Another look a Miller's *Myth*

Erhard Scholz, Wuppertal (Germany)

In 1972 A. Miller questioned the view that C.F. Gauss may have used the data of the large triangle Brocken – Hohehagen – Inselsberg as a basis for a precision test of the validity of Euclidean geometry for physical space (Miller 1972). He contrasted this view with a discussion of the same triangle in *Gauss' Disquisitiones generales circa superficies curvas.* Since then, the doubts among historians of mathematics have been deepened by adding arguments of principle in the sense that Gauss even would have been unable to use the data for a test as reported by Sartorius of Waltershausen (Breitenberger 1984).

On the other hand it seems clear that Gauss' angle sum theorem of non-Euclidean geometry (NEG) could very well be used in the framework of Gauss' knowledge of the early 1820s to establish precision bounds for the reliability of Euclidean geometry as an empirical theory of physical space. To understand the whole problem constellation one has to be aware that geodesists use light rays (if necessary, after correction of atmospheric perturbations) as empirical representations of straight lines. Theodolite measurements of sufficiently large triangles do not determine the angles α, β, γ of the light ray triangle itself, but rather measure angles $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ projected into the horizontal plane, that is in three (generally different) tangent planes to the earth figure (here to be approximated by the sphere, as was shown by Gauss in his *Disquisitiones generales*). That means that measurement data are essentially those of a triangle on a sphere in Euclidean or in non-Eculidean space, depending on the underlying physical geometry of (light ray) space.

Gauss applied the technique of closure data of triangles as a quality criterion for his precision measurements. That means he compared the angles sum with the value expected in a spherical triangle of Euclidean geometry:

$$\hat{\alpha} + \hat{\beta} + \hat{\gamma} = \pi + \frac{1}{R^2} A + \epsilon. \tag{1}$$

Here A denotes the area of the triangle and R the earth radius. The term ϵ denotes the *closure error* of the measurement. In principle ϵ might contain a systematic error derived from a hypothetical non-Euclidean curvature κ of the physical space (notation adapted to terminology of Gauss' surface theory which he carefully avoided in his remarks on NEG). In NEG Gauss used formulations of a characteristic *constant* C, describing the deviation of

NEG from the Euclidean case, with $C \longrightarrow \infty$ for the latter. It relates to κ by $C = \frac{1}{\sqrt{|\kappa|}}$. If physical space should be non-Euclidean with characteristic constant C, Gauss could expect a deviation $\delta := \kappa A = -\frac{A}{C^2}$ of the angle sum of the light ray triangle from π. Simple approximation considerations then show that (in very good linearized approximation) the same deviation δ would appear as a systematic contribution to the closure error ϵ in equ. (1).[1]

Gauss was very careful in reducing closure errors in his total net of the degree measurement campaign linking Göttingen to Hamburg Altona. There he could link his triangulation to the one of his Danish colleague and friend Schumacher. Schumacher had gauged his net by a direct base line measurement of the best available precision. After error equalization Gauss arrived at a mean square error $\sigma \approx 0.48''$ (empirical standard deviation) of single angles in his total net from Göttingen to Altona. That was an extremely good result. Comparable measurements of the time had a mean square error one order of magnitude larger and worked with much smaller triangles. The triangulation of the Netherlands by C. von Krayenhoff, e.g., achieved a precision of $\sigma \approx 2.7''$ only. With a typical area $A \approx 300\, km^2$ even the *spherical excess* of singles triangles, $\frac{A}{R^2} \approx 1.5''$, was hidden *deep inside the measurement error.*

On the other hand, Gauss kept the largest triangle between the three mountains Brocken – Hohehagen – Inselsberg, the *large triangle* $\triangle BHI$ (with side lengths $\overline{BH} \approx 69\, km, \overline{HI} \approx 85\, km, \overline{BI} \approx 107\, km$, area $A \approx 2920\, km^2$) distinct from the error equalization procedures of the main net. In this triangle the spherical excess was

$$\frac{A}{R^2} \approx 14.86'',$$

two orders of magnitude *above his* mean square error.

A search for possible systematic errors of closure data became now meaningful for the first time. Gauss calculated the closure error of this large triangle seperately and found $\epsilon \approx 0.6''$, *far below the mean square error* of the total net. Therefore any *systematic contribution* δ to the error (independent of the sign) could reasonably be expected to have smaller absolute value. In terms of the constant of non-Euclidean geometry,

$$|\delta| = \frac{A}{C^2} < 0.6'' \,, \quad C^2 > \frac{A}{0.6''} \;.$$

With Gauss' calculation of the spherical excess of the large triangle $\frac{A}{R^2} \approx$ 14.86", another comparison could easily be made:

$$C^2 > \frac{A}{0.6''} \approx \frac{14.86}{0.6} R^2 \approx 25R^2.$$

[1] For a detailed exposition of the argument see (Scholz 2004).

Thus the constant of NEG had to be

$$C > 5R. \tag{2}$$

Gauss never stated the result in this form, probably because it was ridiculously small in comparison with what he expected anyhow (astronomical orders of magnitude for C). But reliable astronomical data were difficult or even impossible to acquire in the 1820s. As an experienced empirical mathematician and astronomer Gauss would probably not have tried to draw precise quantitative conclusions from astronomical evidence before the first stable parallax measurement of fixed stars were available (Bessel in 1838).[2] This leads to another story, the determination of curvature bounds of astronomical space by parallax measurements in the 19th century, in which Gauss and his Göttingen environment (in particular B. Riemann) again played their part.

The geodetical data, on the other hand, were determined with unpeccable precision already in the 1820s. Sartorius von Waltershausen reported that in his inner circle Gauss sometimes referred to his result on the large triangle when he wanted to give bounds for the validity of Euclidean geometry. Then he referred to an error bound of "two tenths of a second per angle" as measured for the large triangle of his Hannover degree campaign.

If one looks at the arguments given in (Miller 1972) from hindsight, it appears surprising why such a harsh and weakly founded criticism of Sartorius' report as in *Miller's myth* was so easily accepted not only among historians of science but also among historians of mathematics, and why Sartorius von Waltershausen's report was discredited as a reliable source.[3] The reasons have probably to be sought for in the community constellation of the history of science at that time and a certain weakness of history of mathematics in it on an international level, rather than in the epistemic achievements of the original publication.

References

Breitenberger, Ernst. 1984. "Gauss's geodesy and the axiom of parallels." *Archive for History of Exact Sciences* 31:273–289.

[2]That was different for N.I. Lobachevsky. Being a mathematician not as close to astronomical practice as Gauss, he dared to use the unreliable parallax data announced by the Belgian astronomer d'Assad Montdardier in his estimations of a bound for the curvature of physical space in his *Načala geometrie* (Lobatschewsky 1829-30/1898, 22ff.). His courage was rewarded by an impressive bound which was not essentially superceded by astronomical measurements of the 19th century up to K. Schwarzschild's in 1900.

[3]A clear cut rejection of Miller's myth came from B.L. van der Waerden (van der Waerden 1974), but was apparently recieved with reservation from the side of professional historians of mathematics.

Gauss, Carl Friedrich. 1828. “Disquisitiones generales circa superficies curvas.” *Commentationes Societatis Regiae Scientiarum Gottingensis* 6 (1828):99–146. In *Werke* IV, 217–258. Deutsch von A. Wangerin *Allgemeine Flächentheorie* Leipzig: Engelmann 1889.

Gauss, Carl Friedrich. Werke. *Werke* 12 Bände. Göttingen: Akademie der Wissenschaften: 1863–1927. Nachdruck Hildesheim: Olms 1973.

Lobatschewskij, Nikolaj I. 1829-30/1898. “O natschalach geometrij.” *Kasanskij Vestnik* 25–28. Deutsche Übersetzung F. Engel: Ueber die Anfangsgründe der Geometrie. In *Zwei geometrische Abhandlungen*. Leipzig: Teubner, 1–66.

Miller, Arthur. 1972. “The myth of Gauss’ experiment on the Euclidean nature of physical space.” *Isis* 63:345–348.

Sartorius von Waltershausen, Wolfgang. 1856. *Gauss zum Gedächtnis.* Leipzig: Hirzel. Nachdruck Wiesbaden: Sändig 1956.

Scholz, Erhard. 2004. C.F.Gauß’ Präzisionsmessungen terrestrischer Dreiecke und seine Überlegungen zur empirischen Fundierung der Geometrie in den 1820er Jahren. In: Folkerts, Menso; Hashagen, Ulf; Seising, Rudolf; (Hrsg.): *Form, Zahl, Ordnung. Studien zur Wissenschafts- und Technikgeschichte. Ivo Schneider zum 65. Geburtstag.* Stuttgart: Franz Steiner Verlag, 355–380. [http://arxiv.org/math.HO/0409578]

Schwarzschild, Karl. 1900. “Ueber das zulässige Krümmungsmaass des Raumes.” *Vierteljahresschrift der Astronomischen Gesellschaft Leipzig* 35:337–347.

van der Waerden, Baartel L. 1974. “Comment II” [on (Miller 1972)]. *Isis* 65:85.