Twist deformations of module homomorphisms and connections

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(joint work with Paolo Aschieri)

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and

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Workshop on Noncommutative Field Theory and Gravity

Corfu Summer Institute

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Motivation from noncommutative gravity

Noncommutative gravity à la Wess et al.

◇ Basic idea: [Aschieri,Blohmann,Dimitrijević,Meyer,Schupp,Wess]

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  - $h\star k = \bar{f}^{\alpha}(h)\,\bar{f}_{\alpha}(k)$ “ $= h\, e^{\frac{i\lambda}{2}\overleftarrow{\partial_\mu}\Theta^{\mu\nu}\overrightarrow{\partial_\nu}}\, k$ ”

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◇ NC Einstein equation

$$R^\star{}_{ab} - \frac{1}{2}g_{ab}\star\mathfrak{R}^\star = 8\pi G_N\, T^\star{}_{ab}$$

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  - extension (lift) to tensor product modules $V \otimes_A W$

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  - extension (lift) to tensor product modules $V \otimes_A W$
  - NC gravity solutions from a global point-of-view (not using $X^{toomany}_{strangeletters}$)

Twist deformation of homomorphisms and connections

Twist deformation preliminaries

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Theorem (textbook)

- *Given a twist, there is a new Hopf algebra $H^\mathcal{F}$ with coproduct $\Delta^\mathcal{F}(\cdot) = \mathcal{F}\,\Delta(\cdot)\,\mathcal{F}^{-1}$ and antipode $S^\mathcal{F}(\cdot) = \chi\,S(\cdot)\,\chi^{-1}$, where $\chi = f^\alpha S(f_\alpha)$.*

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- *Given also an H-module A-bimodule V, there is an $H^\mathcal{F}$-module $A_\star$-bimodule $V_\star$ with $a\star v = (\bar{f}^\alpha\triangleright a)\cdot(\bar{f}_\alpha\triangleright v)$ and $v\star a = (\bar{f}^\alpha\triangleright v)\cdot(\bar{f}_\alpha\triangleright a)$.*

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NB: $D_{\mathcal{F}}$ is a **quantization isomorphism**, mapping one-to-one classical endomorphisms $P(v \cdot a) = P(v) \cdot a$ to deformed ones $P_\star(v \star a) = P_\star(v) \star a$.

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Ex: Consider the dual module $V' := \mathrm{Hom}_A(V,A)$. Then $D_{\mathcal{F}}$ ensures that $(V_\star)'\simeq (V')_\star$.

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- Second try: ☺

$$\begin{tikzcd} V_\star \arrow[rr, "\widetilde{D}_{\mathcal{F}}(\nabla)"] \arrow[drr, "D_{\mathcal{F}}(\nabla)"'] & & V_\star \otimes_{A_\star} \Omega^1_\star \\ & & (V \otimes_A \Omega^1)_\star \arrow[u, "\iota^{-1}"'] \end{tikzcd}$$

With isomorphism

$$\iota(v \otimes_{A_\star} \omega) = (\bar{f}^\alpha \triangleright v) \otimes_A (\bar{f}_\alpha \triangleright \omega)$$

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- First try: Consider $D_{\mathcal{F}}(\nabla) : V_\star \to \left(V \otimes_A \Omega^1\right)_\star$

  ☹ does not give (yet) a connection $\nabla_\star : V_\star \to V_\star \otimes_{A_\star} \Omega^1_\star$

- Second try: ☺

$$\begin{tikzcd}
V_\star \arrow[r, "\widetilde{D}_{\mathcal{F}}(\nabla)"] \arrow[dr, "D_{\mathcal{F}}(\nabla)"'] & V_\star \otimes_{A_\star} \Omega^1_\star \\
& (V \otimes_A \Omega^1)_\star \arrow[u, "\iota^{-1}"']
\end{tikzcd}$$

With isomorphism

$$\iota(v \otimes_{A_\star} \omega) = (\bar{f}^\alpha \triangleright v) \otimes_A (\bar{f}_\alpha \triangleright \omega)$$

Theorem

The map $\widetilde{D}_{\mathcal{F}} : \mathrm{Con}_A(V) \to \mathrm{Con}_{A_\star}(V_\star)$ is an isomorphism between connections on V and connections on $V_\star$.

Product module homomorphisms and connections

Lifting homomorphisms

? Given $P \in \mathrm{Hom}_A(V, \widetilde{V})$ is there a lift to $\mathrm{Hom}_A(V \otimes_A W, \widetilde{V} \otimes_A W)$ and $\mathrm{Hom}_A(W \otimes_A V, W \otimes_A \widetilde{V})$?

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Product module connections

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$$\nabla_V \oplus_R \nabla_W := (\mathrm{id} \otimes \tau_R) \circ (\nabla_V \otimes \mathrm{id}) + (\mathrm{id} \otimes_R \nabla_W)~,$$

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NB: Our product connections differ from the ones of Madore

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◇ For *central* connections $\mathrm{id} \otimes_R \nabla_W = \mathrm{id} \otimes \nabla_W$, but for noncentral connections the $\otimes_R$ is important!

Noncommutative gravity solutions revisited

Existing results

- Schupp and Solodukhin [arXiv:0906.2724]:
  - NC black hole solution with $[x^i, x^j] = i\,\lambda\,\epsilon^{ijk} x^k$
  - Uses local *central* basis of vector fields and one-forms

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If the twist is constructed by sufficiently many Killing vector fields, then a classical solution is also a solution of the NC Einstein equation.

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  - localization is i.g. incompatible with non-formal deformations

Global approach to NC gravity solutions

We use the methods developed above, in particular the isomorphisms

- $D_{\mathcal{F}} : \big(\mathsf{Hom}_A(V,W)\big)_\star \to \mathsf{Hom}_{A_\star}(V_\star,W_\star)\,,\ P \mapsto (\bar{f}^\alpha \blacktriangleright P)\circ(\bar{f}_\alpha \triangleright\ )$
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Theorem

Let $\big(\mathcal{M}, g \in \Omega^1 \otimes_A \Omega^1\big)$ be an Einstein manifold, i.e. $\mathrm{Ric} = \Lambda\, g$, with Killing Lie algebra $\mathfrak{K}$ and let $\mathcal{F}^{-1} - 1 \otimes 1 \in U\Xi\mathfrak{K} \otimes U\Xi + U\Xi \otimes U\Xi\mathfrak{K}$. Then the deformed manifold $\big(\mathcal{M}, g_\star = \iota^{-1}(g), \mathcal{F}\big)$ is deformed Einstein, i.e. $\mathrm{Ric}_\star = \Lambda\, g_\star$.

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  - quantization of the classical Levi-Civita connection ∇, i.e. $\nabla_\star := \widetilde{D}_{\mathcal{F}}(\nabla)$, yields deformed LC connection, $(\nabla_\star \oplus_R \nabla_\star)(g_\star) = 0$ and $T_\star = 0$.

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- Proof sketch:
  - quantization of the classical Levi-Civita connection ∇, i.e. $\nabla_\star := \widetilde{D}_{\mathcal{F}}(\nabla)$, yields deformed LC connection, $(\nabla_\star \oplus_R \nabla_\star)(g_\star) = 0$ and $T_\star = 0$.
  - deformed curvature $R_\star := \nabla_\star \circ \nabla_\star : \Omega^1_\star \to \Omega^1_\star \otimes_{A_\star} \Omega^2_\star$ is related to $R = \nabla \circ \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^2$ via $R_\star = \iota^{-1} \circ D_{\mathcal{F}}(R)$
  - deformed Ricci tensor $\mathrm{Ric}_\star := \mathrm{Tr}_\star(R_\star) \in \Omega^1_\star \otimes_{A_\star} \Omega^1_\star$ is related to $\mathrm{Ric} \in \Omega^1 \otimes_A \Omega^1$ via $\mathrm{Ric}_\star = \iota^{-1}(\mathrm{Ric})$

  $\Rightarrow$ $\mathrm{Ric}_\star = \iota^{-1}(\mathrm{Ric}) = \iota^{-1}(\Lambda\, g) = \Lambda\, g_\star$

Conclusions and outlook

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- $D_{\mathcal{F}} : \big(\mathsf{Hom}_A(V, W)\big)_\star \to \mathsf{Hom}_{A_\star}(V_\star, W_\star)$
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◇ Open issues:

- existence and uniqueness of NC Levi-Civita connection
- introducing $*$-structures and reality conditions