OPERATORS BETWEEN FRÉCHET SPACES

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The present paper, essentially based on the author's lectures during the Analysis Conference in Manila 1987, gives a survey on results about operators between Fréchet spaces which have been obtained in the last years. In particular results on the splitting of exact sequences of Fréchet spaces with and without tame bounds on the continuity estimates are reported. Here we present the general theorems in the (DN) and (Ω)–case, and explain the results in the special, but very important case of power series spaces. In this case we also give some results on the behaviour of the characteristics of continuity of operators in power series spaces.

Throughout this paper we use the common notation on Fréchet spaces and refer for that to [1], [6], [15].

1. Fréchet Spaces

A Fréchet space E is a complete topological vector space, whose topology can be given by a sequence of seminorms $\| \ \|_1 \leq \| \ \|_2 \leq \dots$ in the following way: a basis of neighborhoods of zero are the sets $U_{k,\varepsilon} = \{x \in E : \| x \|_k \leq \varepsilon\}$. Such a system is called a fundamental system of seminorms. It is by no means uniquely determined by the topology. In fact, two systems $\| \ \|_1 \leq \| \ \|_2 \leq \dots$ and $\| \ \|_1^\sim \leq \| \ \|_2^\sim \leq \dots$ give the same topology if and only if there exist constants C_k and $n(k) \in \mathbb{N}$ such that

$$\| \ \|_k \leq C_k \| \ \|_{n(k)}^\sim \quad \text{and} \quad \| \ \|_k^\sim \leq C_k \| \ \|_{n(k)}$$

for all k. In this case the systems of seminorms are called equivalent.

A Fréchet space equipped with a fixed fundamental system of seminorms is called a graded Fréchet space. This concept is important in connection with many problems in analysis, where the index of a norm indicates e.g. the order of derivatives involved.

As examples of Fréchet spaces we mention spaces $C^\infty(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open, $\mathcal{D}(K)$, $C^\infty(K)$ for $K \subset \mathbb{R}^n$ compact (think of $K = \bar{\Omega}$, Ω open with sufficiently regular boundary), $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{H}(\Omega)$ for $\Omega \subset \mathbb{C}^n$ open (holomorphic functions).

A special class are the Köthe sequence spaces, which we define now. Many of the spaces in analysis are by means of series expansions with respect to certain systems of special functions (e.g. Fourier expansion) isomorphic to sequence spaces.

Let $A = (a_{j,k})$ be an infinite matrix with $0 \leq a_{j,k} \leq a_{j,k+1}$, $\sup\limits_k a_{j,k} > 0$ for all j, k, then the Köthe sequence space $\lambda(A)$ is defined by

$$\lambda(A) = \left\{ x = (x_1, x_2, \ldots) : \| x \|_k = \sum_j |x_j| a_{j,k} < \infty \text{ for all } k \in \mathbb{N} \right\} .$$

Equipped with the seminorms $\| \|_k$ it is a graded Fréchet space.

If the matrix has the special form $a_{j,k} = e^{k\alpha_j}$ (resp. $a_{j,k} = e^{-\frac{1}{k}\alpha_j}$) for some sequence α: $\alpha_1 \leq \alpha_2 \leq \ldots$, $\lim\limits_n \alpha_n = +\infty$, then we put $\lambda(A) = \Lambda_\infty(\alpha)$ (resp. $\lambda(A) = \Lambda_o(\alpha)$) and call it a power series space of infinite (resp. of finite) type. Both are considered as graded Fréchet spaces. As linear spaces we have for $r \in \{0, \infty\}$

$$\Lambda_r(\alpha) = \left\{ x = (x_1, x_2, \ldots) : |x|_t = \sum_j |x_j| e^{t\alpha_j} < +\infty \text{ for all } t < r \right\} .$$

For any increasing sequence $t_k \to r$ the system of seminorms $(| \, |_{t_k})_k$ is easily seen to be equivalent to the norms $(\| \, \|_k)_k$. For many considerations it is more useful to use the one parameter family of norms $(| \, |_t)_{t>0}$.

If E is a graded Fréchet space we define the local Banach spaces by $E_k = (E/\ker \| \, \|_k)^\wedge$ for $k \in \mathbb{N}$, $^\wedge$ denoting the completion with respect to the norm $\| \bar{x} \|_k = \| x \|_k$ for $\bar{x} = x + \ker \| \, \|_k \in E/\ker \| \, \|_k$. Identity gives rise to a canonical connecting map $\iota_l^k : E_l \to E_k$ for $l > k$. Clearly $\iota_l^k = \iota_{k+1}^k \circ \ldots \circ \iota_l^{l-1}$ for $l > k$, $\iota_k^k = \mathrm{id}_E$. Hence we have a sequence

$$E_1 \xleftarrow{\iota_2^1} E_2 \xleftarrow{\iota_3^2} E_3 \xleftarrow{\iota_4^3} \ldots$$

of Banach spaces with connecting maps. Such a sequence is called a projective spectrum of Banach spaces. We call it the canonical projective spectrum.

E is called a Fréchet–Schwartz space if for every k there is an $l > k$ such that ι_l^k is a compact map. E is called nuclear if for every k there is an $l > k$ such that the map ι_l^k is nuclear. Hence every nuclear Fréchet space is a Fréchet–Schwartz space.

We recall that a continuous linear map $A : X \to Y$, X and Y denoting Banach spaces, is called compact if it maps the unit ball into a relatively compact set. It is nuclear, if it admits an expansion

$$Ax = \sum_n g_n(x) y_n$$

where $g_n \in X'$, $y_n \in Y$ for all n and

$$\sum_n \| g_n \|^* \| y_n \| < +\infty$$

where $\| \, \|^*$ denotes the norm in X'.

A Köthe space is a Fréchet–Schwartz space iff for every k there is an l such that

$$\lim_n \frac{a_{n,k}}{a_{n,l}} = 0 \,.$$

It is nuclear iff for every k there is an l such that

$$\sum_n \frac{a_{n,k}}{a_{n,l}} < +\infty \,.$$

Therefore every power series space is Fréchet–Schwartz. $\Lambda_r(\alpha)$ is nuclear if and only if

$$\limsup_n \frac{\log n}{\alpha_n} < +\infty \quad \text{for } r = +\infty$$

$$\lim_n \frac{\log n}{\alpha_n} = 0 \qquad \text{for } r = 0 \,.$$

For $\Lambda_r(\alpha)$ we shall always assume nuclearity.

2. Linear operators and characteristics of continuity

A linear map $A : E \to F$ from a graded Fréchet space E to a graded Fréchet space F is continuous if for every k there is $C > 0$ and $\sigma \in \mathbb{N}$ such that

$$\| Ax \|_k \leq C \| x \|_\sigma$$

for all $x \in E$. We put

$$\sigma(k) = \min \left\{ \sigma : \sup_{\|x\|_\sigma \leq 1} \| Ax \|_k < +\infty \right\},$$

i.e. the minimal σ making the above estimate valid, and call it the characteristic of continuity of A.

If e.g. $E = F = C^\infty(\mathbb{R}^n)$ equipped with the seminorms

$$\| f \|_k = \sup \left\{ |f^{(\alpha)}(x)| : |\alpha| \leq k,\ |x| \leq k \right\}$$

and $A = P(D) : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ a partial differential operator with constant coefficients, then $\sigma(k) = k + m$, where m is the order of the polynomial $P(z)$.

A linear map is called tame if $\sigma(k) \leq k + b$ for some $b \in \mathbb{N}$, it is called linear–tame if $\sigma(k) \leq ak + b$ for some $a, b \in \mathbb{N}$. The above example shows a tame map.

It is an interesting question which functions $\sigma(\cdot)$ for given E, F really occur as characteristic of continuity of some operator, whether there are bounds etc.. Notice that an operator is bounded, i.e. sends some neighbourhood of zero into a bounded set, if and only if its characteristic of continuity is bounded. Hence the question, for wich pairs E, F all continuous linear maps are bounded is equivalent to the question for which E, F characteristics of continuity are bounded, hence eventually constant.

As an (in fact very important) example we will treat maps between power series spaces. So let

$$A : \Lambda_r(\alpha) \to \Lambda_\rho(\beta) \; (r, \rho \in \{0, +\infty\})$$

be a continuous map, $\sigma(\cdot)$ its characteristic of continuity. We will use also the following function:

$$S(t) = \inf \left\{ s < r : \sup_{|x|_s \leq 1} |Ax|_t < \infty \right\} .$$

2.1 Lemma. *$S(t)$ is convex, increasing, $S(t) < r$ for all t.*

The convexity follows from an elementary interpolation argument ([22], Lemma 5.1), the rest is clear. This Lemma, however, has far reaching consequences, for instance the following theorem of V.P. Zaharjuta [33]:

2.2 Theorem. *Every continuous linear map from $\Lambda_o(\alpha)$ to $\Lambda_\infty(\beta)$ is bounded.*

PROOF: $S(t)$ is a nondecreasing convex function on $\mathbb{R}$, which is bounded from above, hence constant. Therefore $\sigma(\cdot)$ is eventually constant.

2.3 Theorem (cf. [2])**.** *Every continuous linear map from $\Lambda_o(\alpha)$ to $\Lambda_o(\beta)$ is linearly tame.*

PROOF: From an elementary geometrical argument using convexity of $S(t)$ one sees that there is an $a \in \mathbb{N}$ such that $S(t) < \frac{1}{a}t$ for $t \geq -1$. Hence $S(-\frac{1}{k}) < -\frac{1}{ak}$ and therefore $\sigma(k) \leq ak$ for all $k \in \mathbb{N}$.

We call (E, F) a tame pair if there exist functions $S_\alpha : \mathbb{N} \to \mathbb{N}$ for $\alpha = 1, 2, \ldots$ such that for every $A \in L(E, F)$ there is α with $\sigma(k) \leq S_\alpha(k)$ for all k. If every continuous linear map from E to F is bounded then trivially (E, F) is a tame pair. So we have:

2.4 Corollary. *$(\Lambda_o(\alpha), \Lambda_\rho(\beta))$ is a tame pair for all α, β and $\rho \in \{0, +\infty\}$.*

A characterization of all pairs (E, F), for which every continuous linear map is bounded, is given in [23], a characterization of all tame pairs in [4].

Of particular interest in connection with the unsolved problem of the structure of complemented subspaces of infinite type power series spaces is the case $(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$. It is completely described in [3], [4]. For more general cases see [10].

2.5 Theorem ([3], [4])**.** *The following are equivalent:*

(1) $(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$ *is a tame pair.*

(2) The set of finite limit points of $\left(\frac{\alpha_n}{\alpha_m}\right)_{n,m}$ *is bounded.*

(3) *Up to equivalence α is of the following form: there are an increasing sequence $n(k)$ in $\mathbb{N}$ $(n(1)=1)$ and a nondecreasing sequence $(\beta_k)_k$ with*

$$\lim_k \frac{\beta_{k+1}}{\beta_k} = +\infty$$

such that $\alpha_n = \beta_k$ for $n(k) \leq n < n(k+1)$.

(4) *Every continuous linear operator in $\Lambda_\infty(\alpha)$ is linearly tame.*

(5) *There is an equivalent sequence $\widetilde{\alpha}$ (namely the one from (3)) such that in $\Lambda_\infty(\widetilde{\alpha})$ every continuous linear operator is tame.*

Equivalence of sequences α and $\widetilde{\alpha}$ means the existence of a constant $C>0$ with $\frac{1}{C}\alpha_n < \widetilde{\alpha}_n \leq C\alpha_n$ for all n. This is equivalent to $\Lambda_r(\alpha) = \Lambda_r(\widetilde{\alpha})$ setwise or topologically. They are then even linear–tamely equivalent, i.e. identity is linearly tame in both directions.

Since, however, power series spaces occuring in analysis do in general not belong to the class described in Theorem 2.5, we cannot expect to find bounds on the characteristics of continuity. This constitutes a principal difficulty in the investigation of operators in power series spaces of infinite type and, more general, of Fréchet spaces and leads to the restriction on the consideration of tame operators in certain investigations.

Other principal difficulties we see by the following two examples:

Example. (1) $E = F = C_o^\infty[0,+\infty) = \left\{ f \in C^\infty[0,+\infty) : f^{(p)}(0) = 0 \text{ for all } p \right\}$, $\| f \|_k = \sup\{|f^{(p)}(t)| : p \leq k,\ t \in [0,k]\}$, $T = \frac{d}{dx}$, then $(R_\lambda f)(x) = e^{\lambda x} \int\limits_0^x e^{-\lambda t} f(t)\, dt$ inverts $T - \lambda\,\mathrm{id}$ for all $\lambda \in \mathbb{C}$. Hence the spectrum of T is empty.
(2) $E = F = H(\mathbb{C})$, $(Tf)(f) = zf(z)$. Then the spectrum of T is $\mathbb{C}$.

3. Exact sequences of Fréchet spaces

Let E_1, E_2, E_3 be Fréchet spaces, $\phi_1 \in L(E_1, E_2)$, $\phi_2 \in L(E_2, E_3)$, then $E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} E_3$ is called exact in E_2 if $\operatorname{im}\phi_1 = \ker\phi_2$. In this case $\operatorname{im}\phi_1$ is closed, hence a Fréchet space, therefore ϕ_1 is an open map onto $\ker\phi_2$. A short exact sequence is a sequence

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

which is exact in F, G, E, i.e. j is topologically injective and q is surjective. So $F \cong \ker q$, $E \cong G \big/ \ker q$.

We say that the sequence splits if q has a right inverse, equivalently: j has a left inverse or, again equivalently: There exists a continuous linear projection from G onto $\operatorname{im} j = \ker q$.

Let $\| \,\|_0 \leq \| \,\|_1 \leq \ldots$ be a fundamental system of seminorms on G, $0 \to F \xrightarrow{j} G \xrightarrow{q} H \to 0$ exact. On F we may use the seminorms $\| x \|_k = \| jx \|_k$, on H the quotient seminorms $\| x \|_k = \inf\{\| \xi \|_k : \xi \in G,\ q\xi = x\}$. Then for every k we obtain an exact sequence of Banach spaces

$$0 \to F_k \xrightarrow{j_k} G_k \xrightarrow{q_k} H_k \to 0$$

where j_k and q_k are continuous extensions of j and q.

It is an important fact, that even if all these "local exact sequences" split, the given exact sequence of Fréchet spaces need not split (see the first of the following examples).

The investigation of this phenomenon and of conditions under which $0 \to F \to G \to H \to 0$ splits will be content of the following sections. Before we give examples of short exact sequences of Fréchet spaces which do not split, i.e. of surjective continuous linear maps which do not admit right inverses.

Examples: (1) We put for $k \in \mathbb{N} \cup \{\infty\}$

$$C_o^k[-1,+1] = \left\{ f \in C^k[-1,+1] : f^{(p)}(0) = 0 \text{ for all } p \leq k \text{ (resp. } p < +\infty) \right\}$$

and set $\omega = \mathbb{R}^{\mathbb{N}_0}$ equipped with the seminorms $\| x \|_k = \sup\limits_{j=0,\ldots,k} |x_j|$ for $x = (x_o, x_1, \ldots)$.

If $\Delta_o f = (f(0), f'(0), \ldots)$ then the classical theorem of E. Borel tells that the sequence

$$0 \to C_o^\infty[-1,+1] \to C^\infty[-1,+1] \xrightarrow{\Delta_0} \omega \to 0$$

is exact. Clearly the local sequences

$$0 \to C_o^k[-1,+1] \to C^k[-1,+1] \xrightarrow{q_k} \mathbb{R}^{k+1} \to 0$$

where $q_k f = (f(0), f'(0), \ldots, f^{(k)}(0))$ are exact and do even split, e.g. by

$$R_k(x_o, \ldots, x_k) = \sum_{p=0}^{k} x_p \frac{x^p}{p!} .$$

However Δ_o cannot have a continuous right inverse (see [7]). For, assume R is such a right inverse and

$$\| Rx \|_o \leq C \| x \|_k , \; x \in \omega ,$$

then we apply R to $e_{k+1} = (\delta_{j,k+1})_j$ and obtain $\| e_{k+1} \|_k = 0$ hence $Re_{k+1} = 0$, hence $e_{k+1} = \Delta_o R e_{k+1} = 0$, which is a contradiction.

(2) Let $\Omega_1 = \{z \in \mathbb{C} : \Re z > -1\}$, $\Omega_2 = \{z \in \mathbb{C} : \Re z < +1\}$. Then we have a short exact sequence of Fréchet spaces (see [8])

$$(*) \qquad 0 \to H(\mathbb{C}) \xrightarrow{j} H(\Omega_1) \oplus H(\Omega_2) \xrightarrow{q} H(\Omega_1 \cap \Omega_2) \to 0$$

where $jf = (f|_{\Omega_1}, f|_{\Omega_2})$, $q(f_1, f_2) = f_1|_{\Omega_1 \cap \Omega_2} - f_2|_{\Omega_1 \cap \Omega_2}$. With $\alpha_n = n$ for all n we have $H(\mathbb{C}) \cong \Lambda_\infty(\alpha)$ and, due to the Riemann mapping theorem, $H(\Omega_1) \cong H(\Omega_2) \cong H(\Omega_1 \cap \Omega_2) \cong H(D) \cong \Lambda_o(\alpha)$, where $D = \{z \in \mathbb{C} : |z| < 1\}$.

By $((x_o, x_1, \ldots), (y_o, y_1, \ldots)) \to (x_o, y_o, x_1, y_1, \ldots)$ we have $\Lambda_o(\alpha) \times \Lambda_o(\alpha) \cong \Lambda_o(\alpha)$.

So by use of all these isomorphisms we obtain an exact sequence of power series spaces.

$$0 \to \Lambda_\infty(\alpha) \xrightarrow{j} \Lambda_o(\alpha) \xrightarrow{q} \Lambda_o(\alpha) \to 0 .$$

Assume, that the exact sequence splits, then there exists a left inverse L of j. However, due to Zaharjuta's theorem (Theorem 2.2) L would be compact, hence also $\mathrm{id}_{\Lambda_\infty(\alpha)} = Lj$, so $\Lambda_\infty(\alpha)$ would be finite dimensional.

This proves that the exact sequence $(*)$ does not split. We even have somewhat more.

3.1 Proposition. *No exact sequence of the form*

$$0 \to \Lambda_\infty(\alpha) \to \Lambda_o(\beta) \to H \to 0$$

or of the form

$$0 \to F \to \Lambda_\infty(\alpha) \to \Lambda_o(\beta) \to 0$$

can split.

4. Splitting of short exact sequences of Fréchet spaces

We start with an analysis of the problem and refer for that to [13], [14], [24], [25].

Let $0 \to F \overset{j}{\to} G \overset{q}{\to} H \to 0$ be an exact sequence of Fréchet spaces, E a Fréchet space and $\phi \in L(E, H)$. We ask for the existence of ψ such that $q \circ \psi = \phi$.

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "j"] & G \arrow[r, "q"] & H \arrow[r] & 0 \\ & & & E \arrow[u, "\phi"'] \arrow[ul, dashed, "\psi"] & \end{tikzcd}$$

Given any fundamental system $\| \ \|_o \leq \| \ \|_1 \leq \ \ldots$ of seminorms in G we can choose fundamental systems in F and H, as described in section 1, such that we have exact sequences of the local Banach spaces for all k and in total obtain the following picture for every k:

$$\begin{tikzcd} 0 \arrow[r] & F_k \arrow[r, "j_k"] & G_k \arrow[r, "q_k"] & H_k \arrow[r] & 0 \\ & & & E \arrow[u, "\phi_k"'] \arrow[ul, dashed, "\psi_k"] & \end{tikzcd}$$

where j_k, q_k, ϕ_k are the continuous extensions of j, q, ϕ and we hope to find ψ_k.

We *assume* that:

(1) For every k we can find ψ_k such that $q_k \circ \psi_k = \phi_k$.

(2) For every sequence $A_k \in L(E, F_k)$, $k = 0, 1, \ldots$ we can find a sequence $B_k \in L(E, F_k)$, $k = 0, 1, \ldots$ such that $A_k = B_k - \iota^k_{k+1} \circ B_{k+1}$ for all k.

In this case we find for the sequence ψ_k, $k = 0, 1, \ldots$ obtained from (1) maps $A_k \in L(E, F_k)$ such that: $j_k \circ A_k = \psi_k - \iota^k_{k+1} \circ \psi_{k+1}$. This is possible, since q_k applied to the right hand side is zero. Then we find B_k, $k = 0, 1, \ldots$ according to (2). We set

$$\Psi_k = \psi_k - j_k \circ B_k$$

and obtain $\Psi_k = (j_k \circ A_k + \iota^k_{k+1} \circ \psi_{k+1}) - j_k \circ (A_k + \iota^k_{k+1} \circ B_{k+1}) = \iota^k_{k+1} \circ \Psi_{k+1}$ for all k. Therefore there exists a $\Psi \in L(E, G)$ such that $\Psi_k = \iota^k \circ \Psi$ for all k, where $\iota^k : G \to G_k$ is the quotient map.

Obviously we have for all k:

$$\iota^k \circ (q \circ \Psi) = q_k \circ \Psi_k = q_k \circ \psi_k = \phi_k$$

hence $q \circ \Psi = \phi$, which was to be proved.

We have now to discuss the assumptions.

Assumption (1) is e.g. satisfied if G is a hilbertizable Fréchet space, i.e. admits a fundamental system of seminorms $\| \ \|_o \leq \| \ \|_1 \leq \ldots$ such that $\| \ \|_k^2 = < \, , \, >_k$ for all k, $< \, , \, >_k$ a semiscalar product. In this case the G_k are Hilbert spaces and all local exact sequences $0 \to F_k \to G_k \to H_k \to 0$ split.

It is also satisfied if E admits a fundamental system of seminorms such that every $E_k \cong l_1$. We may assume $\phi_k \in L(E_k, H_k)$ for all k. Let $I_k : E_k \to l_1$ be an isomorphism, then we find for every j a vector $x_j \in G_k$ such that $\phi_k \circ I_k^{-1}(e_j) = q_k(x_j)$ $(e_j = (\delta_{j,\nu})_\nu)$. Setting $\psi_k(x) = \sum_j \xi_j x_j$ where $I_k x = (\xi_o, \xi_1, \ldots)$ we have $\psi_k \in L(E_k, G_k)$ and $q_k \circ \psi_k(x) = \phi_k \circ I_k^{-1}(\sum_j \xi_j e_j) = \phi_k x$.

The preceding is always satisfied E if is nuclear or a Köthe sequence space (see section 1). This is fulfilled in the following *special situation*, in which we investigate assumption (2).

We assume that $E = \Lambda_r(\alpha)$, $F = \Lambda_\rho(\beta)$. If $\rho_k \nearrow \rho$ then we may assume the fundamental system of seminorms on G, leading to the local exact sequences of Banach spaces, chosen in such way that $|x|_{\rho_k} \leq \| jx \|_k$ for all k and $x \in F = \Lambda_\rho(\beta)$. It is quite obvious that we may replace the F_k then by $\Lambda_k = \left\{ \xi = (\xi_o, \xi_1, \ldots) : |\xi|_{\rho_k} = \sup_j |\xi_j| e^{\rho_k \beta_j} < +\infty \right\}$. For the A_k we have continuity assumptions of the form

$$|A_k x|_{\rho_k} \leq C_k |x|_{r_k}, \; x \in E = \Lambda_r(\alpha) \, .$$

We obtain the following fundamental lemma (cf. [30]):

4.1 Lemma. *If* $\frac{r_{k+1} - r_k}{\rho_{k+1} - \rho_k}$, $k = 0, 1, \ldots$, *is nondecreasing, then there exist* $B_k \in L(E, \Lambda_k)$, $k = 1, 2, \ldots$ *such that* $A_k = B_k - B_{k+1}$ *for all* k.

Proof: Let

$$A_k x = \left(\sum_{\nu=1}^{\infty} x_\nu a_{\nu,j}^{(k)} \right)_j$$

for $x = (x_1, x_2, \ldots)$ where $\left(a_{\nu,j}^{(k)} \right)_{\nu,j}$ is a matrix which satisfies the estimates

$$|a_{\nu,j}^{(k)}| e^{\rho_k \alpha_j} \leq e^{d(k) + r_k \beta_\nu}$$

for all j, ν, k.

We determine inductively matrices $\left(u_{\nu,j}^{(k)} \right)$ for $k = 2, 3, \ldots$, $\left(v_{\nu,j}^{(k)} \right)$ for $k = 1, 2, \ldots$ and constants $D(k) \geq d(k)$ such that

(1) $|u_{\nu,j}^{(k)}| \leq e^{-k + \min\{r_n \beta_\nu - \rho_n \alpha_j : n = 1, 2, \ldots, k-2\}}$

(2) $|v_{\nu,j}^{(k)}| \le e^{D(k)+r_k\beta_\nu-\rho_k\alpha_j}$

(3) $u_{\nu,j}^{(k+1)} + v_{\nu,j}^{(k+1)} = a_{\nu,j}^{(k)} + v_{\nu,j}^{(k)}$.

We set $v_{\nu,j}^{(1)} = 0$ for all ν and j, $D(1) = d(1)$. Let $v_{\nu,j}^{(k)}$ be determined. Then we set $M(k) = k + 2 + D(k) = \frac{r_k - r_{k-1}}{r_{k+1}-r_k} N(k)$ and

$$\begin{aligned} I_o &= \{(\nu, j) : (r_{k+1} - r_k)\beta_\nu + N(k) \le (\rho_{k+1} - \rho_k)\alpha_j\} \\ I_1 &= \{(\nu, j) : (r_{k+1} - r_k)\beta_\nu + N(k) > (\rho_{k+1} - \rho_k)\alpha_j\} \ . \end{aligned}$$

We define

$$\begin{aligned} u_{\nu,j}^{(k+1)} &= \begin{cases} a_{\nu,j}^{(k)} + v_{\nu,j}^{(k)} & \text{for } (\nu,j) \in I_o \\ 0 & \text{for } (\nu,j) \in I_1 \end{cases} \\ v_{\nu,j}^{(k+1)} &= \begin{cases} 0 & \text{for } (\nu,j) \in I_o \\ a_{\nu,j}^{(k)} + v_{\nu,j}^{(k)} & \text{for } (\nu,j) \in I_1 \ . \end{cases} \end{aligned}$$

For $(\nu, j) \in I_o$ and $n = 1, 2, \dots, k-1$ we obtain

$$\begin{aligned} |u_{\nu,j}^{(k+1)}| &\le |a_{\nu,j}^{(k)}| + |v_{\nu,j}^{(k)}| \\ &\le e^{d(k)+r_k\beta_\nu-\rho_k\alpha_j} + e^{D(k)+r_k\beta_\nu-\rho_k\alpha_j} \\ &\le e^{1+D(k)+r_k\beta_\nu-\rho_k\alpha_j} \\ &\le e^{M(k)+(r_k-r_n)\beta_\nu-(\rho_k-\rho_n)\alpha_j} e^{-(k+1)+r_n\beta_\nu-\rho_n\alpha_j} \end{aligned}$$

and we get the estimate

$$\begin{aligned} &M(k) + (r_k - r_n)\beta_\nu - (\rho_k - \rho_n)\alpha_j \\ &\le M(k) + \frac{r_k - r_n}{r_{k+1} - r_k}(r_{k+1} - r_k)\beta_\nu - (\rho_k - \rho_n)\alpha_j \\ &\le M(k) + \frac{r_k - r_n}{r_{k+1} - r_k}(\rho_{k+1} - \rho_k)\alpha_j - \frac{r_k - r_n}{r_{k+1} - r_k}N(k) - (\rho_k - \rho_n)\alpha_j \\ &\le \sum_{m=n}^{k-1} \left[(r_{m+1} - r_m)\frac{\rho_{k+1} - \rho_k}{r_{k+1} - r_k} - (\rho_{m+1} - \rho_m) \right] \alpha_j \\ &\le 0 \ . \end{aligned}$$

For $(\nu, j) \in I_1$ we obtain

$$\begin{aligned} |v_{\nu,j}^{(k+1)}| &\le |a_{\nu,j}^{(k)}| + |v_{\nu,j}^{(k)}| \\ &\le e^{1+D(k)+r_k\beta_\nu-\rho_k\alpha_j} \\ &\le e^{1+D(k)+r_k\beta_\nu+(\rho_{k+1}-\rho_k)\alpha_j-\rho_{k+1}\alpha_j} \\ &\le e^{1+D(k)+N(k)+r_{k+1}\beta_\nu-\rho_{k+1}\alpha_j} \\ &= e^{D(k+1)+r_{k+1}\beta_\nu-\rho_{k+1}\alpha_j} \end{aligned}$$

with $D(k+1) = 1 + D(k) + N(k)$.

Finally we put

$$b_{\nu,j}^{(k)} = v_{\nu,j}^{(k)} - \sum_{l=k+1}^{\infty} u_{\nu,j}^{(l)} = v_{\nu,j}^{(k+1)} - a_{\nu,j}^{(k)} - \sum_{l=k+2}^{\infty} u_{\nu,j}^{(l)} ,$$

where the second equation follows from (3). The series converges because (1) implies

$$|u_{\nu,j}^{(k)}| \le e^{-k} \cdot e^{r_1\beta_\nu - \rho_1\alpha_j}$$

for all k.

We obtain the following estimate

$$\begin{aligned} |b_{\nu,j}^{(k)}| e^{\rho_k\alpha_j} &\le e^{D(k+1)+r_{k+1}\beta_\nu} + e^{d(k)+r_k\beta_\nu} + \sum_{l=k+2}^{\infty} e^{-k+r_k\beta_\nu} \\ &\le C e^{r_{k+1}\beta_\nu} \end{aligned}$$

with suitable $C = C(k)$. Obviously we have

$$b_{\nu,j}^{(k+1)} - b_{\nu,j}^{(k)} = v_{\nu,j}^{(k+1)} - v_{\nu,j}^{(k)} + u_{\nu,j}^{(k+1)} = a_{\nu,j}^{(k)}$$

for all k, ν, j.

We are investigating exact sequences of the form

$$\text{(I)} \qquad 0 \to \Lambda_\rho(\beta) \to G \to \Lambda_r(\alpha) \to 0 \, .$$

In Section 3, Example (2) we have seen an example, where $\rho = \infty$, $r = 0$ which did not split. If, however $r = \infty$, then the condition in Lemma 4.1 can (by possibly increasing, the r_k inductively) always be fulfilled. Hence we proved (see [21]):

4.2 Theorem. *If $r = +\infty$ then the exact sequence always splits.*

Example: We consider for $a < b$ the exact sequence

$$0 \to \mathcal{D}[a-1,a] \times \mathcal{D}[b,b+1] \xrightarrow{j} \mathcal{D}[a-1,b+1] \xrightarrow{q} C^\infty[a,b] \to 0$$

where q is the restriction map. Then by the above–mentioned E. Borel theorem q is surjective. By a classical result (expansion into a series using transformed Hermite functions, resp. Tchebyshev polynomials) $\mathcal{D}[a,b]$ and $C^\infty[a,b]$ are isomorphic to $s = \Lambda_\infty(\alpha)$, $\alpha_n = \log(n+1)$ for all n. We obtain (see [7], [18]):

4.3 Theorem (Mityagin–Seeley)**.** *There exists a continuous linear extension operator $C^\infty[a,b] \to C^\infty(\mathbb{R})$.*

If we replace Theorem 4.2 by the more advanced Theorem 6.1 then this method has been frequently used to prove the existence of extension operators (eg. [20]).

It remains now the case $r = 0$, $\rho = 0$. We give an *Example* (cf [22], 5.4).

Let $(\zeta_k)_k$ and $(z_k)_k$ be sequences in the unit disc in $\mathbb{C}$, $\lim_k |\zeta_k| = 1$, $\lim_k |z_k| = 1$. We assume that $|\zeta_0| < |\zeta_1| < \ldots$ and $|z_o| \leq |z_1| \leq \ldots$ and put $\rho_k = \log|\zeta_k|$, $r_k = \log|z_k|$. Then $\rho_k \nearrow 0$, $r_k \nearrow 0$.

We define with $D = \{z \in \mathbb{C} : |z| < 1\}$

$$G = \{(f,g) \in H(D) \times H(D) : f(\zeta_k) = g(z_k) \text{ for all } k \ \}$$

and $q(f,g) = g$ for $(f,g) \in G$. Then we obtain an exact sequence

$$\text{(II)} \qquad 0 \to K \overset{j}{\to} G \overset{q}{\to} H(D) \to 0$$

Where

$$K = \{(f,0) : f \in H(D),\ f(\zeta_k) = 0 \text{ for all } k\ \} \cong f_o H(D)$$

with $f_o \in H(D)$ suitably chosen.

First we assume, that there exists a right inverse $g \mapsto Rg = (R_1 g, g) \in G$. R_1 defines a continuous linear map $H(D) \to H(D)$. On $H(D)$ we use the norms

$$\| f \|_t = \sup_{|z| \leq e^t} |f(z)|\,, \quad |f|_t = \sum_{j=0}^{\infty} |x_j| e^{tj}$$

for $t < 0$, where x_j are the Taylor coefficients of f (i.e. using the $(|\ |_t)_{t<0}$ we identify $H(D)$ with $\Lambda_o(\alpha)$, $\alpha_n = n$ for all n). Clearly we have $\|\ \|_t \leq |\ |_t \leq \frac{1}{1-e^{t-s}} \|\ \|_s$ for $s > t$.

We set as in section 2

$$S(t) = \inf \left\{ s : \sup_{|f|_s \leq 1} |R_1 f|_t < +\infty \right\}.$$

Then S is convex, increasing and $S(t) < 0$ for all t. For $s > S(\log|\zeta_k|)$ for all $g \in H(D)$ with $f = R_1 g$

$$|g(z_k)| = |f(\zeta_k)| \leq D \| g \|_s \ .$$

This implies $|z_k| \leq e^s$ for all such s, hence

$$|z_k| \leq e^{S(\log|\zeta_k|)}$$

for all k and, due to the properties of S,

$$|z_k| \leq |\zeta_k|^{\varepsilon}$$

for all k, with suitable $\varepsilon > 0$.

If, on the other hand, this is satisfied, then we put $\rho_k = -2^{-k-1}$ and $r_k = -\varepsilon 2^{-k}$. We use the (equivalent) exact sequence

$$0 \to H(D) \overset{J}{\to} G \overset{q}{\to} H(D) \to 0$$

where $Jh = (f_o h, 0)$. On the left space we use the norms $(|\ |_{\rho_k})_k$, on the right space the norms $(|\ |_{r_k})_k$ and on the middle one their maximum.

For q_k we get a right inverse by $\psi_k(g) = (L_k g, g)$ where L_k is given by the Lagrange–interpolation formula

$$L_k g = \sum_{\substack{j \\ |\zeta_j| \le e^{\rho_k}}} g(z_j) \prod_{\substack{\nu \neq j \\ |\zeta_j| \le e^{\rho_k} \\ |\zeta_\nu| \le e^{\rho_k}}} \frac{z - \zeta_\nu}{\zeta_j - \zeta_\nu}$$

and therefore

$$A_k g = \frac{1}{f_o} (L_k g - L_{k+1} g) \ .$$

With $0 > \rho_{k+1} > \rho_k$ chosen n such that f_o has no zero on $\{|z| : |z| = e^{\rho_k}\}$ we get

$$\begin{aligned} |A_k g|_{\rho_k} &\le C' \parallel A_k g \parallel_{\rho_k} \le C'' \parallel L_k g - L_{k+1} g \parallel_{\rho_k} \\ &\le D |g|_{r_k} \ . \end{aligned}$$

The last inequality since $|\zeta_j| \le e^{\rho_{k+1}}$ implies $|z_j| \le e^{\varepsilon \rho_{k+1}} = e^{r_k}$.

Since $\frac{r_{k+1} - r_k}{\rho_{k+1} - \rho_k} = 2\varepsilon$ for all k, Lemma 4.1 implies that the given exact sequence splits.

We proved the following Theorem:

4.4 Theorem. *The exact sequence* (II) *splits if and only if there exists* $\varepsilon > 0$ *such that* $|z_k| \le |\zeta_k|^\varepsilon$ *for all* k.

Hence we have plenty of examples for exact sequences

$$0 \to \Lambda_o(\alpha) \to G \to \Lambda_o(\alpha) \to 0$$

with $\alpha_n = n$ for all n, which do not split. On the other hand we have the following theorem which we quote from [22], 4.5.

4.5 Theorem. *If* $\sup \frac{\alpha_{2n}}{\alpha_n} < +\infty$, $\limsup_n \frac{\log n}{\alpha_n} < +\infty$ *and the same holds for* β, *if moreover*

$$0 \to \Lambda_o(\alpha) \to G \to \Lambda_o(\alpha) \to 0$$

is an exact sequence, then $G \cong \Lambda_o(\beta) \oplus \Lambda_o(\alpha)$.

Since these assumptions are fulfilled for $\alpha_n = \beta_n = n$ we have shown that $G \cong \Lambda_o(\beta) \oplus \Lambda_o(\alpha)$, however the exact sequence needs not to split, i.e. the isomorphism does not necessarily come from the exact sequence.

Finally, since in analysis many spaces are isomorphic to power series spaces, the following scheme is of interest. If

(III) $\qquad 0 \to \Lambda_\rho(\beta) \to \Lambda_\sigma(\gamma) \to \Lambda_r(\alpha) \to 0$

is exact, then the exact sequence splits for:

$r =$ / $\rho =$	0	$+\infty$
0	sometimes	
$+\infty$	never	always

So in two of three cases, it does *only* depend on the structure of the spaces *not* on the special maps. The case $\rho = 0$, $r = +\infty$ is impossible, due to Zaharjuta's theorem (Theorem 2.2).

5. Tame exact sequences and tame splitting

To complete our scheme, we will now investigate the exact sequence (III) in the case, where $\rho = 0$ and $r = 0$. Theorem 2.2 (Zaharjuta's theorem) then implies that $\sigma = 0$, otherwise $\Lambda_o(\beta)$ had to be finite dimensional.

We consider $\Lambda_o(\alpha)$ (resp. $\Lambda_o(\beta)$, $\Lambda_o(\gamma)$) as graded Fréchet space equipped with the system of norms

$$\| x \|_k = \sum_{j=0}^{\infty} |x_j| e^{-\frac{1}{k+1}\alpha_j}$$

(resp. the analogous definition for $\Lambda_o(\beta)$, $\Lambda_o(\gamma)$).

A continuous linear map $T : E \to F$, where E, F are graded Fréchet spaces is called tame (resp. linear–tame) if $\sigma(k) \leq k + b$ for suitable b (resp. $\sigma(k) \leq ak + b$ for suitable a, b). Two systems of seminorms $\| \, \|_o \leq \| \, \|_1 \leq \ldots$ and $\| \, \|'_o \leq \| \, \|'_1 \leq \ldots$ are called tamely equivalent (resp. linear–tamely equivalent) if they satisfy estimates $\| \, \|_k \leq C_k \| \, \|'_{k+b}$ and $\| \, \|'_k \leq C_k \| \, \|_{k+b}$ (resp. $\| \, \|_k \leq C_k \| \, \|'_{ak+b}$ and $\| \, \|'_k \leq C_k \| \, \|_{ak+b}$) for all k. A sequence $E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} E_3$ of graded Fréchet spaces and continuous linear maps is called tame exact (resp. linear–tame exact) in E_2 if it is exact and the system of seminorms on $\operatorname{im} \phi_1 = \ker \phi_2$ inherited from E_2 and that given by $\| x \|_k = \inf \{ \| \xi \|_k : \phi_1 \xi = x \}$ are tamely (resp. linear–tamely) equivalent. In particular ϕ_1 is supposed to be tame (resp. linear–tame).

A short exact sequence

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

of graded Fréchet spaces is tame exact (resp. linear–tame exact) if and only if it is tame exact in F, G, E, which means that j and q are tame maps and by a tame (resp. linear–tame) change of seminorms on F, E we may assume that j is seminormwise isometric and q is seminormwise a quotient map.

5.1 Theorem. *The exact sequence*

$$0 \to \Lambda_o(\beta) \xrightarrow{j} \Lambda_o(\gamma) \xrightarrow{q} \Lambda_o(\alpha) \to 0$$

splits if and only if it is linear–tame exact. In this case the right inverse of q (resp. left inverse of j) is linear–tame.

PROOF: By use of Theorem 2.3 j and q are linear–tame. If the exact sequence splits then q has a right inverse R and j a left inverse L. Again by use of Theorem 2.3 R and L are linear–tame. This proves that the sequence is linear–tame exact.

If, on the other hand, the sequence is linear–tame exact then we choose the seminorms on $F = \Lambda_o(\beta)$ and $H = E = \Lambda_o(\alpha)$ leading to the local exact sequences (as in §2) starting from the $\| \, \|_k$ on $\Lambda_o(\gamma)$. We call these seminorms $||| \, |||_k$.

We then have $\| \ \|_k \leq C_k ||| \ |||_{ak+b}$ on $\Lambda_o(\beta)$. Hence the analysis of §2 applied to id : $\Lambda_o(\alpha) = E \to H = \Lambda_o(\alpha)$ leads to maps A_k with $\| A_k x \|_k \leq C_k \| x \|_{Ak+B}$ with suitable A, B. So we may apply Lemma 4.1 with $\rho_k = -\frac{1}{k+1}$ and $r_k = -\frac{1}{Ak+B+1}$ and proceed along the argument of §2.

Because of Theorem 2.3 the right inverse R will automatically be linear–tame. However, we can get this also from the construction of the right inverse in §2 and the concrete estimates obtained in Lemma 4.1.

The proof, just described, in fact, gives a lot more.

A pair (E, F) of graded Fréchet spaces is called a tame (linear–tame) splitting pair, if every tame (linear–tame) exact sequence

$$0 \to F \to G \to E \to 0$$

splits tamely (linear–tamely).

For the following Theorem cf. [11], [30].

5.2 Theorem. *$(\Lambda_r(\alpha), \Lambda_r(\beta))$ is a tame and linear–tame splitting pair for every r, α, β.*

6. Splitting theorems

In the present section we report on two of the most important generalizations of Theorem 4.2. In particular Theorem 6.1 has found many applications. We need some more notations.

A Fréchet space E has property (DN) (see [21]), if there exists p such that for every k there is K and $C > 0$ such that

$$\| \ \|_k^2 \leq C \| \ \|_p \| \ \|_K \ .$$

It has property (Ω) (see [31]), if for every p there exists q such that for every k there is $0 < t < 1$ and $C > 0$ such that

$$\| \ \|_q^* \leq C \| \ \|_p^{*\ t} \| \ \|_k^{*\ 1-t} \ .$$

Here $\| y \|_k^* = \sup \{|y(x)| : \| x \|_k \leq 1\}$ denotes the dual (extended real valued) norm on E'.

Example: (1) (see [21], [31]) $\lambda(A)$ has property (DN) if and only if it admits an equivalent matrix (i.e. defining the same space) with:

$$a_{j,k}^2 \leq a_{j,k-1} a_{j,k+1} \text{ for all } j, k.$$

It has property (Ω), if and only if it admits an equivalent matrix such that

$$a_{j,k}^2 \geq a_{j,k-1} a_{j,k+1} \text{ for all } j, k.$$

Finally it has both properties if it admits an equivalent matrix of the form $a_{j,k} = a_j^k$, i.e. in the nuclear case if $\lambda(A) = \Lambda_\infty(\alpha)$ for suitable α.

(2) $\Lambda_r(\alpha)$ has always property (Ω). It has property (DN) if and only if $r = +\infty$.

6.1 Theorem. *If $0 \to F \to G \to E \to 0$ is an exact sequence of Fréchet spaces, one of which is nuclear, if E has property (DN) and F has property (Ω), then the sequence splits.*

This Theorem was first proved in [21], [31]. It is closely related to the structure theory of nuclear Fréchet spaces as we will see below. Other proofs can be found in [14] and [25].

A useful variant is also the following which avoids nuclearity (see [29]).

6.2 Theorem. *If $0 \to F \to G \to E \to 0$ is an exact sequence of hilbertizable Fréchet spaces, if E has property (DN) and F has property (Ω) then the sequence splits.*

To discuss the close connection to the structure theory of nuclear Fréchet spaces (see [21], [31], [32]) we assume now all spaces as nuclear. Then refering to the assumptions in the discussion of §4, assumption (1) can always be assumed to be satisfied. Moreover we know that every exact sequence

$$0 \to s \to G \to s \to 0$$

splits.

If $E \subset s$, then the maps $A_k : E \to s_k$ have the form $A_k x = \sum_j x_j^*(x) y_j$ where $\sum_j \| x_j^* \|_m^* \| y_j \|_k < +\infty$ for suitable m, hence can be extended to maps $\widetilde{A}_k : s \to s_k$. Now we are in the situation to apply Lemma 4.1. Similarly we may argue for a quotient F of s. Hence we have:

6.3 Proposition. *If E is a subspace of s, F a quotient of s, $0 \to F \to G \to E \to 0$ exact then the sequence splits.*

Note that the assumptions imply that F, G, E are nuclear. Proposition 6.3 which is an immediate consequence of §4, is also a special case of Theorem 6.1 (or 6.2). Now the structure theory tells us the following (see [21], [31]):

6.4 Theorem. *(a) A nuclear Fréchet space E has property (DN) if and only if it is isomorphic to a subspace of s.*

(b) A nuclear Fréchet space F has property (Ω) if and only if it is isomorphic to a quotient of s.

Therefore 6.1 and 6.3 are, in the nuclear case, the same. Proposition 6.3 on the other hand serves to prove Theorem 6.4 (see [21], [31]).

Finally we report analogous theorems in the case of graded Fréchet spaces (see [26], for more general situations [16]).

For that we need the following notation. A graded Fréchet space is said to admit smoothing operators (see [9]) if there exist continuous linear operators $(S_\theta)_{\theta \in \mathbb{R}}$ such that with suitable constants $C_{k,m}$

$$\| S_\theta x \|_k \le C_{k,m} \theta^{k+p-m} \| x \|_m \quad \text{for all } m < k+p,\ \theta > 0,\ x \in E$$
$$\| x - S_\theta x \|_k \le C_{k,m} \theta^{k+p-m} \| x \|_m \quad \text{for all } m \ge k+p,\ \theta > 0,\ x \in E\,.$$

We call it tame, if it admits a family of smoothing operators. This notation is more general than in [5], [19].

6.5 Theorem. *A tame exact sequence $0 \to F \to G \to E \to 0$ of tame Fréchet spaces, one of which is nuclear, splits tamely.*

6.6 Theorem. *Every nuclear tame Fréchet space is tamely isomorphic to a power series space $\Lambda_\infty(\alpha)$.*

Theorem 6.6 is proved in [27], Theorem 3.1. If in the proof of Theorem I in [26] we replace Theorem II by the present Theorem 6.6 we obtain Theorem 6.5.

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