UNITARY ENDOMORPHISMS OF POWER SERIES SPACES

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Dedicated to Professor Mikhail Mikhaylovich Dragilev on the occasion of his 90th birthday

Abstract. We present a systematic study of the properties of unitary endomorphism, that is, of endomorphisms of spaces $\Lambda_\infty(\alpha)$ which are unitary in ℓ_2. We describe their role in the investigation of complemented subspaces of spaces $\Lambda_\infty(\alpha)$, in particular, of the open problem whether all these complemented subspaces have bases.

In the present paper we consider, not necessarily nuclear, power series spaces of infinite type $\Lambda_\infty(\alpha)$ over ℓ_2. We study endomorphisms and, in the last section, automorphisms of these spaces which are unitary in ℓ_2. They are a special form of 'local imbeddings' which have been introduced by Aytuna-Krone-Terzioğlu [1] in their study of the basis problem for complemented subspaces of nuclear power series spaces. Their fundamental result was based on the fact that they could show that these maps are invertible on a enough big complemented subspace. We extend their construction to the nonnuclear case and to a bigger class of power series spaces, and produce partial inverses on complemented subspaces of the range of the unitary endomorphism. This leads to the proof of the existence of complemented subspaces $\Lambda_\infty(\beta)$, where β can be explicitly given, in Fréchet-Hilbert spaces of class (DN) and (Ω). These always can be represented as range of a projection in some space $\Lambda_\infty(\alpha)$ (see [12, Theorem 5] or Theorem 6.5 below). That the range of a projection in a nuclear power series space $\Lambda_\infty(\alpha)$ always has a infinite dimensional complemented subspace with basis has been shown in Schrubba [9].

A special case are the unstable spaces, introduced by Dragilev [2] for general Köthe spaces and studied for nuclear power series spaces in Dragilev-Kondakov [3], Dubinsky-Vogt [4], [5], Kondakov [7], Wagner [13] and for Fréchet-Hilbert power series spaces in [12]. For these spaces any unitary endomorphism of $\Lambda_\infty(\alpha)$ is invertible and this results in the existence of bases in related spaces.

In a second part we consider unitary automorphisms T of $\Lambda_\infty(\alpha)$ with $T^2 = \mathrm{id}$. These occur as $T = P - Q$, $Q = I - P$, in the study of projections in $\Lambda_\infty(\alpha)$ which always can be assumed to be orthogonal in ℓ_2 (see Theorem 7.2). We show that for any space E with dominating norm $\| \; \|$, which is isomorphic to

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$\Lambda_\infty(\alpha)$, there is an isomorphism $U : E \to \Lambda_\infty(\alpha)$ such that $|Ux|_0 = \|x\|$. In consequence of this we show that a space $\Lambda_\infty(\alpha)$ has the property that all of its complemented subspaces have a basis if, and only if, every unitary automorphism of $\Lambda_\infty(\alpha)$ can be diagonalized by means of a unitary automorphism. This leads to a reformulation of the problem posed by Mityagin, whether ever complemented subspace of s has a basis, in terms of a problem on Hilbert space operators.

The paper uses the notation and many results of [12]. For further unexplained notation, in particular, concerning topological linear invariants (DN) and (Ω) (or $\mathcal{D}_1$ and Ω_1 in Zaharjuta's notation) we refer to [8]. For the definition of a dominating norm see [8, p. 359].

1. Power series spaces

Let α be a sequence $0 < \alpha_0 \leq \alpha_1 \leq \cdots \nearrow +\infty$. We set

$$\Lambda_\infty(\alpha) := \Big\{x = (x_0, x_1, \dots) \,:\, |x|_t^2 = \sum_j |x_j|^2 e^{2t\alpha_j} < \infty \text{ for all } t \in \mathbb{R}\Big\}.$$

Equipped with the norms $|\cdot|_t$, $t \in \mathbb{R}$, this is a Fréchet-Hilbert space.

We denote by

$$\Lambda_t^\alpha = \Big\{x = (x_0, x_1, \dots) \,:\, |x|_t^2 = \sum_j |x_j|^2 e^{2t\alpha_j} < \infty\}$$

the local Banach spaces which are, of course, Hilbert spaces. By $\iota_T^t : \Lambda_T^\alpha \hookrightarrow \Lambda_t^\alpha$, for $T > t$, we denote the identical imbedding.

$\langle\cdot,\cdot\rangle_t$ denotes the scalar product of $|\cdot|_t$. By definition $\Lambda_0^\alpha = \ell_2$ and $|\cdot|_0 = |\cdot|$ where the latter is the norm of ℓ_2, analogous notation for the scalar products.

Another definition could be the following: Let H be a separable Hilbert space, A the inverse of a compact, positive self-adjoint operator and $\Lambda_\infty(A) = \bigcap_t D(e^{tA}) = \bigcap_t R(e^{-tA})$. Then $\Lambda_\infty(A) = \Lambda_\infty(\alpha)$ where α is the eigenvalue sequence of A.

If $\Lambda_\infty(\alpha)$ is given we use this notation with $H = \ell_2$ and A the diagonal operator defined by the sequence α.

The group $z \mapsto e^{zA}$ operates continuously on $\Lambda_\infty(\alpha)$, we have $|x|_t = |e^{zA}x|$ for $x \in \Lambda_\infty(\alpha)$ and $z = t + is$. $\xi \mapsto e^{i\xi A}$ defines unitary group on every Λ_t^α.

$\Lambda_\infty(\alpha)$ is called stable if $\Lambda_\infty(\alpha) \oplus \Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha)$ and this is equivalent to $\sup \alpha_{2n}/\alpha_n < +\infty$, it is called shift-stable if $\sup \alpha_{n+1}/\alpha_n < +\infty$. For further stability conditions see Section 3.

$\Lambda_\infty(\alpha)$ is called tame if, up to equivalence, α has the following form: there are strictly increasing sequences $n(k)$ in $\mathbb{N}_0$ with $n(0) = 0$ and $\beta_k > 0$ such that

(1) $\alpha_n = \beta_k$ for $n(k) \leq n < n(k+1)$
(2) $\lim_k \beta_k/\beta_{k+1} = 0$.

If α has this form (without equivalence) then $\Lambda_\infty(\alpha)$ is called blockwise unstable. Tame spaces are characterized by the fact that all maps $A \in L(\Lambda_\infty(\alpha))$ are linearly tame, that is, have continuity estimates of the form $|Ax|_k \leq C_k |x|_{ak+b}$, $k \in \mathbb{N}_0$ (see [4, Proposition 1], [5, Theorem 1.3] or [12, Theorem 8]).

For the following result, which is a generalization of a result in Dragilev-Kondakov [3], see [5, Lemma 2.1] or [12, Lemma 25].

Lemma 1.1. *If α is blockwise unstable, $A \in L(\Lambda_\infty(\alpha))$ and $|Ax|_0 \leq C|x|_0$, then for each $\varepsilon > 0$ the set $A^1 U_\varepsilon$ is relatively compact in $\Lambda_\infty(\alpha)$, where*

$$U_\varepsilon = \{x \in \Lambda_\infty(\alpha) \,:\, |x|_\varepsilon \leq 1\}.$$

2. Unitary endomorphisms

Let $\Lambda_\infty(\alpha)$ be a power series space.

Definition 1. *A unitary endomorphism of $\Lambda_\infty(\alpha)$ is a unitary map in ℓ_2 which maps $\Lambda_\infty(\alpha)$ into $\Lambda_\infty(\alpha)$.*

Lemma 2.1. *If T is a unitary endomorpism of $\Lambda_\infty(\alpha)$ then $T \in L(\Lambda_\infty(\alpha))$*

Proof. This follows from the closed graph theorem. □

Of course a unitary endomorphism T is injective and invertible in ℓ_2. This does not imply that T is invertible on $\Lambda_\infty(\alpha)$, that is, $T^{-1}(\Lambda_\infty(\alpha)) \subset \Lambda_\infty(\alpha)$. If $\Lambda_\infty(\alpha)$ is shift stable we have the following well-known example:

Example 1. We choose a sequence $k_0 = 0 < k_1 < k_2 < \dots$ of integers and define T by setting $Te_{k_{j+1}} = e_{k_j+1}$ for $j \in \mathbb{N}_0$, $Te_0 = e_0$ and $Te_k = e_{k+1}$ otherwise. Here e_k denote the canonical unit vectors in ℓ_2. Then T obviously is unitary and for $x \in \Lambda_\infty(\alpha)$ we have $Tx = x_0 e_0 + \sum_{j=0}^\infty x_{k_{j+1}} e_{k_j+1} + \sum_k' x_k e_{k+1}$ and therefore

$$|Tx|_t^2 = |x_0|^2 e^{2t\alpha_0} + \sum_{j=0}^\infty |x_{k_{j+1}}|^2 e^{2t\alpha_{k_j+1}} + {\sum_k}' |x_k|^2 e^{2t\alpha_{k+1}} \leq |Tx|_{dt}$$

where $\alpha_{k+1} \leq d\alpha_k$ and $\sum'$ denotes the sum over the remaining terms. Therefore $T \in L(\Lambda_\infty(\alpha))$.

On the other hand we have $|Te^{-\alpha_{k_{j+1}}} e_{k_{j+1}}|_t = e^{-\alpha_{k_{j+1}} + t\alpha_{k_j+1}}$ and $|e^{-\alpha_{k_{j+1}}} e_{k_{j+1}}|_1 = 1$ for all j. If the sequence k_j is chosen such that $\lim_{j\to\infty} \alpha_{k_j+1}/\alpha_{k_{j+1}} = 0$, the map T is not invertible in $\Lambda_\infty(\alpha)$.

Notice that the matrix of T is upper triangular plus one subdiagonal.

The situation is different for tame power series spaces.

Theorem 2.2. *If $\Lambda_\infty(\alpha)$ is tame then every unitary endomorphism T is invertible in $\Lambda_\infty(\alpha)$.*

Proof. We may assume $\Lambda_\infty(\alpha)$ to be blockwise unstable. Then we have in a natural way

$$\Lambda_\infty(\alpha) \cong \Big\{x = (x_0, x_1, \ldots) \in \prod_k \ell_2(m(k)) \,:\, |x|_t^2 = \sum_k |x_k|^2 e^{2t\beta_k} < +\infty \text{ for all } t \in \mathbb{R}\Big\}$$

where $m(k) = n(k+1) - n(k)$ and $\ell_2(m(k))$ is the $m(k)$-dimensional Hilbert space with norm $|\ |$.

Then T can be written as a matrix of block-maps, $T = (T_{k,j})_{k,j}$ where

$$T_{k,j} \in L(\ell_2(m(j), \ell_2(m(k)).$$

By T_0 we denote the block-diagonal map given by the $T_{j,j}$. Then, by Lemma 1.1, $T_1 := T - T_0$ defines a compact map $\Lambda_0^\alpha = \ell_2 \to \Lambda_\infty(\alpha)$. Therefore (or because $\|T_{j,j}\| \leq 1$ for all j) we have $T_0 \in L(\Lambda_\infty(\alpha)) \cap L(\ell_2)$. Since T_1 is a compact operator in ℓ_2 the operator T_0 is a Fredholm operator in ℓ_2. This implies that $N(T_{j,j}) \neq \{0\}$ for all $j \notin J$ where $J \subset \mathbb{N}_0$ is finite. Since T_0 has closed range in ℓ_2 there must exist a constant $C > 0$ such that $|T_{j,j}x| \geq \frac{1}{C}|x|$ for all $j \notin J$ and $x \in \ell_2(m(j))$. We define a block diagonal operator S_0 by $S_{j,j} := T_{j,j}^{-1}$ for $j \notin J$ and $S_{j,j} = 0$ otherwise. Then $S_0 \in L(\Lambda_\infty(\alpha)) \cap L(\ell_2)$ and $T_0 S_0 x = S_0 T_0 x = x - \sum_{j \in J} x_j e_j$. So T_0 is a Fredholm operator in $L(\Lambda_\infty(\alpha))$ and therefore also $T = T_0 + T_1$. T is injective therefore it is an isomorphism onto a finite codimensional closed subspace of $\Lambda_\infty(\alpha)$. Since $\Lambda_\infty(\alpha)$ is unstable this means that $R(T) = \Lambda_\infty(\alpha)$. We have shown that T is an isomorphism. □

This does not mean that any $\Lambda_\infty(\alpha)$ which is not shift-stable fulfills the assertion of Theorem 2.2. For that we modify Example 1.

Example 2. We set $\alpha_k = 2^{k+j^2}$ for $j^2 < k \leq (j+1)^2$. We define T by $Te_{(j+1)^2} = e_{j^2+1}$ for $j \in \mathbb{N}_0$, $Te_0 = e_0$ and $Te_k = e_{k+1}$ otherwise. Then $\limsup \alpha_{k+1}/\alpha_k = +\infty$, that is, $\Lambda_\infty(\alpha)$ is not shift-stable. The proof that T is a unitary endomorphism which is not invertible works like in Example 1.

One can, of course, generalize our concept. A unitary map in ℓ_2 which maps $\Lambda_\infty(\alpha)$ into $\Lambda_\infty(\beta)$ we call a unitary map from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$. By use of the closed range theorem we see that such a map is in $L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ and we have:

Lemma 2.3. *If there is a unitary map from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$ then $\Lambda_\infty(\alpha) \subset \Lambda_\infty(\beta)$.*

Proof. Let T be the unitary map in the assumption. Since T is continuous there are $t > 0$ and $\varepsilon > 0$ such that $T(\varepsilon U_t^\alpha) \subset U_1^\beta$ where in both spaces $U_t^* = \{x \in \Lambda_\infty(*) : |x|_t \leq t\}$. Therefore

$$e^{-\beta_n} = \delta_n(U_1^\beta, U_0) \geq \delta_n(T(\varepsilon U_t^\alpha), U_0) = \varepsilon\, \delta_n(U_t^\alpha, U_0) = \varepsilon\, e^{-t\alpha_n}.$$

This implies $\beta_n \leq t\,\alpha_n + \log(1/\varepsilon)$ which implies the result. □

If $\Lambda_\infty(\alpha) \subset \Lambda_\infty(\beta)$ then the identical imbedding is a unitary map from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$.

We conclude this section with a simple remark.

Remark 1. *Let T be a unitary endomorphism in $L(\Lambda_\infty(\alpha))$ then $R(T)$ has a basis.*

Proof. Let $x = T\xi \in R(T)$. Then we have

$$x = T\xi = \sum_{n=0}^{\infty} \xi_n T e_n = \sum_{n=0}^{\infty} \langle \xi, e_n \rangle T e_n = \sum_{n=0}^{\infty} \langle x, T e_n \rangle T e_n.$$

That means, for $x \in R(T)$ the expansion with respect to the orthonormal basis $(Te_n)_{n \in \mathbb{N}_0}$ converges in $\Lambda_\infty(\alpha)$. □

While for shift stable power series spaces a unitary endomorphism needs not to be invertible on $\Lambda_\infty(\alpha)$, for a certain class of shift stable power series spaces it has an inverse at least on certain infinite dimensional subspaces.

3. A class of power series spaces

Definition 2. *A power series space $\Lambda_\infty(\alpha)$ is called partially stable if α fulfills the following condition*

(1) $$\exists C\, \forall n\, \exists m : \alpha_{n+m} \leq C \alpha_m.$$

In the definition $C > 1$ can be chosen arbitrarily. We have:

Lemma 3.1. *For every $p > 1$ the following is equivalent to the definition:*

(2) $$\forall n\, \exists m : \alpha_{n+m} \leq p \alpha_m.$$

Proof. Obviously the negation of (1) is the following

(3) $$\forall C\, \exists n\, \forall m : \alpha_{n+m} > C \alpha_m.$$

And again it is obvious that for every $p > 1$ the following is equivalent to condition (3)

(4) $$\exists n\, \forall m : \alpha_{n+m} \geq p \alpha_m$$

which shows the claim. □

We assume without restriction of generality that $\alpha_0 \geq 1$. For given $p > 1$ we set $M_k = \{j : p^k \leq \alpha_j < p^{k+1}\}$ and we can write every $\Lambda_\infty(\alpha)$ isomorphically equivalent in the form

$$\begin{aligned}\Lambda_\infty(\alpha) &= \Big\{(\xi_j)_{j\in\mathbb{N}_0} \,:\, |\xi|_t^2 = \sum_{k=0}^{\infty}\Big(\sum_{j\in M_k}|\xi_j|^2\Big)e^{2tp^k} < \infty \text{ for all } t\in\mathbb{R}\Big\}\\ &\cong \Big\{(x_k)_{k\in\mathbb{N}_0} \in \prod_{k=0}^{\infty}\ell_2(n(k)) \,:\, |x|_t^2 = \sum_{k=0}^{\infty}|x_k|^2 e^{2tp^k} < \infty \text{ for all } t\in\mathbb{R}\Big\}\end{aligned}$$

where we have put $n(k) = \#M_k$. Here $|\ |$ denotes the norm in $\ell_2(\cdot)$. We call this the standard p-block representation.

Lemma 3.2. *$\Lambda_\infty(\alpha)$ is partially stable, if and only if, the sequence $n(k)$ is unbounded.*

Proof. If the sequence $n(k)$ is unbounded then $\Lambda_\infty(\alpha)$ ist partially stable by Lemma 3.1. If $\Lambda_\infty(\alpha)$ is partially stable then we choose m such that $\alpha_{2n+m} \le p\alpha_m$. We find k such that $m \in M_k$. Then $\alpha_{m+2n} \in M_k \cup M_{k+1}$. Therefore either $n(k) = \#M_k \ge n$ or $n(k+1) = \#M_{k+1} \ge n$. □

Therefore $\Lambda_\infty(\alpha)$ is not partially stable if, and only if, there is $n \in \mathbb{N}$ such that $n(k) \le n$ for all k. We set $\beta_j = p^{\frac{j}{n}}$. Then

$$\Lambda_\infty(\beta) \cong \Big\{(x_k)_{k\in\mathbb{N}_0} \in \prod_{k=0}^{\infty}\ell_2(n) \,:\, |x|_t^2 = \sum_{k=0}^{\infty}|x_k|^2 e^{2tp^k} < \infty \text{ for all } t\in\mathbb{R}\Big\}.$$

$n(k) \le n$ for all k is equivalent to $\Lambda_\infty(\alpha)$ being a block-subspace of $\Lambda_\infty(\beta)$. This means that there is a subsequence $(k_j)_{j\in\mathbb{N}_0}$ and $C > 0$ such that

$$\frac{1}{C}\,p^{k_j/n} \le \alpha_j \le C\,p^{k_j/n}.$$

for all $j \in \mathbb{N}_0$. We have shown:

Lemma 3.3. *$\Lambda_\infty(\alpha)$ is not partially stable if, and only if, there is $q > 1$, $C > 0$ and a subsequence $(k_j)_{j\in\mathbb{N}_0}$ such that*

$$\frac{1}{C}\,q^{k_j} \le \alpha_j \le C\,q^{k_j} \tag{5}$$

for all $j \in \mathbb{N}_0$.

In particular, estimate (5) implies that $\liminf_j \alpha_j^{1/j} > 1$. It is easily seen that the reverse implication does not hold.

So we have:

Corollary 3.4. *If $\liminf_j \alpha_j^{1/j} = 1$ then $\Lambda_\infty(\alpha)$ is partially stable.*

Example 3. Every stable space $\Lambda_\infty(\alpha)$ is partially stable. If $\alpha_j = e^{f(j)}$ and $\lim_j f(j)/j = 0$ then $\Lambda_\infty(\alpha)$ is partially stable.

Definition 3. *$\Lambda_\infty(\alpha)$ is called strongly shift-stable if it has an infinite codimensional complemented subspace which is isomorphic to $\Lambda_\infty(\alpha)$.*

Lemma 3.5. *$\Lambda_\infty(\alpha)$ is strongly shift-stable if, and only if, there exists $p > 0$ such that for every $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ with $\alpha_{m+n} < p\,\alpha_m$ for all $m \geq m(n)$.*

Proof. If F is infinite dimensional, $F \oplus \Lambda_\infty(\alpha) = \Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ is the associated power series space of F then obviously $\Lambda_\infty(\alpha) = \Lambda_\infty(\beta) \oplus \Lambda_\infty(\alpha)$ as sets and therefore as Fréchet spaces . This implies that there is $p > 0$ and a subsequence $(k(n))_{n\in\mathbb{N}_0}$ such that $k(n) - n$ is nondecreasing and unbounded and $\alpha_{k(n)} < p\,\alpha_n$ for all n. If, on the other hand such a subsequence exists then $\Lambda_\infty(\alpha)$ is obviously strongly shift-stable.

It is easily seen that the existence of such a sequence $(k(n))_{n\in\mathbb{N}_0}$ is equivalent to our condition on α. □

From this we see that $\Lambda_\infty(\alpha)$ is strongly shift-stable if, and only if, the sequence $n(k)$ from the standard p-block representation is nondecreasing and unbounded. From this we see the following:

Example 4. Every strongly shift-stable space $\Lambda_\infty(\alpha)$ is partially stable.

However there are also examples of partially stable spaces which are far from being stable or even shift stable. In fact, they are strongly unstable.

If $\Lambda_\infty(\alpha)$ is blockwise unstable, then we have in a natural way

$$\Lambda_\infty(\alpha) \cong \Big\{ x = (x_0, x_1, \dots) \in \prod_k \ell_2(m(k)) \;:\; |x|_t^2 = \sum_k |x_k|^2 e^{2t\beta_k} < +\infty \text{ for all } t \in \mathbb{R} \Big\}$$

where $m(k) = n(k+1) - n(k)$ and $\ell_2(m(k))$ is the $m(k)$-dimensional Hilbert space. From this we see immediately:

Proposition 3.6. *If $\Lambda_\infty(\alpha)$ is blockwise unstable and $\{m(k) : k \in \mathbb{N}_0\}$ is unbounded, then $\Lambda_\infty(\alpha)$ is partially stable.*

The definition of partial stability we can also rewrite in the form: There is a nondecreasing unbounded sequence of integers $(m(n))_{n\in\mathbb{N}_0}$ such that

$$\limsup_{n\to\infty} \frac{\alpha_{n+m(n)}}{\alpha_{m(n)}} < \infty. \tag{6}$$

$(m(n))_{n\in\mathbb{N}_0}$ will denote from now on always such a sequence.

4. Partial inverses

We reformulate [12, Lemma 13] in a way suitable for our purposes.

Lemma 4.1. *If $A \in L(\ell_2)$ and $a_{k,j} = 0$ for $\alpha_k > \beta_j$ then $A \in L(\Lambda_\infty(\beta), \Lambda_\infty(\alpha))$.*

The method for the construction of S in the following Lemma goes back to Aytuna, Krone and Terzioğlu [1]. We modify the construction for the nonnuclear case, as given in [12]. The difference here is, that we replace stability by the much weaker condition of partial stability.

Theorem 4.2. *Let $\Lambda_\infty(\alpha)$ be partially stable and T a unitary endomorphism of $\Lambda_\infty(\alpha)$ Let $(m(n))_{n\in\mathbb{N}_0}$ be a nondecreasing unbounded sequence of integers fulfilling (6). Then there is $S \in L(\Lambda_\infty(\alpha))$, such that $P = T \circ S$ is a projection in $\Lambda_\infty(\alpha)$, orthogonal in ℓ_2, and $R(P) \cong \Lambda_\infty(\beta)$ where $\beta_n = \alpha_{m(n)}$.*

Proof. Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \Lambda_\infty(\alpha)$ and $f_j = Te_j$. We choose inductively vectors $g_n \in \Lambda_\infty(\alpha)$ with following properties:

(1) $g_n \in \operatorname{span}\{f_0, \ldots, f_{n+m(n)}\}$
(2) $g_n \perp g_0, \ldots, g_{n-1}$ in ℓ_2
(3) $g_n \perp e_0, \ldots, e_{m(n)-1}$ in ℓ_2
(4) $|g_n|_0 = 1$.

This is possible since $\dim \operatorname{span}\{f_0, \ldots, f_{n+m(n)}\} = n + m(n) + 1$. Due to (1) we have

$$g_n := \sum_{k=0}^{n+m(n)} \mu_{k,n} f_k = T\left(\sum_{k=0}^{n+m(n)} \mu_{k,n} e_k\right).$$

We set

$$h_n = \sum_{k=0}^{n+m(n)} \mu_{k,n} e_k$$

and obtain an orthonormal system $(h_n)_{n\in\mathbb{N}_0}$. We set $\mu_{k,n} = 0$ for $k > n + m(n)$.

We define

$$Sx := \sum_{n=0}^{\infty} \langle x, g_n \rangle h_n.$$

This means $S = T^{-1} \circ P$ where P is the orthogonal projection onto $\overline{\operatorname{span}\{g_0, g_1, \ldots\}}$. We have to show that S defines a map in $L(\Lambda_\infty(\alpha))$.

We do that in two steps. First we define a map $\psi \in L(\ell_2)$ by

$$\psi(x) = \sum_{n=0}^{\infty} \langle x, e_n \rangle h_n.$$

The matrix elements are $\psi_{k,j} = \langle \psi e_j, e_k \rangle = \langle h_j, e_k \rangle$. Therefore $\psi_{k,j} = 0$ for $k > j + m(j)$. By Lemma 4.1 we obtain that $\psi \in L(\Lambda_\infty(\gamma), \Lambda_\infty(\alpha))$ where $\gamma_n = \alpha_{n+m(n)}$.

Now we define a map $\varphi \in L(\ell_2)$ by

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x, g_n \rangle e_n.$$

The matrix elements are $\varphi_{k,j} = \langle \varphi e_j, e_k\rangle = \langle e_j, g_k\rangle$. This means that $\varphi_{k,j} = \langle \varphi e_j, e_k\rangle = 0$ if $m(k) > j$. Therefore, by Lemma 4.1, $\varphi \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ where $\beta_n = \alpha_{m(n)}$.

By assumption we have $\Lambda_\infty(\beta) = \Lambda_\infty(\gamma)$.

Since obviously $S = \psi\circ\varphi$ we have shown that $S \in L(\Lambda_\infty(\alpha))$. Therefore $P = T\circ S$ is a continuous projection in $\Lambda_\infty(\alpha)$. We show that $R(P) \cong \Lambda_\infty(\gamma)$.

The map $T\circ\psi \in L(\Lambda_\infty(\gamma), R(P))$ is injective and, because of $(T\circ\psi)\circ\varphi = T\circ S = P$, also surjective, hence it is an isomorphism. □

5. Examples

To consider a concrete non stable case we assume that $\alpha_n = e^{f(n)}$ where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuously differentiable, increasing and strictly concave for large t. We assume moreover that $\lim_{t\to\infty} f'(t) = 0$ and we put $h(t) = 1/f'(t)$.

Lemma 5.1. *In this case the space $\Lambda_\infty(\alpha)$ is strongly shift-stable.*

Proof. We verify the criterion in Lemma 3.5 with $p = e$. For every n we have to find $m(n)$ such that $f(m+n) - f(m) \le 1$ for $m \ge m(n)$. Since to $f(m+n) - f(m) \le f'(m)\, n$ and $f'(m) \to 0$ this is possible. □

So $\Lambda_\infty(\alpha)$ is also partially stable and we have:

Proposition 5.2. *$m(n)$ in Lemma 4.2 may be chosen as $h^{-1}(cn)$ where $c > 0$. This means P is a projection onto a subspace isomorphic to $\Lambda_\infty(\beta)$ with $\beta_n = e^{f(h^{-1}(cn))}$.*

Proof. We fix $C > 1$ and choose $m(n)$ so that $f(n+m(n)) - f(m(n)) \le C$ for large n. With the choices we have made this follows from the argument in the proof of Lemma 5.1. □

Example 5. If $\alpha_n = e^{n^{\frac{1}{s}}}$ with $s > 1$ then we may choose $\beta_n = e^{n^{\frac{1}{s-1}}}$.

Example 6. If $\alpha_n = e^{(\log(n+1))^s}$ with $s > 1$ then we may choose

$$\beta_n = e^{(\log(n+1)+(s-1)\log\log(n+1))^s}.$$

Proof. With $f(t) = (\log(t+1))^s$ we have

$$f'(t) = s\,\frac{(\log(t+1))^{s-1}}{t+1}.$$

With $m(n) = (n+1)(\log(n+1))^{s-1}$ we obtain

$$nf'(m(n)) = sn\,\frac{(\log(n+1) + (s-1)\log\log(n+1))^{s-1}}{(n+1)(\log(n+1))^{s-1}+1}.$$

Since $\lim_{n\to\infty} nf'(m(n)) = s$ the sequence $m(n)$ may be used in Lemma 4.2. This implies the result. □

6. Complemented subspaces

Unitary endomorphisms of power series spaces occur in various contexts. One of the most important is described in this section. The following notation goes back to Terzioğlu [10].

Definition 4. *Let E be a Fréchet-Hilbert-Schwartz space with properties* (DN)*and* (Ω)*. Let $\|\cdot\|_0$ be a Hilbertian dominating norm, $\|\cdot\|_1$ chosen for $\|\cdot\|_0$ according to (Ω) and*

$$\alpha_n = -\log \delta_n(U_1, U_0)$$

where $U_j = \{x \in E \,:\, \|x\|_j \leq 1\}$. Then $\Lambda_\infty(\alpha)$ is called the associated power series space of s.

We will use the following fundamental Lemma (see [12, Corollary 5]):

Lemma 6.1. *Let E be a Fréchet-Hilbert-Schwartz space with properties* (DN)*and* (Ω) *and $\Lambda_\infty(\alpha)$ its associated power series space. Then there exist maps $\psi \in L(\Lambda_\infty(\alpha), E)$ and $\varphi \in L(E, \Lambda_\infty(\alpha))$ such that ψ extends to a unitary map $\psi_0 : \ell_2 \to E_0$ and φ extends to a unitary map $\varphi_0 : E_0 \to \ell_2$.*

In this case the combination $T := \varphi \circ \psi$ obviously extends to a unitary endomorphism of $\Lambda_\infty(\alpha)$.

Theorem 6.2. *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated power series space $\Lambda_\infty(\alpha)$ is partially stable, then E has a complemented subspace isomorphic to $\Lambda_\infty(\beta)$, β like in Theorem 4.2.*

Proof. From Lemma 6.1 we get $\varphi \in L(E, \Lambda_\infty(\alpha))$, $\psi \in L(\Lambda_\infty(\alpha), E)$ such that $T := \varphi \circ \psi$ extends to a unitary map in $L(\ell_2)$. Then by Theorem 4.2 we get $S \in L(\Lambda_\infty(\alpha))$, such that $P = T \circ S$ is a projection in $\Lambda_\infty(\alpha)$ with $R(P) \cong \Lambda_\infty(\beta)$.

We set $\pi := \psi \circ S \circ P \circ \varphi \in L(E)$ and obtain a projection. $P \circ \varphi \in L(R(\pi), R(P))$ is an isomorphism, since $\psi \circ S|_{R(P)}$ is its inverse. □

Example 7. If E is a Fréchet-Hilbert-Schwartz space with properties (DN) and (Ω) and its associated power is one of the spaces $\Lambda_\infty(\alpha)$ as considered in Section 5 the E has a complemented subspace isomorphic to $\Lambda_\infty(\beta)$ chosen as in Section 5.

The most important example, of course, is that of a space E with stable associated power series space. In this case we can choose $m(n) = n$, that is $\Lambda_\infty(\beta) = \Lambda_\infty(\alpha)$ and we have:

Lemma 6.3. *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated power series space $\Lambda_\infty(\alpha)$ is stable then E has a complemented subspace isomorphic to $\Lambda_\infty(\alpha)$.*

And we obtain the following fundamental theorem which was proved for the nuclear case in Aytuna-Krone-Terzioğlu [1], and for general Fréchet-Hilbert spaces in [12].

Theorem 6.4. *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated power series space $\Lambda_\infty(\alpha)$ is stable then $E \cong \Lambda_\infty(\alpha)$.*

An easy corollary of this theorem is the following theorem, which follows also from [12, Theorem 5] which is based on a different line of arguments.

Theorem 6.5. *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *then there is a power series space $\Lambda_\infty(\beta)$ such that E is a complemented subspace of $\Lambda_\infty(\beta)$.*

Proof. Let $\Lambda_\infty(\alpha)$ be the associated power series space of E. Let $\Lambda_\infty(\gamma)$ be another power series space, then we denote by β the increasing reordering of $\alpha_0, \gamma_0, \alpha_1, \gamma_1, \dots$. Now it is easy to find γ, such that β is stable. Since obviously $\Lambda_\infty(\beta)$ is the associated power series space of $E \oplus \Lambda_\infty(\gamma)$, we obtain $E \oplus \Lambda_\infty(\gamma) \cong \Lambda_\infty(\beta)$. Hence E is isomorphic to a complemented subspace of $\Lambda_\infty(\beta)$. □

The following Theorem was shown in a somewhat different way in [12, Theorem 7], for the nuclear case see Wagner [13, Theorem 5] and Kondakov [7].

Theorem 6.6. *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated power series space $\Lambda_\infty(\alpha)$ is tame then $E \cong \Lambda_\infty(\alpha)$.*

Proof. This is an immediate consequence of Lemma 6.1 and Theorem 2.2. □

7. Projections

Another important setting where unitary endomorphisms, even isomorphisms, appear is described in the following. Let $\Lambda_\infty(\alpha)$ be a power series space and P a continuous projection in $\Lambda_\infty(\alpha)$.

We will use the following Lemma which is shown in [12, Lemma 9]. In the nuclear case it follows from Kondakov [6].

Lemma 7.1. *If $\| \;\|$ is a Hilbert norm on $\Lambda_\infty(\alpha)$ and $|\;|_0 \leq \|\;\| \leq C |\;|_\tau$, $C > 1$. Then there is an automorphism U of $\Lambda_\infty(\alpha)$ such that $|Ux|_0 = \|x\|$ and*

$$|x|_t \leq |Ux|_t \leq C\,|x|_{t+\tau}$$

for all $x \in \Lambda_\infty(\alpha)$, $t \geq 0$.

Theorem 7.2. *If P is a continuous projection in $\Lambda_\infty(\alpha)$ then there is an automorphism U of $\Lambda_\infty(\alpha)$ such that UPU^{-1} is an orthogonal projection in ℓ_2.*

Proof. We set $\|x\|^2 = |Px|_0^2 + |Qx|_0^2$ where $Q = I - P$. Then $\|\ \|$ fulfills the assumption of Lemma 7.1. Hence there is an automorphism U of $\Lambda_\infty(\alpha)$ such that $|Ux|_0 = \|x\|$ for all $x \in \Lambda_\infty(\alpha)$. We set $\widetilde{P} = UPU^{-1}$, then $\widetilde{P}$ is a continuous projection in $\Lambda_\infty(\alpha)$ and we obtain $|\widetilde{P}x|_0 = \|PU^{-1}x\| = |PU^{-1}x|_0 \leq \|U^{-1}x\| = |x|_0$ and $\langle \widetilde{P}x, \widetilde{Q}y\rangle_0 = (PU^{-1}x, QU^{-1}y) = 0$ where $\langle\cdot\,,\cdot\rangle_t$ denotes the scalar product belonging to $|\ \ |_t$, $(\cdot\,,\cdot)$ that belonging to $\|\ \ \|$ and $\widetilde{Q} = I - \widetilde{P}$. □

From now on we assume that our projection P is orthogonal with respect to $\langle\cdot,\cdot\rangle_0$ which is the scalar product of ℓ_2 and we set

$$T := P - Q.$$

Then T is a unitary isomorphism of $\Lambda_\infty(\alpha)$. It is self-adjoint in ℓ_2 and $T^2 = I$.

We will need the following interpolation lemma which we quote in a slightly more precise form from [8, 29.17].

Lemma 7.3. *Let seminorms $\|\ \|_0 \leq \|\ \|_1 \leq \|\ \|_2$ be given on the vector spaces G and H, such that for suitable numbers $0 < \tau < \theta < 1$ and $C > 0$ the following hold:*

(1) $\|\ \|_1^* \leq C \|\ \|_0^{*\,1-\theta} \|\ \|_2^{*\,\theta}$ *on* $(G, \|\ \|_0)'$.
(2) $\|\ \|_1 \leq C \|\ \|_0^{1-\tau} \|\ \|_2^{\tau}$ *on* H.

Then for every linear map $A : G \to H$ satisfying $\|Ax\|_j \leq C_j \|x\|_j$ for all $x \in G$ and $j = 0, 2$ the following also holds:

$$\|Ax\|_1 \leq DCC_0^{1-\tau} C_2^{\tau} \|x\|_1 \text{ for all } x \in G$$

where $D = D(\tau/\theta) = 4\Big(1 - 2^{-(1-\frac{\tau}{\theta})}\Big)^{-1}$.

In particular, the constant D can be chosen uniformly for given $0 < \gamma < 1$ and $\tau/\theta \leq \gamma$.

And we will need the following fact:

Lemma 7.4. *Let $\|\ \|$ be a norm on $\Lambda_\infty(\alpha)$ with $|x|_0^2 \leq \|x\|\, |x|_1$ then*

$$|x|_k \leq \|x\|^{\frac{K-k}{1+K}} |x|_K^{\frac{1+k}{1+K}}$$

for all $x \in \Lambda_\infty(\alpha)$ and $1 \leq k < K$.

Proof. We use that

$$|x|_1 \leq |x|_0^{\frac{t-1}{t}} |x|_t^{\frac{1}{t}}$$

for all $t > 0$ and $x \in \Lambda_\infty(\alpha)$ hence

$$\frac{|x|_1}{|x|_0} \leq \left(\frac{|x|_t}{|x|_0}\right)^{\frac{1}{t}}.$$

This implies

$$\frac{|x|_0}{\|x\|} \le \frac{|x|_1}{|x|_0} \le \left(\frac{|x|_t}{|x|_0}\right)^{\frac{1}{t}}.$$

So we have

$$|x|_0 \le \|x\|^{\frac{t}{1+t}} |x|_t^{\frac{1}{1+t}} \tag{7}$$

and we obtain using this for $t = K$

$$|x|_k \le |x|_0^{\frac{K-k}{K}} |x|_K^{\frac{k}{K}} \le \|x\|^{\frac{K}{1+K}\frac{K-k}{K}} |x|_K^{\frac{1}{1+K}\frac{K-k}{K}} |x|_K^{\frac{k}{K}} = \|x\|^{\frac{K-k}{1+K}} |x|_K^{\frac{1+k}{1+K}}$$

for all $x \in \Lambda_\infty(\alpha)$ and $1 \le k < K$. □

Lemma 7.5. *If $\| \ \|$ is a Hilbert norm on $\Lambda_\infty(\alpha)$ with $\|x\| \le C\,|x|_0$ and $|x|_0^2 \le \|x\|\ |x|_1$ for all $x \in \Lambda_\infty(\alpha)$. Then there is an automorphism U of $\Lambda_\infty(\alpha)$ such that $|Ux|_0 = \|x\|$ and*

$$|Ux|_t \le C\,|x|_t$$

for all $x \in \Lambda_\infty(\alpha)$, $t \ge 0$.

Proof. We denote by H_0 the Hilbert space generated by $\| \ \|$ and by $(\ , \ )$ its scalar product. For every $K > \tau$ we consider the canonical map $\imath_K^0 \colon \Lambda_K^\alpha \hookrightarrow H_0$. It is compact, let s_n be the singular numbers. We set

$$\beta_n = -\frac{1}{K}\log s_n.$$

Then the Schmidt representation takes the form

$$\imath_K^0 x = \sum_{n=0}^{\infty} e^{-K\beta_n} \langle x, e_n\rangle_K f_n, \tag{8}$$

where $(e_n)_n, (f_n)_n$ are orthonormal bases in Λ_K^α and H_0, respectively. If we set $U_t = \{x : |x|_t \le 1\}$, $V = \{x : \|x\| \le 1\}$ then $\frac{1}{C}U_0 \subset V$ leads to

$$\delta_n(U_K, V) \le C\,\delta_n(U_K, U_0)$$

i.e.

$$e^{-K\beta_n} \le C\,e^{-K\alpha_n}. \tag{9}$$

We put

$$u_K x = ((x, f_n))_{n\in\mathbb{N}_o}.$$

Then we have

$$|u_K x|_0 = \|x\| \le C\,|x|_0$$

and, by use of (9),

$$\begin{aligned} \frac{1}{C}|u_K x|_K &= \frac{1}{C}\left(\sum_{n=0}^{\infty} e^{2K\alpha_n} |(x, f_n)|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=0}^{\infty} e^{2K\beta_n} |(x, f_n)|^2\right)^{\frac{1}{2}} = |x|_K. \end{aligned}$$

By use of [12, Lemma 9] we obtain

$$|u_K x|_t \le C|x|_t$$

for all $0 \le t \le K$.

Now we have to show lower estimates for the norms $|u_K x|_k$.

By estimate (7) we have

$$|x|_0 \le \|x\|^{\frac{K}{1+K}} |x|_K^{\frac{1}{1+K}}.$$

For the following calculation see Terzioğlu [10] (cf. [11]).

We set $B_k = U_k^\circ$, $B = V^\circ$ (the respective polar sets) and $\tau = \frac{1}{1+K}$. We obtain

$$B_0 \subset C_\tau^{-1}(r^\tau B + \frac{1}{r^{1-\tau}} B_K)$$

where $C_\tau = \inf_{s>0}(s^\tau + s^{\tau-1})$.

For $d > \delta_n(B, B_K)$ we find an at most n-dimensional subspace F such that $B \subset d\, B_K + F$ and we obtain

$$B_0 \subset C_\tau^{-1}(r^\tau d + \frac{1}{r^{1-\tau}}) B_K + F.$$

Since this holds for all $r > 0$ we get

$$\delta_n(B_0, B_K) \le C_\tau^{-1} \inf_{r>0}(r^\tau d + \frac{1}{r^{1-\tau}}) = d^{1-\tau}$$

and we have shown that

$$\delta_n(B_0, B_K) \le \delta_n(B, B_K)^{1-\tau}$$

and therefore that

$$e^{-K\alpha_n} \le e^{-(1-\tau)K\beta_n}.$$

This yields, using that $\frac{K}{1-\tau} = 1 + K$,

$$|x|_K = \left(\sum_n e^{2K\beta_n} |(x, f_n)|^2\right)^{\frac{1}{2}} \le \left(\sum_n e^{2\frac{K}{1-\tau}\alpha_n} |(x, f_n)|^2\right)^{\frac{1}{2}} = |u_K x|_{K+1}.$$

Due to equation (8) we have $f_n \in \Lambda_K^\alpha$ and, by definition, $u_K f_n = e_n$. Therefore $\Lambda_{K+1}^\alpha \subset u_K(\Lambda_K^\alpha)$ and $u_K^{-1} : \Lambda_{K+1}^\alpha \to \Lambda_K^\alpha$ is continuous with norm ≤ 1.

We want to use Lemma 7.3. We have

$$|x|_k^* \le |x|_0^{*\,\frac{K+1-k}{K+1}} |x|_{K+1}^{*\,\frac{k}{K+1}}$$

and, by Lemma 7.4,

$$|x|_k \le \|x\|^{\frac{K-k}{1+K}} |x|_K^{\frac{1+k}{1+K}}.$$

Since $\frac{1+k}{3k} \le \gamma := \frac{2}{3} < 1$ we have with a universal constant C

$$|x|_k \le C\, |u_K x|_{3k}$$

for all $k \in \mathbb{N}$ with $3k \le K + 1$.

For every $k \in \mathbb{N}$ the set $\{u_K : K \geq k\}$ is an equicontinuous subset of $L(\Lambda_k^\alpha)$. Since $\Lambda_t^\alpha \longrightarrow \Lambda_s^\alpha$ is compact for $t > s$ the set is relatively compact in $L(\Lambda_{k+1}^\alpha, \Lambda_{k-1}^\alpha)$ for every $k \in \mathbb{N}$. Therefore we may, by use of a diagonal procedure, find a subsequence u_{K_n}, such that $(u_{K_n})_n$ converges in $L(\Lambda_{k+1}^\alpha, \Lambda_{k-1}^\alpha)$ for every $k \in \mathbb{N}$.

Moreover, we have that for every $k \in \mathbb{N}$ the set $\{u_K^{-1} : K + 1 \geq 3k\}$ is an equicontinuous subset of $L(\Lambda_{3k}^\alpha, \Lambda_k^\alpha)$. By the same argument as before we may choose the subsequence so that also $(u_{K_n}^{-1})_n$ converges in $L(\Lambda_{3k+1}^\alpha, \Lambda_{k-1}^\alpha)$ for all $k \in \mathbb{N}$, and we set for $x \in \Lambda_\infty(\alpha)$:

$$Ux = \lim_{n\to\infty} u_{K_n} x, \quad Vx = \lim_{n\to\infty} u_{K_n}^{-1} x.$$

and certainly $U, V \in L(\Lambda_\infty(\alpha))$. Of course, we first take the limits in the local Banach spaces separately and then see that those results define elements $Ux \in \Lambda_\infty(\alpha)$, $Vx \in \Lambda_\infty(\alpha)$, respectively.

It can easily be seen that $UV = VU = \mathrm{id}$, hence U is an automorphism. We have

$$|Ux|_0 = \lim_n |u_{K_n} x|_0 = \|x\|$$

and we have for any $t > 0$

$$|Ux|_t = \lim_n |u_{K_n} x|_t \leq C|x|_t.$$

This proves the result. □

Remark 2. By [12, Lemma 11] we see that the matrix of U is blockwise upper triangular where the blocks are given by the sets of indices on whic α is constant. If, in particular, α is strictly increasing, then the matrix of U is upper triangular.

Corollary 7.6. *If $\| \ \|$ is a dominating norm on $\Lambda_\infty(\alpha)$ then there is an automorphism U of $\Lambda_\infty(\alpha)$ such that $|Ux|_0 = \|x\|$ for all $x \in \Lambda_\infty(\alpha)$.*

Proof. There is $t \in \mathbb{R}$ and $C > 0$ such that $\|x\| \leq C\,|x|_t$ for all $x \in \Lambda_\infty(\alpha)$. Then we choose $s \geq 0$ and $C' > 0$ such that $|x|_t^2 \leq C'\,\|x\|\,|x|_{t+s}$ for all $x \in \Lambda_\infty(\alpha)$.

We set $|||x||| := C'\,\|e^{-tA}x\|$ and $\beta = s\alpha$. Then $\Lambda_\infty(\beta) = \Lambda_\infty(\alpha)$, $|||\ |||$ is a dominating norm in $\Lambda_\infty(\beta)$, $|||x||| \leq C\,C'\,|x|_0$ and $|x|_0 \leq |||x||| \cdot |x|_1$ for all $x \in \Lambda_\infty(\beta)$. By Lemma 7.5 there is an automorphism $\widetilde{U}$ of $\Lambda_\infty(\beta)$ such that

$$|\widetilde{U}x|_0 = |||x|||. \tag{10}$$

We set $U = \frac{1}{C'}\widetilde{U}e^{tA}$ and insert $e^{tA}x$ into equation (10). This gives $|Ux|_0 = \|x\|$ for all $x \in \Lambda_\infty(\beta) = \Lambda_\infty(\alpha)$. Obviously U is an automorphism of $\Lambda_\infty(\alpha) = \Lambda_\infty(\beta)$. □

Corollary 7.7. *Let E be a Fréchet-Hilbert space. If $\| \ \|_0$ is a dominating Hilbert norm on E and $E \cong \Lambda_\infty(\alpha)$, then the isomorphism can be chosen so that it is unitary between E_0 and $\Lambda_0^\alpha = \ell_2$.*

Here E_0 denotes the local Hilbert space belonging to $\| \ \|_0$.

Proof. Let $T : \Lambda_\infty(\alpha) \to E$ be an isomorphism. We set $\|x\| := \|Tx\|_0$ on $\Lambda_\infty(\alpha)$. Then $\| \ \|$ is a dominating norm on $\Lambda_\infty(\alpha)$ and, by Corollary 7.7, we obtain an automorphism U of $\Lambda_\infty(\alpha)$ such that $|Ux|_0 = \|x\| = \|Tx\|_0$ for all $x \in \Lambda_\infty(\alpha)$. Then $\widetilde{T} := T \circ U^{-1}$ is an isomorphism $\Lambda_\infty(\alpha) \to E$ and $|x|_0 = \|\widetilde{T}x\|_0$ for all $x \in \Lambda_\infty(\alpha)$. □

We return to the study of a projection P in $\Lambda_\infty(\alpha)$. We may assume that P is orthogonal in ℓ_2. We set $E = P(\Lambda_\infty(\alpha))$ and $F = Q(\Lambda_\infty(\alpha))$. If E and F have bases then there exist β and γ such that $E \cong \Lambda_\infty(\beta)$ and $F \cong \Lambda_\infty(\gamma)$. On E and F we consider the subspace topology inherited from $\Lambda_\infty(\alpha)$, then $| \ |_0$ is a dominating norm on both spaces. By use of Corollary 7.7 we may assume that we have isomorphisms $V_1 : \Lambda_\infty(\beta) \to E$ and $V_2 : \Lambda_\infty(\gamma) \to F$ which are unitary with respect to the zero norms.

If we choose an increasing reordering of $\beta_0, \gamma_0, \beta_1, \gamma_1, \ldots$ the we obtain, up to equivalence, α. In this way $\Lambda_\infty(\alpha) = \Lambda_\infty(\beta) \oplus \Lambda_\infty(\gamma)$ with diagonal projections P_β and P_γ onto the respective subspaces. We set $S = P_\beta - P_\gamma$. This is a diagonal map with only 1 and -1 as entries.

We consider the map

$$V : x \mapsto V_1 \circ P_\beta x + V_2 \circ P_\gamma x.$$

It is a unitary automorphism of $\Lambda_\infty(\alpha)$ which sends $\Lambda_\infty(\beta)$ to E and $\Lambda_\infty(\gamma)$ to F. So we have $V^{-1} \circ T \circ V = S$, where $T = P - Q$.

Setting $U := V^{-1}$ we have shown (1) ⇒ (2) of the following theorem. The other implication is obvious.

Theorem 7.8. *The following are equivalent:*

(1) *Every complemented subspace of $\Lambda_\infty(\alpha)$ has a basis.*
(2) *For every unitary automorphism T of $\Lambda_\infty(\alpha)$ with $T^2 = \mathrm{id}$ there is a unitary automorphism of $\Lambda_\infty(\alpha)$ such that UTU^{-1} is diagonal.*

This is special relevance for $\Lambda_\infty(\alpha) = s$, because all nuclear spaces with (DN) and (Ω), that is, all complemented subspaces of any nuclear power series space are isomorphic to a complemented subspace of s.

Theorem 7.9. *The following assertions are equivalent:*

(1) *Every complemented subspace of s has a basis.*
(2) *For every unitary automorphism T of s with $T^2 = \mathrm{id}$ there is a unitary automorphism of s such that UTU^{-1} is diagonal.*

From Theorem 7.9 we can derive the following equivalence of problems. We call it "Theorem" because it states the equivalence of two unknown conjectures.

"Theorem" 7.10. *The following assertions are equivalent:*

(1) *Every complemented subspace of a space* $\Lambda_\infty(\alpha)$ *has a basis.*
(2) *For every unitary automorphism* T *of any space* $\Lambda_\infty(\alpha)$ *with* $T^2 = \mathrm{id}$ *there is a unitary automorphism of* $\Lambda_\infty(\alpha)$ *such that* UTU^{-1} *is diagonal.*

We should remark that (2) in Theorem 7.9 describes the following problem for operators in ℓ_2. Let A be a positive self-adjoint operator in ℓ_2 with compact inverse, $\mathcal{G}_A := \{e^{-tA},\ t \geq 0\}$ the one parameter semigroup generated by $-A$ and $R(A^\infty) = \bigcap_{t\geq 0} R(e^{-tA})$ the asymptotic range space. By $\mathcal{U}(A^\infty)$ we denote the group of all unitary operators on ℓ_2 such U and U^* leave $R(A^\infty)$ invariant. Problem: Does for every $T \in \mathcal{U}(A^\infty)$ exist $U \in \mathcal{U}(A^\infty)$ such that UTU^* commutes with $\mathcal{G}_A$?

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