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Distinguished Köthe Spaces

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In [1] (cf. [2]) Bierstedt and Bonet gave a characterization of distinguished Köthe sequence spaces by means of a condition (D) for the defining matrix. This was done in the framework of a general theory involving S. Heinrich's density condition [4]. In the present note we give a direct and completely elementary characterization and derive a scheme for constructing counterexamples, which is used in [6]. It is, in fact, a generalization of the well known counterexample of Köthe and Grothendieck (see [5], § 31, 7).

We use standard terminology and notation from [5]. $A=(a_{j,k})_{j,k\in\mathbb{N}}$ always denotes an infinite matrix with

$$0 \leqq a_{j,k} \leqq a_{j,k+1}, \qquad \sup_k a_{j,k} > 0 \qquad \text{for } j \quad \text{and} \quad k.$$

We put

$$\lambda(A) = \{\xi = (\xi_1, \xi_2, \ldots) : \|\xi\|_k = \sum_j |\xi_j|\, a_{j,k} < +\infty \text{ for all } k\}.$$

Equipped with the seminorms $\| \;\|_k$, $k=1, 2, \ldots$, this is a Fréchet space.

We use the following result of Bierstedt, Meise and Summers [3].

1. Lemma. *$\lambda(A)$ has a fundamental system of bounded subsets of the form*

$$B = \{x \in \lambda(A) : \sum_j |x_j|\, \lambda_j \leqq 1\}$$

where $\lambda_j = \sup_k \dfrac{a_{j,k}}{C_k}$ for some sequence of positive numbers C_k. (B is said to be of standard form).

Let E be a Fréchet space, $\| \;\|_2 \leqq \| \;\|_2 \leqq \ldots$ a fundamental system of seminorms, $U_k = \{x : \|x\|_k \leqq 1\}$, E'_k the Banach space generated by the polar set U_k^0 of U_k. Then E is distinguished (cf. [5], § 29, 4(3)), if the strong dual E'_b is bornological, i.e. the inductive limit of the E'_k. Hence we have

2. Lemma. *E is distinguished if and only if for every sequence $\lambda_k > 0$ there is a bounded set $B \subset E$ such that $B^0 \subset \Gamma_{k=1}^{\infty} \lambda_k U_k^0$.*

Here Γ denotes the absolutely convex hull.

3. Theorem. *$\lambda(A)$ is distinguished if and only if for every sequence $D_k > 0$ there is a sequence $C_k > 0$ such that for every $C > 0$ and $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with*

$$\min\left\{C a_{j,n}, \sup_k \frac{a_{j,k}}{C_k}\right\} \leqq \sup_{k=1,\ldots,m} \frac{a_{j,k}}{D_k}$$

for all j.

Proof. Let $\lambda(A)$ be distinguished and $D_k > 0$, $k \in \mathbb{N}$ be given. For $\lambda_k = \frac{1}{D_k}$ we choose B according to Lemma 2 which may be assumed of standard form. This yields a sequence $C_k > 0$. For any $C > 0$, $n \in \mathbb{N}$ we denote by ξ_j the left hand side of the inequality to be proved. Since obviously $\xi = (\xi_j)_j \in B^0$ there exist $\xi^{(k)}$, $k = 1, \ldots, m$, $\xi^{(k)} \in \frac{1}{D_k} U_k^0$ and numbers ε_k, $k = 1, \ldots, m$, such that

$$\sum_{k=1}^{m} |\varepsilon_k| \leqq 1 \quad \text{and} \quad \xi = \sum_k \varepsilon_k \xi^{(k)}.$$

Therefore

$$\xi_j \leqq \sum_{k=1}^{m} |\varepsilon_k| \frac{a_{j,k}}{D_k} \leqq \sup_{k=1,\ldots,m} \frac{a_{j,k}}{D_k}$$

which proves the inequality.

To prove the converse, we assume $\lambda_k > 0$, $k \in \mathbb{N}$, to be given and put $D_k = \frac{2^k}{\lambda_k}$. The condition gives a sequence C_k which defines a bounded set B in standard form. Let $\xi \in B^0$. Then there is C and n such that $|\xi_j| \leqq$ left hand side of the inequality for all j. Hence for some $m \in \mathbb{N}$ we have

$$|\xi_j| \leqq \sup_{k=1,\ldots,m} \frac{a_{j,k}}{D_k} \quad \text{for all } j.$$

For every j we determine $k = k(j)$ such that

$$\frac{a_{j,k(j)}}{D_{k(j)}} = \sup_{k=1,\ldots,m} \frac{a_{j,k}}{D_k}$$

and put $\xi_j^{(k)}=\xi_j$ if $k=k(j)$, $\xi_j^{(k)}=0$ otherwise. Then $2^k\xi^{(k)}\in\lambda_k U_k^0$ and

$$\xi=\sum_{k=1}^{m}\xi^{(k)}=\sum_{k=1}^{m}2^{-k}(2^k\xi^{(k)})\in\overset{\infty}{\underset{k=1}{\Gamma}}\lambda_k U_k^0.$$

4. Corollary. *Let $A=(a_{i,j;k})$ be a doubly indexed Köthe matrix such that*

(1) $a_{i,j;k}=a_{i,j;1}$ *for* $k\leqq i$,

(2) $\lim\limits_{j}\dfrac{a_{m,j;m}}{a_{m,j;m+1}}=0$.

Then the doubly indexed Köthe space $\lambda(A)$ is not distinguished.

Proof. Assuming $\lambda(A)$ distinguished we apply Theorem 3 to $D_k=1$ for all k and obtain $C_k>0$, $k=1,2,\ldots$. We put $C=2$, $n=1$ and obtain m such that

$$\min\left\{2a_{i,j;1},\ \sup_k\frac{a_{i,j;k}}{C_k}\right\}\leqq a_{i,j;m}$$

for all i,j. We choose $i=m$ and get by use of (1)

$$\frac{a_{m,j;m+1}}{C_{m+1}}\leqq\sup_k\frac{a_{m,j;k}}{C_k}\leqq a_{m,j;m}$$

for all j, which contradicts (2).

Example:

$$a_{i,j;k}=\begin{cases} j^i & \text{for } k\leqq i\\ j^k & \text{for } k>i.\end{cases}$$

In this case $\lambda(A)$ is not distinguished, whereas

$$a_{i,j;k}^2\leqq a_{i,j;k-1}\,a_{i,j;k+1}$$

i.e. A is of type (d_1), or $\lambda(A)$ has property (DN). So $\lambda(A)$ is center of a normal scale, contained in some $\Lambda^\infty(M,a)$ (see [7]) and, since its bidual has also (DN), $\lambda(A)''$ admits a continuous norm (cf. [6]).

Clearly the Köthe-Grothendieck example (see [5], § 31,7) is also contained in Corollary 4.

Note added in proof. Conditions in the same spirit as our condition in 3. Theorem are used in: Bastin, F.: On bornological spaces $C\bar{V}(X)$. Arch. Math. (to appear)

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