ON THE MORAVA K-THEORY OF SOME FINITE 2-GROUPS

Björn Schuster

Abstract. We compute the Morava K-theories of finite nonabelian 2-groups having a cyclic maximal subgroup, i.e., dihedral, quaternion, semidihedral and quasidihedral groups of order a power of two.

1. Introduction

For any fixed prime p and any nonnegative integer n there is a $2(p^n-1)$-periodic generalized cohomology theory $K(n)^*$, the n^{th} Morava K-theory. Let G be a finite group and BG its classifying space. For some time now it has been conjectured that $K(n)^*(BG)$ is concentrated in even dimensions. Standard transfer arguments show that a finite group enjoys this property whenever its p-Sylow subgroup does. It is easy to see that it holds for abelian groups, and it has been proved for some nonabelian groups as well, namely groups of order p^3 ([7]) and certain wreath products ([3], [2]). In this note we consider finite (nonabelian) 2-groups with cyclic maximal subgroup, i.e., dihedral, semidihedral, quasidihedral and generalized quaternion groups of order a power of two.

Theorem. *Let G be a nonabelian 2-group with cyclic maximal subgroup and $n > 1$. Then the Morava K-theory of BG is muliplicatively generated by three classes in dimensions 2, 2 and 4, respectively, modulo some explicit relations.*

The Morava K-theories of the dihedral and quaternion group of order eight have been known for a while (see [7], with a correction in [8]); the relations mentioned in the statement of the theorem are written down explicitly in section 2.

2. Preliminaries

The theorem is proved by calculating a spectral sequence of Atiyah-Hirzebruch-Serre type ("AHSSS"). For any extension of finite groups $H \to G \to K$ and any generalized cohomology theory h^*, there is a spectral sequence (see [5])

$$E_2 = H^*(BH; h^*(BK)) \Longrightarrow h^*(BG)\,,$$

1991 *Mathematics Subject Classification.* 55N20, 55T99.
Key words and phrases. Classifying spaces, Morava K-theory.

natural both in the cohomology theory and in the extension. It generalizes the Atiyah-Hirzebruch as well as the Serre spectral sequence: for $H = 1$ it degenerates into the Atiyah-Hirzebruch spectral sequence ("AHSS") for BK, whereas for h^* being ordinary cohomology one has the Serre spectral sequence. It is well known (see e.g. [9]) that the Atiyah-Hirzebruch spectral sequence for $K(n)^*$ has first differential ("AHSS-differential") $d_{2(p^n-1)} = v_n \otimes Q_n$, where Q_n is Milnor's operation defined inductively by $Q_0 = \beta$ (resp. Sq^1) and $Q_i = [Q_{i-1}, \mathcal{P}^{p^i}]$ (resp. $Q_i = [Q_{i-1}, Sq^{2^i}]$ if $p = 2$); this fact will be used repeatedly below.

3. Proof of the theorem

The proof splits into two cases. The first case covers dihedral, semidihedral and generalized quaternion groups: let

$$G \ = < s, t \ \mid \quad s^{2^{N+1}} = 1, \quad t^2 = s^e, \quad tst^{-1} = s^r >$$

where $N \geq 1$, and either $e = 2^N$, $r = 1$ or $e = 0$, $r = -1$, $2^n - 1$. For $e = 0$, $r = -1$ the group G is the dihedral group $D_{2^{N+2}}$ of order 2^{N+2}, for $e = 2^N$, $r = -1$ we get the generalized quaternion group $Q_{2^{N+2}}$, while for $N \geq 2$, $e = 0$ and $r = 2^N - 1$ gives the semidihedral group $SD_{2^{N+2}}$. The center Z of G is a $\mathbf{Z}/2$, generated by s^{2^N}, with quotient isomorphic to the dihedral group $D_{2^{N+1}}$. Thus we get a central extension

$$1 \longrightarrow Z \xrightarrow{i} G \xrightarrow{\pi} D_{2^{N+1}} \longrightarrow 1 \tag{3.1}$$

(where $D_4 = \mathbf{Z}/2 \times \mathbf{Z}/2$).

The second case are the quasidihedral groups $QD_{2^{N+2}}$, which can be presented as

$$QD_{2^{N+2}} \ = < s, t \ \mid \quad s^{2^{N+1}} = t^2 = 1, \quad tst^{-1} = s^{2^N+1} >$$

where $N \geq 2$. The center of $QD_{2^{N+2}}$ is the cyclic subgroup generated by s^2 with quotient $\langle \bar{s}, t \rangle$ isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$, i.e., we have a central extension

$$1 \longrightarrow \mathbf{Z}/2^N \xrightarrow{i} QD_{2^{N+2}} \xrightarrow{\pi} \mathbf{Z}/2 \times \mathbf{Z}/2 \longrightarrow 1\,. \tag{3.2}$$

We shall compute the AHS spectral sequences of the extensions (3.1) and (3.2). We start with extension (3.1). The images $\bar{s}$ and $\bar{t}$ of s and t under π are generators for the quotient $D_{2^{N+1}}$. For $N = 1$, G is isomorphic to either the dihedral or the quaternion group of order 8; this computation has been carried out in [7]. However, since we will need some details of this computation in the course of this section, we want to summarize it briefly. Let a and b denote the multiplicative generators of $H^*(B(\mathbf{Z}/2 \times \mathbf{Z}/2); \mathbf{F}_2)$ dual to $\bar{s}$ and $\bar{t}$. Then the extension class q is either $a^2 + ab$ if G is dihedral or $a^2 + ab + b^2$ if G is quaternion. The E_2-term of the spectral sequence can thus be identified with $K(n)^*[z]/z^{2^n} \otimes \mathbf{F}_2[a, b]$, and the only nontrivial differentials are d_3 and $d_{2^{n+1}-1}$, acting as $d_3 z = a^2b + ab^2$, $d_{2^{n+1}-1}a = v_n a^{2^{n+1}}$ and $d_{2^{n+1}-1}b = v_n b^{2^{n+1}}$. The $E_{2^{n+1}}$-page is even dimensional, thus $E_{2^{n+1}} = E_\infty$. There is also a nontrivial extension problem, see below. Now assume $N > 1$. Let K denote the subgroup of $D_{2^{N+1}}$ generated by $\bar{s}^{2^{N-1}}$ and $\bar{t}$, T the subgroup generated by $\bar{s}^{2^{N-1}}$ and $\bar{s}\bar{t}$, and finally $C \cong \mathbf{Z}/2^N$ the cyclic subgroup generated by $\bar{s}$. Then both K and T are isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$, and it is known that K and T detect $D_{2^{N+1}}$ in mod 2 cohomology (see [1], VI, §3). The inclusions of K, T, and C into $D_{2^{N+1}}$ produce three induced extensions

$$1 \longrightarrow Z \longrightarrow G_8 \longrightarrow K \longrightarrow 1 \tag{3.3}$$

with $G_8 \cong D_8$ for G dihedral or semidihedral and $G_8 \cong Q_8$ for G quaternion;

$$1 \longrightarrow Z \longrightarrow G_8 \longrightarrow T \longrightarrow 1 \tag{3.4}$$

with $G_8 \cong D_8$ for G dihedral, $G_8 \cong Q_8$ for G semidihedral or quaternion; and

$$1 \longrightarrow Z \longrightarrow \mathbf{Z}/2^{N+1} \longrightarrow \mathbf{Z}/2^N \longrightarrow 1 \tag{3.5}$$

for any of the three types of G. We will use the associated maps of spectral sequences in the computations below. Recall the mod 2 cohomology of $D_{2^{N+1}}$ ([1], VI, § 3):

$$H^*(BD_{2^{N+1}}; \mathbf{F}_2) = \mathbf{F}_2[x, y, w]/(x^2 + xy)$$

where $x, y \in H^1(BD_{2^{N+1}}; \mathbf{F}_2) = \mathrm{Hom}(D_{2^{N+1}}, \mathbf{Z}/2)$ are defined by $\langle x, \bar{s}\rangle = 1$, $\langle y, \bar{t}\rangle = 1$, $\langle x, \bar{t}\rangle = \langle y, \bar{s}\rangle = 0$, and w is the second Stiefel-Whitney class of the standard representation of $D_{2^{N+1}}$ in $O(2)$. Moreover, the image of $H^*(BD_{2^{N+1}}; \mathbf{F}_2)$ in $H^*(BK; \mathbf{F}_2)$, $H^*(BT; \mathbf{F}_2)$ and $H^*(BC; \mathbf{F}_2)$ is given by the following table (see [4]):

	x	y	w
i_K^*	0	b	$a^2 + ab$
i_T^*	b	b	$a^2 + ab$
i_C^*	u	0	v

Here i_K^*, i_T^* and i_C^* are the restrictions of $H^*(BD_{2^{N+1}}; \mathbf{F}_2)$ to the mod 2 cohomology of K, T and C, respectively, a and b denote the one dimensional polynomial generators of $H^*(BK; \mathbf{F}_2)$ (or$H^*(BT; \mathbf{F}_2)$) dual to $\bar{s}^{2^{N-1}}$ and $\bar{t}$ (or $\bar{s}^{2^{N-1}}$ and $\bar{s}\bar{t}$ for T), u is the exterior generator of $H^*(BC; \mathbf{F}_2)$ in degree one and v its 2^Nth Bockstein. The extension class q of (3.1) will play a significant role. It is given by (see [6])

$$\begin{cases} w & \text{if } G \text{ is dihedral} \\ w + y^2 & \text{if } G \text{ is quaternion} \\ w + x^2 & \text{if } G \text{ is semidihedral} \end{cases}$$

The spectral sequence calculation for (3.3) and (3.4) has been indicated above, and we have chosen our notation as to coincide with the one used there. We also want to sketch the calculation for (3.5). It is an almost immediate consequence of the fact that we know what the Morava K-theory of $B\mathbf{Z}/2^{N+1}$ is, namely $K(n)^*[\tilde{z}]/\tilde{z}^{2^{(N+1)n}}$ with $\tilde{z}$ in dimension two. Thus we have

$$E_2 = \Lambda(u) \otimes \mathbf{F}_2[v] \otimes K(n)^*[z]/z^{2^n} \Longrightarrow K(n)^*(B\mathbf{Z}/2^{N+1})\,;$$

with the only nonzero differential being the one responsible for the "truncation", i.e., $d_{r(N)}u = v_n^{k(N)}v^{2^{Nn}}$ where $r(N) = 2^{Nn+1} - 1$ and $k(N) = (2^{Nn} - 1)/(2^n - 1)$. There is also a nontrivial extension, $v_n z^{2^n} = v$, and z represents the generator $\tilde{z}$ of $K(n)^*(B\mathbf{Z}/2^{N+1})$. We now return to the extension (3.1). The E_2-term of its associated spectral sequence is

$$E_2 = H^*(BD_{2^{N+1}}; \mathbf{F}_2) \otimes K(n)^*(BZ) \cong \mathbf{F}_2[x, y, w]/(x^2 + xy) \otimes K(n)^*[z]/z^{2^n}\,.$$

Lemma 1. *(a)* $d_3 z = yw$.
(b) z^2 *is a permanent cycle.*

Proof. Let $k(n)$ denote the connective analogue of $K(n)$. We analyze first the differentials in the AHSSS converging to $k(n)^*(BG)$ by comparing it to the mod 2-cohomology version: the Thom map $k(n) \longrightarrow H\mathbf{F}_2$ induces a natural transformation of cohomology theories and hence a map of spectral sequences. Let

$$\overline{E}_2 = H^*(BD_{2^{N+1}}; \mathbf{F}_2) \otimes H^*(BZ; \mathbf{F}_2) \cong \mathbf{F}_2[x, y, w]/(x^2 + xy) \otimes \mathbf{F}_2[u] \Rightarrow H^*(BG; \mathbf{F}_2)$$

be the E_2-term of the Serre spectral sequence in mod 2-cohomology, where u denotes the one dimensional generator of the fiber. In $\overline{E}_r$, u transgresses to q, hence $u^2 = Sq^1 u$ to $Sq^1 q$. An easy argument shows that the map

$$k(n)^*(BZ) \cong k(n)^*[z]/v_n z^{2^n} \longrightarrow \mathbf{F}_2[u] \cong H^*(BZ; \mathbf{F}_2)$$

(we are abusing notation again by denoting z the generator for the connective Morava K-theory of BZ as well as for the nonconnective case) is given by $z \mapsto u^2$ and $v_n \mapsto 0$. Therefore

$$d_3 z \equiv Sq^1 q \bmod v_n$$

in $k(n)$-theory, hence in $K(n)$-theory. (Note that d_2 is zero since the fiber is concentrated in even dimensions.) We want to show that this equation holds on the nose, not only mod v_n. Suppose $d_3 z = Sq^1 q + v_n T$ for some class $T \in E_2^{3,-|v_n|}$. For dimensional reasons T has to be of the form $z^{2^n-1}R$, where $R \in E_2^{3,0}$ is a class in $H^3(bD_{2^{N+1}}; \mathbf{F}_2)$. Then $d_3 d_3 = 0$ implies

$$\begin{aligned} 0 &= d_3(Sq^1 q) + (d_3 z) v_n z^{2^n-2} R + v_n z^{2^n-1} d_3 R \\ &= d_3(Sq^1 q) + v_n z^{2^n-2} R Sq^1 q + v_n^2 z^{2^{n+1}-3} R^2 + v_n z^{2^n-1} d_3 R\,. \end{aligned}$$

As long as $n > 1$, $z^{2^{n+1}-3} = 0$ and comparison to the AHSS for $BD_{2^{N+1}}$ shows that d_3 is zero on classes not divisible by z, hence $R = 0$. (Recall that in the AHSS, $d_{2^{n+1}-1}$ is the first differential.) Now $Sq^1 x^2 = Sq^1 y^2 = 0$ and $Sq^1 w = yw$, which implies (a). Furthermore, let θ be the following two dimensional complex representation of G:

$$\theta(s) = \begin{pmatrix} e & 0 \\ 0 & e^r \end{pmatrix} \quad , \quad \theta(t) = \begin{pmatrix} 0 & 1 \\ (-1)^\sigma & 0 \end{pmatrix}$$

where $e = \exp(\pi i/2^N)$, $r = -1$ if G is dihedral or quaternion, $r = 2^N - 1$ if G is semidihedral, $\sigma = 0$ if G is dihedral or semidihedral and $\sigma = 1$ for G quaternion. Then $\theta(s^{2^N}) = -I_2$ (where I stands for the identity matrix), i.e., θ restricted to the center is twice the nontrivial representation η of $\mathbf{Z}/2$. Thus the second Chern class of θ restricts to the class z^2 of the fiber. $\square$

Note that yw is not a zero divisor in the cohomology of $D_{2^{N+1}}$ whence the E_4-page remains a tensor product. Since $x^2 = xy$, all classes of the form $x^i w$, $i > 1$, and $y^j w$, $j > 0$ are eliminated; xw however is not. Also observe that the E_4-term is detected by the E_4-terms of the spectral sequences associated to (3.3) - (3.5). Thus the next potentially nonzero differential is $d_{2^{n+1}-1} = v_n \otimes Q_n$,

$$d_{2^{n+1}-1} x = v_n x^{2^{n+1}} \quad , \quad d_{2^{n+1}-1} y = v_n y^{2^{n+1}} .$$

Detection by the spectral sequences of (3.3) - (3.5) also shows $d_{2^{n+1}-1} w = 0$ and $d_{2^{n+1}-1} xw = (d_{2^{n+1}-1} x) w = x^{2^{n+1}} = 0$. Finally, comparison to (3.5) shows that the remaining odd dimensional classes, xw^k, are killed by the differential $d_{r(N)}$. More precisely, one has $d_{r(N)} xw = v_n^{k(N)} w^{2^{Nn}+1}$, where $r(N) = 2^{Nn+1} - 1$ and $k(N) = (2^{Nn} - 1)/(2^n - 1)$ as above. To obtain the ring structure of $K(n)^*(BG)$ we still have to solve extension problems. This can be done, up to some indeterminacy, by comparison arguments once again. The indeterminacy arises from the fact that the E_∞-page is no longer detected by the E_∞-pages of (3.3) - (3.5), but the only classes eluding detection are those divisible by $w^{2^{Nn}}$ (for $N = 1$ the class ab plays the role of w in this respect). The filtration drop occuring for (3.5) was described above; for (3.3) and (3.4), that is the case $N = 1$, it can be worked out by restricting further to the three subgroups $\mathbf{Z}/2$ of the base. Thus

$$v_n z^{2^n} = \begin{cases} q + \varepsilon v_n (ab)^{2^n} & \text{if } N = 1 \\ q + \varepsilon v_n w^{2^{Nn}} & \text{if } N > 1 \end{cases}$$

where ε is either 0 or 1. Let y_1, y_2 and c_2 denote the classes represented by x^2, y^2 and z^2, respectively. (The reader should be warned that despite its suggestive name, c_2 is *not* the second Chern class of the representation θ, but only up to lower filtration. Thus the choice of c_2 is not canonical; this is the price we pay to obtain explicit relations.) Then $K(n)^*(BG) \cong K(n)^*[y_1, y_2, c_2]/R$ where the relations R can be derived from the relations introduced in the spectral sequence by the differentials d_3 and $d_{2^{n+1}-1}$, and the filtration drop described above. Recall that q is either w, $w + x^2$ or $w + y^2$, depending on whether G is dihedral, semidihedral or quaternion, and note that it does not make a difference whether ε is 0 or 1, since at E_∞, the class $w^{2^{Nn}}$ is annihilated by any class of positive horizontal degree. Thus we obtain the following list of relations (where we include the case $N = 1$ for completeness, see also [7] and [8]):

(1) $y_1^{2^n}$, $y_2^{2^n}$, $v_n y_1 c_2^{2^{n-1}} = v_n y_2 c_2^{2^{n-1}} = y_1 y_2 = v_n^2 c_2^{2^n}$ *if* $G = D_8$

(2) $y_1^{2^n}$, $y_2^{2^n}$, $v_n y_1 c_2^{2^n} + y_1^2$, $v_n y_2 c_2^{2^n} + y_2^2$, $v_n^2 c_2^{2^n} + y_1^2 + y_1 y_2 + y_2^2$ *if* $G = Q_8$

(3) $y_1^{2^n}$, $y_2^{2^n}$, $c_2^{2^{n-1}(2^{Nn}+1)}$, $y_1 y_2 = y_1^2$ *if* $N > 1$ *and all three types, and*

(a) $y_1 c_2^{2^{n-1}}, y_2 c_2^{2^{n-1}}$ *if* G *is dihedral*

(b) $y_1^2 + v_n y_1 c_2^{2^{n-1}}, y_2^2 + v_n y_2 c_2^{2^{n-1}}$ *if* G *is quaternion*

(c) $y_1^2 + v_n y_1 c_2^{2^{n-1}}, y_1^2 + v_n y_2 c_2^{2^{n-1}}$ *if* G *is semidihedral*

We now turn to extension (3.2), i.e., quasidihedral groups. We have

$$E_2 = H^*(B(\mathbf{Z}/2 \times \mathbf{Z}/2); K(n)^*(B\mathbf{Z}/2^N)) \cong \mathbf{F}_2[x, y] \otimes K(n)^*[z]/z^{2^{Nn}},$$

with x and y dual to $\overline{s}$ and t, respectively, and z (as usual) the first Chern class of the standard complex character of $\mathbf{Z}/2^N$. As before, we will use comparison arguments throughout the computation: let i_1, i_2 and i_3 denote the inclusions of $\langle\overline{s}\rangle$, $\langle\overline{s}t\rangle$ and $\langle t\rangle$ into $\langle\overline{s}, t\rangle$, respectively. Then we get three induced extensions

$$1 \longrightarrow \mathbf{Z}/2^N \longrightarrow H_k \longrightarrow \mathbf{Z}/2 \longrightarrow 1\,.$$

H_1 and H_2 are cyclic of order 2^{N+1}, whereas $H_3 = \mathbf{Z}/2^N \times \mathbf{Z}/2$.

Lemma 2. *(a)* $d_3 z = (x^2 y + x y^2) U$ *where* U *is a unit in* E_3 *with* $d_3 U = 0$.
(b) z^2 *is a permanent cycle.*

Proof. We start with (b). It suffices to show that there is a complex representation of $QD_{2^{N+2}}$ restricting to twice the standard representation on the fiber. Set

$$\theta(s) = \begin{pmatrix} e & 0 \\ 0 & e^{2^N+1} \end{pmatrix} \quad , \quad \theta(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $e = \exp(2\pi i/2^{N+1})$.
To prove (a), observe that z cannot be a permanent cycle: under the map induced by the natural transformation $k(n) \longrightarrow H\mathbf{F}_2$, z maps to the polynomial generator of $H^*(B\mathbf{Z}/2^N; \mathbf{F}_2)$, which has to transgress to the zero line (otherwise the dimension of $H^3(BQD_{2^{N+2}}; \mathbf{F}_2)$ becomes too big). To actually compute $d_3 z$, we compare

the spectral sequence of (3.2) to both the mod 2 cohomology spectral sequence as above and the Morava K-theory spectral sequences of the three extensions induced by i_1, i_2 and i_3. In every one of them, the only nontrivial differential is $d_{2^{n+1}-1}$; in particular, d_3z restricts to zero on all three of them. Since the E_3-term of (3.2) is a tensor product, d_3z is a sum of products of classes in $H^3(B\mathbf{Z}/2 \times \mathbf{Z}/2; \mathbf{F}_2)$ and elements of the Morava K-theory of the fiber of degree zero. Let $\alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3$ be an arbitrary class in $H^3(B\mathbf{Z}/2 \times \mathbf{Z}/2; \mathbf{F}_2)$ and let u denote the one dimensional generator of $H^*(B\mathbf{Z}/2; \mathbf{F}_2)$. The following table lists the images of x, y and $\alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3$ under the restrictions i_k^*:

	x	y	$\alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3$
i_1^*	u	0	αu^3
i_2^*	u	u	$(\alpha+\beta+\gamma+\delta)u^3$
i_3^*	0	u	δu^3

Now $\alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3$ restricts to zero under all three inclusions if and only if $\alpha = \delta = 0$ and $\beta = \gamma = 1$ or $\alpha = \beta = \gamma = \delta = 0$. In other words, d_3z is divisible by $x^2y + xy^2$. Together with the above observation that d_3z is not zero in the cohomology spectral sequence, that is not zero modulo v_n, this shows $d_3z = (x^2y+xy^2)(1+R)$, where $R \neq 1$ is an element of degree zero in $K(n)^*[z]/z^{2^{Nn}}$. Using $d_3d_3 = 0$, we can conclude that there are no summands of R containing an odd power of z as a factor (since d_3 is nontrivial on odd powers one otherwise gets relations in the E_3-term, contradicting its structure). Hence, $1 + R$ is a cycle for d_3 and, since R is nilpotent, a unit in $K(n)^*[z]/z^{2^{Nn}}$. □

As a consequence of part (a) of the lemma, zU^{-1} hits $x^2y + xy^2$, so all the mixed powers of x and y of degree at least three become identified from the E_4-page on. As in the previous calculations, E_4 is detected by the E_4-pages of the spectral sequences of the extensions induced by i_1, i_2 and i_3; thus the next nontrivial differential is $d_{2^{n+1}-1}$. This differential kills all remaining odd dimensional classes, hence the AHSSS collapses at the $E_{2^{n+1}-1}$-page. Note that xy is a permanent cycle. The only remaining question is that of possible extension problems. Using comparison to the three induced extensions again yields $v_n^{k(N)}z^{2^{Nn}} = x^2+\varepsilon v_n(xy)^{2^n}$ where $k(N) = (2^{Nn}-1)/(2^n-1)$ and $\varepsilon = 0$ or 1. Note that x^2 is the "extension class" of (3.2) in the sense that it is the class the exterior generator of the mod 2 cohomology of the fiber transgresses to in the mod 2 cohomology version of the spectral sequence. Finally, let y_1, y_2 and c_2 denote the classes represented by xy, y^2 and z^2, respectively; again, c_2 is a Chern class only modulo lower filtration. Then as a consequence of $x^2y = xy^2$ and the filtration drop of $z^{2^{Nn}}$ we obtain the following relations:

(1) $y_1^{2^n+1}$, $y_2^{2^n}$, $c_2^{2^{(N+1)n-1}}$

(2) $y_1^2 = y_1y_2 = v_n^{k(N)}y_1c_2^{2^{Nn-1}} = v_n^{k(N)}y_2c_2^{2^{Nn-1}}$ *where* $k(N) = (2^{Nn}-1)/(2^n-1)$.

References

[1] Z. Fiedorowicz and S. Priddy, *Homology of Classical Groups over Finite Fields and Their Associated Infinite Loop Spaces*, Lecture Notes in Mathematics **674**, Springer-Verlag, Berlin, Heidelberg, New York.

[2] M. J. Hopkins, N. J. Kuhn and D. C. Ravenel, *Morava K-Theories of Classifying Spaces and Generalized characters for Finite Groups*, Algebraic Topology, Barcelona 1990, Lecture Notes in Mathematics **1509**, Springer-Verlag, Berlin, Heidelberg, New York, pp. 186–209.

[3] J. R. Hunton, *The Morava K-theories of wreath products*, Math. Proc. Cambridge Phil. Soc. **107** (1990), 309–318.

[4] S. Mitchell and S. Priddy, *Symmetric product spectra and splittings of classifying spaces*, Amer. J. Math. **106** (1984), 219–232.

[5] D. C. Ravenel, *Morava K-theories and finite groups*, Contemp. Math. **12** (1982), 289–292.

[6] D. Rusin, *The mod 2 cohomology of metacyclic 2-groups*, J. Pure Appl. Algebra **44** (1987), 315–327.

[7] M. Tezuka and N. Yagita, *Cohomology of finite groups and Brown-Peterson cohomology*, Algebraic Topology, Arcata 1986, Lecture Notes in Mathematics **1370**, Springer-Verlag, Berlin, Heidelberg, New York, pp. 396–408.

[8] M. Tezuka and N. Yagita, *Complex K-theory of* $BSL_3(\mathbf{Z})$, K-Theory **6** (1992), 87–95.

[9] N. Yagita, *On the Steenrod algebra of Morava K-theory*, J. London Math. Soc. **22** (1980), 423–438.

Centre de Recerca Matemàtica, Apartat 50, E-08193 Bellaterra, Spain

Current address: FB 7 Mathematik, Universität Wuppertal, Gaußstr. 20, 42097 Wuppertal, Germany

E-mail address: schuster@math.uni-wuppertal.de