Hadamard type operators on spaces of real analytic functions in several variables

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Abstract

We consider multipliers on the space of real analytic functions of several variables $\mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, i.e., linear continuous operators for which all monomials are eigenvectors. If zero belongs to Ω these operators are just multipliers on the sequences of Taylor coefficients at zero. In particular, Euler differential operators of arbitrary order are multipliers. We represent all multipliers via a kind of multiplicative convolution with analytic functionals and characterize the corresponding sequences of eigenvalues as moments of suitable analytic functionals. Moreover, we represent multipliers via suitable holomorphic functions with Laurent coefficients equal to the eigenvalues of the operator. We identify in some standard cases what topology should be put on the suitable space of analytic functionals in order that the above mentioned isomorphism becomes a topological one when the space of multipliers inherits the topology of uniform convergence on bounded sets from the space of all endomorphisms on $\mathscr{A}(\Omega)$. We also characterize in the same cases when the discovered topology coincides with the classical topology of bounded convergence on the space of analytic functionals. We provide several examples of multipliers and show surjectivity results for multipliers on $\mathscr{A}(\Omega)$ if $\Omega \subset \mathbb{R}^d_+$.

1 Introduction

By a (Hadamard type) *multiplier* on the space of real analytic functions $\mathscr{A}(\Omega)$ we mean each linear continuous map $M : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ for which the monomials $x^\alpha := x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ are eigenvectors with a corresponding multiple sequence of eigenvalues $(m_\alpha)_{\alpha\in\mathbb{N}^d}$. Here Ω is an open nonempty subset of $\mathbb{R}^d$. It can be easily seen that if zero $\mathbb{0} := (0, \dots, 0)$ belongs to Ω then the map just multiplies the sequence of Taylor coefficients $(f_\alpha)_{\alpha\in\mathbb{N}^d}$ at zero of the function f by the *multiplier sequence* $(m_\alpha)_{\alpha\in\mathbb{N}^d}$. Nevertheless multipliers are not simply diagonal operators since the monomials never form a basis of $\mathscr{A}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^d$, see [13]. There are several natural examples of such operators, among them variable coefficient linear partial differential operators of Euler type (for more examples, see Section 9). Let us observe that the class of multipliers $M(\Omega)$ forms a closed commutative subalgebra of the algebra of all linear continuous operators on $\mathscr{A}(\Omega)$ equipped with the topology of uniform convergence on bounded subsets (= the strong topology).

In the holomorphic case our multipliers are called Hadamard multipliers since the holomorphic function whose Taylor coefficients sequence is just the coefficientwise product of the Taylor

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coefficient sequences of two other holomorphic functions f and g is called the Hadamard product $f \star g$. Moreover, such product is related to the Hadamard multiplication theorem [3, Ch. 1.4].

In the present paper we consider three main problems.

First, we find a representation of all multipliers on the space $\mathscr{A}(\Omega)$ for arbitrary open nonempty sets $\Omega \subset \mathbb{R}^d$ via analytic functionals $T \in \mathscr{A}(V(\Omega))'$, i.e., those analytic functionals $T \in \mathscr{A}(\mathbb{R}^d)'$ with

$$\operatorname{supp} T \subset V(\Omega) := \{x \in \mathbb{R}^d \mid x\Omega \subset \Omega\},$$

here multiplication is meant coordinatewise, see Theorem 2.2. An analogous result was proved for the one variable case (i.e., $\Omega \subset \mathbb{R}$) in [8] but the several variable case is essentially different.

In the one variable case it was proved in [8] that

$$\mathscr{B} : \mathscr{A}(V(\Omega))' \to M(\Omega), \quad \mathscr{B}(T)(g)(y) := \langle g(y\cdot), T\rangle$$

is surjective observing that every $M \in M(\Omega)$ corresponds to a functional $T \in \mathscr{A}(\mathbb{R})'$ with

$$\operatorname{supp} T \subset \widetilde{V}(\Omega) := \{x \in \mathbb{R} \mid x(\Omega \cap \mathbb{R}_*) \subset \Omega\},$$

where from now on $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$. In the one variable case this is enough since $\widetilde{V}(\Omega) = V(\Omega)$ but in the several variable case the latter is no longer true, see Remark 2.5 (b). Moreover, in the one-variable proof it was used extensively the so-called Köthe-Grothendieck duality which represents an analytic functional as a holomorphic function on the complement of its support. This is not available for the several variable case in general and that is why in the present proof of the representation theorem (i.e. bijectivity of $\mathscr{B}$) we substitute this by a new elementary approach which allows to avoid any representation of the analytic functional. Since the dilation set $V(\Omega)$ plays here and in further considerations a fundamental role, we collect in Section 4 topological and geometrical properties of the dilation sets.

The second problem considered here is to characterize sequences corresponding to multipliers in $M(\Omega)$, $\Omega \subset \mathbb{R}^d$ an arbitrary open nonempty set. Please note that since polynomials are dense in $\mathscr{A}(\Omega)$ the multiplier sequence uniquely determines the multiplier. In fact, the result mentioned in the previous paragraph gives a required description. Namely a sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$ is a multiplier sequence if and only if it is a moment sequence of some analytic functional $T \in \mathscr{A}(V(\Omega))'$ (see Theorem 2.2). In the one variable case we found also a characterization as coefficient sequences of Laurent series representations at zero of holomorphic functions on the complement of $V(\Omega)$ (see [8, Theorem 2.8]) using Köthe-Grothendieck representation of functionals. An analogous representation via Taylor coefficients was also given there. In the several variable case this does not work. Nevertheless in Section 3 (Theorem 3.5 and 3.8) we provide another representation of multipliers in $M(\Omega)$ with a multiplier sequence $(m_\alpha)_\alpha$ in terms of holomorphic functions with Laurent or Taylor coefficients $(m_\alpha)_\alpha$.

The third problem is the question which topology on $\mathscr{A}(V(\Omega))'$ is induced by $\mathscr{B}$ from $M(\Omega) \subset L_b(\mathscr{A}(\Omega))$, i.e., for which topology t on $\mathscr{A}(V(\Omega))'$ the map

$$\mathscr{B} : (\mathscr{A}(V(\Omega))', t) \to M(\Omega)$$

is a topological isomorphism, where $M(\Omega)$ is always equipped with the topology of uniform convergence on bounded sets inherited from the space of all linear continuous operators $L_b(\mathscr{A}(\Omega))$ on $\mathscr{A}(\Omega)$. This problem is considered in Section 7. In [8] considering the one variable case we propose a somehow naive conjecture that

$$\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega) \tag{1}$$

is always a topological isomorphism where b means the natural topology of uniform convergence on bounded sets. Here we prove that this conjecture is false in general (even in the one dimensional case!), see Theorem 8.7 and the remarks below. In fact, we show that a more promising candidate is a weaker topology: the so-called k-topology on $\mathscr{A}(V(\Omega))'$, i.e.,

$$\mathscr{A}(V(\Omega))'_k := \operatorname{proj}_{K \Subset \Omega} \mathscr{A}(V_K(\Omega))'_b$$

where K runs through all compact subsets of Ω and $V_K(\Omega) := \{x \in \mathbb{R}^d \mid xK \subset \Omega\}$. We prove (Theorem 7.2) that

$$\mathscr{B} : \mathscr{A}(V(\Omega))'_k \to M(\Omega) \tag{2}$$

is always continuous. In the natural cases like if either Ω is a convex set or $\Omega \subset \mathbb{R}^d_*$ or $\dim \Omega = 1$ then $\mathscr{B}$ as above in (2) is even a topological isomorphism (Theorem 7.14) but the conjecture that this is always the case remains open. The detailed information on the topology of $\mathscr{A}(V(\Omega))$ as well as on useful topologies on $\mathscr{A}(V(\Omega))'$ are collected in Sec. 5. Instead of the Köthe-Grothendieck duality so useful in the one-dimensional case we have to use here the so-called Tillmann-Grothendieck duality, i.e., a representation of analytic functionals via suitable chosen harmonic functions (or, more precisely, classes of such functions). Again for readers convenience we collect information on the Tillmann-Grothendieck duality in Sec. 6. A surprising consequence of the above theory (Proposition 8.11) means that $M(\mathbb{R}^d)$ is complemented in $L_b(\mathscr{A}(\mathbb{R}^d))$!

In general, the b-topology, the k-topology and the topology induced by $\mathscr{B}$ from $M(\Omega)$ are very close to each other: they have the same bounded sets and convergent sequences (Theorem 7.11). In Sec. 8 we show when these topologies are identical, i.e., when $\mathscr{B}$ in (1) is a topological isomorphism. If either $\dim \Omega = 1$ or $\Omega \subset \mathbb{R}^d_*$ or Ω is convex it holds if and only if $\partial V(\Omega) \cap V(\Omega)$ is compact and in the latter two cases if and only if $V(\Omega)$ is either compact or open (Theorems 8.7, 8.9). Sec. 8 is completed by several examples and simple results explaining when this is the case.

In Section 9, we collect examples of multipliers on spaces of analytic functions of several variables (Euler differential operators, integral multiplier operators, dilation operators and superposition multipliers) together with their basic properties and explain the role they play in the theory (see, for example, Proposition 9.1 or Theorem 9.3).

Finally, in Section 10, we consider the case $\Omega \subset \mathbb{R}^d_+$ where one can translate the problems on multipliers to problems on classical convolution operators. After explaining this translation we get some results on surjectivity of finite order Euler partial differential operators following the results of Hörmander and Langenbruch on partial differential operators.

The one variable case of multipliers on spaces of real analytic functions was studied in [8] (and further analyzed in [9], [10]). In [22], [23], Euler differential operators (which are special cases of multipliers) were considered also on $\mathscr{A}(\Omega)$. Korobeinik considered this type of variable coefficients linear differential operators in [25], [26]. The topic of Taylor coefficient multipliers is in fact very classical, already Hadamard considered such operators in [19, page 158 ff.]. There is an extensive literature on Hadamard type multipliers acting on spaces of holomorphic functions on open complex sets: see, for instance, [4], [5], [17], a series of papers of Müller and Pohlen [32], [33], [34] as well as a series of papers of Render (where the algebraic structure of $M(\Omega)$ is studied) see for example the survey paper [37].

The third named author [41] (comp. also [42]) considered the analogon of multipliers on the space of smooth functions and some of the ideas explained there are clearly inspiring for us.

Let us recall that the space of real analytic functions $\mathscr{A}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^d$ (or an *arbitrary* set Ω!) is endowed with its natural topology $\operatorname{ind}_U H(U)$, i.e., the locally convex inductive limit topology, where $U \subset \mathbb{C}^d$ runs through all neighborhoods of Ω in $\mathbb{C}^d$. With this topology it is clear that for any set $S \subset \mathbb{R}^d$ the space $\mathscr{A}(S)$ contains continuously $\mathscr{A}(\mathbb{R}^d)$ and, the latter is dense in the former space. This allows to identify any linear continuous functional $T \in \mathscr{A}(S)$ with a unique element of $\mathscr{A}(\mathbb{R}^d)'$. Moreover, by the very definition of the inductive topology any $T \in \mathscr{A}(S)'$ has to be continuous on any $H(U)$, U an open neighborhood of S in $\mathbb{C}^d$. Thus $\operatorname{supp} T \subset U \cap \mathbb{R}^d$ for any U as above, so $\operatorname{supp} T \subset S$. Hence if $T \in \mathscr{A}(S)'$ then $T \in \mathscr{A}(\mathbb{R}^d)'$ and $\operatorname{supp} T \subset S$ and the converse holds as well. By $\mathscr{A}(S)'_b$ we denote the dual equipped with the strong topology, i.e., the topology τ_b of uniform convergence on bounded subsets of $\mathscr{A}(S)$.

For more details and definitions related to analytic functionals see [38], for more information on the space of real analytic functions see the survey [6] or [30].

In the present paper we always use the coordinatewise multiplication:

$$xy := (x_1y_1, \ldots, x_dy_d), \qquad \text{if } x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d.$$

We define $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$. Clearly, if $y \in \mathbb{R}_*^d$ we may define

$$\frac{1}{y}\Omega = \left\{ \left(\frac{x_1}{y_1}, \ldots, \frac{x_d}{y_d} \right) : (x_1, \ldots, x_d) \in \Omega \right\}.$$

By $\mathbb{O}$, $\mathbb{1} \in \mathbb{R}^d$ we denote $\mathbb{O} := (0, \ldots, 0)$ and $\mathbb{1} := (1, \ldots, 1)$. For a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we set $|x| := \left(x_1^2 + \cdots + x_d^d\right)^{1/2}$ and $d(x, y) := |x - y|$. Moreover, $B_\varepsilon(x) = \{y \mid d(x, y) < \varepsilon\}$. We will use multiindices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and set $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \ldots \alpha_d!$. We denote $D := \{1, \ldots, d\}$. The space of holomorphic functions $H(U)$ is always equipped with the compact open topology.

For non-explained notions from Functional Analysis and Harmonic Function Theory see [31] and [1], respectively.

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2 The Representation Theorem

First, we introduce the so-called dilation sets (see [8, Sec. 2 and 3]). Let $\Omega \subset \mathbb{R}^d$ be an open set. Then we define the *dilation set* as follows:

$$V(\Omega) := \{x : x\Omega \subset \Omega\} = \bigcap_{y \in \Omega} \{x : xy \in \Omega\}.$$

It is very useful to have the following notation $S_\eta = \{x \mid x\eta \in S\}$ hence $V(\Omega) = \bigcap_{\eta \in \Omega} \Omega_\eta$. Clearly, for non-empty Ω holds $\mathbb{1} \in V(\Omega)$ so $V(\Omega)$ is non-empty.

Let us formulate the following observation.

Proposition 2.1 *For $\eta \in \mathbb{R}^d$ we set $I_\eta := \{j \leq d \mid \eta_j = 0\}$. Then*

$$S_\eta = \left\{ y = (y_1, \ldots, y_d) \mid \ \forall\, j \notin I_\eta : y_j = \frac{x_j}{\eta_j} \text{ where } x = (x_1, \ldots, x_d) \in S \text{ and } x_j = 0 \ \forall\, j \in I_\eta \right\}.$$

Hence S_η is a product of a subset $A \subset \mathbb{R}^{I'_\eta}$, $I'_\eta = \{1, \ldots, d\} \setminus I_\eta$, and of the space $\mathbb{R}^{I_\eta}$ such that A is open (closed, compact, resp.) whenever S is open (closed, compact, resp). In particular, for open (closed) S also S_η is open (closed).

More on dilation sets will be explained in Section 4.

The following is the fundamental representation theorem for multipliers via analytic functionals. Notice that multipliers are a kind of convolution over $\mathbb{R}^d$ with coordinatewise multiplication.

The Representation Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$ be an open set. The map*

$$\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega) \subset L_b(\mathscr{A}(\Omega)),$$
$$\mathscr{B}(T)(g)(y) := \langle g(y\cdot)\,, T\rangle, \qquad T \in \mathscr{A}(V(\Omega))', g \in \mathscr{A}(\Omega),$$

is a bijective linear map and the multiplier sequence of $\mathscr{B}(T)$ is equal to the sequence of moments of the analytic functional T, i.e. to $(\langle x^\alpha, T\rangle)_{\alpha\in\mathbb{N}^d}$.

Moreover, if M is a multiplier on $\mathscr{A}(\Omega)$ then for any $y \in \Omega \cap \mathbb{R}^d_*$, the analytic functional $T \in \mathscr{A}(\mathbb{R}^d)'$ defined as

$$T = \delta_y \circ M \circ M_{1/y} : \mathscr{A}((1/y)\Omega) \to \mathbb{C}, \tag{3}$$

(where $M_y(g)(\xi) := g(y\xi)$ and δ_y denotes the point evaluation at y) does not depend on y, its support is contained in $V(\Omega)$ and $M = \mathscr{B}(T)$.

In Section 7 we will show that $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega) \subset L_b(\mathscr{A}(\Omega))$ is always continuous and in Section 8 we will discuss the problem when this map is a topological isomorphism.

The proof of the representation theorem will be contained in the following two lemmas.

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^d$ be an open set. For any $T \in \mathscr{A}(V(\Omega))'$ and $f \in \mathscr{A}(\Omega)$ we set:*

$$M_T f(x) := \langle f(x\cdot), T\rangle.$$

Then $M_T \in M(\Omega)$ and the multiplier sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$ is equal to the sequence of moments of T, i.e., $m_\alpha = \langle y^\alpha, T_y\rangle$.

Proof: Fix an open subset $\Omega' \subset \Omega$ which is relatively compact in Ω and choose the open set

$$U := \{x \mid x\overline{\Omega'} \subset \Omega\}.$$

For any $f \in \mathscr{A}(\Omega)$ the function $(x, y) \mapsto f(xy)$ is real analytic on $\Omega' \times U$. It is well known that then the map $x \mapsto f(x\cdot)$ is real analytic from Ω' to $\mathscr{A}(U)$. Therefore for any $T \in \mathscr{A}(V(\Omega))' \subset \mathscr{A}(U)'$ the map

$$x \mapsto \langle f(x\cdot), T\rangle$$

is real analytic on Ω'. Since $\Omega' \Subset \Omega$ was chosen arbitrarily we have proved that $M_T f \in \mathscr{A}(\Omega)$.

It is easy to observe that

$$M_T x^\alpha = \langle y^\alpha, T_y\rangle x^\alpha.$$

Now, it remains to show that $M_T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is continuous. We apply de Wilde's closed graph theorem which holds for $\mathscr{A}(\Omega)$ ([31, 24.31], comp. [6, Cor. 1.28]). Let us fix $x \in \Omega$ then we get that

$$f \mapsto f(x\cdot)$$

is a continuous linear map $\mathscr{A}(\Omega) \to \mathscr{A}(\Omega_x)$, where $\Omega_x = \{y \mid xy \in \Omega\} \subset \mathbb{R}^d$. Clearly, Ω_x is an open neighborhood of $V(\Omega)$, thus $T \in \mathscr{A}(\Omega_x)'$ and

$$f \mapsto \langle f(x\cdot), T\rangle \in \mathscr{A}(\Omega)'.$$

This shows that the graph of $M_T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is closed. $\square$

We denote by $C(\Omega)$ the space of continuous functions on Ω with the compact open topology. Let us recall

$$\frac{1}{y}\Omega = \left\{ \left(\frac{x_1}{y_1}, \ldots, \frac{x_d}{y_d} \right) \mid (x_1, \ldots, x_d) \in \Omega \right\} \text{ if } y \in \mathbb{R}^d_*.$$

Let $I \subset \{1, \ldots, d\} =: D$ we set

$$\{0\}^I \times \mathbb{R}_*^{D \setminus I} := \{x = (x_1, \ldots, x_d) : x_j = 0 \text{ for every } j \in I \text{ and } x_j \neq 0 \text{ for every } j \notin I\}.$$

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^d$ be an open set. For every continuous linear map $M : \mathscr{A}(\Omega) \to C(\Omega)$ with all monomials as eigenvectors there is $T \in \mathscr{A}(V(\Omega))'$ such that $M = M_T$ as defined in Lemma 2.3. In particular, $M \in M(\Omega)$.*

Proof: For any $\eta \in \Omega \cap \mathbb{R}^d_*$ we define

$$T_\eta \in \mathscr{A}(\mathbb{R}^d)', \qquad T_\eta f := (M f_\eta)(\eta),$$

where $f_\eta(x) := f\left(\frac{x}{\eta}\right)$.

Let us note that

$$T_\eta x^\alpha = m_\alpha \qquad \text{for every } \alpha \in \mathbb{N}^d \text{ and } \eta \in \Omega \cap \mathbb{R}^d_*.$$

Since polynomials are dense in $\mathscr{A}(\Omega)$ (see [6, Th. 1.16, Conclusion (1) p. 10]) the functional T_η does not depend on η and $T_\eta = T \in \mathscr{A}(\mathbb{R}^d)'$ for every $\eta \in \Omega \cap \mathbb{R}^d_*$. Moreover, if we show that $\operatorname{supp} T \subset V(\Omega)$ then

$$M x^\alpha = M_T x^\alpha \qquad \text{for any } \alpha \in \mathbb{N}^d$$

and so $M = M_T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ continuously by Lemma 2.3.

Now, we start to prove $\operatorname{supp} T \subset V(\Omega)$. Since $\mathscr{A}(\Omega) = \operatorname{proj}_{K \Subset \Omega} H(K)$ for every compact set $L \Subset \Omega$ there is a compact set $K \Subset \Omega$ such that $M : H(K) \to C(L)$ is continuous. Since $M_{1/\eta} : H(K_\eta) \to H(K)$, $f \mapsto f_\eta$, is a linear continuous map thus for $\eta \in L \cap \mathbb{R}^d_*$ we have

$$T_\eta \in \mathscr{A}(K_\eta)' \qquad \text{i.e., } \operatorname{supp} T_\eta \subset K_\eta. \tag{4}$$

Take any $\eta \in \Omega$, $\eta \neq \mathbb{O}$. Clearly, for some $I \subset \{1, \ldots, d\}$ we have $\eta \in \Omega \cap (\{0\}^I \times \mathbb{R}_*^{D \setminus I})$. Without loss of generality we assume that $I = \{j+1, \ldots, d\}$. Thus

$$\eta = (\eta_1, \ldots, \eta_j, 0, \ldots, 0), \qquad \text{where } \eta_1, \ldots, \eta_j \neq 0$$

and for some $\varepsilon_0 > 0$ and every $0 < \varepsilon < \varepsilon_0$ we have

$$\eta_\varepsilon := (\eta_1, \ldots, \eta_j, \varepsilon, \ldots, \varepsilon) \in L \cap \mathbb{R}^d_*$$

for a fixed compact set $L \Subset \Omega$. We take $K \Subset \Omega$ such that $M : H(K) \to C(L)$ is continuous.

If $x \notin K_\eta$ there is a compact ball $B \subset \mathbb{R}^d \setminus K_\eta$ with the center x since K_η is closed (see Proposition 2.1). Then $\eta B \cap K = \emptyset$ and for $\varepsilon > 0$ small enough also $\eta_\varepsilon B \cap K = \emptyset$, i.e., $B \cap K_{\eta_\varepsilon} = \emptyset$. By (4)

$$\operatorname{supp} T_{\eta_\varepsilon} \cap B \subset K_{\eta_\varepsilon} \cap B = \emptyset.$$

Since $x \notin K_\eta$ was chosen arbitrarily we have proved

$$\operatorname{supp} T \subset K_\eta \subset \Omega_\eta.$$

If $\mathbb{O} \in \Omega$ this holds also for $\eta = \mathbb{O}$ since then $\Omega_\eta = \mathbb{R}^d$, and so $\operatorname{supp} T \subset V(\Omega)$. $\square$

Remark 2.5 (a) In fact, we have proved that in the definition of the multiplier we can relax some conditions. Every linear continuous map $M : \mathscr{A}(\Omega) \to C(\Omega)$ with monomials as eigenvectors is automatically a multiplier since it maps $\mathscr{A}(\Omega)$ continuously into $\mathscr{A}(\Omega)$.

(b) One of the difficulties in the proof of Theorem 2.2 is to show that the analytic functional T has a support contained in $V(\Omega)$. It is relatively easy to show that $\operatorname{supp} T \subset \bigcap_{y\in\Omega\cap\mathbb{R}^d_*} \frac{1}{y}\Omega =: \widetilde{V}(\Omega)$. Now, in the one dimensional case it holds always that $V(\Omega) = \widetilde{V}(\Omega)$. In the many variable case this is true for convex sets (see Proposition 4.3 below) but in general it is true neither for open $V(\Omega)$ (take $\Omega = \mathbb{R}^d \setminus \{\mathbb{O}\}$, then $\widetilde{V}(\Omega) = \mathbb{R}^d \setminus \{\mathbb{O}\} \neq \mathbb{R}^d_* = V(\Omega)$) nor for compact $V(\Omega)$ (see Example 8.10, also Section 4, Proposition 4.3).

In Theorem 2.2 we have established a linear isomorphism $\mathscr{B} : \mathscr{A}(V(\Omega))' \to M(\Omega)$. By this isomorphism the algebra $M(\Omega)$ induces the following multiplication on $\mathscr{A}(V(\Omega))'$

$$(T \star S)f = T_x(S_y f(xy)).$$

This is true, because by definition $\mathscr{B}(T \star S)f = (M_T \circ M_S)f$. Therefore $\mathscr{B} : (\mathscr{A}(V(\Omega))', \star) \to M(\Omega)$ is an algebra isomorphism.

So the situation is the following: $\mathscr{A}(\mathbb{R}^d)'$ is a commutative algebra and $\mathscr{A}(V(\Omega))'$ is a subalgebra for any open $\Omega \subset \mathbb{R}^d$. If $m_\alpha(T)$ and $m_\alpha(S)$ are the moment sequences of T and S, then $m_\alpha(T \star S) = m_\alpha(T)\, m_\alpha(S)$.

This observation will help us in the following section, where we represent the algebras $M(\Omega) \cong A(V(\Omega))'$ by algebras of holomorphic functions equipped with Hadamard multiplication of Laurent, resp. Taylor coefficients.

3 Representation via Hadamard Multiplication of Holomorphic Functions

In this section we will first represent $\mathscr{A}(V(\Omega))' \cong M(\Omega)$ by an algebra of holomorphic functions with Hadamard multiplication of Laurent coefficients. This will be done by the Cauchy transform, so the essential content of the following will be an exact description of the Cauchy transforms of elements in $\mathscr{A}(V(\Omega))'$. Most of it is well known (see e.g. [38]) but we will make it as self-contained as possible for the convenience of the reader.

For $T \in \mathscr{A}'(\mathbb{R}^d)$ and $z \in \mathbb{C}^d$, $z_j \neq 0$ for all j we set $\mathscr{C}(z) = \prod_{j=1}^d \frac{-1}{z_j}$. For any subset $B \subset \mathbb{R}^d$ we define

$$\mathscr{W}(B) = \{z \in \mathbb{C}^d \,:\, \xi_j \neq z_j \text{ for all } \xi \in B \text{ and } j = 1, \ldots, d\}.$$

For $z \in \mathscr{W}(\operatorname{supp} T)$ we define $\mathscr{C}_T(z) = T_\xi(\mathscr{C}(\xi - z)) = (T * \mathscr{C})(z)$.

If $\operatorname{supp} T \subset \{\xi \in \mathbb{R}^d \,:\, |\xi|_\infty \leq R\}$ then $(\mathbb{C} \setminus [-R, +R])^d \subset \mathscr{W}(\operatorname{supp} T)$ and $\mathscr{C}_T$ extends to a holomorphic function on $(\widehat{\mathbb{C}} \setminus [-R, +R])^d$, $\widehat{\mathbb{C}}$ denoting the Riemann sphere.

For $\min_j |z_j| > R$ the function $\mathscr{C}_T(z)$ is defined and holomorphic and it has the expansion

$$\mathscr{C}_T(z) = \frac{1}{z_1 \cdots z_d} \sum_{\alpha\in\mathbb{N}_0^d} T_\xi(\xi^\alpha) \frac{1}{z^\alpha} = \sum_{\alpha\in\mathbb{N}_0^d} m_\alpha \frac{1}{z^{\alpha+\mathbb{1}}}.$$

We obtain the following

Proposition 3.1 *$T \mapsto \mathscr{C}_T$ is an algebra isomorphism from the algebra $(\mathscr{A}'(\mathbb{R}^d), \star)$ to the algebra of all holomorphic functions on $(\widehat{\mathbb{C}} \setminus [-R, +R])^d$ for some $R > 0$, regular with value 0 in all infinite points of $\widehat{\mathbb{C}}^d$, equipped with Hadamard multiplication of the coefficients of the Laurent expansion around $(\infty, \ldots, \infty)$.*

Proof: Only surjectivity has to be shown. Let a function g of this type be given. In each variable z_j we use the polygonal path γ_j passing through the points $r+i\varepsilon$, $-r+i\varepsilon$, $-r-i\varepsilon$, $r-i\varepsilon$, $r+i\varepsilon$ and set S_j to be the convex hull of γ_j. Here $r>R$, and $\varepsilon>0$ is chosen so small that $\prod_{j\le d} S_j$ is contained in the domain of definition of a given function $f\in\mathscr{A}(\mathbb{R}^d)$. We define

$$T(f) := \Big(\frac{1}{2\pi i}\Big)^d \int_{\gamma_1}\dots\int_{\gamma_d} f(\zeta_1,\dots,\zeta_d)\, g(\zeta_1,\dots,\zeta_d)\, d\zeta_1\dots d\zeta_d. \tag{5}$$

Since this can be done for every $\varepsilon>0$ and is independent of ε we have defined $T\in\mathscr{A}'([-r,+r]^d)$. Clearly $\mathscr{C}_T = g$. □

To determine the Cauchy transforms of the subalgebras $\mathscr{A}'(V(\Omega))$ of $\mathscr{A}'(\mathbb{R}^d)$ we need the following definition:

Definition 3.2 *For any holomorphic function f on $(\mathbb{C}\setminus\mathbb{R})^d$ we define the closed set $\sigma(f)\subset\mathbb{R}^d$ in the following way: $x\in\mathbb{R}^d$ is not in $\sigma(f)$ if there exist neighborhoods $U_j\subset\mathbb{R}$ of x_j and holomorphic functions f_j on $(\mathbb{C}\setminus\mathbb{R})^d\cup((\mathbb{C}\setminus\mathbb{R})^{j-1}\times U_j\times(\mathbb{C}\setminus\mathbb{R})^{d-j})$ such that $f=f_1+\dots+f_d$ on $(\mathbb{C}\setminus\mathbb{R})^d$.*

Let us remark that $\sigma(T)$ is the support of the hyperfunction determined by f (see [38]).

Proposition 3.3 $\operatorname{supp} T = \sigma(\mathscr{C}_T)$.

Proof: Let $Q := \prod_{j=1}^d]a_j,b_j[$ and $Q\cap\operatorname{supp}T=\emptyset$. By [38, Théorème 121,(4)], there are $T_j\in\mathscr{A}'(\{x\in\mathbb{R}^d : x_j\notin]a_j,b_j[\})$ such that $T=T_1+\dots+T_d$. So we have $\{x : x_j\in]a_j,b_j[\}\subset\mathscr{W}(\operatorname{supp}T_j)$ and $\mathscr{C}_T=\mathscr{C}_{T_1}+\dots+\mathscr{C}_{T_d}$ where $\mathscr{C}_{T_j}$ is holomorphic on $(\mathbb{C}\setminus\mathbb{R})^d\cup((\mathbb{C}\setminus\mathbb{R})^{j-1}\times]a_j,b_j[\times(\mathbb{C}\setminus\mathbb{R})^{d-j})$ which shows that $Q\cap\sigma(\mathscr{C}_T)=\emptyset$. Therefore $\sigma(\mathscr{C}_T)\subset\operatorname{supp}T$.

It remains to show that $\operatorname{supp}T\subset\sigma(\mathscr{C}_T)$. Assume $x\notin\sigma(\mathscr{C}_T)$. Then there is a neighborhood Q of x such that $Q\cap\sigma(\mathscr{C}_T)=\emptyset$ and we may assume that $\mathscr{C}_T=f_1+\dots+f_d$, where by Proposition 3.1, $f_j=\mathscr{C}_{T_j}$, $j=1,\dots,d$ and $T=T_1+\dots+T_d$. We choose r large enough, arbitrary $\delta>0$ and replace γ_j in (5) applied to T_j instead of T with the union of two polygonal paths going through the points $r+i\varepsilon$, $b_j-\delta+i\varepsilon$, $b_j-\delta-i\varepsilon$, $r-i\varepsilon$, $r+i\varepsilon$ and $a_j+\delta+i\varepsilon$, $-r+i\varepsilon$, $-r-i\varepsilon$, $a_j+\delta-i\varepsilon$, $a_j+\delta+i\varepsilon$ resp. which shows that $\{x : x_j\in]a_j,b_j[\}\cap\operatorname{supp}T_j=\emptyset$ for all j. Hence $Q\cap\operatorname{supp}T=\emptyset$. □

Definition 3.4 *Let $X\subset\mathbb{R}^d$ be closed under multiplication. Then let $\mathscr{H}_C(X)$ denote the algebra of all holomorphic functions f on $(\widehat{\mathbb{C}}\setminus[-R,+R])^d$ for some $R>0$, regular with value 0 in all infinite points of $\widehat{\mathbb{C}}^d$, such that $\sigma(f)\subset X$, equipped with Hadamard multiplication of the coefficients of the Laurent expansion around $(\infty,\dots,\infty)$.*

We have shown:

Theorem 3.5 *$M_T\mapsto\mathscr{C}_T$ defines an algebra isomorphism $M(\Omega)\to\mathscr{H}_C(V(\Omega))$.*

In a next step we want to change the equivalence into one with Hadamard multiplication of power series. We use the automorphism $r:(z_1,\dots,z_d)\to(1/z_1,\dots,1/z_d)$ of $\widehat{\mathbb{C}}$. We set

$$\mathscr{R}f(z) := \frac{1}{z_1\dots z_d} f\Big(\frac{1}{z_1},\dots,\frac{1}{z_d}\Big)$$

and

$$\mathscr{H} := \operatorname{ind}_U H(((\mathbb{C}\setminus\mathbb{R})\cup U)^d)$$

where U runs through all open neighborhoods of zero in $\mathbb{C}$. Then $\mathscr{R}$ is a linear isomorphism from $\mathscr{H}_C(\mathbb{R}^d)$ onto $\mathscr{H}$. If $f \in \mathscr{H}_C$ and $f(z) = \sum_{\alpha \in \mathbb{N}_0^d} m_\alpha z^{-\alpha - \mathbb{1}}$ its Laurent expansion around $(\infty, \ldots, \infty)$ then $\mathscr{R}f(z) = \sum_{\alpha \in \mathbb{N}_0^d} m_\alpha z^\alpha$ is the Taylor expansion of $\mathscr{R}f$ around $(0, \ldots, 0)$.

In particular $\mathscr{H}$ is an algebra with respect to Hadamard multiplications, which means the following: if $f, g \in \mathscr{H}$ and $f(z) = \sum_\alpha b_\alpha z^\alpha$, $g(z) = \sum_\alpha c_\alpha z^\alpha$ are their Taylor expansions, then there is a unique $f \star g \in \mathscr{H}$, such that $(f \star g)(z) = \sum_\alpha b_\alpha c_\alpha z^\alpha$ is its Taylor expansion. For $T \in \mathscr{A}(\mathbb{R}^d)'$ we set $C_T = \mathscr{R}(\mathscr{C}_T)$ then we obtain by obvious calculations

$$C_T(z) = T_\xi\Big(\prod_j \frac{1}{1 - \xi_j z_j}\Big).$$

We have shown:

Proposition 3.6 *$T \mapsto C_T$ defines an algebra isomorphism from $(\mathscr{A}(\mathbb{R}^d)', \star)$ onto $(\mathscr{H}, \star)$.*

Since r is an automorphism of $\widehat{\mathbb{C}}^d$ which maps $\mathbb{R}^d$ to $\mathbb{R}^d$ we have $\sigma(\mathscr{R}(f)) = r(\sigma(f))$ and therefore

$$\operatorname{supp} T = \sigma(\mathscr{C}_T) = r(\sigma(C_T)).$$

This leads to the definition:

Definition 3.7 *Let $X \subset \mathbb{R}^d$ be closed under multiplication. Then we define*

$$\mathscr{H}(X) = \{f \in \mathscr{H} \,:\, r(\sigma f) \subset X\}.$$

Finally obtain:

Theorem 3.8 *The map $M_T \mapsto C_T$ is an algebra isomorphism from $M(\Omega)$ to $\mathscr{H}(V(\Omega))$ equipped with Hadamard multiplication. If $0 \in \Omega$ and Ω is connected, then $M_T \in M(\Omega)$ acts on $\mathscr{A}(\Omega)$ by Hadamard multiplication with C_T.*

Proof: Only the last part has to be shown. It is enough to show it for monomials which is obvious by the definition. □

4 Dilation Sets

We have seen in the preceding sections that understanding the dilation set $V(\Omega)$ and its topological properties is of central importance for the theory of Hadamard type operators. In the present section we summarize some basic properties of dilation sets (for the one variable case comp. [8, Proposition 2.1]). Note that trivially $\mathbb{1} \in V(\Omega)$ always.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

- *If $\Omega \subset \mathbb{R}^d$ is bounded (convex) then $V(\Omega)$ is bounded (convex).*
- *If $V(\Omega)$ is bounded then $V(\Omega) \subset [-1, 1]^d$.*
- *If Ω is convex, bounded and symmetric with respect to all hyperplanes of the coordinate system then $V(\Omega) = [-1, 1]^d$.*

Proof: The boundedness statement is obvious. To prove the convexity statement we define the linear map $h_y : \mathbb{R}^d \to \mathbb{R}^d$ by $h_y(x) = yx$. The set $h_y^{-1}(\Omega)$ is convex. Hence $V(\Omega) = \bigcap_{y\in\Omega} h_y^{-1}(\Omega)$ is also convex. The second statement follows from the obvious fact that $V(\Omega)\cdot V(\Omega) \subset V(\Omega)$.

In the last case symmetry means that every vector of the form $x = (\pm1, \pm1, \ldots, \pm1)$ belongs to $V(\Omega)$. Hence $[-1,1]^d = \mathrm{conv}\{(\pm1,\pm1,\ldots,\pm1)\} \subset \mathrm{conv}(V(\Omega)) = V(\Omega) \subset [-1,1]^d$. □

The above proposition implies immediately that if Ω is an open ball of a finite dimensional space ℓ_p for $1 \le p \le \infty$ then $V(\Omega)$ is the closed unit ball of the finite dimensional space ℓ_∞.

The following instructive examples are verified by direct calculation.

Example 4.2 (Menagerie of dilation sets).

1. If $\Omega = \{x \in \mathbb{R}^2 : d(x,(1,10)) < 2\}$ then $V(\Omega) = \{\mathbb{1}\}$.
2. If $\Omega = \{x \in \mathbb{R}^2 : 1 < x_1 < 2\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : x_1 = 1\}$.
3. If $\Omega = \{x \in \mathbb{R}^2 : (1/2)x_1 < x_2 < 3x_1, 0 < x_1\}$ then $V(\Omega) = \{(t,t) \in \mathbb{R}^2 : 0 < t < \infty\}$.
4. If $\Omega = \{x \in \mathbb{R}^2 : d(x,(0,3)) < 2\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : -1 \le x_1 \le 1, x_2 = 1\}$.
5. If $\Omega = \{x \in \mathbb{R}^2 : -x_1 < x_2 < -x_1 + (1/2)\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : 0 < x_1 = x_2 \le 1\}$.
6. If $\Omega = \{x \in \mathbb{R}^2 : 0 < x_2 < 2x_1\}$ then $V(\Omega) = \{x \in \mathbb{R}^2 : 0 < x_2 \le x_1\}$.

Many very sophisticated examples of $V(\Omega)$ are provided in Section 8 where the geometry of $V(\Omega)$ is related to continuity and openness of the representation map $\mathscr{B}$ defined in the Representation Theorem 2.2. All these examples show a variety of forms of the possible sets $V(\Omega)$. Is there any pattern behind these strange examples? We will collect some answers to that question.

Let us recall that $\widetilde{V}(\Omega) := \bigcap_{y\in\Omega\cap\mathbb{R}^d_*} \frac{1}{y}\Omega$. Let $x \in \widetilde{V}(\Omega)$. We set $L_x := \{y : y_j = 0 \text{ whenever } x_j = 0\}$. Since $x(\Omega \cap \mathbb{R}^d_*) \subset \Omega \cap L_x$ we have $x\Omega \subset \overline{\Omega \cap L_x}$. Moreover $x\Omega$ is open in L_x, Therefore $x\Omega \subset \mathrm{interior}_{L_x}(\overline{\Omega \cap L_x})$. An open subset U of a topological space is called *regular open* if interior $\overline{U} = U$. Examples are convex open subsets of a locally convex space. We have shown that if $\Omega \cap L_x$ is regular open for every $x \in \widetilde{V}(\Omega)$ then $V(\Omega) = \widetilde{V}(\Omega)$.

Proposition 4.3 *If Ω is a non-empty open convex set then $V(\Omega) = \widetilde{V}(\Omega)$.*

Proof: For any linear subspace L of $\mathbb{R}^d$ the set $\Omega \cap L$ is convex and open in L, hence regular open in L. □

Let $I \subset \{1,\ldots,d\} := D$ then $V_I(\Omega) := V(\Omega) \cap (\{0\}^I \times \mathbb{R}^{D\setminus I})$. We show that $V(\Omega) \cap \mathbb{R}^d_*$ is always closed in $\mathbb{R}^d_*$. In particular, if $V(\Omega) \subset \mathbb{R}^d_*$ then it must be closed.

Proposition 4.4 *For every non-empty open set $\Omega \subset \mathbb{R}^d$ and every set $I \subset \{1,\ldots,d\}$ we have $V_I(\Omega)\cdot V(\Omega) \subset V_I(\Omega)$ and $\mathbb{1} = (1,\ldots,1) \in V(\Omega)$. Moreover, $V(\Omega) \cap (\{0\}^I \times \mathbb{R}^{D\setminus I}_*)$ is closed in $\{0\}^I \times \mathbb{R}^{D\setminus I}_*$. If Ω is convex then $V_I(\Omega)$ is convex as well.*

Proof: The first statement is obvious.

For the second claim, let $(x_n) \subset V(\Omega) \cap (\{0\}^I \times \mathbb{R}^{D\setminus I}_*)$ be a sequence convergent to $x \in \{0\}^I \times \mathbb{R}^{D\setminus I}_*$, $x = (\tilde{x}_1,\ldots,\tilde{x}_d)$, $x_n = (x_{n1},\ldots,x_{nd})$. For any $y = (y_1,\ldots,y_d) \in \Omega$ we define $y_n := (y_{n1},\ldots,y_{nd})$,

$$y_{nj} := \begin{cases} y_j & \text{for } j \in I; \\ y_j \cdot \frac{\tilde{x}_j}{x_{nj}} & \text{for } j \notin I. \end{cases}$$

Clearly, $y_n \to y$, hence for n sufficiently big $y_n \in \Omega$. On the other hand, since $x_n \in V(\Omega)$, $y_n x_n \in \Omega$ but $x_n y_n = xy$. We have proved that $xy \in \Omega$ for any $y \in \Omega$ hence $x \in V(\Omega)$.

The third statement follows from Proposition 4.1. □

There is another case when $V(\Omega)$ is closed.

Proposition 4.5 *If Ω is open, convex and $\mathbb{0} \in \Omega$ then $V(\Omega)$ is closed.*

Proof: We denote $V(A,B) := \{x \in \mathbb{R}^d : xA \subset B\}$. We need two elementary facts:
1. If $\gamma \in \mathbb{R}_+$ (more general $\gamma \in \mathbb{R}^d_*$) and $A, B \subset \mathbb{R}^d$ then $V(\gamma A, \gamma B) = V(A,B)$.
2. If $\omega \subset \mathbb{R}^d$ is open, convex, $\mathbb{0} \in \omega$ and $0 < \gamma < 1$ then $\overline{\gamma\omega} \subset \omega$.

For $0 < s < 1$ and $R > 0$ we set $K_{s,R} = s\overline{\Omega} \cap B_R$ where $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$. Then for $0 < \tau < t < 1$ we obtain

$$V(K_{s,R},\Omega) = V(K_{ts,tR}, t\Omega) \supset V(K_{ts,tR}, \overline{\tau\Omega}) \supset V(K_{ts,tR}, \tau\Omega) = V(K_{\frac{t}{\tau}s,\frac{t}{\tau}sR},\Omega). \tag{6}$$

Since $V(\Omega) = \bigcap_{0<s<1,R>0} V(K_{s,R},\Omega)$ and the middle term in (6) is closed we see that $V(\Omega)$ is closed. □

The set $V(\Omega)$ is open only in very special cases. We define $V_K := \{x \mid xK \subset \Omega\}$.

Proposition 4.6 *For open $\Omega \subset \mathbb{R}^d$ the following are equivalent:*

(1) $V(\Omega)$ is open.

(2) $\mathbb{R}^d_+ \subset V(\Omega)$.

(3) There is $0 \leq k \leq d$ and a subset J of $\{+1,-1\}^{d-k}$ such that, after a permutation of variables,

$$V(\Omega) = \mathbb{R}^k \times \bigcup_{e \in G} e \cdot \mathbb{R}^{d-k}_+, \tag{7}$$

where G is a subgroup of $\{-1,+1\}^{d-k}$.

(4) There is a finite set $K \subset \Omega$ such that $V_K(\Omega) = V(\Omega)$.

Proof: (1) $\Rightarrow$ (2): Assume there is $a \in \mathbb{R}^d_+$, $a \notin V(\Omega)$. Then the interval $[\mathbb{1}, a]$ meets $\partial V(\Omega)$ in a point in $\mathbb{R}^d_+ \subset \mathbb{R}^d_*$, which by Proposition 4.4 must belong to $V(\Omega)$; a contradiction with (1).

(1)$\wedge$(2) $\Rightarrow$ (3): We set $\sigma = \{j : \exists a \in V(\Omega), a_j = 0\}$. We may assume $\sigma = \{1,\ldots,k\}$. Since $V(\Omega)$ is multiplicatively closed (in particular, square closed) and $\mathbb{R}^d_+ \subset V(\Omega)$ we obtain $(0,\ldots,0,1,\ldots,1) \in V(\Omega)$ with k zeros. Since $V(\Omega)$ is open there is $\varepsilon > 0$ such that $[-\varepsilon,+\varepsilon]^k \times \{\mathbb{1}_{d-k}\} \in V(\Omega)$. We use again that $\mathbb{R}^d_+$ operates on $V(\Omega)$ and obtain $\mathbb{R}^k \times \{\mathbb{R}^{d-k}_+\} \in V(\Omega)$.

Let now $a = (a', a'') \in V(\Omega)$ where $a' = (a_1,\ldots,a_k)$, $a'' = (a_{k+1},\ldots,a_d)$. Then, by the same openness argument as before, we may assume that $a_j \neq 0$ for $j = 1,\ldots,k$. Since $\mathbb{R}^k \times \{\mathbb{R}^{d-k}_+\} \in V(\Omega)$ operates on $V(\Omega)$ we obtain $\mathbb{R}^k \times e \cdot \mathbb{R}^{d-k}_+ \subset V(\Omega)$ where $e_j = \operatorname{sign} a_j$ for $j = k+1,\ldots,d$.

If $G = \{e \in \{-1,+1\}^{d-k} : (\mathbb{1}_k, e) \in V(\Omega)\}$ then G is a group and, with this group we get the representation (7).

(3) $\Rightarrow$ (1): Obvious.

(2) $\Rightarrow$ (4): We set $K = \Omega \cap \{-1,0,+1\}^d$. Since $\mathbb{R}^d_+ \subset V(\Omega)$ every $x \in \Omega$ can be written as $x = ex_+$ where $e \in K$ and $x_+ \in \mathbb{R}^d_+$. That is, $\Omega = K \cdot \mathbb{R}^d_+$. Since $\mathbb{R}^d_+$ operates on $V_K(\Omega)$ we see that $V_K(\Omega) = V(\Omega)$.

(4) $\Rightarrow$ (1). This is again obvious since $V_K(\Omega)$ is open. □

Corollary 4.7 *Let $\Omega \subset \mathbb{R}^d$ be open, then $V(\Omega) = \Omega$ if, and only if Ω has the form (7).*

Now, we explain what happens when $V(\Omega)$ has empty interior.

Proposition 4.8 *Let Ω be an open convex non-empty subset of $\mathbb{R}^d$. Then $V(\Omega)$ has empty interior if and only if one of the following two conditions holds:*

(i) there is j such that for every $x = (x_1, \dots, x_d) \in V(\Omega)$ holds $x_j = 1$;

(ii) there are j, k, $j \neq k$, such that for every $x = (x_1, \dots, x_d) \in V(\Omega)$ holds $x_j = x_k$.

Proof: It is enough to prove necessity only. Assume that $V(\Omega)$ has empty interior. Since it is a convex set (see Prop.4.1) it is contained in a hyperplane given by $a_1 x_1 + \cdots + a_d x_d = b$ with suitable a_j not all zero and b.

If $V(\Omega) \subset \bigcup_{j<k}(\{x : x_j = x_k\} \cup \{x : x_j = -x_k\})$ then it is contained in one of the hyperplanes. Since $\mathbb{1} \in V(\Omega)$ it must be of the form $\{x : x_j = x_k\}$

Otherwise there is $x \in V(\Omega)$ such that all $|x_j|$ are different. We may assume $|x_1| < \cdots < |x_d|$. Then, since $V(\Omega) \cdot V(\Omega) \subset V(\Omega)$, we have $a_1 x_1^n + \cdots + a_d x_d^n = b$ for all $n \in \mathbb{N}_0$. Dividing through x_d^n and letting $n \to +\infty$ we obtain $a_d = 0$. Repeating this we end up with $a_1 x_1 = b$, $a_1 \neq 0$. Since $\mathbb{1} \in V(\Omega)$ we get $a_1 = b$ and $x_1 = 1$ for all $x \in V(\Omega)$. □

Proposition 4.9 *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set containing $\mathbb{0}$. Then $V(\Omega)$ is bounded if and only if Ω contains no axis.*

Proof: If Ω contains an axis then $V(\Omega)$ contains the same axis.

If $V(\Omega)$ is unbounded there is a coordinate j such that $\{x_j : x = (x_1, \dots, x_j, \dots, x_d) \in V(\Omega)\}$ is unbounded. Since $\mathbb{0} \in \Omega$ so there is $\varepsilon > 0$ such that for any $|x_j| < \varepsilon$ the vector $\hat{x}_j := (0, \dots, 0, x_j, 0, \dots, 0)$ belongs to Ω. Multiplying the elements of $V(\Omega)$ and $\hat{x}_j$ we will get all elements of the j-th axis. So Ω contains this axis. □

Clearly there are plenty of unbounded convex open sets $\Omega \ni \mathbb{0}$ with compact $V(\Omega)$.

Proposition 4.10 *Let $\Omega \subset \mathbb{R}^d$ be an open nonempty set. Then $V(\Omega) \subset \mathbb{R}^d_*$ and compact iff*

$$V(\Omega) \subset \{\pm 1\}^d. \tag{8}$$

Proof: $V(\Omega) \subset [-1, 1]^d$ by Proposition 4.1. For $x = (x_1, \dots, x_d) \in V(\Omega)$ the sequence $y_n := x^n \in V(\Omega)$ by Proposition 4.4 and it has a subsequence converging to $y \in V(\Omega)$ since $V(\Omega)$ is closed (Proposition 4.4. Then $y \notin \mathbb{R}^d_*$ if $|x_j| < 1$ for some j, a contradiction. □

The condition (8) holds for instance, if each of the intersections Ω_j of Ω with the j^{th} coordinate axis is non-void and satisfies $\Omega_j \subset]-C_2, -C_1[\cup]C_1, C_2[$ for some $0 < C_1 < C_2 < \infty$ or if Ω is bounded and $\overline{\Omega} \subset \mathbb{R}^d_*$.

Later on a special role will be played by those Ω where $V(\Omega)$ has a countable basis of open neighborhoods. The following observation is well known but for the convenience of the reader we provide a proof.

Proposition 4.11 *An arbitrary set $S \subset \mathbb{R}^d$ has a countable basis of open neighborhoods if and only if $\partial S \cap S$ is compact (for instance, empty).*

Proof: Let $S \subset \mathbb{R}^d$ be a set with $\partial S \cap S$ compact. Then we write $S = \operatorname{Int} S \cup (\partial S \cap S)$. The second summand has a countable neighborhood basis $(V_n)_{n \in \mathbb{N}}$ so $(U_n)_{n \in \mathbb{N}}$ is a countable neighborhood basis for S where $U_n = V_n \cup \operatorname{Int} S$.

On the other hand, assume that $\partial S \cap S$ is not compact but S has a countable open neighborhood basis $(U_n)_{n\in\mathbb{N}}$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\partial S \cap S$ such that either $x_n \to x \in \partial S \setminus S$ or $(x_n)_{n\in\mathbb{N}}$ has no accumulation point. Let us take $y_n \in U_n \setminus S$ such that $d(x_n, y_n) \underset{n\to\infty}{\longrightarrow} 0$ and the interval $[x_n, y_n]$ is contained in U_n. Now, the open set

$$U := \mathbb{R}^d \setminus (\{y_n \mid n \in \mathbb{N}\} \cup \{x\}) \quad (\text{if } x_n \to x) \qquad \text{or} \qquad U := \mathbb{R}^d \setminus \{y_n \mid n \in \mathbb{N}\} \quad (\text{otherwise})$$

does not contain any U_n but it is a neighborhood of S; a contradiction. □

In the one dimensional case there is an easy characterization of Ω with $V(\Omega)$ having a countable basis of open neighborhoods.

Proposition 4.12 *Let $\Omega \subset \mathbb{R}$ be an open nonempty set. Then $\partial V(\Omega) \cap V(\Omega)$ is compact if and only if one of the following conditions holds:*

- $0 \in V(\Omega)$ *(i.e., $0 \in \Omega$);*
- $V(\Omega) \subset \{1, -1\}$*;*
- $V(\Omega)$ *has a nonempty interior.*

Proof: Necessity: If $V(\Omega)$ has an empty interior, $\mathbb{0} \notin V(\Omega)$ and $V(\Omega)$ contains an element $x \neq \mathbb{0}$ with $|x| \neq 1$ then $\partial V(\Omega) \cap V(\Omega)$ contains x^n for every $n \in \mathbb{N}$. Then clearly $\partial V(\Omega) \cap V(\Omega)$ is not compact.

Sufficiency: If $\mathbb{0} \in V(\Omega)$ then $V(\Omega)$ is closed by Prop. 4.4. So it is either bounded (then compact) or $\Omega = \mathbb{R}$ and $\partial V(\Omega) = \emptyset$ (see [8, Prop. 2.1 (c)]). If $V(\Omega) \subset \{1, -1\}$ then $V(\Omega)$ is compact. If $V(\Omega)$ has a nonempty interior then using semigroup property it is easily seen that its boundary is finite. □

For $d > 1$ we know much less.

Proposition 4.13 *Let $d > 1$. If $\partial V(\Omega) \cap V(\Omega)$ is compact and not empty, then $V(\Omega)$ is bounded.*

Proof: Since $V(\Omega)$ is square closed it suffices to show that $V(\Omega) \cap (\mathbb{R}_+ \cup \{0\})^d$ is bounded. If $V(\Omega) \cap \mathbb{R}_+^d$ is bounded but $V(\Omega) \cap (\mathbb{R}_+ \cup \{0\})^d$ is unbounded then $\partial V(\Omega) \cap V(\Omega)$ is unbounded. So it suffices to show that $V(\Omega) \cap \mathbb{R}_+^d$ is bounded.

First, observe that $V(\Omega) \cap \mathbb{R}_+^d$ and $\mathbb{R}_+^d \setminus V(\Omega)$ cannot be both unbounded. Indeed, let $(x_n)_{n\in\mathbb{N}} \subset V(\Omega) \cap \mathbb{R}_+^d$ and $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+^d \setminus V(\Omega)$ be unbounded sequences. Then on the interval $[x_n, y_n]$ there is a point $z_n \in \partial V(\Omega) \cap \mathbb{R}_+^d$. Clearly, the sequence $(z_n)_{n\in\mathbb{N}}$ is unbounded. Since $V(\Omega) \cap \mathbb{R}_+^d$ is closed in $\mathbb{R}_+^d$ (Proposition 4.4), we get $(z_n)_{n\in\mathbb{N}} \subset \partial V(\Omega) \cap V(\Omega)$; a contradiction.

Secondly, observe that for $d > 1$ if $V(\Omega) \cap \mathbb{R}_+^d$ is unbounded then $V(\Omega) \supset \mathbb{R}_+^d$, hence $V(\Omega)$ is open by Proposition 4.6. Indeed, by the previous observation $\mathbb{R}_+^d \setminus V(\Omega)$ is bounded. Take any point $(x_1, \ldots, x_d) \in \mathbb{R}_+^d$. Define

$$a_\varepsilon := (\varepsilon, \ldots, \varepsilon, x_d/\varepsilon), \qquad b_\varepsilon := (x_1/\varepsilon, \ldots, x_{d-1}/\varepsilon, \varepsilon),$$

where $\varepsilon > 0$. For $\varepsilon > 0$ small enough we have $a_\varepsilon, b_\varepsilon \in V(\Omega) \cap \mathbb{R}_+^d$ and therefore $a_\varepsilon \cdot b_\varepsilon = (x_1, \ldots, x_d) \in V(\Omega)$. □

It is easy to see that Proposition 4.13 does not hold for $d = 1$ (take $\Omega = \mathbb{R} \setminus ([0, 1/2] \cup \{1\})$ and hence $V(\Omega) = \{1\} \cup [2, \infty[$, compare [8, Proposition 2.1]). We cannot improve Proposition 4.13 to show that $V(\Omega)$ must be compact, see Example 8.5 below.

Proposition 4.14 *Let $\Omega \subset \mathbb{R}^d$, $d > 1$, be an open nonempty set and either $\Omega \subset \mathbb{R}_*^d$ or Ω is convex. Then the set $\partial V(\Omega) \cap V(\Omega)$ is compact if and only if $V(\Omega)$ is either compact or open.*

Proof: We need to prove necessity only.

From Proposition 4.13 we know that $V(\Omega)$ is either open or bounded.

If it is bounded and $\Omega \subset \mathbb{R}^d_*$, also $V(\Omega) \subset \mathbb{R}^d_*$ and, by Proposition 4.4, $\partial V(\Omega) \cap \mathbb{R}^d_* \subset V(\Omega)$. Thus for any $I \subset \{1, \ldots, d\}$ the boundary of $\overline{V(\Omega)} \cap (\{0\}^I \times \mathbb{R}_*^{D\setminus I})$ in $\{0\}^I \times \mathbb{R}_*^{D\setminus I}$ belongs to the closure of $\partial V(\Omega) \cap V(\Omega)$ but does not belong to $V(\Omega)$, hence it is empty. Thus the set $\overline{V(\Omega)} \cap (\mathbb{R}^d \setminus \mathbb{R}^d_*)$ is either empty or unbounded. The second case is impossible because $V(\Omega)$ is bounded. Summarizing, $\partial V(\Omega) \subset V(\Omega)$ so $V(\Omega)$ is compact.

It remains to show that $V(\Omega)$ is closed, if it is convex and bounded. By Proposition 4.1, $V(\Omega)$ is contained in the closed unit ball of the d-dimensional space ℓ_∞.

Let $x \in \partial V(\Omega) \setminus V(\Omega)$. By Proposition 4.4, $x \in \{0\}^I \times \mathbb{R}^{D\setminus I}$ for some set $I \subset \{1, \ldots, d\} = D$ but there is a sequence $(x_n)_{n\in\mathbb{N}} \subset (V(\Omega) \cap (\{0\}^J \times \mathbb{R}_*^{D\setminus J})) \setminus \{0\}^I \times \mathbb{R}^{D\setminus I}$ for some $J \subsetneq I$ tending to x. Choose an axis Y indexed by an element of $I \setminus J$.

Now, we take a line ℓ_n parallel to $\{0\}^I \times \mathbb{R}^{D\setminus I}$ going through x_n and crossing Y at some point v_n. Except the point v_n the whole ℓ_n is contained in $\{0\}^J \times \mathbb{R}_*^{D\setminus J}$. Thus there is a point $w_n \in \ell_n \cap \partial V(\Omega) \cap (\{0\}^J \times \mathbb{R}_*^{D\setminus J})$ (and thus by Proposition 4.4, $w_n \in V(\Omega)$) such that $x_n \in [w_n, v_n]$. Choosing a subsequence of $(x_n)_{n\in\mathbb{N}}$ without loss of generality we may assume that $w_n \to w \in \{0\}^I \times \mathbb{R}^{D\setminus I}$. Since $\partial V(\Omega) \cap V(\Omega)$ is closed, $w \in V(\Omega)$.

Again without loss of generality we may assume that either for every $n \in \mathbb{N}$ the interval $[w_n, v_n)$ is contained in $V(\Omega)$ or for every $n \in \mathbb{N}$ there exists $\tilde{v}_n \in (x_n, v_n)$ such that $\tilde{v}_n \in \partial V(\Omega)$.

In the first case, we take a line p_n parallel to Y, perpendicular to $\{0\}^I \times \mathbb{R}^{D\setminus I}$ going through a point $z_n \in [x_n, v_n)$, $d(z_n, v_n) < 1/n$. Then there is a point $u_n \in p_n \cap \partial V(\Omega)$. Again by Proposition 4.4, $u_n \in \partial V(\Omega) \cap V(\Omega)$ and their accumulation point $u \in \partial V(\Omega) \cap V(\Omega) \cap Y$. Clearly,

$$\mathbb{0} = u \cdot w \in V(\Omega) \cdot V(\Omega) \subset V(\Omega)$$

and therefore $\mathbb{0} \in \Omega$. By Proposition 4.5, $V(\Omega)$ is closed.

In the second case, by Proposition 4.4, $\tilde{v}_n \in \partial V(\Omega) \cap V(\Omega)$. The sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ has an accumulation point $\tilde{v} \in (\{0\}^I \times \mathbb{R}^{D\setminus I}) \cap \partial V(\Omega) \cap V(\Omega)$, since $\partial V(\Omega) \cap V(\Omega)$ is closed. It is clear that $x \in [w, \tilde{v}]$, so by convexity of $V(\Omega)$ holds $x \in V(\Omega)$. □

5 Topologies on $\mathscr{A}(V(\Omega))'$

Before we start our further investigation we need more detailed information about the natural topology on $\mathscr{A}(V(\Omega))$ and three useful topologies on $\mathscr{A}(V(\Omega))'$, namely τ_b, τ_k and τ_{k_*}. The motivation comes from the fact that we will show in Section 7 that

$$\mathscr{B} : (\mathscr{A}(V(\Omega))', \tau_k) \to L_b(\mathscr{A}(\Omega)) \qquad \text{and} \qquad \mathscr{B}^{-1} : L_b(\mathscr{A}(\Omega)) \to (\mathscr{A}(V(\Omega))', \tau_{k_*})$$

are continuous and there are reasons to conjecture that $L_b(\mathscr{A}(\Omega))$ induces via $\mathscr{B}$ the topology τ_k on $\mathscr{A}(V(\Omega)')$ (see Theorem 7.14, Example 7.10 and cf. [41]).

Let S be an arbitrary subset of $\mathbb{R}^d$. Then there are two natural ways to define topologies on $\mathscr{A}(S)$:

$$\mathscr{A}_I(S) = \operatorname{ind}_U H(U) \text{ or } \mathscr{A}_P(S) = \operatorname{proj}_K \mathscr{A}(K),$$

where U runs through the neighborhoods of S in $\mathbb{C}^d$ and $H(U)$ is the space of holomorphic functions on U, and K runs through the compact subsets of S. Here $\mathscr{A}(K)$ denotes the space of germs of holomorphic functions around the compact set K with the natural inductive limit topology for the family $H(U)$, where U runs over all neighborhood of K in $\mathbb{C}^d$ (see [31, Example

24.37 (2)]). These topologies coincide (see [30, Théorème 1.2.a)]; for the precise result cf. proof p. 69) and define, what we consider to be the natural topology on $\mathscr{A}(S)$.

$\mathscr{A}(S)$ is nuclear (as a projective limit of nuclear spaces $\mathscr{A}(K)$, see [31, Cor. 28.8, Example 28.9 (4)]), and, by the first version, ultrabornological (see [31, Th. 24.16, Remark 24.15 (c)]), by the second version, complete as $\mathscr{A}(K)$ are complete. Moreover, by [31, 23.23, 24.14, 24.16], $\mathscr{A}(S)$ is also barreled. A set $B \subset \mathscr{A}(S)$ is bounded if there is an open neighborhood U of S such that B is bounded in $H(U)$, see [30, Th. 1.2, Proposition 1.2], comp. [6, Fact 1.21, Th. 1.27].

Therefore topologically

$$\mathscr{A}(S)'_b = \operatorname{proj}_U H(U)'_b, \tag{9}$$

here b-topology (denoted by τ_b) means the strong topology, i.e., the topology of uniform convergence on bounded sets in $\mathscr{A}(S)$. This implies immediately that topologically

$$\mathscr{A}(S)'_b = \operatorname{proj}_{W \supset S} \mathscr{A}(W)'_b$$

where W runs through all open neighborhoods of S in $\mathbb{R}^d$. Since $\mathscr{A}(S)$ is ultrabornological $\mathscr{A}(S)'_b$ is complete [31, 24.11]. Since $\mathscr{A}(S)$ is a complete Schwartz space its strong dual is ultrabornological by [31, 24.23]. Please note that $H(U)$ is Montel [31, Section 24] so by [31, 24.25] $H(U)'_b$ is also Montel so all bounded sets in $\mathscr{A}(S)'_b$ are relatively compact.

As linear space we have $\mathscr{A}(S)'_b = \operatorname{ind}_{K \Subset S} \mathscr{A}(K)'_b$ and the topology of this inductive limit is finer than τ_b on $\mathscr{A}(S)$. If S has a fundamental sequence of compact subset (i.e., it is hemicompact), in particular if S is locally closed or open then $\operatorname{ind}_{K \Subset S} \mathscr{A}(K)'_b$ is an (LF)-space, hence webbed, and, by de Wilde's Theorem (see [31, 24.30]), we have $\mathscr{A}(S)'_b = \operatorname{ind}_{K \Subset S} \mathscr{A}(K)'_b$ topologically and $\mathscr{A}(S)'_b$ is an (LFN)-space, i.e., a countable inductive limit of nuclear Fréchet spaces.

In particular, we have shown:

Proposition 5.1 *For every nonempty open set $\Omega \subset \mathbb{R}^d$ the space $\mathscr{A}(V(\Omega))'_b$ is an ultrabornological complete space with $\mathscr{A}(V(\Omega))'_b = \operatorname{proj}_{W \supset V(\Omega)} \mathscr{A}(W)'_b$, where $W \subset \mathbb{R}^d$ runs through all open neighborhoods of $V(\Omega)$.*

To define τ_k and τ_{k_*} we need some auxiliary definitions. Let $\Omega \subset \mathbb{R}^d$ be a non-void open set. For a compact set $K \subset \Omega$ let us define

$$V_K(\Omega) := \{\xi \mid \xi K \subset \Omega\}.$$

Notice that $V_K(\Omega)$ is open since for $\xi \in V_K(\Omega)$

$$(\xi + B_\gamma(0))K \subset \xi K + B_\gamma(0)K \subset \xi K + B_\varepsilon(0) \subset \Omega$$

if $\varepsilon > 0$ and then $\gamma > 0$ are chosen suitably. Clearly, $V_K(\Omega)$ contains $V(\Omega)$ and

$$\bigcap_{K \Subset \Omega} V_K(\Omega) = \{\xi \mid \xi\Omega \subset \Omega\} = V(\Omega).$$

The two last "natural" topologies on $\mathscr{A}(V(\Omega))'$ we call *k-topology* and *k_*-topology*, denoted by τ_k and τ_{k_*} respectively, and they are by definition:

$$\mathscr{A}(V(\Omega))'_k := \operatorname{proj}_{K \Subset \Omega} \mathscr{A}(V_K(\Omega))'_b, \qquad \mathscr{A}(V(\Omega))'_{k_*} := \operatorname{proj}_{K \Subset \Omega \cap \mathbb{R}^d_*} \mathscr{A}(V_K(\Omega))'_b,$$

where K runs through all compact subsets of Ω and of $\Omega \cap \mathbb{R}^d_*$ respectively. Please note that τ_k and τ_{k_*} depend not only on $V(\Omega)$ but also on Ω itself. These topologies are analogous to the t-topology introduced in [41].

Proposition 5.2 *For every open nonempty set $\Omega \subset \mathbb{R}^d$ the spaces $\mathscr{A}(V(\Omega))'_k$ and $\mathscr{A}(V(\Omega))'_{k_*}$ are complete countable projective limits of LFN-spaces (i.e., countable locally convex inductive limits of nuclear Fréchet spaces). In particular, $\mathscr{A}(V(\Omega))'_k$ and $\mathscr{A}(V(\Omega))'_{k_*}$ are webbed.*

Clearly, $\tau_{k_*} \leq \tau_k \leq \tau_b$.

We will need the following simple remark. For $K \Subset \Omega$ we denote by $V^0_K(\Omega)$ the union of all connected components of $V_K(\Omega)$ which have a nonempty intersection with $V(\Omega)$, and we have

$$\mathscr{A}(V(\Omega))'_k := \operatorname{proj}_{K\Subset\Omega} \mathscr{A}(V^0_K(\Omega))'_b, \qquad \mathscr{A}(V(\Omega))'_{k_*} := \operatorname{proj}_{K\Subset\Omega\cap\mathbb{R}^d_*} \mathscr{A}(V^0_K(\Omega))'_b.$$

We can now compare the topologies τ_b, τ_k and τ_{k_*}. We recall the fact, which is due to the Cartan-Grauert Theorem in the version of [14, Lemma 1.1.(b)], that for any open $U \subset \mathbb{R}^d$ there is a real analytic function on U which cannot be extended beyond U.

Proposition 5.3 *Let $\Omega \subset \mathbb{R}^d$ be on open nonempty set.*

(a) $\tau_b = \tau_k$ *on $\mathscr{A}(V(\Omega))'$ if and only if the sets $V^0_K(\Omega)$, $K \subset \Omega$ compact, form a basis of neighborhoods for $V(\Omega)$.*

(b) $\tau_b = \tau_{k_*}$ *on $\mathscr{A}(V(\Omega))'$ if and only if the sets $V^0_K(\Omega)$, $K \subset \Omega \cap \mathbb{R}^d_*$ compact, form a neighborhood basis for $V(\Omega)$.*

(c) $\tau_k = \tau_{k_*}$ *on $\mathscr{A}(V(\Omega))'$ if and only if for every compact $L \subset \Omega$ there is a compact $K \subset \Omega\cap\mathbb{R}^d_*$ such that $V^0_K(\Omega) \subset V_L(\Omega)$.*

Proof: (a): Since one implication is trivial, it remains to show that from equality of the topologies follows: if U is an open neighborhood of $V(\Omega)$ then there is $K \Subset \Omega$ such that $V^0_K \subset U$.

We choose a function $f \in \mathscr{A}(U)$ which cannot be extended beyond U. $y_f : T \mapsto T(f)$ is a linear form on $\mathscr{A}(V(\Omega))'_b = \mathscr{A}(V(\Omega))'_k$. Therefore there exists $K \Subset \Omega$ such that $y_f \in (\mathscr{A}(V^0_K(\Omega))'_b)'$ which means that there is $g \in \mathscr{A}(V^0_K(\Omega))$ such that $y_f(T) = T(g)$ for all $T \in \mathscr{A}(V^0_K(\Omega))'$. Applying this to $T = \delta^{(\alpha)}_x$ for all $x \in V(\Omega)$ and $\alpha \in \mathbb{N}^d_0$ we obtain that $f = g$ in a neighborhood of $V(\Omega)$. If $V^0_K(\Omega) \not\subset U$ there must be $x \in \partial U \cap V^0_K(\Omega)$, that means g extends f into a neighborhood of x, which contradicts the choice of f.

(b) and (c): The proof is analogous. □

Notice that the sets $V^0_K(\Omega)$ can be very different from the sets $V_K(\Omega)$ and, with the latter, Proposition 5.3 would be false, as the following example shows.

Example 5.4 *Let $\Omega := \mathbb{R}^2_+ \setminus \{\mathbb{1}\}$. Then $V(\Omega) = \{\mathbb{1}\}$ and $\tau_b = \tau_k$ by Proposition 5.3 while all $V_K(\Omega)$ are unbounded.*

Proof: Let $K_n := \{(x,y) : 1/n \leq x, y \leq n\} \setminus \{(x,y) : n/(n+1) < x, y < (n+1)/n\}$ then $V_{K_n}(\Omega) = \{(x,y) : n/(n+1) < x, y < (n+1)/n\} \cup (\mathbb{R}^2 \setminus \{(x,y) : 1/n \leq x, y \leq n)\}$ hence $V^0_{K_n}(\Omega) = \{(x,y) : n/(n+1) < x, y < (n+1)/n\}$. □

6 Elements of Harmonic Function Theory

For a deeper study of topological features of the Representation Theorem 2.2 in Section 7 we will need the following elements of the harmonic function theory. For a compact set $K \subset \mathbb{R}^n$ let $C_\Delta(K)$ (and $C_{\Delta,0}(\mathbb{R}^n \setminus K)$, respectively) denote the family of all harmonic germs near K (and the harmonic functions on $\mathbb{R}^n \setminus K$ vanishing at ∞, respectively). It is well known that every continuous linear functional T on $C_\Delta(K)$ corresponds to a harmonic function $f_T \in C_{\Delta,0}(\mathbb{R}^n \setminus K)$

via the so-called *Tillmann-Grothendieck duality* (TG duality, see [39, Satz 6] and also [2]; a general version for zero solutions of hypoelliptic partial differential operators is contained in [27]). To be precise, let

$$G(\xi) := \frac{-1}{c_n(n-2)}|\xi|^{2-n}, \quad \xi \neq 0,$$

be the canonical elementary solution for the Laplacian in n variables (for $n \geq 3$, see e.g. [1, p. 193], c_n is the area of the unit sphere). The function f_T corresponding to T is then defined by

$$f_T(\xi) := \langle G(\xi \ - \cdot\ ), T\rangle\,, \xi \in \mathbb{R}^n \setminus K. \tag{10}$$

The correspondence of f_T and T is given by the TG duality (see [2, (4)])

$$T(h) = (f_T, h) := \langle \Delta(\varphi h), f_T\rangle := \int \Delta(\varphi h)(\xi) f_T(\xi)\, d\xi, \qquad h \in C_\Delta(K), \tag{11}$$

where $\varphi \in C_0^\infty(U)$ is chosen such that $\varphi = 1$ near K if $h \in C_\Delta(U)$ — the class of all harmonic functions on U and U is an open neighborhood of K. By Green's formula, we also get (compare [39, (48)])

$$(f_T, h) = \int_{\partial A} f_T(\xi)\frac{\partial}{\partial\nu}h(\xi) - h(\xi)\frac{\partial}{\partial\nu}f_T(\xi)\; d\sigma(\xi), \qquad h \in C_\Delta(K), \tag{12}$$

where A is a compact set with smooth boundary such that $K \subset \text{interior}\, A \subset A \subset U$ if $h \in C_\Delta(U)$ and σ is the Lebesgue-surface measure.

We will apply the duality (11) to represent analytic functionals supported in a compact set $K \subset \mathbb{R}^d$ as harmonic functions on $\mathbb{R}^{d+1}\backslash K$. For this we write the points in $\mathbb{R}^{d+1}$ as $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. For $K \subset \mathbb{R}^d$ compact let $\widetilde{C}_\Delta(K)$ denote the class of all harmonic germs (in $(d+1)$ variables) near K which are even with respect to the variable t. Notice that $\mathscr{A}(K)$ is topologically isomorphic to $\widetilde{C}_\Delta(K)$ via the mapping $S : \mathscr{A}(K) \to \widetilde{C}_\Delta(K)$, where $S(g)$ is the harmonic function near K with Cauchy data (existing also by the Cauchy-Kovalevska theorem)

$$S(g)(x,0) = g(x), \qquad \partial_t S(g)(x,0) = 0.$$

An explicit formula for $S(g)$ is provided by

$$S(g)(x,t) := \sum_{k=0}^{\infty}(-\Delta_x)^k g(x)\frac{t^{2k}}{(2k)!}. \tag{13}$$

It is easily seen that

$$S : H(\mathbb{C}^d) \to \widetilde{C}_\Delta(\mathbb{R}^{d+1}) \text{ is continuous.} \tag{14}$$

Even more holds true (see [7, Prop. 2.3]), fix an open set $V \subset \mathbb{R}^d$, then for every $\widetilde{V}$ an open neighborhood of V in $\mathbb{R}^{d+1}$ there is U an open neighborhood of V in $\mathbb{C}^d$ and for every U_1 an open neighborhood of V in $\mathbb{C}^d$ there is $\widetilde{V}_1$ an open neighborhood of V in $\mathbb{R}^{d+1}$ such that

$$S : H(U_1) \to \widetilde{C}_\Delta(\widetilde{V}_1) \quad \text{and} \quad S^{-1} : \widetilde{C}_\Delta(\widetilde{V}) \to H(U) \quad \text{are continuous.} \tag{15}$$

We also need the following Cauchy type estimate: there is $C > 0$ such that for any $\delta > 0$ and any $\beta \in \mathbb{N}^{d+1}$ the following holds if f is harmonic near $\overline{B_\delta(0)} \subset \mathbb{R}^{d+1}$

$$|\partial^\beta f(0)| \leq \beta!(C/\delta)^{|\beta|} \sup_{\xi \in B_\delta(0)} |f(\xi)| \tag{16}$$

(see [15, Theorem 2.2.7]). From (16) we obtain the following precise estimate for the derivatives of G: there is $C > 0$ such that

$$\sup_{(x,t)\neq 0, \alpha\in\mathbb{N}^{d+1}} \frac{|\partial^\alpha G(x,t)||(x,t)|^{|\alpha|+d-1}}{C^{|\alpha|}\alpha!} < \infty. \tag{17}$$

Indeed, this estimate follows for $|(x,t)| = 1$ from (16) (with $\delta := 1/2$), and for general $(x,t) \neq 0$ by the homogeneity of Δ (consider $G_\tau(\xi) := G(\tau\xi)$ for $\tau > 0$).

For $K \subset \mathbb{R}^d$ let $\widetilde{C}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K)$ denote the class of harmonic functions on $\mathbb{R}^{d+1} \setminus K$ which are even with respect to the variable t and vanish at ∞. The TG duality (11) then shows that $\mathscr{A}(K)'_b$ is topologically isomorphic to $\widetilde{C}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K)$ via

$$T(g) = (f_T, S(g)) = \langle \Delta(\varphi S(g)), f_T\rangle, \qquad T \in \mathscr{A}(K)', g \in \mathscr{A}(K), \tag{18}$$

where

$$f_T(x,t) = \langle G(x \ - \cdot\ , t), T\rangle\,, \qquad (x,t) \in \mathbb{R}^{d+1} \setminus K, \tag{19}$$

by (10). Notice that $f_T \in \widetilde{C}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K)$.

7 Topological Representation

We will study the topological aspects of the Representation Theorem 2.2, in particular, for which topologies on $\mathscr{A}(V(\Omega))'$ the map $\mathscr{B} : \mathscr{A}(V(\Omega))' \to M(\Omega) \subset L_b(\mathscr{A}(\Omega))$ is continuous and, more sophisticated, which topology is induced on $\mathscr{A}(V(\Omega))'$ via $\mathscr{B}$ from $L_b(\mathscr{A}(\Omega))$. More precisely, we ask what is the relation of the topology induced from $M(\Omega)$ via $\mathscr{B}$ on $\mathscr{A}(V(\Omega))'$ (we denote this topology here as τ_m) with respect to topologies τ_b, τ_k and τ_{k_*} (see Section 5). We prove that τ_m is between τ_k and τ_{k_*} (see Theorems 7.5 and 7.2), then we show that τ_m, τ_b and τ_k have the same families of bounded sets and convergent sequences (Theorem 7.11). Finally we describe cases when $\tau_k = \tau_m = \tau_{k_*}$ (Theorem 7.14).

Now, we analyze continuity of the map $\mathscr{B}$. We assume without restriction of generality that $\mathbb{1} \in \Omega$. Please note that dilation by a factor $a \in \mathbb{R}^d_*$ is a homeomorphism of Ω which does not change $V(\Omega)$. We set for compact $K \subset \Omega$

$$\begin{aligned} M(\Omega,K) &= \{M \in L_b(\mathscr{A}(\Omega), \mathscr{A}(K)) \,:\, M \text{ admits all monomials as 'eigenvectors'}\},\\ MC(\Omega,K) &= \{M \in L_b(\mathscr{A}(\Omega), C(K)) \,:\, M \text{ admits all monomials as 'eigenvectors'}\}. \end{aligned}$$

Here $C(K)$ carries the sup-norm topology. Obviously $M(\Omega,K) \subset MC(\Omega,K)$ with continuous embedding.

We assume from now on that $\mathbb{1} \in K$, then $V_K(\Omega) \subset \Omega$, and we assume that $K \cap \mathbb{R}^d_*$ is dense in K.

Proposition 7.1 *$M(\Omega,K) = MC(\Omega,K)$ as sets and their equicontinuous sets coincide. The map $T \mapsto M_T$ (as in the Representation Theorem 2.2) defines a continuous isomorphism from $\mathscr{A}(V_K(\Omega))'_b$ onto $M(\Omega,K)$. Its inverse map is $M \mapsto T_M$ where $T_M f = (Mf)(\mathbb{1})$. Both maps send equicontinuous sets to equicontinuous sets.*

As a direct consequence of Proposition 7.1 we obtain:

Theorem 7.2 *For every open $\Omega \subset \mathbb{R}^d$ the map $\mathscr{B} : \mathscr{A}(V(\Omega))'_k \to M(\Omega)$ is continuous.*

Proof: It follows from Proposition 7.1 by going to the projective limit over K_n where $K_n = \overline{\omega_n}$ and $\omega_1 \subset\subset \omega_2 \subset\subset \dots$ is an open exhaustion of Ω since $\mathscr{A}(V(\Omega))'_k = \operatorname{proj}_n \mathscr{A}(V_{K_n}(\Omega))'_b$ and $M(\Omega) = \operatorname{proj}_n M(\Omega, K_n)$. □

Proof of Proposition 7.1: First we show that $T \mapsto M_T$ maps equicontinuous sets into equicontinuous sets. We fix a compact set $L \subset V_K(\Omega)$. By the result of the third named author [40], a standard seminorm on $\mathscr{A}(L)$ is given by

$$\|f\|_{L,\delta} := \sup_{\alpha\in\mathbb{N}_0^d, y\in L} \frac{|f^{(\alpha)}(y)|}{\alpha!}\delta_{|\alpha|}^{|\alpha|}, \tag{20}$$

where $\delta = (\delta_k)_k$, $\delta_k > 0$ and $\delta_k \to 0$, which, without restriction of generality, may be assumed to be decreasing. We assume $|Tf| \le \|f\|_{L,\tilde{\delta}}$ for a suitable fixed sequence $\tilde{\delta}$.

We fix δ and set $r = \sup\{|x| + |y| + \delta_0 \,:\, x \in K,\, y \in L\}$. We obtain for $M = M_T$

$$\|Mf\|_{K,\delta} = \sup_{\alpha\in\mathbb{N}_0^d, x\in K} \left| T_y\Big(\frac{y^\alpha f^{(\alpha)}(xy)}{\alpha!}\Big)\delta_{|\alpha|}^{|\alpha|}\right| \le \sup_{\alpha\in\mathbb{N}_0^d, x\in K}\ \sup_{\beta\in\mathbb{N}_0^d, y\in L} \frac{|\partial_y^\beta y^\alpha f^{(\alpha)}(xy)|}{\alpha!\beta!}\delta_{|\alpha|}^{|\alpha|}\tilde{\delta}_{|\beta|}^{|\beta|}.$$

We estimate the derivatives in the last term:

$$\begin{aligned}
\left|\frac{1}{\alpha!\beta!}\partial_y^\beta(y^\alpha f^{(\alpha)}(xy))\right| &= \left|\frac{1}{\alpha!\beta!}\sum_{0\le\gamma\le\min(\alpha,\beta)}\binom{\beta}{\gamma}\frac{\alpha!}{(\alpha-\gamma)!}y^{\alpha-\gamma}x^{\beta-\gamma}f^{(\alpha+\beta-\gamma)}(xy)\right| \\
&\le 2^{|\beta|}\sup_{0\le\gamma\le\min(\alpha,\beta)} r^{|\alpha+\beta-2\gamma|}\binom{\alpha+\beta-\gamma}{\beta}\frac{|f^{(\alpha+\beta-\gamma)}(xy)|}{(\alpha+\beta-\gamma)!} \\
&\le \sup_{0\le\gamma\le\min(\alpha,\beta)} (4r)^{|\alpha+\beta-\gamma|}\frac{|f^{(\alpha+\beta-\gamma)}(xy)|}{(\alpha+\beta-\gamma)!}\,r^{-|\gamma|}.
\end{aligned}$$

We may assume $\delta \ge \tilde{\delta}$ and δ decreasing. We put

$$\tilde{c}_M = \sup_{n+m=M} \delta_n^{n/M}\tilde{\delta}_m^{m/M}.$$

Then $\tilde{c}_M$ is a strictly positive null-sequence. We denote by $(c_M)_M$ its decreasing majorant. Then we have for α, $\beta \in \mathbb{N}_0^d$ and $0 \le \gamma \le \min(\alpha,\beta)$ the estimate

$$r^{-|\gamma|}\delta_{|\alpha|}^{|\alpha|}\tilde{\delta}_{|\beta|}^{|\beta|} \le \delta_{|\alpha-\gamma|}^{-|\gamma|}\delta_{|\alpha-\gamma|}^{|\alpha|}\tilde{\delta}_{|\beta|}^{|\beta|} \le c_{|\alpha+\beta-\gamma|}^{|\alpha+\beta-\gamma|}.$$

Since $KL \Subset \Omega$ we obtain:

$$\|Mf\|_{K,\delta} \le \|f\|_{KL,4rc}.$$

We have shown that $T \mapsto M_T$ maps equicontinuous subsets of $\mathscr{A}(V_K(\Omega))'_b$ into equicontinuous, hence bounded, subsets of $M(\Omega, K)$. Since $\mathscr{A}(V_K(\Omega))'_b$ is barreled every bounded set is equicontinuous [31, 23.27]. So $T \mapsto M_T$ maps bounded sets into bounded sets. Since $\mathscr{A}(V_K(\Omega))'_b$ is bornological the map $T \mapsto M_T$ is continuous from $\mathscr{A}(V_K(\Omega))'_b$ to $M(\Omega, K)$.

To show the reverse direction, we assume that $\mathcal{M} \subset MC(\Omega, K)$ is equicontinuous, that is, we find a compact set $L \subset \Omega$ and a null-sequence δ such that

$$\sup_{\eta\in K}|(Mf)(\eta)| \le \|f\|_{L,\delta} = \sup_{\alpha,\, x\in L}\frac{|f^{(\alpha)}(x)|}{\alpha!}\,\delta_{|\alpha|}^{|\alpha|}$$

for all $M \in \mathcal{M}$ and $f \in \mathscr{A}(\Omega)$, in particular, for all $f \in \mathscr{A}(\mathbb{R}^d)$.

Given $f \in \mathscr{A}(\mathbb{R}^d)$, this applies to all f_η, $f_\eta(x) = f\left(\frac{x}{\eta}\right)$, $\eta \in K \cap \mathbb{R}^d_*$, and we obtain for all these η

$$|Tf| = |(Mf)(\mathbb{1})| = |(Mf_\eta)(\eta)| \le \|f_\eta\|_{L,\delta} = \|f\|_{L_\eta,\delta(\eta)}, \tag{21}$$

where

$$L_\eta = \frac{1}{\eta}\cdot L, \quad \delta(\eta)_m := \frac{\delta_m}{\min_{j=1,\dots,d}|\eta^{(j)}|}, \quad \eta = (\eta^{(1)},\dots,\eta^{(d)}).$$

We conclude, using the assumption that $K \cap \mathbb{R}^d_*$ is dense in K, that

$$\operatorname{supp} T_M \subset \bigcap_{\eta\in K\cap\mathbb{R}^d_*} L_\eta = \{y \in \mathbb{R}^d \,:\, yK \subset L\} \Subset V_K(\Omega)$$

and, since all L_η are compact, we can find $\eta_1,\dots,\eta_m \in K \cap \mathbb{R}^d_*$ such that

$$\operatorname{supp} T_M \subset \hat{L} := \bigcap_{j=1}^{m} L_{\eta_j} \Subset V_K(\Omega). \tag{22}$$

This holds for all $M \in \mathcal{M}$. From (21) it follows that $\{T_M \mid M \in \mathcal{M}\}$ are equicontinuous in $\mathscr{A}(L_\eta)'$. By (22) and using the approach in [38, Th. I11] one gets easily that $\{T_M \mid M \in \mathcal{M}\}$ are equicontinuous in $\mathscr{A}(\hat{L})'$, i.e., there is a null-sequence γ such that

$$|T_M f| \le \|f\|_{\hat{L},\gamma}$$

for all $M \in \mathcal{M}$ and $f \in \mathscr{A}(\Omega)$.

Finally, we have continuous maps $\mathscr{A}(V_K(\Omega))'_b \to M(\Omega, K) \hookrightarrow MC(\Omega, K)$ and the composition is surjective. Therefore $M(\Omega, K) = MC(\Omega, K)$ as sets. □

Remark 7.3 (a) Let us recall that $\mathscr{A}(\Omega)'_b$ and $\mathscr{A}(L)'_b$, $L \Subset \Omega$, are nuclear, thus (see [24, Ch. 21]) we have topological isomorphisms:

$$L_b(\mathscr{A}(\Omega), C(K)) \cong \mathscr{A}(\Omega)'_b \hat{\otimes} C(K), \qquad L_b(\mathscr{A}(L), C(K)) \cong \mathscr{A}(L)'_b \hat{\otimes} C(K).$$

Algebraically $L_b(\mathscr{A}(\Omega), C(K)) = \operatorname{ind}_{L\Subset\Omega} L_b(\mathscr{A}(L), C(K))$ and, by [18, I §1no. 3, Cor. p. 47], the topologies coincide as well. Hence $L_b(\mathscr{A}(\Omega), C(K))$ is an LF-space.

(b) Also the spaces $\mathscr{A}(V_K(\Omega))'_b$ are (LF)-spaces and the step spaces are $\mathscr{A}(L)'_b$, L compact in $V_K(\Omega)$. $\mathscr{B} : \mathscr{A}(V_K(\Omega))'_b \to L_b(\mathscr{A}(\Omega), C(K))$ is a continuous, injective map, its range $M(\Omega, K)$ is closed in $L_b(\mathscr{A}(\Omega), C(K))$. It is a correspondence between the bounded (= equicontinuous) sets in $\mathscr{A}(V_K(\Omega))'_b$ and $M(\Omega, K)$. For every compact $L \subset \Omega$ there is a compact $\hat{L} \subset V_K(\Omega)$ such that $\mathscr{B}^{-1}(L(\mathscr{A}(L), C(K))) \subset \mathscr{A}'(\hat{L})$. To show the last assertion we use the proof of Proposition 7.1 with δ and γ being constant or Grothendieck's Factorization Theorem.

The question whether $\mathscr{B} : \mathscr{A}(V_K(\Omega))'_b \to M(\Omega, K)$ is a topological isomorphism is, by Remark 7.3, a classical problem of well-locatedness (see e.g. [16]), i.e., the question if closed subspace $M(\Omega, K)$ in $L_b(\mathscr{A}(\Omega), C(K))$ is a topological inductive limit of $\mathscr{B}(\mathscr{A}(L)'_b)$, $L \Subset V_K(\Omega)$ (or, equivalently, of $M(\Omega, K) \cap L_b(\mathscr{A}(L), C(K))$, $L \Subset \Omega$). For a stronger assumption on K, however, we can show it.

Proposition 7.4 *If $K \subset \Omega \cap \mathbb{R}^d_*$, then $\mathscr{B} : \mathscr{A}(V_K(\Omega))'_b \to MC(\Omega, K)$, $\mathscr{B}(T) = M_T$ (see Theorem 2.2) is a linear topological isomorphism. In particular, $M(\Omega, K) = MC(\Omega, K)$ as topological linear spaces.*

As a direct consequence we get:

Theorem 7.5 *The map $\mathscr{B}^{-1} : M(\Omega) \to \mathscr{A}(V(\Omega))'_{k_*}$ is a continuous map, i.e., the topology τ_{k_*} on $\mathscr{A}(V(\Omega))'$ is weaker than the topology on $\mathscr{A}(V(\Omega))'$ induced by $\mathscr{B}$ from $M(\Omega) \subset L_b(\mathscr{A}(\Omega))$.*

Proof: It follows from Proposition 7.4 that

$$\mathscr{B} : \mathscr{A}(V(\Omega))'_{k_*} = \operatorname{proj}_{K \Subset \Omega \cap \mathbb{R}^d_*} \mathscr{A}(V_K(\Omega))'_b \to \operatorname{proj}_{K \Subset \Omega \cap \mathbb{R}^d_*} M(\Omega, K)$$

is a topological isomorphism. The locally convex topology $\operatorname{proj}_{K \Subset \Omega \cap \mathbb{R}^d_*} M(\Omega, K)$ is weaker than the original topology on $M(\Omega) = \operatorname{proj}_{K \Subset \Omega} M(\Omega, K)$. □

The proof of Proposition 7.4 is based on the harmonic representation of analytic functionals (see Section 6) combined with some ideas and results from [40] which are introduced now:

Lemma 7.6 *(See [40]) Let $(X, \| \ \|)$ be a Banach space and let $F := \operatorname{ind}_{k \to \infty} F_k$ where*

$$F_k := \{(x_\alpha)_{\alpha \in \mathbb{N}_0^n} \in X^{\mathbb{N}_0^n} \mid \|x\|_k := \sup_\alpha \|x_\alpha\| k^{-|\alpha|} < \infty\}.$$

Then a fundamental system of seminorms on F is given by

$$|x|_\delta := \sup_\alpha \|x_\alpha\| \delta_{|\alpha|}^{|\alpha|},$$

where $\delta = (\delta_j)_{j \in \mathbb{N}}$ is any strictly positive sequence tending to 0.

Notice that a fundamental system of bounded sets in F is given by the closed unit balls B_k in F_k. Indeed, a fundamental system is provided by the closures $\overline{B_k}$ (in F, see [31, 25.16]) which coincide with B_k since the identity mapping $id : F \to X^{\mathbb{N}_0^n}$ is continuous (and so the closure in F is contained in the coordinatewise closure).

Let us introduce the following space:

$$\widetilde{C}_{\Delta,c}(\widetilde{V}) := \operatorname{ind}_{K \Subset \widetilde{V}} \widetilde{C}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K).$$

In [40] the Lemma 7.6 was used to determine a canonical fundamental system on the space $\mathscr{A}(K)$ for compact K. A variant of this proof gives the following basic result:

Theorem 7.7 *Let $\widetilde{V} \subset \mathbb{R}^{d+1}$ be open such that $t \to 0$ if $\widetilde{V} \ni (x,t) \to \infty$. Then a fundamental system of seminorms on $\widetilde{C}_{\Delta,c}(\widetilde{V})$ is given by*

$$|f|_\delta := \sup_{\xi \in \partial \widetilde{V}, \alpha \in \mathbb{N}_0^{d+1}} \frac{|f^{(\alpha)}(\xi)|}{\alpha!} \delta_{|\alpha|}^{|\alpha|}$$

where $\delta = (\delta_j)_{j \in \mathbb{N}}$ is any positive sequence tending to zero.

Proof: We define F as in Lemma 7.6 using $X := C_0(\partial \widetilde{V}) := \{f \in C(\partial \widetilde{V}) \mid \lim_{\widetilde{V} \ni \xi \to \infty} f(\xi) = 0\}$ endowed with the sup-norm. Let $A(f) := (\frac{1}{\alpha!} f^{(\alpha)}|_{\partial \widetilde{V}})_{\alpha \in \mathbb{N}_0^{d+1}}$. Then $A : \widetilde{C}_{\Delta,c}(\widetilde{V}) \to F$ is defined and continuous by Lemma 7.6. We will prove that A is an injective topological homomorphism using Baernstein's Lemma [31, 26.26]. By Lemma 7.6 this will show the theorem.

Notice that $\widetilde{C}_{\Delta,c}(\widetilde{V})$ is a (DFS)-space (in particular Montel). F is a (DF)-space by [31, 25.16]. By the remark after Lemma 7.6 we have to show that $A^{-1}(B_k)$ is bounded in $\widetilde{C}_{\Delta,c}(\widetilde{V})$. Clearly, the functions in $A^{-1}(B_k)$ are uniformly bounded on $U_\varepsilon := \partial \widetilde{V} + B_\varepsilon(0)$ for some $\varepsilon > 0$ by Taylor expansion. $K := \widetilde{V} \setminus U_\varepsilon$ is a compact set contained in $\widetilde{V}$ since for any $\varepsilon > 0$ there is $\gamma > 0$ such that $|t| < \varepsilon$ if $(x,t) \in \widetilde{V}$ and $|x| \geq \gamma$ by the assumption on $\widetilde{V}$. Hence $A^{-1}(B_k)$ is bounded in $\widetilde{C}_{\Delta,c}(\widetilde{V})$ since it is well known that for compact $K \subset \widetilde{V}$ the topology of $\widetilde{C}_{\Delta,0}(\mathbb{R}^{d+1} \setminus K)$ is induced by $\widetilde{C}_\Delta(\widetilde{V} \setminus K)$. □

Remark 7.8 Let us observe that $V = V_K(\Omega)$ is open and has the following property

$$\forall\, x \in \partial V \;\; \exists\, y_x \in K \quad xy_x \notin \Omega. \tag{23}$$

Proof of Proposition 7.4: The claim clearly holds for $\Omega = \mathbb{R}^d$ since then $\mathscr{B}^{-1}(M) = \delta_{\mathbb{1}} \circ M$ by the Representation Theorem 2.2. Let $\Omega \neq \mathbb{R}^d$.

By Proposition 7.1, it suffices to show that $\mathscr{B}^{-1} : MC(\Omega, K) \to \mathscr{A}(V)'_b$ is continuous for $V = V_K(\Omega)$. Clearly, by (9) and (15),

$$\mathscr{A}(V)'_b = \operatorname{proj}_{\widetilde{V}} \widetilde{C}_\Delta(\widetilde{V})'_b,$$

where the limits run over the neighborhoods $\widetilde{V}$ of V in $\mathbb{R}^{d+1}$, $\widetilde{V} \cap \mathbb{R}^d = V$. By the TG duality it is therefore sufficient to prove the continuity of

$$\mathscr{B}^{-1} : MC(\Omega, K) \to \widetilde{C}_{\Delta,c}(\widetilde{V}) \simeq \widetilde{C}_\Delta(\widetilde{V})'_b$$

where $\widetilde{V}$ is as above. Choose a continuous function $t : V \to]0, \infty[$ such that $t(x) \to 0$ if $x \to \partial V$ or $x \to \infty$ and put

$$\widetilde{V} := \{(x, t) \in \mathbb{R}^{d+1} \mid x \in V, |t| < t(x)\}.$$

The set $\widetilde{V}$ satisfies the assumption of Theorem 7.7. Since $V \neq \mathbb{R}^d$ we may assume that $\partial V \neq \emptyset$. Set $t(x) := 0$ if $x \in \partial V$. For $x \in \overline{V}$ choose $\widehat{x} \in \partial V$ such that $|x - \widehat{x}| = \operatorname{dist}(x, \partial V)$ (especially, $x = \widehat{x}$ if $x \in \partial V$) and set $y_x := y_{\widehat{x}} \in K$ chosen for $\widehat{x}$ by (23).

Let $M_T := \mathscr{B}(T)$ for $T \in \mathscr{A}(V)' \subset \widetilde{C}_\Delta(\widetilde{V})'$ and let $f_T \in \widetilde{C}_{\Delta,c}(\widetilde{V})$ be the representation of T by the TG duality. Let $\delta = (\delta_j)_{j\in\mathbb{N}}$ be a strictly positive sequence tending to 0. By the definition of $\widetilde{V}$ we have

$$\partial\widetilde{V} = \{(x, \pm t(x)) \mid x \in \overline{V}\}.$$

Notice that f_T is even in t. By Theorem 7.7 we thus have to estimate

$$\begin{aligned} |f_T|_\delta &= \sup_{x\in\overline{V}, \alpha\in\mathbb{N}_0^{d+1}} \frac{|f_T^{(\alpha)}(x, t(x))|}{\alpha!}\delta_{|\alpha|}^{|\alpha|} = \sup_{x\in\overline{V}, \alpha\in\mathbb{N}_0^{d+1}} \frac{|\langle_\xi T, G^{(\alpha)}(x - y_x(\xi/y_x), t(x))\rangle|}{\alpha!}\delta_{|\alpha|}^{|\alpha|} \\ &= \sup_{x\in\overline{V}, \alpha\in\mathbb{N}_0^{d+1}} |M_T(h_{x,\alpha})(y_x)| \le \sup_{y\in K, h\in B} |M_T(h)(y)| \end{aligned} \tag{24}$$

where

$$B := \left\{ h_{x,\alpha}(\xi) := \frac{G^{(\alpha)}(x - \xi/y_x, t(x))}{\alpha!}\delta_{|\alpha|}^{|\alpha|} \mid x \in \overline{V}, \alpha \in \mathbb{N}_0^{d+1} \right\}.$$

Since $K \subset \Omega$ is compact by (23), the right hand side of (24) is a continuous seminorm on $L_b(\mathscr{A}(\Omega), C(K))$ if we show that B is bounded in $\mathscr{A}(\Omega)$. To prove this, let $J \subset \Omega$ be compact. Notice that

$$C_J := \inf_{x\in\overline{V}, \xi\in J} |(x - \xi/y_x, t(x))| > 0. \tag{25}$$

If not, then there are sequences $x_n \in \overline{V}$ and $\xi_n \in J$ such that $|(x_n - \xi_n/y_{x_n}, t(x_n))| \to 0$. Since $K \subset \mathbb{R}^d_*$ and J are compact we may assume that $y_{x_n} \to y \in K \subset \mathbb{R}^d_*$ and $\xi_n \to \xi \in J$, hence

$$x_n y_{x_n} - \xi_n = y_{x_n}(x_n - \xi_n/y_{x_n}) \to 0. \tag{26}$$

If the sequence $(x_n)_n$ is unbounded then $x_{n_k} y_{x_{n_k}} \to \infty$ for a subsequence since $y_{x_n} \to y \in \mathbb{R}^d_*$. This is a contradiction to (26). Hence $(x_n)_n$ is bounded and we can assume that $x_n \to x_0 \in \partial V$

since $t(x_n) \to 0$ and the function t is continuous on $\overline{V}$ and strictly positive on V. Hence we get by (26) that

$$\Omega \supset J \ni \xi = \lim_n \xi_n = \lim_n x_n y_{x_n} = \lim_n (\widehat{x_n} y_{x_n} + (x_n - \widehat{x_n}) y_{x_n}) = \lim_n \widehat{x_n} y_{x_n} \notin \Omega$$

since $|x_n - \widehat{x_n}| = \operatorname{dist}(x_n, \partial V) \to 0$ and $(y_{x_n})_n$ is convergent, a contradiction.

For any $\gamma > 0$ we may choose C_γ such that $\delta_j^j \le C_\gamma \gamma^j$ for any j since $\delta_j \to 0$. Also notice that $\eta := \min(1, \inf\{|y_j| \mid y \in K, j \le d\}) > 0$ since $K \subset \mathbb{R}^d_*$ is compact. Using (25) and (17) we thus get

$$\sup_{x\in\overline{V},\alpha\in\mathbb{N}_0^{d+1}} \sup_{\xi\in J,\beta\in\mathbb{N}_0^d} \left|\partial_\xi^\beta h_{(x,\alpha)}(\xi)\right| \frac{\gamma^{|\beta|}}{\beta!} \le \sup_{x\in\overline{V},\alpha\in\mathbb{N}_0^{d+1},\xi\in J,\beta\in\mathbb{N}_0^d} \left|(\partial^\alpha \partial_x^\beta G)\left(x - \frac{\xi}{y_x}, t(x)\right)\right| \frac{\gamma^{|\beta|}\delta_{|\alpha|}^{|\alpha|}}{|y_x^\beta|\beta!\alpha!}$$

$$\le C_0 C_\gamma \sup_{x\in\overline{V},\alpha\in\mathbb{N}_0^{d+1},\xi\in J,\beta\in\mathbb{N}_0^d} |(x - \xi/y_x, t(x))|^{-|\beta|-|\alpha|-d+1} \left(\frac{2C\gamma}{\eta}\right)^{|\beta|+|\alpha|}$$

$$\le C_0 C_\gamma C_J^{1-d} \sup_{j\in\mathbb{N}_0} \left(\frac{2C\gamma}{C_J\eta}\right)^j \le C_0 C_\gamma C_J^{1-d}$$

if $\gamma < C_J\eta/(2C)$. The theorem is proved because we have proved that B is bounded in every $\mathscr{A}(J)$, J compact subset of Ω, so it is bounded in $\mathscr{A}(\Omega) = \operatorname{proj}_{J\Subset\Omega} \mathscr{A}(J)$. □

From Proposition 5.3 and Theorem 7.2 and the fact that the topology τ_k is weaker than the topology τ_b we obtain:

Corollary 7.9 *$\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to L_b(\mathscr{A}(\Omega))$ is continuous. If it is open onto its image, then $\tau_b = \tau_k$ on $\mathscr{A}(V(\Omega))'$ and, in consequence, $V(\Omega)$ has a countable neighborhood basis.*

This shows that, in general, the b-topology is not the "natural" topology, induced via $\mathscr{B}$ on $\mathscr{A}(V(\Omega))'$.

Example 7.10 Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, 1 < y < 2\}$ then $V(\Omega) = \{(x, 1) \in \mathbb{R}^2 : x > 0\}$. Since $\Omega \subset \mathbb{R}^d_*$ thus, by definition, $\tau_k = \tau_{k_*}$. By Theorems 7.2 and 7.5, the topology induced on $\mathscr{A}(V(\Omega))'$ via $\mathscr{B}$ is the topology τ_k. Hence $\tau_b \ne \tau_k$ on $\mathscr{A}(V(\Omega))'$ since $V(\Omega)$ does not admit a countable neighborhood basis (see Proposition 5.3). However $\mathscr{A}(V(\Omega))'_b$ is an (LF)-space since $V(\Omega)$ is locally closed (see remarks before Proposition 5.1). Then, due to de Wilde's Theorem, $\mathscr{A}(V(\Omega))'_k$ cannot be bornological (see Theorem 7.11 below).

Next we show that all topologies under consideration (except may be τ_{k_*}) have the same bounded sets.

Theorem 7.11 *A set $B \subset \mathscr{A}(V(\Omega))'_b$ is bounded if and only if $\{M_T := \mathscr{B}(T) \mid T \in B\}$ is bounded in $L_b(\mathscr{A}(\Omega))$. In particular:*

(a) a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathscr{A}(V(\Omega))'_b$ is convergent if and only if $(\mathscr{B}(T_n))_{n\in\mathbb{N}} \subset M(\Omega)$ is convergent;

(b) the topologies τ_b and τ_k on $\mathscr{A}(V(\Omega))'$ have the same bounded sets and the same convergent sequences.

Hence $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a topological isomorphism if and only if $M(\Omega)$ equipped with the topology inherited from $L_b(\mathscr{A}(\Omega))$ is (ultra)bornological.

Let us note that if $V(\Omega)$ is compact then $\mathscr{B}:\mathscr{A}(V(\Omega))'_b\to M(\Omega)$ is a topological isomorphism if and only if $M(\Omega)$ is metrizable (since metrizable spaces are bornological [31, 24.13]).

Proof: As we explained in the introduction to the Section 5 in $\mathscr{A}(V(\Omega))'_b$ all bounded sets are relatively compact. Hence by Corollary 7.9 we only have to show the "if" part of the first statement.

If $\mathcal{M}\subset M(\Omega)$ is bounded and $\omega\subset\subset\Omega$ open and $K=\overline{\omega}$, then $\mathcal{M}$ is bounded in $M(\Omega,K)$ and therefore equicontinuous (since $\mathscr{A}(\Omega)$ is barreled, as explained in Section 5 and by [31, 23.27]). By Proposition 7.1, the set $\mathcal{T}:=\{T\in\mathscr{A}(V_K(\Omega))' : M_T\in\mathcal{M}\}$ is equicontinuous.

The proof of [38, Th. I.11] can be easily adapted to show that for any equicontinuous set B of analytic functionals on $\mathscr{A}(U)$, $U\subset\mathbb{R}^d$ open, there is a minimal compact set L such that B is also equicontinuous in $\mathscr{A}(L)'$. If we choose L as above for $\mathcal{T}$ then $L\subset V_K(\Omega)$ for every $K\Subset\Omega$. Hence $L\subset\bigcap V_K(\Omega)=V(\Omega)$ and this shows the claim. □

If $\widetilde{V}(\Omega)=V(\Omega)$ (see Section 2) we have $V(\Omega)=\bigcap_{K\Subset\Omega\cap\mathbb{R}^d_*}V_K(\Omega)$ and therefore $\tau_{k_*}\leq\tau_k$ and the topology τ_{k_*} is a complete locally convex topology on $\mathscr{A}(V(\Omega))'$. By $M(\Omega,\Omega')\subset L_b(\mathscr{A}(\Omega),\mathscr{A}(\Omega'))$ we denote as previously the subspace of all maps admitting all monomials as 'eigenvectors'. From Proposition 7.4 we obtain:

Proposition 7.12 *If $\widetilde{V}(\Omega)=V(\Omega)$ then $M(\Omega)=M(\Omega,\Omega\cap\mathbb{R}^d_*)$ and the map $\mathscr{B}:\mathscr{A}(V(\Omega))'_{k_*}\to M(\Omega,\Omega\cap\mathbb{R}^d_*)$ is a topological linear isomorphism.*

The property of $\tau_b=\tau_k$ and $\tau_k=\tau_{k_*}$ are completely independent as we will see later, see Example 8.10 and Theorems 7.14 and 8.9.

The property $\tau_b=\tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ should be compared with the notion of *nice set* introduced in [8] for open sets $\Omega\subset\mathbb{R}$. There we assumed that $V_K(\Omega)\subset V$ for some finite set $K\subset\Omega\cap\mathbb{R}^d_*$. Clearly, if Ω is a nice set then $\tau_b=\tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$. In fact, the class of sets with $\tau_b=\tau_{k_*}$ is strictly larger already in the case of one variable: all examples of not nice sets given in [8] satisfy $\tau_b=\tau_{k_*}$ (see [8, Examples 2.5 and 3.2] and Example 7.13 below).

The following example shows that in the description of $\tau_b=\tau_{k_*}$ we cannot put finite sets instead of compact ones.

Example 7.13 *Let $\Omega:=\mathbb{R}^d_+\setminus\{\mathbb{1}\}$ for $a\in\mathbb{R}^d_+$. Then $V(\Omega)=\{\mathbb{1}\}$ and $\tau_b=\tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ but Ω is not nice for $d=1$ and the K's in the definition of the topology τ_{k_*} cannot be chosen finite for $d\geq 2$.*

Proof: We have $\tau_b=\tau_{k_*}$ by the argument in Example 5.4. i) Let $d=1$. Then Ω is not nice by [8, Examples 3.2]. ii) Let $d\geq 2$ (see Example 5.4). If K is finite then the complement (in $\mathbb{R}^d_+$) of the $V_K(\Omega)$ finite. □

On the basis of the information collected up to now we can identify the topology induced by $M(\Omega)$ on $\mathscr{A}(V(\Omega))'$ via $\mathscr{B}$ for many important cases.

Theorem 7.14 *Let $\Omega\subset\mathbb{R}^d$ be an open nonempty set. In the following cases $\mathscr{B}:\mathscr{A}(V(\Omega))'_k\to M(\Omega)$ is a linear topological isomorphism, that is $M(\Omega)$ induces on $\mathscr{A}(V(\Omega))'$ via $\mathscr{B}$ the topology τ_k:*

(a) $\Omega\subset\mathbb{R}^d_*$;

(b) $d=1$;

(c) Ω *is convex.*

In fact, in these cases $\tau_k=\tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$.

Proof: Observe that if $\tau_k = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ then, by Theorems 7.2 and 7.5, the map

$$\mathscr{B} : \mathscr{A}(V(\Omega))'_k \to M(\Omega)$$

is a topological isomorphism.

(a): $\tau_k = \tau_{k_*}$ just by definition.

In the other cases we use Proposition 5.3 (c).

(b): By [8, Proposition 2.1], if $0 \in V(\Omega)$ then either $V(\Omega)$ is bounded or $V(\Omega) = \Omega = \mathbb{R}$. In the latter case by (c) below $\tau_k = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$. In the former case we can choose $K \Subset \Omega$ such that $V := V_K^0(\Omega)$ is bounded so ∂V is compact. Clearly $V_K(\Omega) \subset \mathbb{R} \setminus \partial V$.

Take $J := K \setminus B_\delta(0)$. Choose $\delta > 0$ so small that $\partial V \cdot B_\delta(0) \subset \Omega$ (this is possible since ∂V is bounded and Ω is an open neighborhood of 0). Now, for any $\xi \in \partial V$ holds $\xi K \not\subseteq \Omega$ but $\xi B_\delta(0) \subset \Omega$. Hence $\xi J \not\subseteq \Omega$ and $V_J(\Omega) \subset \mathbb{R} \setminus \partial V$, so $V_J^0(\Omega) \subset V$. If $0 \notin V(\Omega)$ the set $\Omega \subset \mathbb{R}^d_*$ so we apply (a).

(c): If $\xi J \subset \Omega$ then $\xi \operatorname{conv} J \subset \Omega$. Thus $V_J(\Omega) \subset V_{\operatorname{conv} J}(\Omega)$. It is easy to see that for any compact $K \subset \Omega$ there is compact $L \subset \Omega \cap \mathbb{R}^d_*$ such that $K \subset \operatorname{conv} L$. Hence $V_L(\Omega) \subset V_{\operatorname{conv} L}(\Omega) \subset V_K(\Omega)$. □

Based on this and also on the analogous results in [41] we make the

Conjecture. *For every open non-empty set $\Omega \subset \mathbb{R}^d$ the map $\mathscr{B} : \mathscr{A}(V(\Omega))'_k \to M(\Omega)$ is a topological isomorphism.*

8 Topological Representation in Terms of $\mathscr{A}(V(\Omega))'_b$

In Theorem 7.14 we have solved for many important cases the problem of topological representation of $M(\Omega)$ via $\mathscr{B}$ and shown that the induced topology is the k-topology. It remains the question under which conditions the induced topology is the b-topology, that is, the map $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a linear topological isomorphism. For all the cases covered by Theorem 7.14, in particular for all convex sets, we will solve this problem completely.

Since in all theses cases the sets are shown in Theorem 7.14 to satisfy $\tau_k = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$, our problem means the question, when $\tau_b = \tau_k$ on $\mathscr{A}(V(\Omega))'$. This is, as we already have remarked earlier (see Corollary 7.9), a rather restrictive property.

Proposition 8.1 *If $\tau_b = \tau_k$ on $\mathscr{A}(V(\Omega))'$ for some $\Omega \subset \mathbb{R}^d$ then $V(\Omega)$ has a countable basis of open neighborhoods or, equivalently, $\partial V(\Omega) \cap V(\Omega)$ is compact.*

Proof: The first part follows from Proposition 5.3 (a), the second part follows from Prop. 4.11. □

From Proposition 4.6, (4) and Proposition 5.3 (a) we get:

Corollary 8.2 *If $V(\Omega)$ is open, then $\tau_b = \tau_k$ on $\mathscr{A}(V(\Omega))'$.*

Assume still that $V(\Omega)$ is open. We set $K_0 = \Omega \cap \{-1, +1\}^d$ then for any compact K with $K_0 \subset K \subset \Omega \cap \mathbb{R}^d_*$ we have $V_K(\Omega) = \{x \in \mathbb{R}^d : x \cdot (\Omega \cap \mathbb{R}^d_*) \subset \Omega\} = \widetilde{V}(\Omega)$ and we have shown:

Remark 8.3 If $V(\Omega)$ is open, then $\tau_b = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ if and only if Ω satisfies $\widetilde{V}(\Omega) = V(\Omega)$.

Example 8.4 Let $\Omega = \mathbb{R}^2 \setminus (\{0\} \times [0, +\infty))$. Then $V(\Omega) = (\mathbb{R}_+ \cup \mathbb{R}_-) \times \mathbb{R}_+$ and $\widetilde{V}(\Omega) = V_{K_0}(\Omega) = (\mathbb{R}_+ \cup \mathbb{R}_-) \times \mathbb{R}$. Hence $\tau_b = \tau_k \neq \tau_{k_*}$. Please note that if $\Omega_1 := V(\Omega)$ then $V(\Omega_1) = V(\Omega)$ and, by Corollary 8.2 and Theorem 7.14 $\tau_b = \tau_k = \tau_{k_*}$ on $\mathscr{A}(V(\Omega_1))'$. This means that the topologies τ_{k_*} really depend on Ω and not only on $V(\Omega)$.

Example 8.5 *An example of a set $\Omega \subset \mathbb{R}^2$ with $\tau_b = \tau_k = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ and $\partial V(\Omega) \cap V(\Omega)$ is nonempty compact but $V(\Omega)$ is bounded and non-compact.*

Proof: Let us take Ω as the union of the following sets

$$\{(x,y) \mid -1 < y \le -1/2, -1 < x < 1\}, \qquad \{(x,y) \mid -1/2 < y < 0, -1 < x < \varphi(y)\},$$
$$\{(x,y) \mid 0 \le y < 1/4, -1 < x\}, \qquad \{(x,y) \mid 1/4 \le y \le 3/8, 0 < x\},$$
$$\{(x,y) \mid 3/8 < y < 1/2, -1 < x\}$$

with the sequence of sets

$$\Omega_0 := \{(x,y) \mid 1 < x < 2, 3/4 < y < 1\}, \quad \Omega_n := \{(x,y) \mid 3^{n+1} < x < 3^{n+1} + 1, 3/4 < y < 1\}$$

for $n = 1, 2, \ldots$ and where $\varphi : (-1/2, 0) \to (1, +\infty)$, $\varphi(-1/2) = 1$, is a strictly increasing function tending to infinity at zero from below. It needs some calculations but one see that

$$V(\Omega) = \{(x,y) \mid 0 < x \le 1, 0 \le y \le 1/2\} \cup \{(0,y) \mid -1/4 \le y \le 1/4\} \cup \{(0,1/2)\} \cup \{(1,1)\}.$$

Moreover, $\tau_b = \tau_{k_*}$ — the tedious calculations based on Proposition 5.3 are left to the reader. □

The necessary condition in Prop. 8.1 need not be sufficient, we have the following example:

Example 8.6 *An open set $\Omega \subset \mathbb{R}^2$ such that $V(\Omega)$ is compact but $\tau_b \ne \tau_k$ on $\mathscr{A}(V(\Omega))'$.*

Proof: Apply Proposition 5.3 (a). Let us take Ω to be the union of the following sets:

$$\{(x,y) \mid -1 < y < 1/2, -1 < x, x \ne 1\},$$
$$\{(x,y) \mid 1/2 \le y \le 3/4, 0 < x, x \ne 1, x \ne 2\},$$
$$\{(x,y) \mid 3/4 < y < 1, 0 < x, x \ne 2\}.$$

Then

$$V(\Omega) = \{(0,y) : 0 \le y \le 1/2\} \cup \{(1,1)\}.$$

Take y slightly bigger than $1/2$. Then $(x,y) \notin V(\Omega)$ for x close to 0. But the only $(\tilde{x}, \tilde{y}) \in \Omega$ such that $(\tilde{x}, \tilde{y})(x,y) \notin \Omega$ are $\tilde{x} = \frac{1}{x}$, or $\tilde{x} = \frac{2}{x}$. Hence the set of $(\tilde{x}, \tilde{y})$ cannot be chosen compact if $x \to 0$. □

However, it turns out that in the one dimensional case the necessary condition in Proposition 8.1 is indeed sufficient, which leads to an explicit description in terms of $V(\Omega)$.

Theorem 8.7 *If $\Omega \subset \mathbb{R}$ is a nonempty open set then the following assertions are equivalent:*

(a) the map $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a topological isomorphism;

(b) $\tau_b = \tau_{k_*}$ (or, equivalently, $\tau_b = \tau_k$) on $\mathscr{A}(V(\Omega))'$;

(c) $\partial V(\Omega) \cap V(\Omega)$ is compact;

(d) one of the following conditions holds:

- $0 \in V(\Omega)$ *(i.e., $0 \in \Omega$);*
- $V(\Omega) \subset \{1, -1\}$*;*
- $V(\Omega)$ *has a non-empty interior.*

The result above improves [8, Th. 2.6]. In fact, by Proposition 4.4 and [8, Proposition 2.1], if $0 \in \Omega$ then the conditions above are always satisfied. In case $0 \notin \Omega$, the conditions are not satisfied if and only if $\partial V(\Omega)$ contains either an unbounded sequence or a zero sequence (for instance the condition above is not satisfied for the set $\Omega = (0, +\infty) \setminus \{2^n \mid n \in \mathbb{Z}\}$ where $V(\Omega) = \{2^n \mid n \in \mathbb{Z}\}$). So the above results disprove the conjecture posed in [8] that (a) in Theorem 8.7 always holds.

Before we prove the result we need the following useful lemma showing in some cases that the crucial condition (23) holds locally for $K \subset \Omega \cap \mathbb{R}^d_*$:

Lemma 8.8 *Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set. If $x_0 \in \mathbb{R}^d_* \setminus \widetilde{V}(\Omega)$ or $x_0\Omega \nsubseteq \overline{\Omega}$ then there are a neighborhood U_0 of x_0 and a compact set $K \subset \Omega \cap \mathbb{R}^d_*$ such that*

$$\forall x \in U_0 \ \exists y_x \in K : \quad xy_x \notin \Omega.$$

Proof: i) Let $x_0 \in \mathbb{R}^d_* \setminus \widetilde{V}(\Omega)$ and choose $y_0 \in \Omega \cap \mathbb{R}^d_*$ such that $x_0 y_0 \notin \Omega$. For $|x| \leq \delta$ we set $y(x) := -xy_0/(x + x_0)$. For sufficiently small $\delta > 0$, $y(x)$ is defined (since $x_0 \in \mathbb{R}^d_*$) and $\{y_0 + y(x) \mid |x| \leq \delta\}$ is a compact subset of $\Omega \cap \mathbb{R}^d_*$. Obviously, $(x_0 + x)(y_0 + y(x)) = (x_0 + x)[y_0 - xy_0/(x_0 + x)] = x_0 y_0 \notin \Omega$.

ii) By assumption there is $y_0 \in \Omega$ such that $x_0 y_0 \notin \overline{\Omega}$. Then $x_0(y_0 + y_1) \notin \overline{\Omega}$ and $(y_0 + y_1) \in \Omega$ for small $|y_1|$ and we may choose $(y_0 + y_1) \in \mathbb{R}^d_*$. Hence $x_0 \notin \widetilde{V}(\Omega)$ and $(x_0 + x)(y_0 + y_1) \notin \overline{\Omega}$ for small $|x|$. The Lemma is proved. □

Proof of Theorem 8.7: (a) and (b) are equivalent by Theorem 7.14, and (b) implies (c) by Proposition 8.1.

(c) $\Rightarrow$ (b): If $V(\Omega)$ is closed and $\partial V(\Omega)$ is compact then for every open neighborhood U of $V(\Omega)$ there is a smaller open neighborhood V of $V(\Omega)$ such that $0 \notin \partial V$ and ∂V is compact. By Remark 2.5 (b), $\widetilde{V}(\Omega) = V(\Omega)$ so by Lemma 8.8 we can cover ∂V by a finite family of open sets $(U_i)_{i=1}^m$ such that there are $(K_i)_{i=1}^m$, $K_i \subset \Omega \cap \mathbb{R}^d_*$ with

$$\forall \ x \in U_i \ \exists \ y_x \in K_i \qquad xy_x \notin \Omega.$$

Then $V^0_{\bigcup_{i=1}^m K_i}(\Omega) \subset V$ and $\tau_b = \tau_{k_*}$ by Proposition 5.3 (b).

If $V(\Omega)$ is not closed so $0 \in \partial V(\Omega) \setminus V(\Omega)$ (see Proposition 4.4) but then $0 \notin \Omega$. Moreover, $\partial V(\Omega)$ splits into the two disjoint compact sets $\{0\}$ and $\partial V(\Omega) \cap V(\Omega)$. For any neighborhood U of $V(\Omega)$ there is a smaller open neighborhood V of $V(\Omega)$ such that ∂V is compact and splits into two disjoint compact sets $\{0\}$ and A. As in the case $V(\Omega)$ closed, by Lemma 8.8, there is a compact set $K \subset \Omega \cap \mathbb{R}^d_*$ such that $V_K(\Omega)$ is disjoint with A. Moreover, $0 \cdot K = \{0\}$ and $0 \notin \Omega$. Hence $0 \notin V_K(\Omega)$. This completes the proof (see Proposition 5.3).

(c)$\Leftrightarrow$(d): Proposition 4.12. □

Also in the other cases covered by Theorem 7.14, in particular for convex sets, the necessary condition in Proposition 8.1 turns out to be sufficient and this leads to an explicit description in terms of $V(\Omega)$.

Theorem 8.9 *Let $\Omega \subset \mathbb{R}^d$, $d > 1$, be an open nonempty set and either $\Omega \subset \mathbb{R}^d_*$ or Ω convex. Then the following are equivalent:*

(a) the map $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a topological isomorphism;

(b) $\tau_b = \tau_{k_*}$ (equivalently, $\tau_b = \tau_k$) on $\mathscr{A}(V(\Omega))'$;

(c) the set $\partial V(\Omega) \cap V(\Omega)$ is compact;

(d) the set $V(\Omega)$ is either open or compact;

Proof: (a) $\Leftrightarrow$ (b): By Theorem 7.14 in both cases $\tau_k = \tau_{k_*}$. So (a) holds if and only if (b) holds.

(b) $\Rightarrow$ (c) is Proposition 8.1.

(c) $\Rightarrow$ (d): Proposition 4.14.

(d) $\Rightarrow$ (b): If $V(\Omega)$ is open then, by Corollary 8.2, $\tau_b = \tau_k$. If $\Omega \subset \mathbb{R}^d_*$ and $V(\Omega)$ is compact then $V(\Omega) = \widetilde{V}(\Omega) \subset \mathbb{R}^d_*$. So for every neighborhood U of $V(\Omega)$ we find a neighborhood $V \subset U \cap \mathbb{R}^d_*$ of $V(\Omega)$. Using Lemma 8.8 we can find as in the proof of Theorem 8.7 (c)$\Rightarrow$(b) a compact set $K \subset \Omega \cap \mathbb{R}^d_* = \Omega$ such that $V^0_K(\Omega) \subset V$. The result follows by Proposition 5.3 (b).

So here it remains to show that for convex Ω with compact $V(\Omega)$ holds $\tau_b = \tau_k$.

Now, let $V(\Omega)$ be compact. By Proposition 4.1, $V(\Omega)$ is convex. Let U be an arbitrary open convex neighborhood of $V(\Omega)$ where ∂U is a compact surface in $\mathbb{R}^d$. Let $x \in \partial U$ and let $x_1 \notin V(\Omega)$ be an internal point of some interval connecting x and some point in $V(\Omega)$. By Proposition 4.3, there is $y \in \Omega \cap \mathbb{R}^d_*$ such that $x_1 \notin \frac{1}{y}\Omega$. It is easily seen that there is some open neighborhood of x on ∂U disjoint from $\frac{1}{y}\Omega$.

We have proved that for any point $x \in \partial U$ there is $y_x \in \Omega \cap \mathbb{R}^d_*$ such that for some neighborhood U_x of x in ∂U, $U_x \cap \frac{1}{y_x}\Omega = \emptyset$. Since (U_x) is a covering of ∂U, there are finitely many $x_1, \ldots, x_n$ so $U_{x_1} \cup \cdots \cup U_{x_n}$ covers ∂U and then $\bigcap_{j=1}^m \frac{1}{y_{x_j}}\Omega \subset U$. Hence $\tau_b = \tau_{k_*}$ by Proposition 5.3 (b) and Ω is even nice. $\square$

It was just shown that for convex Ω and compact $V(\Omega)$ we have $\tau_b = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$. This is not true for general Ω. The following example is a variant of [8, Example 3.2]. It shows that might be $\tau_k \neq \tau_{k_*}$ even for compact $V(\Omega)$.

Example 8.10 Swiss cross. *For $0 < a < b$ let $\Omega := \{x \in \mathbb{R}^2 \mid \|x\|_\infty < b\} \setminus \Big[([-a,a] \times \{0\}) \cup (\{0\} \times [-a,a])\Big]$. Then $\widetilde{V}(\Omega) = \{x \in \mathbb{R}^2 \mid 0 < |x_1|, |x_2| \le 1\}$ while $V(\Omega) = \{\pm 1\}^2$, hence $V(\Omega) \cap \mathbb{R}^d_* \neq \widetilde{V}(\Omega) \cap \mathbb{R}^d_*$ and $\tau_b = \tau_k \neq \tau_{k_*}$. We do not know where in between of τ_k and τ_{k_*} there is the topology induced by $\mathscr{B}$ from $M(\Omega)$.*

Proof: Indeed, if $K = \{(x,y) \in \Omega \mid d((x,y), \partial\Omega) > \varepsilon\}$ then for small $\varepsilon > 0$ the set $V_K(\Omega)$ is a small neighborhood of $V(\Omega)$. On the other hand, $\partial B_\varepsilon(\mathbb{1}) \cap \widetilde{V}(\Omega) \neq \emptyset$ for any $\varepsilon > 0$. This completes the proof, comp. Proposition 5.3. $\square$

Substituting the interval $[-a,a]$ by $[0,a]$ in the above example we obtain $V(\Omega) = \{\mathbb{1}\}$.

In case $V(\Omega) = \Omega$ we get even more than topological isomorphism $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$.

Proposition 8.11 *Let $\Omega \subset \mathbb{R}^d$ be an open set. If $\Omega = V(\Omega)$ then $M(\Omega)$ is a complemented subalgebra in $L_b(\mathscr{A}(\Omega))$ with the following continuous projection:*

$$\mathscr{P} : L_b(\mathscr{A}(\Omega)) \to M(\Omega); \quad \mathscr{P}(L) := M_T \text{ where } T = \delta_{\mathbb{1}} \circ L.$$

In particular, $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a topological homomorphism.

Corollary 8.12 *A nonempty open set $\Omega \subset \mathbb{R}^d_+$, $d > 1$, satisfies $\tau_b = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ if and only if either $\Omega = \mathbb{R}^d_+$ or $V(\Omega) = \{\mathbb{1}\}$.*

Proof: By Theorem 8.9, sufficiency follows. For necessity, observe, again by Theorem 8.9, that if $V(\Omega)$ is not open and for $\Omega \subset \mathbb{R}^d_+$ we have $\tau_b = \tau_{k_*}$ then $V(\Omega)$ must be compact. If $\eta \neq \mathbb{1}$ belongs to $V(\Omega)$ then $\eta^n \in V(\Omega)$ for all $n \in \mathbb{N}$, and this sequence is either unbounded or converges to some point outside $\mathbb{R}^d_+$, a contradiction. $\square$

In case Ω contains $\mathbb{1}$ a description of sets with $\tau_b = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ is even more straightforward. From Propositions 4.5, 4.9 and Theorem 8.9 it follows immediately:

Corollary 8.13 *A non-empty open convex set $\Omega \subset \mathbb{R}^d$, $\mathbb{1} \in \Omega$, $d > 1$, satisfies $\tau_b = \tau_{k_*}$ on $\mathscr{A}(V(\Omega))'$ if and only if it contains no axis (then $V(\Omega)$ is compact) or it is equal to the whole space $\mathbb{R}^d$ (then $V(\Omega) = \mathbb{R}^d$).*

The criterion of Lemma 8.8 can also be applied for many open sets with C^1-boundary with $V(\Omega)$ not necessarily contained in $\mathbb{R}^d_*$.

9 Special Classes of Multipliers

In this section we present four important classes of multipliers, the *Euler operators*, the *integral operators*, the *dilation operators* and the *superposition multipliers*. We define η_α for $\alpha \in \mathbb{N}^d$ by

$$\eta_\alpha(x) = x^\alpha.$$

Euler operators. First, we present the so-called Euler partial differential operators (of finite or infinite order). The one variable theory is classical (see [22], [23], [25], [26], [17], for a survey see Section 4 in [8]) but the authors could not find its several variables analogue in the literature. We present the theory in details in the forthcoming paper [11].

Let $\theta_j(f)(x) := x_j \frac{\partial f}{\partial x_j}(x)$ for $j = 1, \dots, d$ denote the *Euler differentials* and set $\theta^\alpha := \theta_1^{\alpha_1}\theta_2^{\alpha_2}\dots\theta_d^{\alpha_d}$. Then Euler differential operators are defined by

$$E(\theta)(g)(x) := \sum_{\beta\in\mathbb{N}^d} a_\beta \theta^\beta(g)(x) \text{ for } g \in \mathscr{A}(\Omega).$$

It is proved in [11] that $E(\theta) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is a multiplier for every open $\emptyset \neq \Omega \subset \mathbb{R}^d$ if the sequence $(a_\beta)_{\beta\in\mathbb{N}^d}$ satisfies

$$\forall \varepsilon > 0 : \quad \sup_\beta |a_\beta| \frac{\beta!}{\varepsilon^{|\beta|}} < \infty. \tag{27}$$

The multiplier sequence $(m_\alpha)_\alpha$ of $E(\theta)$ is given by $m_\alpha = E(\alpha) := \sum_{\beta\in\mathbb{N}^d} a_\beta \alpha^\beta$ and the analytic functional T defining $E(\theta)$ via the Representation Theorem 2.2 is given by

$$\langle g, T\rangle = \sum_{\alpha\in\mathbb{N}^d} a_\alpha \frac{\partial^\alpha g}{\partial x^\alpha}(\mathbb{1}),$$

hence $T \in \mathscr{A}(\{\mathbb{1}\})'_b$. On the other hand, every multiplier whose corresponding analytic functional has support concentrated at $\mathbb{1}$ (or, equivalently, a multiplier which acts on $\mathscr{A}(\Omega)$ for every open set $\Omega \subset \mathbb{R}^d$) is equal to some $E(\theta)$ with coefficients satisfying (27).

If $V(\Omega) = \{\mathbb{1}\}$ then the Euler differential operators are the only multipliers on $\mathscr{A}(\Omega)$. In many cases they form a big subset of all multipliers:

Proposition 9.1 *The Euler differential operators of finite order are dense in the space $M(\Omega)$ of all multipliers on $\mathscr{A}(\Omega)$ if $V(\Omega)$ is connected.*

Proof: By Theorem 2.2 and Corollary 7.9, the map $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a continuous bijective map which maps $\mathrm{lin}\{\delta_{\mathbb{1}}^{(\alpha)} : |\alpha| \leq m\}$ onto $\mathrm{lin}\{\theta^\alpha : |\alpha| \leq m\}$. So it suffices to show that the linear span of $\delta_{\mathbb{1}}^{(\alpha)}$ is dense in $\mathscr{A}(V(\Omega))'_b$. Since $\mathscr{A}(V(\Omega))$ is reflexive, this will follow from the weak-star density. The latter is seen as follows: Let $f \in \mathscr{A}(V(\Omega))$ satisfy $f^{(\alpha)}(\mathbb{1}) = 0$ for every $\alpha \in \mathbb{N}_0^d$. Then $f \equiv 0$ since $V(\Omega)$ is connected. □

Euler differentials θ_j determine multipliers also in another way (see [8, Th.2.13] for $d = 1$).

Proposition 9.2 *Let $\Omega \subset \mathbb{R}^d$ be an open connected non-empty set. A continuous linear operator $T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is a multiplier if and only if it commutes with all operators θ_j, $j = 1, \dots, d$.*

Proof: Since θ_j is a multiplier, it commutes with every multiplier.

Let us assume that T commutes with all θ_j. Thus

$$\theta_j T(\eta_\alpha)(x) = T(\theta_j(\eta_\alpha))(x) = T(\alpha_j \eta_\alpha)(x) = \alpha_j T(\eta_\alpha)(x).$$

We fix $x^0 \in \Omega$. In a neighborhood of x^0 the solution $T(\eta_\alpha)$ of this system of differential equations has the form

$$T(\eta_\alpha)(x) = C\eta_\alpha(x)$$

which then, due to connectedness of Ω and unique analytic continuation holds in all of Ω. We have proved that T is a multiplier. □

Integral operators. These are the typical examples of multipliers using measures T in the Representation Theorem 2.2. We mention only two significant examples:

$$M^{\{1\}}(g)(y) = \int_0^1 g(ty)dt, \quad m_\alpha = \frac{1}{|\alpha| + 1};$$

$$M^{\{2\}}(g)(y) = \int_{[0,1]^d} g(x \cdot y)dx, \quad m_\alpha = \prod_{j=1}^d \frac{1}{\alpha_j + 1}.$$

This type of multipliers already appear in [19] for $d = 1$.

Dilation operators. We define dilation operators M_a with dilation factor $a \in \mathbb{R}^d$ as follows:

$$M_a(g)(y) := g(a \cdot y), \quad m_\alpha = a^\alpha := a_1^{\alpha_1} \cdot \dots \cdot a_d^{\alpha_d}$$

These operators acts on $\mathscr{A}(\Omega)$ if and only if $a \in V(\Omega)$. Dilation operators play an important role in the Representation Theorem 2.2, since this theorem somehow shows that every multiplier is a "combination" of dilation operators with factors belonging to $V(\Omega)$. In that sense dilation operators determine multipliers but there is another reason why dilations determine multipliers:

Theorem 9.3 *Let Ω be an open convex non-empty set. The following assertions are equivalent:*

(a) $V(\Omega)$ has non-empty interior;

(b) A continuous linear operator $T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is a multiplier if and only if it commutes with all dilations M_a for every $a \in V(\Omega)$.

Remark 9.4 Let Ω be convex. Then the dilation set $V(\Omega)$ has a non-empty interior if Ω is bounded and $0 \in \Omega$. A description when $V(\Omega)$ is open was given in Proposition 4.6.

The proof of Theorem 9.3 is more complicated than in the one dimensional case [8, Th. 2.15], we will use Proposition 4.8 which has no non-trivial one-dimensional analogue.

Proof: (a)⇒(b): We may assume that $\mathbb{1} \in \Omega$. Clearly, M_a are multipliers so every multiplier commutes with every M_a. Assume now that T commutes with all dilations M_a for $a \in V(\Omega)$. Then

$$T(\eta_\alpha)[a] = (M_a \circ T)(\eta_\alpha)[\mathbb{1}] = T((a\eta)^\alpha)[\mathbb{1}] = m_\alpha a^\alpha,$$

with $m_\alpha = (T\eta_\alpha)[\mathbb{1}]$. This holds for all $a \in V(\Omega) \subset \Omega$. Since Ω is connected and $V(\Omega)$ has non-void interior, it holds for all $a \in \Omega$.

(b)⇒(a): Assume that $V(\Omega)$ has empty interior. By Proposition 4.8, one of the following two cases holds:

1. there exists $j \in \{1, \dots, d\}$ such that $V(\Omega) \subset \{a \mid a_j = 1\}$;
2. there exist $j, k \in \{1, \dots, d\}$, $j \neq k$, such that $V(\Omega) \subset \{a \mid a_j = a_k\}$.

Case 1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an analytic function. Then

$$T_\varphi(f)(x) := \varphi(x_j) f(x)$$

is a linear continuous map $T_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ which is not necessarily a multiplier. On the other hand,

$$M_a T_\varphi(f)(x) = \varphi(x_j) f(ax) = T_\varphi(M_a(f))(x).$$

Case 2. The map

$$T_{j,k}(f)(x) = x_j \frac{\partial}{\partial x_k} f(x)$$

is a linear continuous map $T_{j,k} : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ which is not a multiplier. On the other hand,

$$M_a T_{j,k}(f)(x) = a_j x_j \frac{\partial}{\partial x_k} f(ax) \quad \text{and} \quad T_{j,k}(M_a f)(x) = x_j a_k \frac{\partial}{\partial x_k} f(ax).$$

Since $a_j = a_k$ we have

$$T_{j,k} M_a = M_a T_{j,k}$$

for any $a \in V(\Omega)$. □

As in Proposition 9.1 we can prove that if $V(\Omega)$ is connected with non-empty interior then the multipliers $M(\Omega)$ are the closure of the linear span of the dilation operators on $\mathscr{A}(\Omega)$.

Superposition multipliers. Take any distribution $T \in C^\infty(]-1,1[^d)'_b$ and a smooth function $f \in C^\infty\left([-1,1]^d\right)$. Then we define a multiplier by

$$S_{f,T}(g)(y) := \sum_{\alpha \in \mathbb{N}^d} \frac{\partial^\alpha g(0)}{\alpha!} y^\alpha \langle R_p^0(f) \circ \widetilde{\eta}_\alpha, T\rangle + \sum_{|\beta| \le p} \frac{\partial^\beta f(0)}{\beta!} \langle g \circ (y\widetilde{\eta}_\beta), T\rangle$$

with the multiplier sequence $(m_\alpha)_{\alpha \in \mathbb{N}^d}$ given by

$$m_\alpha = \langle f \circ \widetilde{\eta}_\alpha, T\rangle \text{ for } \widetilde{\eta_\alpha}(x) := (x_1^{\alpha_1}, \dots, x_d^{\alpha_d}),$$

where $p \in \mathbb{N}$ is chosen so big that the series is absolutely convergent and where R_p^0 means the Taylor remainder of order p at zero. The proof that such a map is well defined (the required p exists and the formula does not depend on p) is similar as in the one variable case (see [10, Example 7.11]). The obtained multiplier sequence explains the name *superposition multiplier*.

Especially interesting is the case when T is a Dirac distribution δ_ε concentrated at $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \subset (-1,1)^d$. Then the multiplier sequence is of the form

$$m_\alpha = f(\varepsilon_1^{\alpha_1}, \ldots, \varepsilon_d^{\alpha_d}).$$

If f does not vanish at $(\varepsilon_1^{\alpha_1}, \ldots, \varepsilon_d^{\alpha_d})$ for any $\alpha \in \mathbb{N}^d$ and at zero this multiplier is invertible: its inverse is just $S_{1/\tilde{f},\delta_\varepsilon}$, where $\tilde{f}$ is a smooth function not vanishing on $[-1,1]^d$ with the same values as f at all points $(\varepsilon_1^{\alpha_1}, \ldots, \varepsilon_d^{\alpha_d})$, $\alpha \in \mathbb{N}^d$.

10 Multipliers on $\mathscr{A}(\mathbb{R}^d_+)$

Since the Euler differential θ_j is singular for $x_j = 0$ it is to be expected that the behavior of Euler differential operators is quite different depending whether $\Omega \subset \mathbb{R}^d_*$ or $\Omega \not\subset \mathbb{R}^d_*$. To see this we start with considering $\Omega \subset \mathbb{R}^d$ open and connected with $\mathbb{0} \in \Omega$.

Let $P(\theta)$ be an Euler differential operator, hence $m_\alpha = P(\alpha)$. Therefore, for $f \in \mathscr{A}(\Omega)$ with Taylor expansion $f(x) = \sum_\alpha c_\alpha x^\alpha$ we know that $P(\theta)f(x) = \sum_\alpha c_\alpha P(\alpha) x^\alpha$ around 0. This and the same argument applied to the dual map $(P(\theta))^*$, acting on $\mathscr{A}(\Omega)' \cong \mathscr{H}_C(\Omega)$ (see Definition 3.4) by Hadamard multiplication, yields for the kernel $\ker P(\theta)$ and the range $\operatorname{im} P(\theta)$ of $P(\theta)$ on $\mathscr{A}(\Omega)$:

Lemma 10.1 *Let $\mathbb{0} \in \Omega \subset \mathbb{R}^d$. Then*

$$\ker P(\theta) = \{f \in \mathscr{A}(\Omega) \ : \ f^{(\alpha)}(\mathbb{0}) = 0 \textit{ whenever } P(\alpha) \neq 0\},$$

$$\overline{\operatorname{im} P(\theta)} \subset \{f \in \mathscr{A}(\Omega) \ : \ f^{(\alpha)}(\mathbb{0}) = 0 \textit{ whenever } P(\alpha) = 0\}.$$

If $P(\alpha) = 0$ has only finitely many integer solutions then the converse of the latter holds as well.

If n is the (finite) number of integer solutions of the equation $P(\alpha) = 0$ then $n = \dim \ker P(\theta)$ and $\operatorname{codim} \overline{\operatorname{im} P(\theta)} = n$.

Example 10.2 *(a) If $P(x) = \sum_j x_j^m$ for $m \in \mathbb{N}$ then* $\ker P(\theta) = \mathbb{C}\{f \equiv 1\}$.

(b) If P is a hypoelliptic polynomial, then $\dim \ker P(\theta)$ *is finite dimensional.*

Specifically, for $P(x) = \sum_j x_j^2$ we have $\ker P(\theta) = \mathbb{C}\{f \equiv 1\}$ if $0 \in \Omega$ while for $\Omega \subset \mathbb{R}^d_+$ then $\ker P(\theta) = \{g(\log x_1, \ldots, \log x_d) \mid g \text{ is harmonic on } \log \Omega\}$ (see also below).

Building a solution theory for Euler differential operators even for $\Omega = \mathbb{R}^d$ may lead to deep problems as is seen by the following example:

Example 10.3 *Let $P(x) := (x_1+1)^m + (x_2+1)^m - (x_3+1)^m$ with $m \geq 3$. Then $P(\theta)$ is injective.*

Proof: By Lemma 10.1 this is equivalent to Fermat's Last Theorem. □

While surjectivity of Euler differential operators on arbitrary open subsets of $\mathbb{R}^d$ will be studied in the forthcoming paper [12], we will in this section treat the case of $\Omega \subset \mathbb{R}^d_*$ and it is easily seen that it suffices to concentrate on open sets $\Omega \subset \mathbb{R}^d_+$.

We define the analytic diffeomorphisms $\log : \Omega \to \log \Omega$ and $\exp : \log \Omega \to \Omega$ by

$$\log(x) = (\log x_1, \ldots, \log x_d), \qquad \exp(x) = (\exp x_1, \ldots, \exp x_d).$$

Then we have for $f \in \mathscr{A}(\Omega)$

$$(P(\theta)f) \circ \exp = P(\partial)(f \circ \exp).$$

Immediate consequences are

Lemma 10.4 *We have for any open $\Omega \subset \mathbb{R}^d_+$:*

$$\ker P(\theta) = \{g \circ \log \,:\, P(\partial)g = 0\}, \qquad \mathrm{im}\, P(\theta) = \{g \circ \log \,:\, g \in \mathrm{im}\, P(\partial)\}.$$

Corollary 10.5 *$P(\theta)$ is surjective on $\mathscr{A}(\Omega)$ if and only if $P(\partial)$ is surjective on $\mathscr{A}(\log \Omega)$.*

Surjectivity of partial differential operators $P(D)$ with constant coefficients on $\mathscr{A}(\omega)$ for convex open ω was characterized by Hörmander [20] by conditions of Phragmén-Lindelöf type valid for plurisubharmonic functions $PSH(Z)$ on the characteristic variety Z of the polynomial P (or its principal part P_m, respectively). These results can immediately be applied to Euler differential operator by setting $\omega = \log \Omega$. Notice that $\log \Omega$ is convex if and only if Ω is multiplicatively convex, that is, with $x, y \in \Omega$ and $0 < t < 1$ also $x^t y^{1-t} \in \Omega$. Using [20] we get

Theorem 10.6 (Euler operators of second order) *A second order pdo $P(\theta)$ is surjective on $\mathscr{A}(\mathbb{R}^d_+)$ iff the principal part P_m is either elliptic, or proportional to a real indefinite quadratic form or to the product of two real linear forms.*

Specifically, $\sum_{j=1}^d \theta_j^2$ is not surjective on $\mathscr{A}(\mathbb{R}^{d+1}_+)$ for $d \geq 2$ (see also Piccinini 72, [35], [36]). Also, the "heat-Euler" operator $\theta_1 - \sum_{j=2}^d \theta_j^2$ is not surjective on $\mathscr{A}(\mathbb{R}^d_+)$ for $d \geq 3$, while the "Laplace-Euler" operator $\sum_{j=1}^d \theta_j^2$ and the "wave-Euler" operator $\theta_1^2 - \sum_{j=2}^d \theta_j^2$ are surjective on $\mathscr{A}(\mathbb{R}^d_+)$ for $d \geq 2$.

For general open ω a characterization of surjective partial differential operators $P(D) : \mathscr{A}(\omega) \to \mathscr{A}(\omega)$ was obtained by Langenbruch [28] using shifted elementary solutions which are real analytic on relatively compact subsets of ω. Also this result can be directly applied to Euler differential operators by direct transfer. Using [20] and [29] we get:

Corollary 10.7 *$P(\theta)$ is surjective on $\mathscr{A}(\mathbb{R}^d_+)$ if it is surjective on $\mathscr{A}(\Omega)$ for some $\emptyset \neq \Omega \subseteq \mathbb{R}^d_+$.*

Problem 10.8 *Let $P(\theta) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ be surjective for $\Omega \subset \mathbb{R}^d_+$. Is $P_m(\theta)$ surjective as well?*

More results on the inheritance of surjectivity for arbitrary (not necessarily convex) open sets ω were proved by Langenbruch [28], [29] giving corresponding results for Euler operators.

Corollary 10.9 *(a) If $P(\theta)$ is surjective on $\mathscr{A}(\mathbb{R}^d_+)$ then for any non-empty $\emptyset \neq \Omega \subset \mathbb{R}^d_+$ there is a smallest $\widetilde{\Omega} \supset \Omega$ such that $P(\theta)$ is surjective on $\mathscr{A}(\widetilde{\Omega})$.*

(b) If $P(\theta)$ is surjective on every $\mathscr{A}(\Omega_j)$, $\Omega_j \subset \mathbb{R}^d_+$ then $P(\theta)$ is surjective on $\mathscr{A}\big((\bigcap \Omega_j)^\circ\big)$.

The same argument as for Lemma 10.4 can be used for general multipliers on $\mathscr{A}(\Omega)$ where $\Omega \subseteq \mathbb{R}^d_+$. By Theorem 7.14,

$$\mathscr{B} : \mathscr{A}(V(\Omega))'_k \to M(\Omega)$$

is a topological isomorphism. Moreover, as in the one-dimensional case (comp. [8, Th. 6.1, 5.3]), we can represent multipliers as convolution operators and, via Fourier-Laplace transform, as entire functions of restricted growth (in particular, of order one).

Define the composition operator $C_\varphi(f) = f \circ \varphi$ for a real analytic map φ. The map

$$\mathscr{E} : L_b(\mathscr{A}(\Omega)\ ) \to L_b(\mathscr{A}(\log\Omega)), \qquad \mathscr{E}(M) = C_{\exp} \circ M \circ C_{\log}$$

is a topological isomorphism onto. Clearly, $\mathscr{E}$ maps θ_j onto ∂_j. Therefore:

Theorem 10.10 *The following conditions are equivalent for connected open $\Omega \subset \mathbb{R}^d_+$:*

(a) M is a multiplier on $\mathscr{A}(\Omega)$.

(b) $\mathscr{E}(M)$ is a convolution operator on $\mathscr{A}(\log\Omega)$.

It is not so obvious to get a *topological* isomorphism of $\mathscr{A}(\log V(\Omega))'_b$ with the set of all convolution operators on $\mathscr{A}(\log\Omega)$ (i.e., operators commuting with all partial derivatives). If $\mathscr{B} : \mathscr{A}(V(\Omega))'_b \to M(\Omega)$ is a topological isomorphism this is so. By Corollary 8.12, this holds for $d > 1$ if and only if either $\Omega = \mathbb{R}^d_+$ (i.e., $\log\Omega = \mathbb{R}^d$) or $V(\Omega) = \{\mathbb{1}\}$ (i.e., $\log V(\Omega) = \{\mathbb{0}\}$).

It is of great interest to describe the transfer via $\mathscr{E}$ of the description of analytic functionals by means of the Laplace transform (see [21, Section 4.5]). For that we need convexity of $\log\Omega$. If $\log\Omega$ is convex then $\log V(\Omega)$ is convex as well. Let us denote the *support function* of a convex compact set K by

$$H_K(y) := \sup_{z\in K}(\operatorname{Re}(z_1y_1 + \cdots + z_dy_d)).$$

Then for any convex compact set K and any convex set Ω we define

$$\operatorname{Exp}(K) := \{f \in H(\mathbb{C}^d) : \ \forall\, \varepsilon > 0 : \ \|f\|_{K,\varepsilon} < \infty\}, \quad \operatorname{Exp}(\Omega) := \bigcup_{K\Subset\Omega} \operatorname{Exp}(K),$$

where $\|f\|_{K,\varepsilon} := \sup_{z\in\mathbb{C}} |f(z)| \exp(-H_K(z) - \varepsilon|z|)$. Finally, we recall the definition of the Laplace transform of an analytic functional μ:

$$\mathscr{L}(\mu)(z) = \langle \exp(z\,\cdot), \mu\rangle, z \in \mathbb{C}^d.$$

Assume that $\log V(\Omega)$ is convex, then $\mathscr{L} : \mathscr{A}(\log V(\Omega))'_b \to \operatorname{Exp}(\log V(\Omega))$ is an algebra isomorphism (see [21, Th. 4.5.3]), a topological algebra isomorphism if $V(\Omega)$ is open or closed. Here $\operatorname{Exp}(\log V(\Omega))$ is an algebra with respect to pointwise multiplication. Define

$$\eta_z(x) := \exp(z_1 \log x_1 + \cdots + z_d \log x_d) = x_1^{z_1} \cdot \cdots \cdot x_d^{z_d}.$$

Summarizing we have (using Corollary 8.12):

Theorem 10.11 *Let $\Omega \subset \mathbb{R}^d_+$, $d > 1$, be an open set and $\log V(\Omega)$ be convex. Then the map $\mathscr{M} : M(\Omega) \to \operatorname{Exp}(\log V(\Omega))$,*

$$\mathscr{M}(M)(z) := \mathscr{L}(\mathscr{E}(M))(z) = \langle \eta_z, \mathscr{B}^{-1}(M)\rangle = \textit{eigenvalue of } M \textit{ for the eigenvector } \eta_z$$

is an algebra homomorphism onto such that

$$\mathscr{M}(M)(\alpha) = m_\alpha \qquad \textit{for every } \alpha \in \mathbb{N}^d$$

and $\mathscr{M}$ is a topological isomorphism if and only if Ω is either $\Omega = \mathbb{R}^d_+$ or $V(\Omega) = \{\mathbb{1}\}$.

Problem 10.12 *Characterize surjective multipliers on $\mathscr{A}(\Omega)$ for open $\Omega \subset \mathbb{R}^d_+$.*

Of course, this problem is equivalent to the surjectivity problem for convolution operators on the sets $\log\Omega \subset \mathbb{R}^d$.

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