such that

$$\text{(ST)} \qquad \begin{array}{ccccccc} 0 \to & \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} & \xrightarrow{\Psi} & \mathcal{Z} & \to 0 \\ & \sim\downarrow \Phi_{\mathcal{X}} & & \sim\downarrow \Phi_{\mathcal{Y}} & & \sim\downarrow \Phi_{\mathcal{Z}} & \\ 0 \to & \tilde{\mathcal{X}} & \xrightarrow{\tilde{\Phi}} & \tilde{\mathcal{Y}} & \xrightarrow{\tilde{\Psi}} & \tilde{\mathcal{Z}} & \to 0 \end{array}$$

commutes up to equivalence and the lower sequence is of the form

$$0 \to \tilde{\mathcal{X}}_n \hookrightarrow \tilde{\mathcal{Y}}_n \to \tilde{\mathcal{Z}}_n \to 0\,,$$

where this sequence is exact, for all n.

A short exact sequence of spectra with these properties is called *exact sequence in standard form*. If we have the diagramm (ST) then $0 \to \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}} \to \tilde{\mathcal{Z}} \to 0$ is called a *standard representation* of $0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$.

For an exact sequence $0 \to \mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z} \to 0$ in standard form we have

$$\begin{aligned} \Phi^0 x &= (\varphi_n^n x_n)_n \\ \Phi^1 a &= (\varphi_n^n a_n)_n + \mathrm{B}(\mathcal{Y}) \end{aligned}$$

and likewise for Ψ.

In this case we define for $z = (z_n)_n \in \mathrm{Proj}^0 \mathcal{Z}$

$$\delta^* z = \left(\iota^n_{n+1} y_{n+1} - y_n\right)_n + \mathrm{B}(\mathcal{X})$$

where $y_n \in Y_n$ is chosen such that $\psi^n_n y_n = z_n$. Note that, because of

$$\psi^n_n \left(\iota^n_{n+1} y_{n+1} - y_n\right) = \iota^n_{n+1} z_{n+1} - z_n = 0,$$

we have $\iota^n_{n+1} y_{n+1} - y_n \in X_n$ for all n. δ^* is a well defined linear map from $\mathrm{Proj}^0 \mathcal{Z}$ to $\mathrm{Proj}^1 \mathcal{X}$.

1.4 Lemma. *If*

$$\begin{array}{ccccccc} 0 \to \mathcal{X}^1 & \to & \mathcal{Y}^1 & \to & \mathcal{Z}^1 & \to 0 \\ \downarrow \Phi_{\mathcal{X}} & & \downarrow \Phi_{\mathcal{Y}} & & \downarrow \Phi_{\mathcal{Z}} & \\ 0 \to \mathcal{X}^2 & \to & \mathcal{Y}^2 & \to & \mathcal{Z}^2 & \to 0 \end{array}$$

is a commutative (up to equivalence) diagram with exact rows in standard form then

$$\begin{array}{ccc} \mathrm{Proj}^0 \mathcal{Z}^1 & \stackrel{\delta^*}{\to} & \mathrm{Proj}^1 \mathcal{X}^1 \\ \downarrow \Phi^0_{\mathcal{Z}} & & \downarrow \Phi^1_{\mathcal{X}} \\ \mathrm{Proj}^0 \mathcal{Z}^2 & \stackrel{\delta^*}{\to} & \mathrm{Proj}^1 \mathcal{X}^2 \end{array}$$

commutes.

From this and Proposition 1.1 (3) we conclude that the following definition is unique.

Definition. For any exact sequence we define $\delta^* : \mathrm{Proj}^0 \mathcal{Z} \to \mathrm{Proj}^1 \mathcal{X}$ by $\delta^* = {\Phi^1_{\mathcal{X}}}^{-1} \circ \delta^* \circ \Phi^0_{\mathcal{Z}}$, where we used a standard representation and (ST).

We obtain and admit without proof (Palamodov [9, p. 542]).

1.5 Theorem. *For every exact sequence*

$$0 \to \mathcal{X} \stackrel{\Phi}{\to} \mathcal{Y} \stackrel{\Psi}{\to} \mathcal{Z} \to 0$$

of spectra the sequence

$$0 \to \mathrm{Proj}^0 \mathcal{X} \stackrel{\Phi^0}{\to} \mathrm{Proj}^0 \mathcal{Y} \stackrel{\Psi^0}{\to} \mathrm{Proj}^0 \mathcal{Z} \stackrel{\delta^*}{\to} \mathrm{Proj}^1 \mathcal{X} \stackrel{\Phi^1}{\to} \mathrm{Proj}^1 \mathcal{Y} \stackrel{\Psi^1}{\to} \mathrm{Proj}^1 \mathcal{Z} \to 0$$

is exact. All maps in this exact sequence depend only on the equivalence class of Φ and Ψ. Moreover this exact sequence depends functorially on the exact sequence of spectra.

The last sentence means the following: If

$$\begin{array}{ccccccc} 0 \to \mathcal{X}^1 & \to & \mathcal{Y}^1 & \to & \mathcal{Z}^1 & \to 0 \\ \downarrow \Phi_{\mathcal{X}} & & \downarrow \Phi_{\mathcal{Y}} & & \downarrow \Phi_{\mathcal{Z}} & \\ 0 \to \mathcal{X}^2 & \to & \mathcal{Y}^2 & \to & \mathcal{Z}^2 & \to 0 \end{array}$$

is a commutative (up to equivalence) diagram with exact rows, then

$$\begin{array}{ccccccccccccc}
0 \to & \mathrm{Proj}^0\,\mathcal{X}^1 & \to & \mathrm{Proj}^0\,\mathcal{Y}^1 & \to & \mathrm{Proj}^0\,\mathcal{Z}^1 & \xrightarrow{\delta^*} & \mathrm{Proj}^1\,\mathcal{X}^1 & \to & \mathrm{Proj}^1\,\mathcal{Y}^1 & \to & \mathrm{Proj}^1\,\mathcal{Z}^1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Proj}^0\,\mathcal{X}^2 & \to & \mathrm{Proj}^0\,\mathcal{Y}^2 & \to & \mathrm{Proj}^0\,\mathcal{Z}^2 & \xrightarrow{\delta^*} & \mathrm{Proj}^1\,\mathcal{X}^2 & \to & \mathrm{Proj}^1\,\mathcal{Y}^2 & \to & \mathrm{Proj}^1\,\mathcal{Z}^2 & \to 0
\end{array}$$

is a commutative diagram with exact rows.

2. Projective spectra of LB–spaces

In this section we assume $\mathcal{X} = (X_n, \iota^n_{n+1})$ to be a spectrum of (LB)–spaces, i.e. every X_n has the form $X_n = \bigcup_k X_{n,k}$, where $X_{n,k}$ is a Banach space with a norm $\| \ \|_{n,k}$ and X_n carries the locally convex inductive limit topology of the $X_{n,k}$. ι^n_{n+1} is assumed to be continuous.

Hence we may assume without loss of generality that $X_{n,k} \subset X_{n,k+1}$ and

$$\| x \|_{n,k} \geq \| x \|_{n,k+1}$$

$$\| \iota^n_{n+1} x \|_{n,k} \leq \| x \|_{n+1,k}$$

for all n, k and $x \in X_{n,k}$ (resp. $x \in X_{n,k+1}$). We put $B_{n,k} = \{x \in X_{n,k} : \| x \|_{n,k} \leq 1\}$ and assume that the $B_{n,k}$, $k = 1, 2, \ldots$ are a fundamental system of bounded sets in X_n. By X (resp. Y, Z) we denote always $\mathrm{Proj}^0\,\mathcal{X}$ (resp. $\mathrm{Proj}^0\,\mathcal{Y}$, $\mathrm{Proj}^0\,\mathcal{Z}$) equipped with the projective topology and by $\iota^n : X \to X_n$ the canonical map.

In this case a necessary and sufficient condition for $\mathrm{Proj}^0\,\mathcal{X} = 0$ has been given by V.S. Retakh [11, Theorem 3].

2.1 Theorem (Retakh). $\mathrm{Proj}^1\,\mathcal{X} = 0$ *if and only if the following holds: For every μ the space X_μ contains a bounded Banach ball B_μ such that*
(1) $\iota^\mu_{\mu+1} B_{\mu+1} \subset B_\mu$ for all μ
(2) for every μ there is $k \geq \mu$ such that $\iota^\mu_k(X_k) \subset \iota^\mu X + B_\mu$.

Remark. Then for every $\varepsilon > 0$ we have even

$$\iota^\mu_k(X_k) \subset \iota^\mu X + \varepsilon B_\mu.$$

We want to put this into a form that makes it ready for evaluation in concrete cases and for closer investigation in general.

2.2 Lemma. *In Theorem 2.1 condition (2) can be replaced by*

(2)' for every μ there is $k \geq \mu$ such that for every $K \geq k$ and $\varepsilon > 0$

$$\iota^\mu_k(X_k) \subset \iota^\mu_K X_K + \varepsilon B_\mu.$$

PROOF. Since (2)⇒(2)′ is clear we have to prove the converse . Given (1), (2)′ we obtain a sequence $k(m)$, $k(0) = \mu$ such that for every m we have

$$\iota^{k(m)}_{k(m+1)} X_{k(m+1)} \subset \iota^{k(m)}_{k(m+2)} X_{k(m+2)} + 2^{-m} B_{k(m)}.$$

We proceed inductively starting with $x = x_1 \in X_{k(1)}$. Let $X_{k(m+1)}$ be chosen. We find $x_{k(m+2)} \in X_{k(m+2)}$ such that

$$\iota^{\nu}_{k(m+1)} x_{k(m+1)} - \iota^{\nu}_{k(m+2)} x_{k(m+2)} \subset 2^{-m} B_\nu$$

for $1 \leq \nu \leq k(m)$.

Then $\lim_{m\to\infty} \iota^{\nu}_{k(m)} x_{k(m)} =: \xi_\nu$ exists for all ν and clearly $\xi = (\xi_\nu)_\nu \in \operatorname{Proj}^0 \mathcal{X}$. We have with $k = k(1)$

$$\begin{aligned} \| \iota^\nu_k x - \iota^\nu \xi \|_\nu &\leq \sum_{m=1}^{\infty} \| \iota^\nu_{k(m)} x_{k(m)} - \iota^\nu_{k(m+1)} x_{k(m+1)} \|_\nu \\ &\leq \sum_{m=1}^{\infty} \| x_{k(m)} - \iota^{k(m)}_{k(m+1)} x_{k(m+1)} \|_{k(m)} \\ &\leq 1 \end{aligned}$$

where $\| \quad \|_\nu$ denotes the Minkowski norm of B_ν. This proves the assertion.

2.3 Lemma. *Condition (2)′ in Lemma 2.2 implies*

(2)″ for every μ there is $k \geq \mu$ such that for every $K \geq k$ and m there are N and S such that

$$\iota^\mu_k B_{K,m} \subset S\,(\iota^\mu_K B_{k,N} + B_\mu)\ .$$

PROOF. This follows from Grothendieck's factorization theorem [4, Théorème A, p.16] and 2.2.

2.4 Lemma. *Condition (2)′ in Lemma 2.2 is implied by*

(2)‴ for every μ there is $k \geq \mu$ such that for every $K \geq k, m$ and ε there are N and S such that

$$\iota^\mu_k B_{k,m} \subset S \iota^\mu_K B_{K,N} + \varepsilon B_\mu\ .$$

The proof is obvious. For the interpretation of this condition see [13, Theorem 4.9], which is a dual version of [11, Theorem 2].

For the following Lemma cf. the Remark after Theorem 2 in [11]. We use the following notation.

Definition. $\mathcal{X}$ is called a (DFS)–spectrum if for every k and m there exists M such that the inclusion $X_{k,m} \hookrightarrow X_{k,M}$ is compact.

2.5 Lemma. *If $\mathcal{X}$ is a (DFS)-spectrum then (2) implies (2)'''.*

Proof. Since (2) implies also (2)'' we may assume with suitable quantifiers

(i) $\iota_k^\mu X_k \subset \iota_K^\mu X_K + \frac{\varepsilon}{2} B_\mu$

(ii) $\iota_k^\mu B_{k,M} \subset S\,(\iota_K^\mu B_{K,N} + B_\mu)$

(iii) $X_{k,m} \hookrightarrow X_{k,M}$ compact.

We put

$$E = \iota_K^\mu X_{K,N} + \operatorname{span} B_\mu.$$

This is in a natural way a Banach space. Because of (ii), (iii) $\iota_k^\mu B_{k,m}$ is relatively compact in E. We consider the sets

$$U_L = \{\iota_K^\mu x + y \;:\, \| x \|_{K,L} < L,\ \| y \|_\mu < \varepsilon\}$$

for $L \geq N$. Because of (i) they are an open covering of the compact set

$$\overline{\iota_k^\mu B_{k,m}} = \iota_k^\mu \overline{B_{k,m}}^{\| \, \|_{k,M}}.$$

This proves the assertion.

2.6 Proposition. *If $\mathcal{X}$ is a (DFS)-spectrum then* $\operatorname{Proj}^1 \mathcal{X} = 0$ *if and only if the following holds: For every μ the space X_μ contains a bounded Banach ball B_μ such that*

(1) $\iota_{\mu+1}^\mu B_{\mu+1} \subset B_\mu$ *for all* μ

(2) $\forall \mu\ \exists k\ \forall K, m\ \exists N, S \;:\; \iota_k^\mu B_{k,m} \subset S\,(B_{K,N} + B_\mu)$

This condition is certainly fulfilled if there is n such that (2) holds with $B_\mu = B_{\mu,n}$. On the other hand (1) and (2) remain true if we replace B_μ by $\varepsilon_\mu B_\mu$, $\varepsilon_\mu \geq \varepsilon_{\mu+1}$ for all μ. Therefore we may assume $B_\mu \subset B_{\mu,n(\mu)}$ for all μ with suitable $n(\mu)$. We obtain the following conditions.

Definition.

(P_1) $\exists n\ \forall \mu\ \ \exists k\ \forall K, m\ \exists N, S \;:\; \iota_k^\mu B_{k,m} \subset S\,(B_{K,N} + B_{\mu,n})$

(P_2) $\forall \mu\ \exists n, k\ \forall K, m\ \exists N, S \;:\; \iota_k^\mu B_{k,m} \subset S\,(B_{K,N} + B_{\mu,n})$

And we obtained:

2.7 Theorem. *If $\mathcal{X}$ is a (DFS)-spectrum, then* $(P_1) \Rightarrow \operatorname{Proj}^1 \mathcal{X} = 0 \Rightarrow (P_2)$.

The advantage of these conditions is that by means of dualization they can be turned into inequalities.

To formulate that we use the following notation $\jmath_\mu^k : X'_\mu \to X'_k$ is the transpose of ι_k^μ for $\mu \le k$. For $y \in X'_\mu$ we set

$$\| y \|_{\mu,n}^* = \sup \{ |y(x)| : \| x \|_{\mu,n} \le 1 \} .$$

This is an extended real valued "norm". We have

$$\begin{aligned} \| y \|_{\mu,n}^* &\le \| y \|_{\mu,n+1}^* \\ \| y \|_{\mu,n}^* &\ge \| \jmath_\mu^{\mu+1} y \|_{\mu+1,n}^* \end{aligned}$$

for all μ, n and $y \in X'_\mu$.

Definition.

(P_1^*) $\exists n \, \forall \mu \, \exists k \, \forall K, m \, \exists N, S \, \forall y \in X'_\mu : \| \jmath_\mu^k y \|_{k,m}^* \le S \left(\| \jmath_\mu^K y \|_{K,N}^* + \| y \|_{\mu,n}^* \right)$

(P_2^*) $\forall \mu \, \exists n, k \, \forall K, m \, \exists N, S \, \forall y \in X'_\mu : \| \jmath_\mu^k y \|_{k,m}^* \le S \left(\| \jmath_\mu^K y \|_{K,N}^* + \| y \|_{\mu,n}^* \right)$

By means of classical duality theory, in particular the bipolar theorem, (P_1^*) and (P_2^*) are equivalent to (P_1) and (P_2) respectively. Therefore we obtain:

2.8 Theorem. *If $\mathcal{X}$ is a (DFS)-space, then (P_1^*)* $\Rightarrow \operatorname{Proj}^1 \mathcal{X} = 0 \Rightarrow$ *(P_2^*).*

Remark. It should be noticed that the condition for $\operatorname{Proj}^1 \mathcal{X} = 0$ in Proposition 2.6 can always be turned into (P_1). One just has to change the fundamental systems $(B_{\mu,k})$ in each X_μ by setting $B_{\mu,0} = B_\mu$, etc. This, however, is not very helpful if the $X_{n,k}$ have a concrete meaning as in most examples.

3. Topological properties of projective limits of (LB)–spaces

We keep the notations and assumptions of §2. We assume additionally, that

$$\bigcup_m B_{k,m} = X_k \text{ for all } k.$$

Then for every bounded set $B \subset X_k$ there is m such that $B \subset B_{k,m}$. We achieve that e.g. by replacing $B_{k,m}$ by $mB_{k,m}$.

We set $B_{k,m}^\infty = (\iota^k)^{-1} B_{k,m}$. Then the sets

$$\mathrm{B} = \bigcap_k B_{k,m(k)}^\infty ,$$

where $(m(k))_k$ runs through all sequences of integers, are a basis of bounded sets in X. This means every bounded set in X is contained in such a set. Clearly these sets are Banach balls.

Remark. X has a basis of bounded sets consisting of Banach balls.

The uniform boundedness principle implies that in X' every weak*– bounded set is bounded in the strong topology.

We call the spectrum $\mathcal{X}$ *reduced* if $X_k = \overline{\iota^k X}$ for all k. From Theorem 2.1 (Retakh's theorem) follows easily:

Remark. If $\mathrm{Proj}^1 \mathcal{X} = 0$ then $\mathcal{X}$ is equivalent to a reduced spectrum.

For a reduced spectrum $\mathcal{X}$ we identify X_k' with $X_k^* := \jmath^k X_k' \subset X'$. Then $X_k^* \subset X_{k+1}^*$ for all k and we obtain an imbedding spectrum of Fréchet spaces. By X^* we denote the dual space X' equipped with the inductive topology. X_b' denotes X' equipped with the strong topology. $\mathrm{id} : X^* \to X_b'$ is continuous.

Remark. Let $\mathcal{X}$ be reduced. For an absolutely convex bounded set $\mathrm{B} \subset X'$ the following are equivalent:

(1) B is equicontinuous

(2) B is relatively weak*–compact

(3) B is contained in a bounded Banach ball

(4) B is contained in some X_k^* and bounded there.

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear, (3) $\Rightarrow$ (4) follows from Grothendieck's factorization theorem, (4) $\Rightarrow$ (1) since X_k is barrelled.

In particular X is a Mackey space, i.e. carries its Mackey topology. The following Lemma is an immediate consequence of the previous Remarks.

3.1 Lemma. *Let $\mathcal{X}$ be reduced. The following are equivalent:*

(1) X barrelled

(2) X quasibarrelled

(3) X_b' sequentially complete (quasi-complete)

(4) X' weak*–sequentially complete (quasi-complete)

(5) Every bounded set $\mathrm{B} \subset X'$ is contained in some X_k^* and bounded there.

From [13] we take the following two crucial Lemmas.

3.2 Lemma. ([13, 5.6]) *If* $\operatorname{Proj}^1 \mathcal{X} = 0$ *then* X *is bornological.*

3.3 Lemma. ([13, 5.9, 5.10]) *If* $\mathcal{X}$ *is reduced and does not satisfy* (P_2^*) *then there is* μ *and a sequence* $(y_n)_n$ *in* X_μ *such that* $\lim_n (2^n \jmath_\mu y_n) = 0$ *in* X^* *but* $\left(\frac{1}{n}\jmath_\mu^k y_n\right)_n$ *does not converge to* 0 *for all* $k \geq \mu$.

Collecting the previous information we obtain the following theorem.

3.4 Theorem. *Let* $\mathcal{X}$ *be reduced. Then (1)* $\Rightarrow$ *(2)* $\Rightarrow$ *(3)* $\Rightarrow$ *(4)* $\Rightarrow$ *(5)* $\Rightarrow$ *(6), where:*

(1) $\operatorname{Proj}^1 \mathcal{X} = 0$

(2) X *bornological*

(3) X_b' *complete*

(4) X *barrelled*

(5) X^* *regular*

(6) Condition (P_2^*).

Here the inductive limit X^* is called *regular* if every bounded set is contained in some X_k^* and bounded there.

If all X_k are reflexive then $X^{*\prime} = X$, hence $X^* = X_b{}'$ (because the topology of X^* is admissible). If $\mathcal{X}$ is a (DFS)–spectrum, then every X_k is reflexive and $X^* = X_b'$ is a Schwartz space. According to a theorem of L. Schwartz ([12, p.43]) completeness of $X^* = X_b'$ implies that $(X_b')_b'$ is bornological (=ultrabornological in our case). Since completeness of X_b' also implies that X is barrelled (see Lemma 3.1) we have X bornological.

Remark. If $\mathcal{X}$ is a (DFS)–spectrum then $X^* = X_b'$ and X is bornological if and only if $X^* = X_b'$ is complete.

3.5 Theorem. *Let* $\mathcal{X}$ *be a reduced (DFS)–spectrum. Then* $X^* = X_b'$ *and (1)* $\Rightarrow$ *(2)* $\Longleftrightarrow$ *(3)* $\Rightarrow$ *(4)* $\Longleftrightarrow$ *(4)'* $\Longleftrightarrow$ *(5)* $\Longleftrightarrow$ *(5)'* $\Rightarrow$ *(6) where*

(1) $\operatorname{Proj}^1 \mathcal{X} = 0$

(2) X *bornological*

(3) X^* *complete*

(4) X *barrelled*   *(4)'* X *reflexive*

(5) X^* *regular*   *(5)'* X^* *reflexive*

(6) Condition (P_2^*).

Proof. According to the previous Remarks and Theorem 3.4 we have only to show (4) $\iff$ (4)′ and (5) $\iff$ (5)′.

In X every bounded set is relatively compact, hence X is semireflexive, so (4) implies (4)′. The converse is clear . If X^* is reflexive every bounded set in X_b' is relatively weak–compact, hence relatively weak*– compact. So (5)′ $\Rightarrow$ (5). If X^* is regular then $(X^*)_b' = X$, since all X_k are reflexive. This proves (5) $\Rightarrow$ (5)′.

4. The case of sequence spaces

Let $(a_{j;k,m})_{j,k,m\in\mathbb{N}}$ be an infinite matrix with

$$\begin{aligned} a_{j;k,m} &> 0 \\ a_{j;k,m} &\geq a_{j;k,m+1} \\ a_{j;k,m} &\leq a_{j;k+1,m} \end{aligned}$$

for all j, k, m. For fixed $1 \leq p < +\infty$ we define

$$X_{k,m} = \{x = (x_1, x_2, \ldots) : \| x \|_{k,m}^p = \sum_j |x_j|^p a_{j;k,m}^p < \infty\},$$

and for $p = +\infty$

$$X_{k,m} = \{x = (x_1, x_2, \ldots) : \lim_j |x_j| a_{j;k,m} = 0\}$$

with the norm $\| x \|_{k,m} = \sup_j |x_j| a_{j;k,m}$.

We put $X_k = \bigcup_m X_{k,m}$, equipped with the inductive topology. Then $X_{k+1} \subset X_k$ and we consider this as a projective spectrum $\mathcal{X}$ with the inclusions as connecting maps. As always in this paper $X = \operatorname{Proj}^0 \mathcal{X}$ equipped with the projective topology.

Dually we define for $1 < p \leq +\infty$

$$X_{k,m}^* = \{y = (y_1, y_2, \ldots) : \| y \|_{k,m}^{*q} = \sum_j |y_j|^q a_{j;k,m}^{-q} < +\infty\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and for $p = 1$

$$X_{k,m}^* = \{y = (y_1, y_2, \ldots) : \| y \|_{k,m}^* = \sup_j |y_j| a_{j;k,m}^{-1} < +\infty\}.$$

We put $X_k^* = \bigcap_m X_{k,m}^*$ with the projective topology, i.e. X_k^* is the Köthe space with the matrix $(a_{j;k,m})_{j,m\in\mathbb{N}}$. Then $X_k^* \subset X_{k+1}^*$ and we consider this as an inductive spectrum $\mathcal{X}^*$ with the inclusions as connecting maps. We set $X^* = \bigcup_k X_k^*$ equipped with the inductive topology.

Obviously $X_{k,m}^* = X_{k,m}'$, $X_k^* = X_k'$ by canonical identification. The topologies are the strong topologies (see [6, p. 406 ff]). Also $X^* = X'$ by the same identification, and the topology is the strong topology for $1 < p < +\infty$, due to reflexivity (see §3).

For $p = 1$ this needs not to be the case. Let $a_{j;k,m} = a_{j,k}$ for all j, k, m, where $(a_{j,k})_{j,k}$ is the matrix of a non–distinguished Köthe space (see Köthe [6, p. 438]). Notice that in this case $\operatorname{Proj}^1 \mathcal{X} = 0$, however $X^* \neq X_b'$.

For $1 \leq p < +\infty$ the $B_{k,m}$ are closed in X_k, hence the $B_{k,m}$ are a fundamental system of bounded sets in X_k (see [6, p. 406 f]). For $p = +\infty$ this needs not to be the case (see [6, p. 437 f]).

To avoid these difficulties we assume for $p = 1, \infty$

$$\forall k, m \,\exists M \quad : \quad \lim_j \frac{a_{j;k,m}}{a_{j;k,M}} = 0\,.$$

Hence we are in the (DFS)–case , so $X^* = X_b'$ (see §3) and $\overline{B_{k,m}}^{X_k} \subset CB_{k,M}$, so the $B_{k,m}$ are a fundamental system of bounded sets in X_k.

Using the ideas of [7] we obtain (see [13, Lemma 6.1]):

4.1 Lemma. *Under the assumptions of this section* $\mathrm{Proj}^1\,\mathcal{X} = 0$ *and* (P_2^*) *coincide with :*

(P) $\forall \mu \,\exists n, k \,\forall m, K \,\exists N, S \,\forall j \quad : \quad \frac{1}{a_{j;k,m}} \leq S \max\left(\frac{1}{a_{j;K,N}}, \frac{1}{a_{j;\mu,n}}\right).$

By use of 3.4 and 3.5 this yields (see [13, Theorem 6.]).

4.2 Theorem. *Under the assumptions of this section* $X^* = X_b'$ *and the following are equivalent:*

(1) $\mathrm{Proj}^1\,\mathcal{X} = 0$

(2) X bornological

(3) X^* complete

(4) X barrelled — *(4)' X reflexive*

(5) X^* regular — *(5)' X^* reflexive*

(6) Condition (P).

Remark. We mention that another condition is equivalent to (1),...,(6) (see [13, Theorem 6.4]):

(7) $X^* = \{y = (y_1, y_2, \ldots) \,:\, \sum_j |x_j y_j| < +\infty$ for all $x \in X\}$ ("Köthe dual").

This Theorem generalizes results of Grothendieck [4, II, §4] and Krone–Vogt [7]. The following special case generalizes results of Vogt–Wagner [14], [15], Hebbecker [5], Nyberg [8]. For the case (ii), $\rho = 0$ and $r = +\infty$ see Braun–Meise–Vogt [1].

We assume that $\alpha = (\alpha_j)_j$, $\beta = (\beta_j)_j$ are nonnegative numerical sequences, $\alpha_j + \beta_j \to +\infty$, and $r, \rho \in \mathbb{R} \cup \{+\infty\}$. For $r_k \nearrow r$, $\rho_k \nearrow \rho$ we consider the matrix

$$a_{j;k,m} = \mathrm{e}^{r_k \alpha_j - \rho_m \beta_j}\,.$$

It is easy to see that $\mathcal{X}$ depends, up to equivalence, only on α, β, r, ρ.

4.3 Theorem. *(P) is satisfied if and only if (i) or (ii), where*

(i) $\rho = +\infty$

(ii) $\mathbb{N} = J_1 \dot\cup J_2$ *such that* $\inf_{j\in J_1} \frac{\alpha_j}{\beta_j} > 0$ *and* $\lim_{j\in J_2} \frac{\alpha_j}{\beta_j} = 0$.

Proof. After taking logarithms and dividing by β_j condition (P) takes the form

$$\forall\mu\,\exists n,k\,\forall m,K\,\exists N,S\,\forall j\;:\;\rho_m - r_k\frac{\alpha_j}{\beta_j} \le \frac{S}{\beta_j} + \max\left(\rho_N - r_K\frac{\alpha_j}{\beta_j}, \rho_n - r_\mu\frac{\alpha_j}{\beta_j}\right)$$

or, taking into account that $\alpha_j + \beta_j \to \infty$, with different k, n and N (using $\frac{S}{\beta_j} - (r_{k+1} - r_k)\frac{\alpha_j}{\beta_j} \le \varepsilon$ for $\alpha_j + \beta_j$ large)

$\forall\mu\,\exists n,k\,\forall m,K\,\exists N$ such that for all but finitely many j

$$\rho_m - r_k\frac{\alpha_j}{\beta_j} \le \max\left(\rho_N - r_K\frac{\alpha_j}{\beta_j}, \rho_n - r_\mu\frac{\alpha_j}{\beta_j}\right).$$

The inequality we can write as

$$\rho_m - \rho_n \le (r_k - r_\mu)\frac{\alpha_j}{\beta_j} \quad\text{or}\quad \rho_N - \rho_m \ge (r_K - r_k)\frac{\alpha_j}{\beta_j}.$$

If $\rho = +\infty$ and $m > n$, $K > k > \mu$ we choose N such that

$$\frac{\rho_N - \rho_n}{\rho_m - \rho_n} \ge \frac{r_K - r_k}{r_k - r_\mu}$$

and obtain for j with $\rho_m - \rho_n > (r_k - r_\mu)\frac{\alpha_j}{\beta_j}$

$$(r_K - r_k)\frac{\alpha_j}{\beta_j} \le \frac{r_K - r_k}{r_k - r_\mu}(\rho_m - \rho_n) \le \rho_N - \rho_m.$$

If $\rho < +\infty$ and (ii) holds, then for given μ we choose $k = \mu + 1$ and n so large that

$$\rho - \rho_n \le (r_k - r_\mu)\frac{\alpha_j}{\beta_j} \text{ for all } j \in J_1.$$

If $N > m$ the second possibility will be satisfied for all $j \in J_2$ up to finitely many.

Now assume $\rho < +\infty$ and (P). For $\mu = 1$ we choose k and n. We put

$$J_1 = \{j\;:\;\rho - \rho_n \le (r_k - r_1)\frac{\alpha_j}{\beta_j}\}$$

and $J_2 = \mathbb{N} \setminus J_1$. For every m the inequality

$$\rho - \rho_m \ge (r_{k+1} - r_k)\frac{\alpha_j}{\beta_j}$$

holds for all $j \in J_2$ up to finitely many. This implies the assertion.

For $J \subset \mathbb{N}$, $1 \le p < +\infty$, α and r we set (different notation for *)

$$\Lambda_r(\alpha, J) = \{(x_i)_{i\in J} \ : \| x \|_t^p = \sum_{j\in J} |x_j|^p e^{pt\alpha_j} < +\infty \text{ for all } t < r\}$$

$$\Lambda_r^*(\alpha, J) = \{(x_j)_{j\in J} \ : \| x \|_t^p = \sum_{j\in J} |x_j|^p e^{pt\alpha_j} < +\infty \text{ for some } t < r\}$$

and obtain (see Braun–Meise–Vogt [1]):

4.4 Corollary. *If $\rho = 0$ then (P) is satisfied if and only if $\mathbb{N} = J_1 \dot\cup J_2$ such that $X = \Lambda_r(\alpha, J_1) \oplus \Lambda_0^*(\beta, J_2)$.*

Important special cases are those used in Braun–Meise–Vogt [1] (cf. [2]), to solve the problem of solvability of convolution equations in Gevrey classes on $\mathbb{R}$, and the cases of matrices

$$a_{\nu,j;k,m} = e^{r_k\alpha_\nu - \rho_m\beta_j}$$

where $\alpha_\nu \to +\infty$, $\beta_j \to +\infty$. They occur in connection with the investigation of tensor-products of (F)– and (DF)–spaces and of Ext^1 (see remarks above).

Restricting us to the nuclear case, i.e. to sequences α, β with

$$\limsup_\nu \frac{\log \nu}{\alpha_\nu} < +\infty \text{ for } r = +\infty$$

$$\lim_\nu \frac{\log \nu}{\alpha_\nu} = 0 \text{ for } r < +\infty$$

and analogous for β, we have $X \cong L(\Lambda_\rho(\beta), \Lambda_r(\alpha))$. The decomposition in Corollary 4.4 gives a decomposition in a "lower triangular" and "upper triangular" part. One maps a fixed neighborhood of zero into a (variable) bounded set, the other some neighborhood of zero into a fixed bounded set. All maps are compact, i.e. $L(\Lambda_\rho(\beta), \Lambda_r(\alpha)) = LB(\Lambda_\rho(\beta), \Lambda_r(\alpha))$ (see Nyberg [8]).

5. Final remarks

The usual way of application of the previous results is the following. We are given an exact sequence

$$0 \to \mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z} \to 0$$

of spectra of complete (LB)–spaces. X, Y, Z are the projective limits, $\varphi = \Phi^0$, $\psi = \Psi^0$. We have the exact sequence

$$0 \to X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

and we want to know whether ψ is surjective.

In many of these applications we have $\mathrm{Proj}^1 \mathcal{Y} = 0$. In this case we have

5.1 Theorem. *If* $\mathrm{Proj}^1 \mathcal{Y} = 0$, *then* ψ *is surjective if and only if* $\mathrm{Proj}^1 \mathcal{X} = 0$.

PROOF. Immediate consequence of Theorem 1.5.

If one investigates surjectivity of a map ψ by this method (see e.g. Braun–Meise–Vogt [2]) then Theorem 4.5 suggests the question whether this is a property of $\mathcal{X}$ or of the space X. This means whether we can replace the investigation of $\mathcal{X}$ by that of any other spectrum of (LB)–spaces generating X.

By means of Grothendieck's factorization theorem one proves (cf. [13, Proposition 3.2])

5.2 Proposition. *If* $\mathcal{X}, \mathcal{Y}$ *are projective spectra of complete (LB)–spaces,* X, Y *the projective limits,* $\varphi : X \to Y$ *continuous linear and* $\mathcal{X}$ *reduced, then there exists* $\Phi : \mathcal{X} \to \mathcal{Y}$ *such that* $\varphi = \Phi^0$.

An immediate consequence is:

5.3 Corollary. *Any two reduced spectra of complete (LB)–spaces generating* X *are equivalent.*

This answers to some extend the question above. However $\mathcal{X}$ occurring as spectrum of kernels of Ψ needs not to be reduced. So the surjectivity of ψ depends in fact on $\mathcal{X}$.

Example. Let $1 \le p \le +\infty$ and

$$\begin{aligned} X_n &= \ell_p \quad , \ \iota^n_{n+1} x = (0, x_1, x_2, \ldots) \text{ for all } n \\ Y_n &= \ell_p \quad , \ \iota^n_{n+1} = \mathrm{id} \ \text{ for all } n \\ Z_n &= \mathbb{K}^n \ , \ \iota^n_{n+1}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n) \text{ for all } n . \end{aligned}$$

We put $\varphi^n_n x = (0, \ldots, 0, x_1, x_2, \ldots)$ writing n zeros at the beginning, $\psi^n_n x = (x_1, \ldots, x_n)$ and obtain an exact sequence

$$0 \to \mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z} \to 0$$

of projective spectra of Banach spaces.

Since $X = \{0\}, Y = \ell_p, Z = \omega := \mathbb{K}^{\mathbb{N}}$, the space X has all good properties, but $\psi : \ell_p \hookrightarrow \omega$ is obviously not surjective.

If we combine the results of this section with the 2. Remark of §3, then we obtain for the question at the beginning of this section.

5.4 Theorem. *If* $\mathrm{Proj}^1 \mathcal{Y} = 0$, *then* ψ *is surjective if and only if* $\mathcal{X}$ *is reduced and* X *admits a spectrum* $\tilde{\mathcal{X}}$ *of complete (LB)–spaces with* $\mathrm{Proj}^1 \tilde{\mathcal{X}} = 0$.

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