MORAVA K-THEORY OF GROUPS OF ORDER 32

BJÖRN SCHUSTER

Abstract. We show that the Morava K-theories of the groups of order 32 are concentrated in even degrees.

Let G be a finite group and BG denote its classifying space. Determining the Morava K-theory of BG is generally difficult, mainly due to the weakness of existing methods of calculation, which all require knowledge of the cohomology of p-groups — in itself a notorious problem. Certain series of groups with particularly simple structure, such as wreath products, or groups having a cyclic maximal subgroup, or minimal non-abelian groups, are quite tractable, see e.g. the work of Hopkins-Kuhn-Ravenel [3], Hunton [4], and Yagita and his coauthors ([7], [10], [11], [12], [14]). A hard example is the 3-Sylow supgroup of $GL_4(\mathbb{F}_3)$: in [5], Kriz computes just enough of its 3-primary Morava K-theory to conclude that there are odd dimensional elements in it, thereby disproving a conjecture of Ravenel. A complete calculation however is still elusive. In later work by Kriz and Lee, this example was generalised to all n and all odd primes p [6].

In this note we shall consider the groups of order 32. In many cases the Morava K-theory is already known, or easily deduced from results in the literature. For the remaining groups, it was established in [8] that for $n = 2$ at least, their Morava K-theory $K(n)^*(BG)$ is generated by transfers of Euler classes of complex representations. In other words, all groups of order 32 are "$K(2)$-good" in the sense of Hopkins-Kuhn-Ravenel. Some of the results however relied on computer calculations. This is to be remedied here, although we only prove a weaker statement:

Theorem. *Let G be a group of order 32. Then $K(n)^{odd}(BG) = 0$.*

When starting this project, our objective of course was not to prove such a result, rather we hoped – rather naively, perhaps – that order 32 would be big enough to find a 2-primary counterexample to the even degree conjecture. In this we have failed and the problem remains open.

We have not tried to determine ring structures. This should be possible in principle, using methods similar to those employed in [1], but since this paper is already so loaded with computations, we have refrained from doing so.

The article is organised as follows. In Section 1, we recall a few old results used in the later calculations. Section 2 contains some technical lemmas. Section 3 collects all we need to know about the Morava K-theories of groups of smaller order. Section 4 lists the 51 groups of order 32 and disposes of those whose Morava K-theory is either in the literature or can easily be read off from known computations. Finally, Section 5 contains the remaining calculations.

Date: November 23, 2010.

2010 *Mathematics Subject Classification.* Primary 55R35, 55N20; Secondary 57T25.

Key words and phrases. Morava K-theory, finite 2-groups.

This paper is very computational, and we found it hard to strike a balance between completeness and readability. We hope to have provided sufficent detail without placing an undue burden on the reader's patience.

1. Preliminaries

Our principal calculational tool shall be the Serre spectral sequence

$$E_2 = H^*\big(BQ; K^*(BH)\big) \Rightarrow K^*(BG) \tag{1.1}$$

associated to a group extension $1 \to H \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$ for K either integral Morava K-theory $\widetilde{K}(n)$ or its mod p version $K(n)$. Here $H^*\big(BQ; K(n)^*(BH)\big)$ is the ordinary cohomology of Q with coefficients in the $\mathbb{F}_p[Q]$-module $K(n)^*(BH)$, the action of Q being induced by conjugation in G as usual. This module structure can be quite messy, even in the simplest cases, since it involves the formal group law. Recall that as any complex oriented cohomology theory, Morava K-theory comes equipped with a formal group law induced from the tensor product of line bundles; we shall write the formal sum as "$+_{K(n)}$".

This spectral sequence is, via π^*, a module over the Atiyah-Hirzebruch spectral sequence for BQ.

In [5], Igor Kriz proved a beautiful theorem about the Serre spectral sequence associated to fibrations over BC_p. This theorem is one of the few practical tools for calculation; Kriz used it to great effect to supply the first counterexample to the even degree conjecture. His result gives a useful criterion to decide whether a group G has even Morava K-theory.

Theorem 1.1 (Kriz [5])**.** *Let G be a p-group and H a normal subgroup of index p with $K(n)^{odd}(BH) = 0$. Then $K(n)^{odd}(BG) = 0$ if and only if the integral Morava K-theory $\widetilde{K}(n)^*(BH)$ is a permutation module for the action of $G/H \cong C_p$.* □

For odd primes p, this condition is equivalent to saying that $K(n)^*(BG)$ is a permutation module, but for $p = 2$ this is trivially false (all $\mathbb{F}_2[C_2]$-modules are permutation modules).

At the other extreme, one could use extensions with a trivial action, such as central extensions. This has the advantage of not needing to know the Morava K-theory of the subgroup as a module for the quotient, which can be hard to determine, but the drawback is that the quotient is usually a large (p-)group, whose mod p cohomology can be quite challenging. So one has to find one's way between two evils, and a combination of both strategies will often be our chosen line of attack.

We conclude the section with a list of nonabelian p-groups whose Morava K-theory is known to be good.

Theorem 1.2 ([3, 4, 5, 11, 12, 13, 14, 15])**.** *If G belongs to any of the following families of p-groups, then $K(n)^{odd}(BG) = 0$:*

(a) *wreath products of the form $H \wr C_p$ with H good* [3, 4];
(b) *metacyclic p-groups* [12];
(c) *minimal non-abelian p-groups, i.e., groups all of whose maximal subgrous are abelian* [13];
(d) *groups of p-rank 2* [14];
(e) *elementary abelian by cyclic groups, i.e., extensions $V \to G \to C$ with V elementary abelian and C cyclic* [15, 5]. □

2. Technical lemmas

The first result describes the Morava K-theory of a central product of a 'good' group with an abelian group. Recall that a group G is called a central product of two subgroups P and Q if $P \cap Q$ contains a central subgroup Z of G such that G is isomorphic to the quotient of $P \times Q$ by the inclusion of Z via the diagonal. We denote the central product by $P \times_Z Q$ or simply by $P \circ Q$.

Theorem 2.1. *Let $G = H \circ C$ be a central product of H with a cyclic group $C \cong C_{p^m}$. If H has even Morava K-theory, then so does G.*

Proof. This is an easy argument with the Rothenberg-Steenrod spectral sequence; alternatively, it follows from Theorem 1.1 and induction. □

The theorem clearly generalises to groups where C is replaced by any finite abelian group.

Remark 2.2. It furthermore follows that the Serre spectral sequence for the extension $1 \to H \to G \to G/H \to 1$ is *simple*, by which we mean that the only differential is the one inherited from the Atiyah-Hirzebruch spectral sequence for G/H (there is indeed only one, since G/H is cyclic). On the other hand, the spectral sequence for the central extension

$$1 \longrightarrow C \longrightarrow G \longrightarrow G/C \longrightarrow 1$$

is not simple unless $G = H \times C$; there is not even reason to believe that G/H has even Morava K-theory.

The other technical result required is a lemma which, under favourable circumstances, describes the Serre spectral sequence associated to a group extension with a dihedral quotient; it is taken from [8].
Suppose G fits into an extension $1 \to N \to G \to D \to 1$, with D isomorphic to a dihedral 2-group. Assume further that the homomorphism $\psi\colon D \to \mathrm{Out}(N)$ defined by the extension is trivial and has a trivial (set theoretic) lift $\phi\colon D \to \mathrm{Aut}(N)$. In other words, every element of D should have a preimage in G which centralises N.

Lemma 2.3. *Let G be as above. Suppose that $K(n)^*(BN)$ is concentrated in even degrees, and that in the Serre spectral sequence*

$$E_2 = H^*(BD;\mathbb{F}_2) \otimes K(n)^*(BN) \Rightarrow K(n)^*(BG)$$

all elements in $E_4^{0,*}$ are permanent cycles. Then $K(n)^*(BG)$ is concentrated in even degrees.

Proof. We first prove the statement when D has order 4; an integral variant of this case can be found in [10]. Consider the inverse images H of any $C_2 \subset D_4$. Such H is either abelian or a central product, thus the associated Serre spectral sequence has only one differential $d_{2^{n+1}-1} = v_n Q_n$, see the remark above. This implies that the first potentially nontrivial differential has to be of the form $d_3 z = \pi \mod v_n$, where $\pi = x_1^2 x_2 + x_1 x_2^2 \in H^*(BD_4;\mathbb{F}_2) = \mathbb{F}_2[x_1, x_2]$. Thus we obtain an isomorphism

$$E_4 \cong K \otimes \mathbb{F}_2[x_1,x_2]/(\pi) \oplus H \otimes \mathbb{F}_2[x_1,x_2]\{\pi\}$$

where $K = \mathrm{Ker}(d_3|_{K(n)^*(BN)})$ and $H = H(K(n)^*(BN), \pi^{-1}d_3)$. By assumption on E_4, the next differential is $d_{2^{n+1}-1} = v_n Q_n$. For $M = H^*(BD_4;\mathbb{F}_2)$ the Q_n-homology $H(M;Q_n)$ is finite and even. Let $M' = M\{\pi\}$ and $M'' = M/(\pi)$.

The short exact sequence $0 \to M' \to M \to M'' \to 0$ gives rise to long exact sequences in Q_n-homology (one for each degree modulo $|v_n|$). The Q_n-homology of M'' is easily seen to be concentrated in even degrees at most 2^{n+1}. Since the map $H(M;Q_n) \to H(M';Q_n)$ is onto, all connecting homomorphisms are trivial, implying that $H(M'';Q_n)$ is even and finite, too. This finishes the proof for this case.

Now let $D = \langle s,t \mid s^{2^m} = t^2 = 1,\ tst = s^{-1}\rangle \cong D_{2^{m+1}}$ with $m > 1$. Let η_1, η_2 be the real representations defined by $\eta_1(s) = \eta_2(s) = -1$, $\eta_1(t) = 1$, $\eta_2(t) = -1$, and $\Delta_{\mathbb{R}}$ the natural representation in $O(2)$. Then

$$H^*(D;\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2,w_2]/(x_1x_2)$$

with x_1, x_2, w_2 the Euler classes of $\eta_1, \eta_1, \Delta_{\mathbb{R}}$, respectively, and $w_1 := w_1(\Delta_{\mathbb{R}}) = x_1 + x_2$. There are two conjugacy classes of maximal elementary abelian subgroups, represented (say) by K and T, both of rank two, and the maximal cyclic subgroup $C = \langle s\rangle$. The extension induced by the inclusion $C \subset D$ is a central product with simple Serre spectral sequence. Thus restricting to C and furthermore to K, T, applying the special case just proved, one sees that d_3 is either trivial, or has image $(w_1w_2) \mod v_n$. If d_3 is trivial, we are done by the assumption on E_4 and the fact that the Atiyah-Hirzebruch spectral sequence for the dihedral quotient produces a finite and even module, since D is a 'good' group. Otherwise, we have

$$E_4 \cong K \otimes M'' \oplus H \otimes M'$$

where K and H are defined as before as kernel and homology of d_3 and $(w_1w_2)^{-1}d_3$ on $K(n)^*(BN)$, respectively. Here $M' = M\{w_1w_2\}$ and $M'' = M/(w_1w_2)$, where $M = H^*(D;\mathbb{F}_2)$ (note that w_1w_2 is not a zero divisor in M). The next differential being $d_{2^{n+1}-1} = v_nQ_n$, we need to calculate the Q_n-homology of M' and M''. We claim

$$H(M;Q_n) = \mathbb{F}_2[x_1^2,x_2^2]/(x_1^2x_2^2, x_2^{2^{n+1}}, x_2^{2^{n+1}}) \otimes \mathbb{F}_2[w_2^2]/(w_2^{2^n}) \oplus \mathbb{F}_2[w_2^2]\{w_2^{2^n}, \zeta\}$$

with $\zeta = \sum_{r=0}^n x_1^{2^{n+1}-2^{r+2}+1} w_2^{2^r} \in M$;

$$\begin{aligned} H(M'';Q_n) &= \mathbb{F}_2[w_2]\{1, x_1w_2\} \oplus \mathbb{F}_2[x_1^2,x_2^2]/(x_1^2x_2^2, x_2^{2^{n+1}}, x_2^{2^{n+1}})\,, \\ H(M';Q_n) &= \mathbb{F}_2[w_2^2]\{x_1^{2^{n+1}},\ Q_n(x_1w_2),\ Q_nw_2\} \\ &\qquad \oplus \mathbb{F}_2\{x_1^{2i}w_2^{2k},\ x_2^{2j}w_2^{2l} \mid 1 \le i,j < 2^n\,,\ 0 \le k,l < 2^{n-1}\}\,. \end{aligned}$$

Assuming this calculation, one finds that the $E_{2^{n+1}}$-page

$$E_{2^{n+1}} \cong K \otimes H(M'';Q_n) \oplus H \otimes H(M';Q_n)$$

is still infinite, and there must be further differentials. The only way to arrive at a finite E_∞-page is when x_1w_2 and Q_nw_2 support differentials, and we obtain an E_∞-page concentrated in even degrees.

It remains to prove the claim on the Q_n-homologies. Recall $Q_n(x_i) = x_i^{2^{n+1}}$; using $Sq^1w_2 = w_1w_2$ one sees inductively $Q_nw_2 = \sum_{r=0}^n w_1^{2^{n+1}-2^{r+2}+1} w_2^{2^r}$. Since $x_1w_1 = x_1^2$, replacing w_1 in this sum with x_1 yields a cycle ζ for Q_n. Comparing coefficients one readily proves that the Q_n-cycles are given by

$$\mathbb{F}_2[x_1^2,x_2^2,w_2^2]/\big((x_1x_2)^2\big)\{1, Q_nw_2\} \oplus \mathbb{F}_2[w_2^2]\{\zeta\}$$

(note $x_1^2\zeta = x_1^2 Q_n w_2$ and $x_2^2\zeta = 0$). Clearly the image of Q_n in odd degrees is given by $\mathbb{F}_2[x_1^2, x_2^2, w_2^2]/\big((x_1x_2)^2\big)\{Q_n w_2\}$. Furthermore, all classes $x_i^{2k+2}w_2^{2m}$ with $k + 2m \geq 2^{n+1}$ are also in the image; the first claim follows.

For M'' the calculation is much simpler: modulo w_1w_2, one has

$$Q_n(w_2) = \sum_{r=0}^{n} w_1^{2^{n+1}-2^{r+1}+1} w_2^{2^r} = 0\,, \quad Q_n(x_1w_2) = 0\,,$$

giving

$$H(M''; Q_n) = \mathbb{F}_2[w_2]\{1, x_1w_2\} \oplus \mathbb{F}_2[x_1^2, x_2^2]/(x_1^2x_2^2, x_2^{2^{n+1}}, x_2^{2^{n+1}})\,.$$

For M', one could either do this directly, or, since we have calculated $H(M; Q_n)$, by means of the short exact sequence of Q_n-modules $0 \to M' \to M \to M'' \to 0$ and the associated long exact sequence(s)

$$\cdots \to H^s(M'; Q_n) \xrightarrow{\iota} H^s(M; Q_n) \xrightarrow{\kappa} H^s(M''; Q_n) \xrightarrow{\delta} H^{s+|Q_n|}(M'; Q_n) \to \cdots\,.$$

We need to determine δ and κ. The formulas for the action of Q_n give

$$\begin{aligned}
\delta(x_i^{2k}) &= \delta(w_2^{2l}) = 0\,,\\
\delta(w_2^{2k+1}) &= w_2^{2k}Q_nw_2 \quad \text{(note that } Q_nw_2 \text{ is not a boundary in } M'\text{)},\\
\delta(x_1w_2^{2k}) &= x_1^{2^{n+1}}w_2^{2k}\,, \quad \delta(x_1w_2^{2k+1}) = Q_n(x_1w_2)w_2^{2k} = \sum_{r=1}^{n} x_1^{2^{n+1}-2^{r+1}+2} w_2^{2^r+2k}\,.
\end{aligned}$$

Reduction modulo w_1w_2 gives

$$\mathrm{Ker}(\kappa) = \mathbb{F}_2\{x_1^{2i}w_2^{2k},\ x_2^{2j}w_2^{2l} \mid 1 \leq i,j < 2^n\,,\ 0 \leq k,l < 2^{n-1}\}\,.$$

Splitting the long exact homology sequences into short exact sequences thus yields an additive isomorphism

$$\begin{aligned}
H(M'; Q_n) \cong\ & \mathbb{F}_2[w_2^2]\{x_1^{2^{n+1}},\ Q_n(x_1w_2),\ Q_nw_2\}\\
& \oplus \mathbb{F}_2\{x_1^{2i}w_2^{2k},\ x_2^{2j}w_2^{2l} \mid 1 \leq i,j < 2^n\,,\ 0 \leq k,l < 2^{n-1}\}\,.
\end{aligned}$$

□

Corollary 2.4. *Suppose in addition that all elements in $E_4^{0,*}$ are restrictions of good elements of $K(n)^*(BG)$. Then $K(n)^*(BG)$ is good.*

Proof. This is a consequence of the following facts: (i) the x_i^2 are clearly represented by Euler classes of one-dimensional complex representations of G; (ii) there is an extension problem identifying w_2 as a polynomial in elements of $E_4^{0,*}$: this can be seen either by restriction to subgroups, or by appealing to the extension class in (ordinary) cohomology. □

Finally, we need one more fact from the wreath product calculation.

Lemma 2.5 ([3]). *Let $G = H \wr C_2$ where H has even Morava K-theory. The Serre spectral sequence of the extension*

$$1 \longrightarrow H \times H \longrightarrow G \longrightarrow C_2 \longrightarrow 1 \tag{2.1}$$

has only one differential $d_{2^{n+1}-1}$ (i.e., is simple). More precisely, if $K(n)^*(BH \times H) \cong K(n)^*(BH) \otimes_{K(n)^*} K(n)^*(BH) = F \oplus T$ is the decomposition into free and *trivial summands, then F^{C_2} and T are in the image of restriction from G and thus consist of permanent cycles.* □

3. Groups of smaller order

The later sections require a few results about smaller groups, in particular concerning the behaviour of certain spectral sequences. The first observation is that all groups of order at most 16 have even Morava K-theory; this follows from the results quoted in the previous section: the only nonabelian groups of order 8 are D_8 and Q_8, and there are 11 nonabelian groups of order 16, namely

(a) $D_8 \times C_2$, $Q_8 \times C_2$,
(b) D_{16}, Q_{16}, the semidihedral group SD_{16}, the quasidihedral group QD_{16}, which are metacyclic,
(c) the central product $C_4 \circ D_8$, also known as almost extraspecial group,
(d) $H_1 = \langle g_1, g_2, g_3 \mid g_1^4 = g_2^2 = g_3^2 = [g_1, g_2] = [g_2, g_3] = 1, g_3 g_1 g_3 = g_1 g_2 \rangle \cong (C_4 \times C_2) \rtimes C_2$,
(e) $H_2 = \langle g_1, g_2 \mid g_1^4 = g_2^4 = 1, g_2^{-1} g_1 g_2 = g_1^{-1} \rangle \cong C_4 \rtimes C_4$.

The groups H_1, H_2 (numbers 9 and 10 in the Hall-Senior list) are minimal nonabelian 2-groups. Although we already know they are both 'good', we shall later use specific calculations for their Morava K-theoies.

H_1 (also known as $16\Gamma_2 c$) has a rank 3 elementary abelian subgroup $E = \langle g_1^2, g_2, g_3 \rangle$, and we consider the corresponding extension

$$1 \longrightarrow E \longrightarrow H_1 \longrightarrow C_2 \longrightarrow 1 \tag{3.1}$$

with associated Serre spectral sequence $E_2 = H^*\big(BC_2; K(n)^*(BE)\big)$.

Lemma 3.1. *The Serre spectral sequence for the extension* (3.1) *is simple.*

Proof. This is very similar to the wreath product calculation 2.5. Let η_1, η_2, η_3 be the linear characters of E with $\eta_1(g_1^2) = \eta_2(g_2) = \eta_3(g_3) = -1$ and $\eta_i = 1$ on the other generators. Then $\eta_i^{g_1} = \eta_i$ for $i = 1, 3$ and $\eta_2^{g_1} = \eta_1\eta_2$. Furthermore, η_3 extends to a character γ of $H = H_1$. Set $y_i = c_1(\eta_i)$, then g_1 acts trivially on y_1, y_3, whereas $g_1^*(y_2) = y_1 +_{K(n)} y_2$. Thus $K(n)^*(BE) \cong M \otimes K(n)^*[y_3]/(y_3^{2^n})$ where M is the module for the switch action as in the wreath product calculation. Clearly $y_3 = \operatorname{Res}^H_E c_1(\gamma)$ is in the image of restriction from H, but so are $(1 + g_1^*)y_2^k = \operatorname{Res}^H_E \operatorname{Tr}^H_E(y_2^k)$ and $\big(y_2 \cdot g_1^*(y_2)\big)^k = \operatorname{Res}^H_E\big(c_2(\operatorname{Ind}^H_E \eta_2)^k\big)$. Thus all invariants are permanent cycles, as they are in $\operatorname{Im}(\operatorname{Res}^H_E)$. □

H_2 has the index two subgroup $B = \langle g_1^2, g_2 \rangle \cong C_2 \times C_4$ with coresponding extension

$$1 \longrightarrow B \longrightarrow H_2 \longrightarrow C_2 \longrightarrow 1 \tag{3.2}$$

and associated Serre spectral sequence $E_2 = H^*\big(BC_2; K(n)^*(BE)\big)$.

Lemma 3.2. *The Serre spectral sequence for the extension* (3.2) *is simple.*

Proof. Instead of using the switch action of C_2 on $C_2 \times C_2$, one can use the action of C_2 on C_4 by inversion to calculate $K(n)^*(BD_8)$; the invariants of the C_2-action on $K(n)^*(BC_4) =: M$ are again restrictions of Chern classes. If y denotes the Euler class of the representation η of B with $\eta(g_1^2) = -1$ and $\eta(g_2) = 1$, then

$E_2 \cong M \otimes K(n)^*[y]/(y^{2^n})$. Since η extends to a complex character of H_2, the class y is in the image of restriction, whence all invariants are. □

4. Groups of order 32 - the easy part

There are 51 groups of order 32, which we shall denote as $G_1 - G_{51}$, the index referring to the Hall-Senior number of the group.

4.1. **Groups 1-15.**

The first 7 groups are abelian, and the next 8 have an abelian factor. Thus their Morava K-theories are generated by Euler classes, and in particular concentrated in even degrees,

4.2. **Groups 16-22.**

$G_{16} - G_{22}$ have a central quotient isomorphic to $C_2 \times C_2$. Thus the technical Lemma 2.3 gives the result, provided one can check the condition on the E_4-page of the Serre spectral sequence. One only has to do this for G_{16} and G_{17}, since G_{18} and G_{20} are minimal non-abelian, whereas G_{19}, G_{21} and $G_{20} \cong QD_{16}$ are split metacyclic.

G_{16} is a semidirect product $(C_4 \times C_4) \rtimes C_2$ with presentation

$$G_{16} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = [b, c] = 1,\ cac = ab^2 \rangle\,;$$

the centre Z ist $\langle a^2, b\rangle$. Consider the Serre spectral sequence of the central extension $1 \to Z \to G_{16} \to V \to 1$ with $E_2 \cong K(n)^*(BZ) \otimes H^*(V; \mathbb{F}_2)$. Then $K(n)^*(BZ)$ is generated by the Euler classes of the two linear characters η, λ defined by $\eta(a^2) = -1$, $\eta(b) = 1$, and $\lambda(a^2) = 1$, $\lambda(b) = i$, respectively. η extends to a representation $\tilde{\eta}$ of G_{16} by setting $\tilde{\eta}(a) = i$ and $\tilde{\eta}(c) = 1$; therefore the Euler class of η is in the image of restriction and thus a permanent cycle. On the other hand, $z := e(\lambda)$ is not a permanent cycle, but it suffices to see that z^2 is. Let $\sigma = \mathrm{Ind}_{\langle a,b\rangle}^{G_{16}}(\mu)$ where $\mu(a) = 1$ and $\mu(b) = i$. Then $\mathrm{Res}_Z^{G_{16}}(\sigma) = 2\lambda$, whence $\mathrm{Res}_Z^g(e(\sigma)) = z^2$.

Secondly,

$$G_{17} = \langle a, b, c \mid a^8 = c^2 = [a, c] = [a, b] = 1,\ b^2 = a^2,\ cbc = a^4 b\rangle$$

with centre $Z = \langle a \rangle$ cyclic of order 8 and central quotient $V \cong C_2 \times C_2$. Consider as above the Serre spectral sequence of the central extension; this time $K(n)^*(BZ)$ is generated by the Euler class of a generator ρ of RZ. Clearly ρ extends to a representation $\tilde{\rho}$ of $\langle a, c\rangle$. Inducing $\tilde{\rho}$ up to G_{17} and restricting to the centre yields 2ρ, implying that $e(\rho)^2$ is a permanent cycle.

4.3. **Groups 23-33.**

The groups $G_{23} - G_{33}$ all have centre of order 4 with quotient D_8 — again a case for Lemma 2.3 and henceforth an easy exercise in representation theory. They also share the same Euler characteristic

$$\chi_{n,2}(G) = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n\,.$$

In fact, one has $G_{23} = C_2 \times D_{16}$, $G_{24} = C_2 \times SD_{16}$, $G_{25} = C_2 \times Q_{16}$, $G_{26} = C_4 \circ D_{16}$, $G_{31} = C_4 \wr C_2$, $G_{33} = (C_2 \times C_2) \wr C_2$, and

$$G_{27} = \langle a, b, c \mid a^8 = b^2 = c^2 = [a, b] = [b, c] = 1,\ cac = a^{-1}b\rangle$$
$$G_{28} = \langle a, b, c \mid a^8 = b^2 = [a, b] = [b, c] = 1,\ c^2 = a^4,\ c^{-1}ac = a^3b\rangle$$
$$G_{29} = \langle a, b \mid a^8 = b^4 = 1,\ b^{-1}ab = a^{-1}\rangle$$
$$G_{30} = \langle a, b \mid a^8 = b^4 = 1,\ b^{-1}ab = a^3\rangle$$
$$G_{32} = \langle a, b, c \mid a^8 = 1,\ b = c^2,\ c^4 = a^4,\ [a, b] = [b, c] = 1,\ c^{-1}ac = a^3\rangle$$

Of these latter groups, G_{29} and G_{30} are split metacyclic and G_{32} is nonsplit metacyclic, so that leaves G_{27} and G_{28}. For both groups, a and b generate a maximal abelian subgroup $A \cong C_8 \times C_2$. As prescribed by Lemma 2.3, we consider the Serre spectral sequence of the central extension

$$1 \longrightarrow \langle a^4, b\rangle \longrightarrow G \longrightarrow \langle \bar{a}, c\rangle \longrightarrow 1$$

with

$$\begin{aligned} E_2 &= H^*(D_8; K(n)^*(BZ)) \\ &\cong \mathbb{F}_2[x_1, w_1, w_2]/(x_1^2 + x_1w_1) \otimes K(n)^*[z_1, z_2]/(z_1^{2^n}, z_2^{2^n})\,. \end{aligned}$$

Here z_1 and z_2 are the Euler classes of λ_1, λ_2 corresponding to a^4 and b, respectively, while we keep the notation for $H^*(D_8; \mathbb{F}_2)$ from Section 2.

Since $[G, G] = \langle a^2b\rangle \cong C_4$, we have a one-dimensional representation β of G with $\beta(b) = -1$ (and $\beta(a) = \beta(c) = 1$); this restricts to λ_2 on the centre. Thus z_2 is a permanent cycle. Now let $A = \langle a, b\rangle \cong C_8 \times C_2$ as above, and define $\rho \in RA$ by $\rho(a) = \exp(\pi i/4)$ and $\rho(b) = 1$. Then ρ^c is either ρ^{-1} (for G_{27}) or ρ^3 (for G_{28}); in any case,

$$\mathrm{Res}^G_Z \mathrm{Ind}^G_A(\rho) = \mathrm{Res}^A_Z(\rho + \rho^c) = 2\lambda_1\,,$$

so z_1^2 is a permanent cycle, too.

4.4. **Groups 34-37.** Presentations of $G_{34} - G_{37}$ are as follows:

$$G_{34} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = 1,\ cac = a^{-1},\ cbc = b^{-1}\rangle$$
$$G_{35} = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2, cac = a^{-1},\ cbc = b^{-1}\rangle$$
$$G_{36} = \langle a, b, c \mid a^4 = b^4 = c^2 = [b, c] = 1,\ a^{-1}ba = b^{-1},\ cac = a^{-1}\rangle$$
$$G_{37} = \langle a, b, c \mid a^4 = c^2 = d^2 = [b, c] = 1,\ d = [a, c],\ b^2 = a^2,\ bab^{-1} = a^{-1}\rangle$$

All four groups have centre $Z \cong C_2 \times C_2$ with quotient C_2^3, and Euler characteristic

$$\chi_{n,2} = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n\,.$$

G_{34} and G_{35} have the maximal abelian subgroup $A = \langle a, b\rangle \cong C_4 \times C_4$, on which the quotient acts (diagonally) by inverting a and b. Since D_8 could be written as a semidirect product $C_4 \rtimes C_2$ with that action, Theorem 1.1 tells us that $M := \widetilde{K}(n)^*(BC_4)$ is a permutation module for the automorphism inverting the generator of the group, thus $\widetilde{K}(n)^*(BA) \cong M \otimes M$ is a permutation module, too. Theorem 1.1 again thus implies that G_{34} and G_{35} are both good.

G_{36} contains the maximal abelian aubgroup $A = \langle b, a^2, c\rangle \cong C_4 \times C_2 \times C_2$. From the relations one reads off that $\widetilde{K}(n)^*(BA) \cong M \otimes N$, where $N = \widetilde{K}(n)^*(BC_2 \times C_2)$

with the switch action, so this is again a permutation module; the situation is similar for G_{37} and the maximal abelian subgroup $A = \langle b, c, d\rangle$.

4.5. **Groups 38-41.**

$$G_{38} = \langle a, b, c \mid a^4 = b^2 = c^4 = [a, b] = 1,\ cac^{-1} = ac^2,\ cbc^{-1} = a^2 b\rangle$$
$$G_{39} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = 1,\ cac = a^3,\ cbc = a^2 b^3\rangle$$
$$G_{40} = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = b^2,\ [a, b] = 1,\ c^{-1}ac = a^3,\ c^{-1}bc = a^2 b^3\rangle$$
$$G_{41} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = 1,\ cac = a^3 b^2,\ cbc = a^2 b\rangle$$

All these groups have centre $C_2 \times C_2$ with quotient C_2^3, a unique index 2 abelian subgroup, and 14 conjugacy classes of elements. This suffices to conclude that they all have the same Euler characteristic

$$\chi_{n,2} = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n\,.$$

The calculation of their Morava K-theories appears to require new arguments and is thus deferred.

4.6. **Groups 42 and 43.** G_{42} and G_{43} are extraspecial and were dealt with in [10], using a variant of Lemma 2.3.

4.7. **Groups 44-48.**

$$G_{44} = \langle a, b, c \mid a^8 = b^2 = c^2 = [b, c] = 1,\ bab = a^{-1},\ cac = a^5\rangle$$
$$G_{45} = \langle a, b, c \mid a^8 = c^2 = 1,\ b^2 = a^4, [b, c] = 1,\ b^{-1}ab = a^{-1},\ cac = a^5\rangle$$
$$G_{46} = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, c]^2 = 1,\ [a, [a, c]] = [b, c] = 1,\ bab = ac\rangle$$
$$G_{47} = \langle a, b, c \mid a^8 = b^2 = c^2 = [b, c] = 1,\ bab = ac,\ cac = a^5\rangle$$
$$G_{48} = \langle a, b, c \mid a^8 = c^2 = 1,\ b^2 = a^4,\ [b, c] = 1,\ b^{-1}ab = ac,\ cac = a^5\rangle$$

For each group, the centre Z is C_2 with quotient either $D_8 \times C_2$ (for G_{44} and G_{45}) or $16\Gamma_2 c$, the group we called H_1 in Section 3, in the other cases. All five groups have Euler characteristic

$$\chi_{n,2} = \frac{7}{4}8^n - \frac{3}{4}4^n\,.$$

G_{44}, G_{46} and G_{47} have a normal rank 3 elementary abelian subgroup with cyclic quotient, so they are covered by Theorem 1.2(e). Furthermore, G_{44} can be written as $\langle a, c\rangle \rtimes \langle b\rangle \cong QD_{16} \rtimes C_2$, and G_{45} is a non-split version of that group, i.e., fits into an extension $1 \to QD_{16} \to G_{45} \to C_2 \to 1$ with the same action as for G_{44}. Similarly, G_{48} is a non-split version of $G_{47} = \langle a, c\rangle \rtimes \langle b\rangle \cong QD_{16} \rtimes C_2$. Thus Theorem 1.1 implies that G_{45} and G_{48} are good, too.

4.8. **Groups 49-51.** G_{49} is dihedral, G_{50} semidihedral, and G_{51} a generalised quaternion group, hence already covered in the literature, e.g. by [14].

We have seen that the theory described in the previous sections covers 47 groups, leaving four to be calculated. Closer inspection also shows that the Morava K-theory of those 47 groups is equidistributed (by which we mean $\mathrm{rank}_{K(n)^*} K(n)^{2i}(BG) = \mathrm{rank}_{K(n)^*} K(n)^0(BG) - 1$ for any $i \not\equiv 0 \mod |v_n|$); we omit the details.

5. The remaining cases

5.1. **Group 38.** The centre of G_{38} is $Z = \langle a^2, c^2\rangle \cong C_2 \times C_2$. The representation ring has three generators α, β, γ of dimension one, inflated from G/Z, with $\alpha(a) = -1$, $\alpha(b) = \alpha(c) = 1$, $\beta(a) = \beta(c) = 1$, $\beta(b) = -1$, and $\gamma(a) = \gamma(b) = 1$, $\gamma(c) = -1$. Let $A = \langle a, b, c^2\rangle \cong C_4 \times C_2 \times C_2$ and $\lambda, \nu \in RA$ be defined by $\lambda(a) = i$, $\lambda(b) = \lambda(c^2) = 1$ and $\nu(a) = \nu(b) = 1$, $\nu(c^2) = -1$. The irreducible representations of G_{38} are $\alpha^r\beta^s\gamma^t$, $r, s, t \in \{0, 1\}$, and $\sigma = \mathrm{Ind}_A^G(\lambda)$, $\alpha\sigma$, $\tau = \mathrm{Ind}_A^G(\nu)$, $\beta\tau$, $\upsilon = \mathrm{Ind}_A^G(\lambda\nu)$, $\alpha\upsilon$.

A is a maximal abelian subgroup, but we found neither the central extension $Z \to G \to G/Z$ nor $A \to G \to G/A$ suitable for calculation: in the spectral sequence for the central extension, the first differential d_3 produces zero divisors, and for the second, we found no easy way to see that all invariants for the action of the quotient on $K(n)^*(BA)$ are permanent cycles, as the rank of $K(n)^*(BG_{38})$ would suggest. Thus we consider the normal subgroup $E = \langle b\rangle \times Z$ with quotient $V \cong C_2 \times C_2$ and consider the Serre spectral sequence of the extension

$$1 \longrightarrow E \longrightarrow G_{38} \longrightarrow V \longrightarrow 1 \tag{5.1}$$

Define nontrivial characters $\xi, \eta, \zeta \in RE$ by the quotients $E/\langle a^2, b\rangle$, $E/\langle a^2, c^2\rangle$, and $E/\langle b, c^2\rangle$, respectively. Then we obtain the following restrictions:

G_{38}	α	β	γ	σ	τ	υ
E	1	η	1	$(1+\eta)\zeta$	2ξ	$(1+\eta)\xi\zeta$

Let $x, y, z \in K(n)^*(BE)$ be the Euler classes of ξ, η, ζ, respectively. Then $K(n)^*(BE) \cong K(n)^*[x, y, z]/(x^{2^n}, y^{2^n}, z^{2^n})$. Conjugation by a is trivial on RE, and $\eta^c = \eta$, $\xi^c = \xi$, but $\zeta^c = \eta\zeta$. This gives the following action of V on $K(n)^*(BE)$:

$$a^*x = x\,,\quad a^*y = y\,,\quad a^*z = z\,,\quad c^*x = x\,,\quad c^*y = y\,,\quad c^*z = y +_{K(n)} z\,.$$

From the above table we furthermore see $x^2 = \mathrm{Res}_E^G\, c_2(\sigma)$.

The Serre spectral sequence has E_2-page

$$E_2 = H^*(BV; K(n)^*(BE)\big)$$

Write $M := K(n)^*(BE)$, then $M = M'[x]/(x^{2^{n+1}})$, where we know M' from the wreath product calculation 2.5. More precisely, M decomposes as $P \oplus T$, where P is a direct sum of $\frac{1}{2}(8^n - 4^n)$ copies of $K(n)^*[V/\langle\bar{a}\rangle]$ and T a sum of 4^n trivial summands $K(n)^*[V/V]$.

Lemma 5.1. (a) *$P^{\langle\bar{c}\rangle}$ consists of permanent cycles, and $P^{\langle\bar{c}\rangle} = F^{\langle\bar{c}\rangle}[x]/(x^{2^n})$ where F is the free part of M';*
(b) *$T = T'[x]/(x^{2^n})$, and $d_3 x = x_1^2x_2 + x_1x_2^2$ where $H^*(BV; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2]$.*

Proof. (a) follows from the fact that $\mathrm{Res}_E^A\colon K(n)^*(BA) \to K(n)^*(BE)$ is onto and the double coset formula, which gives $\mathrm{Res}_E^G \mathrm{Tr}_A^G = \mathrm{Res}_E^A(1 + c^*)$.

For (b), note that $z(y +_{K(n)} z) = \mathrm{Res}_E^G\, c_2(\sigma)$, which implies that $T' \subset \mathrm{Im}(\mathrm{Res}_E^G)$. On the other hand, x cannot be a permanent cycle for size reasons, and looking at the extensions induced by the three inclusions $C_2 \subset V$ tells us that $d_3 x$ has to restrict to zero on all of them: the corresponding subgroups of G_{38} are $C_4 \times C_2 \times C_2$, H_1, and $D_8 \times C_2$, with simple spectral sequences according to 2.5 and 3.1. □

The lemma implies

$$E_4 = P^{\langle\overline{c}\rangle} \otimes H^*(B\langle\overline{a}\rangle; K(n)^*) \oplus T'[x^2]/(x^{2^n}) \otimes \mathbb{F}_2[x_1,x_2]/(x_1^2x_2 + x_1x_2^2)\,.$$

Since x^2 is a permanent cycle, the next differential is Q_n, making $E_{2^{n+1}}$ even and finite (we know the Q_n-homology of $\mathbb{F}_2[x_1,x_2]/(x_1^2x_2+x_1x_2^2)$ from the first part of the proof of Lemma 2.3). This finishes the calculation for this group. A closer look at the distribution of generators shows that $K(n)^*(BG_{38})$ is indeed equidistributed.

5.2. **Group 39.** Let $A = \langle a,b\rangle \subset G_{39}$, then G_{39} is the semidirect product $A \rtimes \langle c\rangle \cong (C_4\times C_4)\rtimes C_2$. The first idea would be to use the extension $A \to G \to C_2$ and show that either all invariants $K(n)^*(BA)^{\langle c\rangle}$ are permanent cycles, or that $\widetilde{K}(n)^*(BA)$ contains no summand -1. Instead, we turn to the central extension

$$1 \longrightarrow C \longrightarrow G_{39} \longrightarrow \langle\overline{a}\rangle \times Q \longrightarrow 1 \qquad (5.2)$$

with $C = \langle a^2\rangle \cong C_2$ and $Q = \langle\overline{b},\overline{c}\rangle \cong C_2 \times D_8$. Then

$$\begin{aligned} E_2 &= K(n)^*(BC) \otimes H^*(BQ;\mathbb{F}_2) \\ &\cong K(n)^*[z]/(z^{2^n}) \otimes \mathbb{F}_2[x_1,w_1,w_2]/(x_1^2+x_1w_1)\otimes \mathbb{F}_2[y] \end{aligned}$$

with x_1, w_i as before and $y \in H^1(B\langle\overline{a}\rangle;\mathbb{F}_2)$.

Lemma 5.2. *$d_3z = w_1^2y + w_1y^2$ mod v_n.*

Proof. Consider subgroups $Q_j < Q$ ($j = 1,2,3$) and the corresponding induced extensions $C \to \widetilde{Q}_i \to Q_i$ in the following cases:

$$\begin{aligned} Q_1 &= \langle\overline{a},\overline{b}\rangle \cong C_2\times C_4\,, & \widetilde{Q}_1 &= A \cong C_4\times C_4\,,\\ Q_2 &= \langle\overline{a},\overline{b}^2,\overline{c}\rangle \cong C_2\times C_2\times C_2\,, & \widetilde{Q}_2 &= \langle a,c\rangle\times\langle b^2\rangle \cong D_8\times C_2\,,\\ Q_3 &= \langle\overline{a},\overline{bc},\overline{b}^2\rangle \cong C_2\times C_2\times C_2\,, & \widetilde{Q}_3 &= \langle a,bc\rangle\times\langle b^2\rangle \cong Q_8\times C_2\,. \end{aligned}$$

Then one has, with y as above,

$$\begin{aligned} &H^*(Q_1;\mathbb{F}_2) \cong \mathbb{F}_2[y]\otimes\Lambda(u)\otimes\mathbb{F}_2[v] \text{ with } \langle u,\overline{a}\rangle = 1,\ v = \beta_2(u),\\ &H^*(Q_2;\mathbb{F}_2) \cong \mathbb{F}_2[y,t_1,t_2] \text{ with } t_1 \text{ dual to } \overline{b}^2 \text{ and } t_2 \text{ dual to } \overline{c},\\ &H^*(Q_3;\mathbb{F}_2) \cong \mathbb{F}_2[y,t_1,t_3] \text{ with } t_1 \text{ dual to } \overline{b}^2 \text{ and } t_3 \text{ dual to } \overline{bc}. \end{aligned}$$

For these extensions, the following table lists restrictions and d_3z, which we know (or can easily deduce) from previous calculations.

Q	x_1	y	w_1	w_2	d_3z mod v_n
Q_1	u	y	0	v	0
Q_2	0	y	t_2	$t_1t_2+t_2^2$	$y^2t_2+yt_2^2$
Q_3	t_3	y	t_3	$t_1^2+t_1t_3$	$t_3^2y+t_3y^2$

The claim follows. ☐

Now let $b = w_1^2y + w_1y^2$, then $d_3z = b(1+U)$ for a unit U, since z is nilpotent, and

$$E_4 \cong K(n)^*[z^2]/(z^{2^n}) \otimes H^*(Q;\mathbb{F}_2)/(b)\,;$$

note that b is not a zero divisor. Comparison to the induced extensions again shows that the next differential is $d_{2^{n+1}-1} = v_n Q_n$, so we have to determine the Q_n-homology of $H^*(Q;\mathbb{F}_2)/(b)$. Note that $x_1^2y^2 = x_1w_1y^2 = x_1w_1^2y = x_1^3y$ etc. There is a splitting

$$H^*(Q;\mathbb{F}_2)/(b) = M \oplus N_1 \oplus N_2$$

with

$$\begin{aligned} M &= H^*(Q;\mathbb{F}_2)/(b,y) = \mathbb{F}_2[x_1,w_1,w_2]/(x_1^2+x_1w_1) \\ N_1 &= \mathbb{F}_2[w_2,y]\{y\} \oplus \mathbb{F}_2[w_2,y]\{w_1y\} \\ N_2 &= \mathbb{F}_2[w_2,y]\{x_1y\} \oplus \mathbb{F}_2[w_2,x]\{x_1^2y\} \end{aligned}$$

and this splitting is respected by the action of Q_n.

Lemma 5.3. (a) *Let* $\zeta = \sum_{j=0}^n x_1^{2^{n+1}-2^{j+1}+1} w_2^{2^j}$. *Then*

$$H(M;Q_n) = \mathbb{F}_2[w_2^2]\{1,\zeta\} \oplus \Big(\mathbb{F}_2\{x_1^{2i}, w_1^{2j} \mid 1 \le i,j < 2^n\} \otimes \mathbb{F}_2[w_2^2]/(w_2^{2^n})\Big).$$

(b) *Let* $\vartheta_1 = (y^2+w_1y)w_2$. *Then*

$$\begin{aligned} H(N_1;Q_n) = {} & \mathbb{F}_2[w_2^2]\{y^{2i}, y^{2j}\vartheta_1 \mid 1 \le i < 2^n, 0 \le j < 2^n\} \\ & \oplus \mathbb{F}_2[w_2^2]/(w_2^{2^n}) \otimes \mathbb{F}_2\{w_1y^{2j+1} \mid 0 \le j < 2^n\}. \end{aligned}$$

(c) *Let* $\vartheta_2 = x_1^2y + x_1y^2$ *and* $\vartheta_3 = \vartheta_2 w_2$. *Then*

$$\begin{aligned} H(N_2;Q_n) = {} & \mathbb{F}_2[w_2^2]\{y^{2i}\vartheta_2, y^{2j}\vartheta_3 \mid 0 \le i < 2^n-1, 0 \le j < 2^n\} \\ & \oplus \mathbb{F}_2[w_2]/(w_2^{2^n})\{x_1^{2k+1}y \mid 1 \le k < 2^n\}. \end{aligned}$$

Proof. The Q_n-homology of M we know from the proof of Lemma 2.3 (it is the Q_n-homology of $H^*(D_8;\mathbb{F}_2)$). For (b), we first show that $H^{\mathrm{odd}}(N_1;Q_n)$ vanishes. Let

$$X = \sum_{i=0}^{d} \lambda_i y^{2d-2i+1} w_2^i + \sum_{i=0}^{d-1} \mu_i y^{2d-2i} w_1 w_2^i$$

be a class in degree $2d+1$. Then $Q_nX = 0$ implies

$$\begin{aligned} 0 = {} & \sum_{i=0}^{d} \lambda_i y^{2^{n+1}+2d-2i} w_2^i + \lambda_d y(Q_nw_2) + \sum_{i=0}^{d-1} \mu_i y^{2d-2i} w_1^{2^{n+1}} w_2^i \\ & + \sum_{i=0}^{d-1} \Big(\big(i\lambda_i y^{2d-2i+1} + i\mu_i w_1 y^{2d-2i}\big) \sum_{j=0}^{n} w_1^{2^{n+1}-2^{j+1}+1} w_2^{2^j+i-1} \Big). \end{aligned}$$

The coefficients of $y^l w_2^k$ tell us that all λ_i must be zero, since the rest of the summands is divisible by w_1. So we only have to consider $\widetilde{N}_1 = w_1N_1$. Let $\mathbb{F}_2[u,v]$ be the graded Q_n-module with u,v in degree 1 and $Q_nu = u^{2^{n+1}}$, $Q_nv = v^{2^{n+1}}$. Then $\varphi(y) := u$, $\varphi(w_1) := u+v$ and $\varphi(w_2) := uv$ defines a monomorphism $\varphi\colon \widetilde{N}_1 \to \mathbb{F}_2[u,v]$ of Q_n-modules. Since $H^{\mathrm{odd}}(\mathbb{F}_2[u,v];Q_n) = 0$, one also has $H^{2d+1}(\widetilde{N}_1;Q_n) = 0$ for $2d+1 \le 2^{n+1}+1$. For larger d, comparing coefficients of w_2^m leads to a system of equations of the form

$$\mu_{2k} + \mu_{2k-1} + \cdots + \mu_{2k-2^r+1} + \cdots + \mu_{2k-2^{n+1}+1} = 0$$

giving $Q_nX = 0$ if and only if $X \in \mathrm{Im}(Q_n)$. One furthermore computes that the even degree cycles are

$$Z^{\mathrm{ev}} = \mathbb{F}_2[w_2^2, y^2]\{w_1 y, \vartheta_1\}\,.$$

Now $\mathbb{F}_2[w_2^2, y^2]\{y^{2^{n+1}}, y^{2^{n+1}}w_1\}$ clearly lies in the image of Q_n, as well as $y^{2^{n+1}}\vartheta_1 = Q_n(y\vartheta_1)$. Finally, $yw_1w_2^{2^n} = \sum_{k=1}^{n-1} y^{2^{n+1}-2^{k+1}+1}w_1w_2^{2^k} + y^{2^{n=1}-2}\vartheta_1 + Q_n(yw_2)$; the claim follows.

The proof of (c) is a similar routine but tedious verification. □

Thus $E_{2^{n+1}} \cong K(n)^*[z^2]/(z^{2^n}) \otimes (H_{\mathrm{inf}} \oplus H_{\mathrm{fin}})$ where

$$H_{\mathrm{inf}} = \mathbb{F}_2[w_2^2]\{y^{2i}, y^{2j}\vartheta_1, y^{2k}\vartheta_2, y^{2l}\vartheta_3, \zeta \mid 0 \le i, j < 2^n, 0 \le k < 2^n - 1, 0 \le l < 2^n\}$$

and

$$H_{\mathrm{fin}} = \mathbb{F}_2[w_2^2]/(w_2^{2^n})\{x_1^{2i}, w_1^{2j}, y^{2k+1}w_1, x_1^{2l+1}y \mid 1 \le i, j, l < 2^n, 0 \le k < 2^n\}\,.$$

denote the infinite and the finite part of the Q_n-homology given by the lemma. All generators in the finite part are squares (recall $x_1^3 y = x_1^2 y^2$) and are represented by Chern classes of complex representations, namely the complexifications of the real representations used to define x_1, w_1, w_2 and y. Furthermore, H_{fin} sits in degrees less than 2^{n+2}, whence no further differential can hit this summand.

For the summand H_{inf}, notice first that w_2^2 is represented by a Chern class and thus a permanent cycle. As modules over $\mathbb{F}_2[w_2^2]$, the even and the odd degree part have the same rank. Since E_∞ has to be finite and ϑ_1 is also a permanent cycle, all odd degree generators have to support a nontrivial differential and we are done.

5.3. **Group 40.** This group is a non-split version of G_{39}, i.e., G_{40} admits a non-split extension $A \to G_{40} \to C_2$ with $A = \langle a, b\rangle \cong C_4 \times C_4$ and the same action of the quotient C_2 as for $G_{39} = A \rtimes C_2$. Thus G_{40} has even Morava K-theory iff G_{39} does.

5.4. **Group 41.** This calculation is similar to the one for G_{38}, so we shall offer less detail. The centre of G_{41} is $Z = \langle a^2, b^2\rangle \cong C_2 \times C_2$, and $A = \langle a, b\rangle \cong C_4 \times C_4$ is an index 2 abelian subgroup. There are 14 irreducible complex representations, $\alpha^r\beta^s\gamma^t$ $(r, s, t = 0, 1)$ of dimension 1 and $\sigma, \alpha\sigma, \tau, \beta\tau, \upsilon, \alpha\upsilon$ of dimension 2, defined as follows. α, β, γ factor through $G/Z \cong C_2^3$, with $\alpha(a) = -1$, $\beta(b) = -1$, $\gamma(c) = -1$, and $\alpha(b) = \alpha(c) = \beta(a) = \beta(c) = \gamma(a) = \gamma(b) = 1$. Let $\lambda, \nu\colon A \to \mathbb{C}^*$ be given by $\lambda(a) = \nu(b) = i$ and $\lambda(b) = \nu(a) = 1$, then $\sigma = \mathrm{Ind}_A^G(\lambda)$, $\tau = \mathrm{Ind}_A^G(\nu)$, and $\upsilon = \mathrm{Ind}_A^G(\lambda\nu)$.

As before, we found it hard to show that the invariants $K(n)^*(BA)^{\langle c\rangle}$ are all in the image of restriction, and also the spectral sequence for the central extension posed problems with zero divisors. Thus let B be the normal subgroup $\langle a^2, b\rangle \cong C_2 \times C_4$ and $V = G/B = \langle \overline{a}, c\rangle \cong C_2 \times C_2$, and consider the extension

$$1 \longrightarrow B \longrightarrow G_{41} \longrightarrow V \longrightarrow 1\,. \tag{5.3}$$

Let ζ, ξ be defined by $\zeta(a^2) = -1$, $\xi(b) = i$, and $\zeta(b) = \xi(a^2) = 1$, then ζ and ξ generate RB. Furthermore, $\xi^c = \xi$, $\zeta^c = \zeta\xi^2$, and $\overline{a}$ acts trivially on RB. The restriction homomorphism $\mathrm{Res}_B^G\colon RG \to RB$ is given by

G_{41}	α	β	γ	σ	τ	υ
B	1	ξ^2	1	$(1+\xi^2)\zeta$	2ξ	$(1+\xi^2)\zeta\xi$

Notice that if we artificially introduce $\eta := \xi^2$, then this table, as well as the action of c, is identical to the case of G_{38}.

The Serre spectral sequence for (5.3) has

$$E_2 = H^*\big(BV; K(n)^*(BB)\big)$$

Set $x = c_1(\xi)$, $z = c_1(\zeta)$, and $y = c_1(\xi^2)$. Then $y = [2]x = v_n x^{2^n}$, and the action of c is given by $c^*(x) = x$ and $c^*(z) = y +_{K(n)} z$. Thus as $K(n)^*[\langle c\rangle]$-module (but certainly not as an algebra), $K(n)^*(BB) \cong M_1 \otimes M_2$ with $M_1 = K(n)^*[y,z]/(y^{2^n}, z^{2^n})$ and $M_2 = K(n)^*[x]/(x^{2^n})$. The rest of the calculation now proceeds as in the case of G_{38}, with one minor difference: since the subgroups of G_{41} corresponding to the three inclusions $C_2 \subset V$ are isomorhpic to $C_4 \times C_4$, H_1, and H_2, we have to appeal to Lemma 3.2 in addition to Lemma 3.1 for the equivalent of Lemma 5.1 (b), i.e., the formula for the differential d_3.

References

[1] M. Bakuradze and V. Vershinin. Morava K-Theory rings for the dihedral, semidihedral and generalized quaternion groups. *Proc. Amer. Math. Soc.* **134** (2006), 3707–3714.

[2] M. Hall and J. K. Senior. *The groups of order* 2^n $(n \leq 6)$. The Macmillan Co., New York; Collier-Macmillan, Ltd., London 1964.

[3] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.* **13** (2000), 553–594.

[4] J. R. Hunton. The Morava K-theories of wreath products. *Math. Proc. Camb. Phil. Soc.* **107** (1990), 309–318.

[5] I. Kriz. Morava K-theory of classifying spaces: some calculations. *Topology* **36** (1997), 1247–1273.

[6] I. Kriz and K. Lee. Odd degree elements in the Morava $K(n)$ cohomology of finite groups. *Topology Appl.* **103** (2000), 229–241.

[7] B. Schuster. On the Morava K-theory of some finite 2-groups. *Math. Proc. Camb. Phil. Soc.* **121** (1997), 7–13.

[8] B. Schuster. *Morava K-theory of classifying spaces.* Habilitationsschrift, 2006. vi+124 pp. `http://www.math.uni-wuppertal.de/~schuster/research/MKBG.pdf`

[9] B. Schuster. $K(n)$ Chern approximations of some finite groups. Preprint, 2010. `http://www.math.uni-wuppertal.de/~schuster/research/CBGex.pdf`

[10] B. Schuster and N. Yagita. Morava K-theory of extraspecial 2-groups. *Proc. Amer. Math. Soc.* **132** (2004), no. 4, 1229–1239.

[11] M. Tezuka and N. Yagita. Cohomology of finite groups and Brown-Peterson cohomology. *Algebraic Topology (Arcata, CA, 1986)*, 396–408, Lecture Notes in Math. **1370**, Springer, Berlin, 1989.

[12] M. Tezuka and N. Yagita. Cohomology of finite groups and Brown-Peterson cohomology II. *Homotopy theory and related topics (Kinosaki, 1988)*, 57–69. Lecture Notes in Math. **1418**, Springer, Berlin, 1990.

[13] N. Yagita. Equivariant BP-cohomology for finite groups. *Trans. Amer. Math. Soc.* **317** (1990), no. 2, 485–499.

[14] N. Yagita. Cohomology for groups of $\mathrm{rank}_p(G) = 2$ and Brown-Peterson cohomology. *J. Math. Soc. Japan* **45** (1993), no. 4, 627–644.

[15] N. Yagita. Note on BP-theory for extensions of cyclic groups by elementary abelian p-groups. *Kodai Math. J.* **20** (1997), no. 2, 79–84.

FB 7 Mathematik, Bergische Universität Wuppertal, Wuppertal, Germany

E-mail address: `schuster@math.uni-wuppertal.de`