A characteristic property of the space s

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Abstract

It is shown that under certain stability conditions a complemented subspace of the space s of rapidly decreasing sequences is isomorphic to s and this condition characterizes s. This result is used to show that for the classical Cantor set X the space $C_\infty(X)$ of restrictions to X of C^∞-functions on $\mathbb{R}$ is isomorphic to s, so completing the theory developed in [7].

1 Introduction

In the present note we study the space s of rapidly decreasing sequences, that is, the space

$$s = \{x = (x_0, x_1, \dots) \,:\, \lim_n x_n n^k = 0 \text{ for all } k \in \mathbb{N}\}.$$

Equipped with the norms $\|x\|_k = \sup_n |x_n|(n+1)^k$ it is a nuclear Fréchet space. It is isomorphic to many of the Fréchet spaces which occur in analysis, in particular, spaces of C^∞-functions.

It is easily seen that instead of the sup-norms we might use the norms

$$|x|_k = \big(\sum_n |x_n|^2 (n+1)^{2k}\big)^{1/2}$$

which makes s a Fréchet-Hilbert space.

More generally, we define for any sequence $\alpha : 0 \le \alpha_0 \le \alpha_1 \le\nearrow +\infty$ the power series space of infinite type

$$\Lambda_\infty(\alpha) := \{x = (x_0, x_1, \dots) \,:\, |x|_t^2 = \sum_{n=0}^{\infty} |x_n|^2 e^{2t\alpha_n} < \infty \text{ for all } t > 0\}.$$

Equipped with the hilbertian norms $|\cdot|_k$, $k \in \mathbb{N}_0$, it is a Fréchet-Hilbert space. It is nuclear if, and only if, $\limsup_n \log n/\alpha_n < \infty$. With this definition $s = \Lambda_\infty(\alpha)$ with $\alpha_n = \log(n+1)$.

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A Fréchet space with the fundamental system of seminorms $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \ldots$ has property (DN) if

$$\exists p\,\forall k\,\exists K,\, C>0 : \|\cdot\|_k^2 \leq C\|\cdot\|_p\,\|\cdot\|_K.$$

In this case $\|\cdot\|_p$ is called a dominating norm.

E has property (Ω) if

$$\forall p\,\exists q\,\forall m\,\exists 0<\theta<1, C>0 : \|\cdot\|_q^* \leq C\|\cdot\|_p^{*\theta}\,\|\cdot\|_m^{*\,1-\theta}.$$

Here we set for any continuous seminorm $\|\cdot\|$ and $y \in E'$ the dual, extended real valued, norm $\|y\|^* = \sup\{|y(x)| : x \in E,\ \|x\| \leq 1\}$.

By Vogt-Wagner [8] a Fréchet space E is isomorphic to a complemented subspace of s if, and only if, it is nuclear and had properties (DN) and (Ω).

It is a long standing unsolved problem of the structure theory of nuclear Fréchet spaces, going back to Mityagin, whether every complemented subspace of s has a basis. If it has a basis then it is isomorphic to some power series space $\Lambda_\infty(\alpha)$. The space $\Lambda_\infty(\alpha)$ to which it is isomorphic, if it has a basis, can be calculated in advance by a method going back to Terzioğlu [4] which we describe now.

Let X be a vector space and $A \subset B$ absolutely convex subsets of X. We set

$$\delta_n(A,B) := \inf\{\delta>0 : \text{exists linear subspace } F \subset X, \dim F \leq n \text{ with } A \subset \delta B + F\}.$$

It is called the n-th Kolmogoroff diameter of A with respect to B.

If now E is a complemented subspace of s, that is, E is nuclear and has properties (DN) and (Ω), then we choose p such that $\|\cdot\|_p$ is a dominating norm and for p we choose $q>p$ according to property (Ω). We set

$$\alpha_n = -\log \delta_n(U_q, U_p)$$

where $U_k = \{x \in E : \|x\|_k \leq 1\}$. The space $\Lambda_\infty(\alpha)$ is called the associated power series space and $E \cong \Lambda_\infty(\alpha)$ if it has a basis.

If $\limsup_n \alpha_{2n}/\alpha_n < \infty$ then, by Aytuna-Krone-Terzioğlu [2, Theorem 2.2], $E \cong \Lambda_\infty(\alpha)$. This is, in particular, the case if E is stable, that is, if $E \oplus E \cong E$.

For all that and further results of structure theory of infinite type power series spaces see [6], for results and unexplained notation of general functional analysis see [3].

2 Main result

Lemma 2.1 *Let E be a complemented subspace of s, $\|\cdot\|_0$ a dominating hilbertian norm and $\|\cdot\|_1$ a hilbertian norm chosen for $\|\cdot\|_0$ according to (Ω). If there is a linear isomorphism $\psi : E \oplus E \to E$ such that*

$$\begin{aligned} \|x\|_0 + \|y\|_0 &\leq C_0\,\|\psi(x\oplus y)\|_0 \\ \|\psi(x\oplus y)\|_1 &\leq C_1(\|x\|_1 + \|y\|_1) \end{aligned}$$

then $E \cong s$.

Proof. For $x \oplus y \in E \oplus E$ we set $|||(x,y)|||_0 := (\|x\|_0^2 + \|y\|_0^2)^{1/2}$ and $|||(x,y)|||_1 := (\|x\|_1^2 + \|y\|_1^2)^{1/2}$. With new constants C_k we have

$$|||x \oplus y|||_0 \leq C_0 \|\psi(x \oplus y)\|_0 \text{ and } \|\psi(x \oplus y)\|_1 \leq C_1 |||x \oplus y|||_1. \tag{1}$$

To calculate the associated power series space for E we set:

$$\begin{aligned} \alpha_n &= -\log \delta_n(U_1, U_0) \text{ where } U_k = \{x \in E \,:\, \|x\|_k \leq 1\}, \\ \beta_n &= -\log \delta_n(V_1, V_0) \text{ where } V_k = \{x \oplus y \in E \oplus E \,:\, |||x \oplus y|||_k \leq 1\}. \end{aligned}$$

Due to the estimates (1) we have

$$\frac{1}{C_1}\psi(V_1) \subset U_1 \subset U_0 \subset C_0 \psi(V_0)$$

and therefore

$$\delta_n(V_1, V_0) = \delta_n(\psi V_1, \psi V_0) \leq C_0 C_1 \delta_n(U_1, U_0)$$

which implies

$$\alpha_n \leq \beta_n + d$$

with $d = \log C_0 C_1$.

By explicit calculation of the Schmidt expansion of the canonical map j_1^0 between the local Hilbert spaces of $|||\cdot|||_1$ and $|||\cdot|||_0$ and by use of the fact that singular numbers and Kolmogoroff diameters coincide, we obtain that $\beta_{2n} = \beta_{2n+1} = \alpha_n$ for all $n \in \mathbb{N}_0$.

Therefore we have $\alpha_{2n} \leq \beta_{2n} + d = \alpha_n + d$ for all $n \in \mathbb{N}_0$ and this implies $\alpha_{2^k} \leq \alpha_1 + k\,d$ for all $k \in \mathbb{N}_0$. For $n \in \mathbb{N}$ we find $k \in \mathbb{N}$ such that $2^{k-1} \leq n \leq 2^k$ and we obtain $\alpha_n \leq \alpha_{2^k} \leq \alpha_1 + k\,d \leq (\alpha_1 + d) + d \log n$.

Since $E \subset s$, which implies the left inequality below, we have shown that there is a constant $D > 0$ such that

$$\frac{1}{D} \log n \leq \alpha_n \leq D \log n$$

for large $n \in \mathbb{N}$. This implies that $\Lambda_\infty(\alpha) = s$. □

A Fréchet-Hilbert space E is called *normwise stable* if it admits a fundamental system of hilbertian seminorms for which there is an isomorphism $\psi : E \oplus E \to E$ such that

$$\frac{1}{C_k}(\|x\|_k + \|y\|_k) \leq \|\psi(x \oplus y)\|_k \leq C_k(\|x\|_k + \|y\|_k)$$

for all k. Since, clearly, s is normwise stable we have shown.

Theorem 2.2 *$E \cong s$ if, and only if, E is isomorphic to a complemented subspace of s and normwise stable.*

We may express Lemma 2.1 also in the following way:

Theorem 2.3 *Let the Fréchet-Hilbert space E be a complemented subspace of s, $\|\cdot\|_0$ a dominating norm and $\|\cdot\|_1$ be a norm chosen according to (Ω). Let P be a linear projection in E, continuous with respect to $\|\cdot\|_0$. We set $E_1 = R(P)$, $E_2 = N(P)$ and assume that there are linear isomorphisms $\psi_j : E \to E_j$, $j = 1, 2$, continuous with respect to $\|\cdot\|_1$ such that ψ^{-1} is continuous with respect to $\|\cdot\|_0$. Then $E \cong s$.*

Proof. We set $\psi(x \oplus y) := \psi_1(x) + \psi_2(y)$ and obtain with suitable constants:

$$\begin{aligned}\|x\|_0 + \|y\|_0 &\leq C'(\|\psi_1(x)\|_0 + \|\psi_2(y)\|_0) \leq C_0\|\psi_1(x) + \psi_2(y)\|_0 = C_0\|\psi(x \oplus y)\|_0 \\ \|\psi(x \oplus y)\|_1 &= \|\psi_1(x) + \psi_2(y)\|_1 \leq \|\psi_1(x)\|_1 + \|\psi_2(y)\|_1 \leq C_0(\|x\|_1 + \|y\|_1).\end{aligned}$$

Lemma 2.1 yields the result. □

3 Application

An interesting application of this result is the following. Let $X \subset [0, 1]$ be the classical Cantor set and $C_\infty(X) := \{f|_X : f \in C^\infty[0, 1]\} = \{f|_E : f \in C^\infty(\mathbb{R})\}$. The space $C_\infty(X)$ equipped with the quotient topology is a nuclear Fréchet space and, since $C^\infty[0, 1] \cong s$ isomorphic to a quotient of s, hence has property (Ω). By a theorem of Tidten [5] it has also property (DN). Therefore it is isomorphic to a complemented subspace of s (see [8]).

We should remark that, due to the fact that X is perfect, we have $C_\infty(X) = \mathcal{E}(X)$ where $\mathcal{E}(X)$ denotes the space of Whitney jets on X, for which Tidten's result is formulated.

By obvious identifications we have

$$C_\infty(X) \cong C_\infty(X \cap [0, 1/3]) \oplus C_\infty(X \cap [2/3, 1]) \cong C_\infty(X) \oplus C_\infty(X)$$

and it is easily seen that this establishes normwise stability. Therefore we have shown

Theorem 3.1 *If X is the classical Cantor set, then $C_\infty(X) \cong s$.*

It should be remarked that in [1] it has been shown that for the Cantor set X the diametral dimensions of $\mathcal{E}(X)$ and s coincide, from where, by means of the Aytuna-Krone-Terzioğlu Theorem, on can derive the same result.

Referring to the terminology of [7] we have also shown that $A_\infty(X) \cong s$ which completes the theory developed in [7].

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