Interpolation of nuclear operators and a splitting theorem for exact sequences of Fréchet spaces

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In the present paper the following theorem is proved:

Theorem: Let $0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0$ be an exact sequence of Fréchet– Hilbert spaces, $E \in (DN)$, $F \in (\Omega)$ then the sequence splits.

Here (DN) denotes the class of all Fréchet spaces admitting a logarithmic convex fundamental system of norms, whereas (Ω) denotes the class of all Fréchet spaces which have a fundamental system of seminorms such that the system of dual (extended real valued) norms is logarithmic convex. These are linear topological invariant classes. The formulation in terms of any given fundamental system of seminorms is contained in Sect. 3.

This theorem has been proved first by Vogt and Wagner in [9], [14](cf. [10], Thm. 7.1) under the assumption, that one of the spaces is nuclear. Other proofs in this or similar cases can be found in Petzsche [6] and Vogt [12]. While the formulation given in the present paper appears to be the most desirable one, it is much harder to prove and has been an open problem for some time. This is because in the nuclear case the crucial decomposition Lemmata 3.1., 3.2. either become easy by reduction to E or F being a power series space, or, as done by Petzsche [6], can be proved by using nuclear expansions of the maps. The present approach was suggested by the observation that the interpolation Lemma 2.1. (cf. [13], 1.4.) for general linear maps is dual to Petzsche's decomposition Lemma [6], 3.4. for nuclear maps. In fact we proceed just the other way in proving an interpolation result for nuclear norms to obtain by dualization the crucial decomposition result (Lemma 1.1) for general continuous linear maps.

As a by-result we obtain the interpolation Theorem 1.2 for nuclear operators. It should be mentioned that by examples of Pietsch [7] nuclear operators between interpolation triples do in general not interpolate as nuclear operators. Theorem 1.2 is also implicitely contained e.g. in the proof of [2], Theorem 13.1, as was indicated to the author by A. Pietsch.

Preliminaries : We use common notation for Fréchet spaces (see [3]). For tensor products and nuclear norms see [8], for scales [4] and for interpolation [1], [2].

A linear operator T from a Banach space G to a Banach space H is called nuclear if it has a representation (called nuclear representation)

$$Tx = \sum_n y_n(x)x_n$$

for all $x \in G$, where $y_n \in G'$ and $x_n \in H$ for all n and

$$\sum_n \|y_n\|^* \|x_n\| < +\infty ,$$

$\| \|^*$ denoting the dual norm. We put

$$\nu(T) = \inf \sum_n \|y_n\|^* \|x_n\|$$

the infimum running through all nuclear representations of T. For instance in case of Hilbert spaces the dual of the normed space (even Banach space) of all nuclear operators equipped with the nuclear norm, can be isometrically identified with the Banach space $L(H,G)$ of all continuous linear operators from H to G. This is done by

$$\langle T , S \rangle = \sum_n y_n(Sx_n) ,$$

where T is represented as above and $S \in L(H,G)$. Obviously the finite dimensional operators and, in case of scales as used in §1 of this paper, the operators with finite matrices are dense in the space of nuclear operators.

A locally convex space is called hilbertizable if it admits a fundamental system of seminorms given by semiscalar products. A Fréchet-Hilbert space is a hilbertizable Fréchet space.

1. For any k we denote by $\langle\ ,\ \rangle$ the usual scalar product on $\mathbb{C}^k$. $|\ |$ is the euclidian distance. For nonnegative $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_n$ and $t \in \mathbb{R}$ we put on $\mathbb{C}^m$ (resp. $\mathbb{C}^n$):

$$\begin{aligned} \langle x , y \rangle_{\alpha,t} &= \sum_{j=1}^m e^{2\alpha_j t} x_j \overline{y_j} , & |x|^2_{\alpha,t} &= \langle x , x \rangle_{\alpha,t} \\ \langle x , y \rangle_{\beta,t} &= \sum_{j=1}^n e^{2\beta_j t} x_j \overline{y_j} , & |x|^2_{\beta,t} &= \langle x , x \rangle_{\beta,t} \end{aligned}$$

For a linear map $T : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ we denote by $\nu(T)$ the nuclear norm with respect to $|\ |$, by $\nu_t(T)$ the nuclear norm of $T : (\mathbb{C}^m, |\ |_{\alpha,t}) \longrightarrow (\mathbb{C}^n, |\ |_{\beta,t})$.

Lemma 1.1 *$\nu_t(T) \leq \nu_o(T)^{1-t}\, \nu_1(T)^t$ for all $t \in \mathbb{R}$.*

Proof: For $x \in \mathbb{C}^m$ (resp. $x \in \mathbb{C}^n$) and $z \in \mathbb{C}$ we put

$$\begin{aligned} A(z)x &= \left(e^{\alpha_j z} x_j \right)_{j=1,\ldots,m} \\ B(z)x &= \left(e^{\beta_j z} x_j \right)_{j=1,\ldots,n} \end{aligned}$$

and $T_z = M_o^{z-1} M_1^{-z} B(z)\, T\, A(-z)$ for $M_t = \nu_t(T)$. We may assume $T \neq 0$, hence $M_t > 0$. Then we obtain for $z = t + i\eta$:

$$\begin{aligned} \nu(T_z) &= \inf \Big\{ \sum_j |\lambda_j| : T_z = \sum_j \lambda_j x_j \otimes y_j \ , \ |x_j| = |y_j|^* = 1 \Big\} \\ &= M_o^{t-1} M_1^{-t} \inf \Big\{ \sum_j |\lambda_j| : T = \sum_j \lambda_j (B(-z) x_j) \otimes (y_j \circ A(z)) \ , \ |x_j| = |y_j|^* = 1 \Big\} \\ &= M_o^{t-1} M_1^{-t} \inf \Big\{ \sum_j |\lambda_j| : T = \sum_j \lambda_j \xi_j \otimes \eta_j \ , \ |\xi_j|_{\beta,t} = |\eta|^*_{\alpha,t} = 1 \Big\} \\ &= M_o^{t-1} M_1^{-t} \nu_t(T) \ . \end{aligned}$$

$\nu(\cdot)$ is a continuous, convex function on the space $M(n,m)$ of all complex $n \times m$ – matrices, $z \longrightarrow T_z$ an entire function with values in $M(n,m)$. Hence $z \longrightarrow \nu(T_z)$ is a continuous subharmonic function on $\mathbb{C}$ which depends only on $\operatorname{Re} z$. Therefore $t \longrightarrow M_o^{t-1} M_1^{-t} \nu_t(T)$ is convex, which implies $M_o^{t-1} M_1^{-t} \nu_t(T) \leq 1$ on $[0,1]$.

For any two index sets I, J and families $(\alpha_i)_{i \in I}$, $(\beta_j)_{j \in J}$ of nonnegative real numbers we put

$$\begin{aligned} G_t &= \Big\{ x = (x_i)_{i \in I} : |x|_t^2 = \sum_{i \in I} e^{2\alpha_i t} |x_i|^2 < +\infty \Big\} \\ H_t &= \Big\{ x = (x_j)_{j \in J} : |x|_t^2 = \sum_{j \in J} e^{2\beta_j t} |x_j|^2 < +\infty \Big\} \end{aligned}$$

These are Hilbert spaces, equipped with their natural scalar products. $\nu_t(\cdot)$ denotes the nuclear norm of a nuclear operator $G_t \longrightarrow H_t$.

Theorem 1.2 *Let $T : G_o \longrightarrow H_o$ be nuclear, $T\, G_1 \subset H_1$ and $T : G_1 \longrightarrow H_1$ nuclear. Then $T\, G_t \subset H_t$ for all t, $T : G_t \longrightarrow H_t$ is nuclear and $\nu_t(T) \leq \nu_o(T)^{1-t} \nu_1(T)^t$ for all $t \in [0,1]$.*

Proof: First assume $T = T_n$, where

$$(T_n x)_j = \begin{cases} \sum\limits_{\nu=1}^{n} t_{\mu,\nu} x_{i_\nu} & \text{for } j = j_\mu,\ \mu = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Then the first and second assertion are clear, the third follows from Lemma 2.1.. For arbitrary T there are sequences j_μ, $\mu = 1, 2, \dots$, i_ν, $\nu = 1, 2, \dots$ and a matrix $(t_{\mu,\nu})$ such that for T_n defined as above for $n = 1, 2, \dots$ holds

$$\begin{aligned} \nu_o(T_n - T) &\longrightarrow 0 \ , \\ \nu_1(T_n - T) &\longrightarrow 0 \ . \end{aligned}$$

In particular $(T_n)_{n \in \mathbb{N}}$ is a $\nu_t(\cdot)$ – Cauchy sequence for $t = 0, 1, \dots$, hence for all $t \in [0,1]$. Therefore, for fixed $t \in [0,1]$, $(T_n)_n$ converges to a nuclear operator $G_t \longrightarrow H_t$, which is obviously the restriction of T to G_t. This shows the first and second assertion. The third follows from

$$\nu_t(T) = \lim_n \nu_t(T_n) \ .$$

Theorem 1.3 *For every continuous linear map $S : H_t \longrightarrow G_t$ and every $r > 0$ there exists $S_o \in L(H_o, G_o)$, $S_1 \in L(H_1, G_1)$ such that*

$$\begin{aligned} \|S_o\| &\leq \frac{1}{r^{\frac{1}{1-t}}}\|S\| \ , \\ \|S_1\| &\leq r^{\frac{1}{t}}\|S\| \end{aligned}$$

and $Sx = S_o x + S_1 x$ for all $x \in H_1$.

Proof: Let E be the space of all complex $(J\times I)$ – matrices with only finitely many nonzero entries, considered as a subspace of $H_t \otimes_\pi G_t'$ for all t. Put $U_t = \{u \in E : \nu_t(u) \leq 1\}$. Lemma 1.1. tells that

$$r^{\frac{1}{1-t}} U_o \ \cap \ \frac{1}{r^{\frac{1}{t}}} U_1 \ \subset \ U_t \tag{1}$$

for all $r > 0$ and $t \in [0,1]$. S defines a linear form σ on E with $|\sigma(u)| \leq \|S\|$ for all $u \in U_t$ hence of the left side of (1). Therefore there are linear forms σ_o, σ_1 on E such that $\sigma_o + \sigma_1 = \sigma$ and

$$\begin{aligned} \sup\left\{|\sigma_o(u)| : u \in r^{\frac{1}{1-t}}\ U_o\right\} &\leq \|S\| \\ \sup\left\{|\sigma_1(u)| : u \in \frac{1}{r^{\frac{1}{t}}}U_1\right\} &\leq \|S\| \ . \end{aligned}$$

σ_o, σ_1 define continuous linear maps $S_o : H_o \longrightarrow G_o$, $S_1 : H_1 \longrightarrow G_1$ with

$$\begin{aligned} \|S_o\| &\leq \frac{1}{r^{\frac{1}{1-t}}}\ \|S\| \ , \\ \|S_1\| &\leq r^{\frac{1}{t}}\ \|S\| \end{aligned}$$

and $S_o x + S_1 x = Sx$ for all x with only finitely many nonzero coordinates, hence for all $x \in H_1$.

2. We shall make use of the following result:

Lemma 2.1 *Let G, H be linear spaces and on each of them seminorms $\| \ \|_o \leq \| \ \|_1 \leq \| \ \|_2$ such that with suitable $0 \leq \tau < \vartheta \leq 1$ and $C \geq 0$*

$$\begin{aligned} \| \ \|_1^* &\leq C\| \ \|_o^{*1-\vartheta}\| \ \|_2^{*\vartheta} \ \text{on } (G, \| \ \|_o)' \\ \| \ \|_1 &\leq C\| \ \|_o^{1-\tau}\| \ \|_2^{\tau} \ \text{on } H \ . \end{aligned}$$

Then there exists D such that for every linear map $A : G \longrightarrow H$ with $\|Ax\|_j \leq C_j\|x\|_j$ for $j = 0, 2$ and $x \in G$, we have $\|Ax\|_1 \leq DC_o^{1-\tau}C_2^{\tau}\|x\|_1$ for all $x \in G$.

PROOF: Instead of the first inequality we may write (possibly increasing C):

$$\|y\|_1^* \le C\Big(r\|y\|_o^* + r^{1-\frac{1}{\vartheta}}\|y\|_2^*\Big)$$

for all $r > 0$ and $y \in (G, \| \|_o)^{'}$. The bipolar theorem yields

$$U_1 \subset C\Big(rU_o + r^{1-\frac{1}{\vartheta}}U_2\Big)$$

for all $r > 0$, where $U_j = \{x \in G : \|x\|_j \le 1\}$, $j = 0, 1, 2$.

Let $x \in U_1$. For every $n \in \mathbb{N}_o$ we have $x_n \in C2^{-n(1-\frac{1}{\vartheta})}U_2$, $y_n \in C2^{-n}U_o$ with $x = x_n - y_n$. Since $x_n - x_{n+1} = y_n - y_{n+1}$ we obtain

$$\begin{aligned}\|Ax_n - Ax_{n+1}\|_1 &\le C^2 C_o^{1-\tau} C_2^{\tau} 2 \cdot 2^{-n(1-\tau)-(n+1)(1-\frac{1}{\vartheta})\tau} \\ &\le D' C_o^{1-\tau} C_2^{\tau} 2^{-n(1-\frac{\tau}{\vartheta})} \ .\end{aligned}$$

Since $C \ge 1$ we may assume $x_o = 0$ and obtain

$$\sup_n \|Ax_n\|_1 \le DC_o^{1-\tau} C_2^{\tau} \ .$$

For n so large that $\|x\|_2 \le C2^{n(\frac{1}{\vartheta}-1)}$ we may assume $y_n = 0$, i.e. $x = x_n$. This proves the assertion.

We prove two consequences of this lemma.

Lemma 2.2 *Let* $F_2 \xrightarrow{\iota_2^1} F_1 \xrightarrow{\iota_1^o} F_o$, $\iota_2^o = \iota_1^o \circ \iota_2^1$ *be continuous linear maps with dense range between Hilbert spaces such that*

$$\|y \circ \iota_1^o\|_1^* \le C\|y\|_o^{*\,1-\vartheta}\|y \circ \iota_2^o\|_2^{*\,\vartheta}$$

for all $y \in F_o^{'}$. *Then there exists a scale* G_t, $t \in \mathbb{R}$, *as is §1 and for every* $0 \le \vartheta_o < \vartheta \le 1$ *continuous linear maps* A_o, A_1, A_2, *which make the following diagram commute*

$$\begin{array}{ccc} F_o & \xrightarrow{A_o} & G_o \\ \uparrow & & \uparrow \\ F_1 & \xrightarrow{A_1} & G_{\vartheta_o} \\ \uparrow & & \uparrow \\ F_2 & \xrightarrow{A_2} & G_1 \end{array}$$

and such that A_o *is invertible and* A_2 *is surjective (hence has a right inverse* R_2*).*

PROOF: After (if necessary) an equivalent change of the scalar product in F_2 we may assume that

$$\iota_2^o x = \sum_{i \in I} e^{-\alpha_i}\langle x\,, e_i\rangle_{i\in I} f_i$$

where $(f_i)_{i\in I}$ is a complete orthonormal system in F_o and $(e_i)_{i\in I}$ is an orthonormal system in F_2, complete in $(\ker \iota_2^o)^{\perp}$ (see [8], 8.3.1.). We set $A_o x = (\langle x\,, f_i\rangle_o)_{i\in I}$ and apply Lemma

2.1. to $G := \operatorname{im} \iota_2^o$ equipped with the norms $\| \|_o$, $\| \|_1$, $\| \|_2$ induced from F_o and, by ι_1^o (resp. ι_2^o), from F_1 (resp. F_2). $\| \|_2$ can be expressed as follows

$$\|x\|_2^2 = \sum_{i \in I} e^{2\alpha_i} |\langle x \,, f_i \rangle_o|^2$$

which means that $|A_o x|_1 = \|x\|_2$ for $x \in G$. We obtain, using $| \, |_{\vartheta_o} \leq | \, |_o^{1-\vartheta_o} | \, |_1^{\vartheta_o}$ that

$$|A_o x|_\vartheta \leq 2D \|x\|_1$$

for all $x \in G$. The maps $A_1 = A_o \circ \iota_1^o$, $A_2 = A_o \circ \iota_2^o$ satisfy the assertions of the Lemma.

Lemma 2.3 *Let* $E_2 \xrightarrow{\iota_2^1} E_1 \xrightarrow{\iota_1^o} E_o$, $\iota_2^o = \iota_1^o \circ \iota_2^1$ *be continuous linear injective maps with dense range between Hilbert spaces such that*

$$\|\iota_2^1 x\|_1 \leq C \|\iota_2^o x\|_o^{1-\tau} \|x\|_2^\tau$$

for all $x \in E_2$. *Then there exists a scale* H_t, $t \in \mathbb{R}$, *as in §1 and for every* $0 \leq \tau < \tau_o \leq 1$ *continuous linear maps* B_o, B_1, B_2 *which make the following diagram commute*

$$\begin{array}{ccc} H_o & \xrightarrow{B_o} & E_o \\ \uparrow & & \uparrow \\ H_{\tau_o} & \xrightarrow{B_1} & E_1 \\ \uparrow & & \uparrow \\ H_1 & \xrightarrow{B_2} & E_2 \end{array}$$

and such that B_o *and* B_2 *are invertible.*

PROOF: We proceed as in the previous proof and first may assume that

$$\iota_2^o x = \sum_{j \in J} e^{-\beta_j} \langle x \,, e_j \rangle_2 f_j$$

where $(e_j)_{j \in J}$ (resp. $(f_j)_{j \in J}$) are complete orthonormal systems in E_2 (resp. E_o). We set

$$B_o(\xi) = \sum_{j \in J} \xi_j f_j$$

for $\xi \in H_o$. Then for $B_2 := B_o|_{H_1}$ we have

$$B_2(\xi) = \sum_{j \in J} e^{\beta_j} \xi_j e_j \ .$$

Hence B_o and B_2 are isomorphisms. We apply Lemma 2.1. to H_1 with the norms $| \, |_o$, $| \, |_{\tau_o}$, $| \, |_1$ and to E_2 with the norms $\|\iota_2^o x\|_o$, $\|\iota_2^1 x\|_1$, $\|x\|_2$. We obtain using $| \, |_{\tau_o}^* \leq | \, |_o^{*1-\tau_o} | \, |^{*\tau_o}$, that

$$\|\iota_2^1 B_2 x\|_1 \leq D |x|_{\tau_o}$$

for all $x \in H_1$, hence for all $x \in H_{\tau_o}$. So $B_1 = \iota_2^1 B_2$ satisfies, together with B_o, B_2, the assertion of the Lemma.

Finally we obtain the following result, which is crucial for the splitting theorem of §3.

Proposition 2.4 *Let* $E_2 \stackrel{\iota_2^1}{\longrightarrow} E_1 \stackrel{\iota_1^o}{\longrightarrow} E_o$, $F_2 \stackrel{\iota_2^1}{\longrightarrow} F_1 \stackrel{\iota_1^o}{\longrightarrow} F_o$ *be as in Lemmata* 2.3., 2.2.. *Assume* $0 \leq \tau < \vartheta \leq 1$, $\varphi \in L(E_1, F_1)$, $\varepsilon > 0$. *Then there exist* $\varphi_o \in L(E_o, F_o)$, $\varphi_2 \in L(E_2, F_2)$ *such that* $\|\varphi_o\| \leq \varepsilon$ *and* $\iota_1^o \circ \varphi \circ \iota_2^1 = \varphi_o \circ \iota_2^o + \iota_2^o \circ \varphi_2$.

Proof: We choose $0 \leq \tau < t < \vartheta \leq 1$ and, in the notation of Lemmata 2.2., 2.3., apply Theorem 1.3. to $\psi = A_1 \circ \varphi \circ B_1 \in L(H_t, G_t)$ and suitable $r > 0$. We obtain $\psi_o \in L(H_o, G_o)$ and $\psi_1 \in L(H_1, G_1)$ such that $\|\psi_o\| \leq \frac{\varepsilon}{\|A_o^{-1}\| \, \|B_o^{-1}\|}$ and $\psi x = \psi_o x + \psi_1 x$ for all $x \in H_1$. We put $\varphi_o = A_o^{-1} \circ \psi_o \circ B_o^{-1}$, $\varphi_2 = R_2 \circ \psi_2 \circ B_o^{-1}$. Then $\varphi_o \in L(E_o, F_o)$, $\varphi_2 \in L(E_2, F_2)$, $\|\varphi_o\| \leq \varepsilon$, and on H_1 we have

$$\begin{aligned} A_o \circ \iota_1^o \circ \varphi \circ \iota_2^1 \circ B_2 &= A_1 \circ \varphi \circ B_1 \\ &= \psi_o + \psi_2 \\ &= A_o \circ (\varphi_o \circ \iota_2^o + \iota_2^o \circ \varphi_2) \circ B_2 \ . \end{aligned}$$

This yields the result.

3. We can now prove the main result of this paper, a splitting theorem for exact sequences of Fréchet–Hilbert spaces. We assume that F, G, E are Fréchet–Hilbert spaces, i.e. there exists a fundamental system $\| \;\|_o \leq \| \;\|_1 \leq \ldots$ of seminorms given as $\|x\|_k^2 = \langle x \,, x \rangle_k$ for some positive semidefinite scalar product $\langle \;,\; \rangle_k$.

Let $0 \longrightarrow F \stackrel{j}{\longrightarrow} G \stackrel{q}{\longrightarrow} E \longrightarrow 0$ be a (topologically) exact sequence. We shall identify $F = \ker j$.

We assume that E has property (DN)

$$\exists \, p \, \forall \, k \, \exists \, K, C \ : \ \| \;\|_k{}^2 \leq C \| \;\|_p \| \;\|_K$$

and F has property (Ω)

$$\forall \, p \, \exists \, q \, \forall \, Q \, \exists \, 0 < \vartheta < 1, \, C \ : \ \| \;\|_q^* \leq C \| \;\|_p^{*\,1-\vartheta} \| \;\|_Q^{*\;\vartheta} \ .$$

Property (DN) is equivalent to the existence of a logarithmically convex fundamental system of seminorms, (Ω) to the existence of a fundamental system of seminorms, such that its dual norms are a logarithmically convex system. Further descriptions can be found in the literature (see [9],[14]). For the above mentioned description see [11]. Obviously (DN) is equivalent to

$$\exists \, p \, \forall \, k \,, \, 0 < \tau < 1 \, \exists \, K, C \, : \, \| \;\|_k \leq C \| \;\|_p^{1-\tau} \| \;\|_K^{\tau} \ .$$

We may assume, as we will do from now on, that p in (DN) is 0.

Moreover we may choose a fundamental system of hilbertian seminorms in G, such that for the induced (hilbertian) seminorms on F we have with suitable $0 < \vartheta_k < 1$, C_k

$$\| \;\|_k^* \leq C_k \| \;\|_{k-1}^{*\;\;1-\vartheta_k} \| \;\|_{k+1}^{\vartheta_k} \ .$$

On E we assume as fundamental system of (hilbertian) seminorms the quotient norms of the $\| \|_k$. This can be done so as to make p in (DN) to be zero.

Let F_k, G_k, E_k be the Hilbert spaces associated to the $\| \|_k$, $\iota_k^l : E_k \longrightarrow E_l$, etc. the connecting map for $k \geq l$, and $\iota^k : E \longrightarrow E_k$, etc. the canonical map. For every k we have an exact sequence

$$0 \longrightarrow F_k \xrightarrow{j_k} G_k \xrightarrow{q_k} E_k \longrightarrow 0$$

of Hilbert spaces, which clearly splits.

On E we define $\|x\|_k^\sim = \inf\{\|\iota^k x + \xi\|_k \ : \ \xi \in \ker \iota_k^o\}$. This is an equivalent fundamental system of hilbertian seminorms on E because, choosing $K > k$ for given k according to (DN), we obtain $\ker\iota_K^o = \ker\iota_K^k$ and therefore for $x \in E$:

$$\|x\|_k^\sim \leq \|x\|_k \leq \inf\{\|\iota^K x + \xi\|_K \ : \ \xi \in \ker\iota_K^k\} = \|x\|_K^\sim \ .$$

Moreover p in (DN) may be chosen again as 0 and all ι_k^o (hence ι_{k+1}^k) are injective. This is easily seen from the fact that the local Banach spaces are $E_k^\sim = E_k/\ker\iota_k^o$ and $\|\cdot\|_0 = \|\cdot\|_0^\sim$.

Lemma 3.1 *For every $\varphi \in L(E, F_k)$ and $\varepsilon > 0$ there are $\psi \in L(E, F_{k-1})$, $\chi \in L(E, F_{k+1})$ such that $\sup_{\|x\|_o \leq 1} \|\psi x\|_{k-1} \leq \varepsilon$, $\iota_k^{k-1} \circ \varphi = \psi + \iota_{k+1}^{k-1} \circ \chi$.*

PROOF: We have $\|\varphi x\|_k \leq C\|x\|_{k'}^\sim$, for suitable C, k'. We choose τ with $0 < \tau < \vartheta_k < 1$ and for k', τ numbers K', C' according to the second form of (DN). Then the assertion is an immediate consequence of Proposition 2.4..

Lemma 3.2 *For every $\varphi \in L(E, F_k)$ and $\varepsilon > 0$ there are $\psi \in L(E, F_{k-1})$, $\chi \in L(E, F)$ such that $\sup_{\|x\|_o \leq 1} \|\psi x\|_{k-1} \leq \varepsilon$, $\iota_k^{k-1} \circ \varphi = \psi + \iota^{k-1} \circ \chi$.*

PROOF: We put $\varphi_o = \varphi$ and choose inductively maps $\varphi_p \in L(E, F_{k+p})$. If φ_p is chosen we find, according to Lemma 3.1., $\psi_p \in L(E, F_{k+p-1})$, $\varphi_{p+1} \in L(E, F_{k+p+1})$, such that $\sup_{\|x\|_o \leq 1} \|\psi_p x\|_{k+p-1} \leq \varepsilon 2^{-p-1}$, $\iota_{k+p}^{k+p-1} \circ \varphi_p = \psi_p + \iota_{k+p+1}^{k+p-1} \circ \varphi_{p+1}$.

We put

$$\chi_{k+p-1} \quad = \quad \iota_{k+p}^{k+p-1} \circ \varphi_p \ - \ \sum_{\nu=p}^{\infty} \iota_{k+\nu-1}^{k+p-1} \circ \psi_\nu \ . \tag{2}$$

Then $\chi_{k+p-1} \in L(E, F_{k+p-1})$, and $\chi_{k+p-1} = \iota_{k+p}^{k+p-1} \circ \chi_{k+p}$. Hence there is $\chi \in L(E, F)$ such that $\chi_{k+p-1} = \iota^{k+p-1} \circ \chi$ for all p.

We set

$$\psi = \sum_{\nu=0}^{\infty} \iota_{k+\nu-1}^{k-1} \circ \psi_\nu \ .$$

Then $\psi \in L(E, F_{k-1})$, $\sup_{\|x\|_o \leq 1} \|\psi x\|_{k-1} \leq \varepsilon$ and the required equality is (2) with $p = 0$.

Lemma 3.3 *For any sequence $\varphi_k \in L(E, F_k)$, $k = 0, 1, \ldots$, there is a sequence $S_k \in L(E, F_k)$, $k = 0, 1, \ldots$, such that $\varphi_k = S_k - \iota_{k+1}^k S_{k+1}$ for all k.*

Proof: For φ_k and $\varepsilon = 2^{-k}$ we choose ψ_k and χ_k according to Lemma 3.2. and put

$$S_k = \varphi_k - \sum_{\nu=0}^{k} \iota^k \circ \chi_\nu + \sum_{\nu=k+1}^{\infty} \iota_{\nu-1}^k \circ \psi_\nu \; .$$

A straightforward calculation shows the result.

Theorem 3.4 *If* $0 \longrightarrow F \xrightarrow{j} G \xrightarrow{q} E \longrightarrow 0$ *is an exact sequence of Fréchet–Hilbert spaces, if moreover* E *has property* (DN) *and* F *has property* (Ω)*, then the sequence splits.*

Proof: Assuming the arrangement described in the beginning of the section, let r_k be a right inverse for q_k and $R_k = r_k \circ \iota^k \in L(E, G_k)$. We put $\varphi_k = \iota_{k+1}^k \circ R_{k+1} - R_k$. Then $\varphi_k \circ q_k = 0$, hence $\varphi_k \in L(E, F_k)$. Notice, that we consider F_k as a subspace of G_k. According to the previous Lemma there are $S_k \in L(E, F_k)$ such that $\varphi_k = S_k - \iota_{k+1}^k S_{k+1}$ for all k. Therefore

$$\iota_{k+1}^k \circ (R_{k+1} + S_{k+1}) = R_k + S_k \; ,$$

which means that there is $R \in L(E, G)$ such that $R_k + S_k = \iota^k \circ R$. Hence R is a right inverse for q.

Remark: It should be noted, that Lemma 3.3. says slightly more, than used in Theorem 3.4.. It states that $\mathrm{Proj}^1(L(E, F_k))_k = 0$, (cf. [6], 3.3.; [5]). In consequence any exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0$ splits, also without the assumption, that G is hilbertizable, if we only know that the local sequences split. However, this clearly implies that G is hilbertizable. Theorem 3.4. can be expressed by saying, that $\mathrm{Ext}^1(E, F) = 0$ in the category of Fréchet–Hilbert spaces.

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