

Twist deformations of module homomorphisms and connections

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(joint work with Paolo Aschieri)

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Workshop on Noncommutative Field Theory and Gravity

Corfu Summer Institute

September 7 – 11, 2011

Motivation from noncommutative gravity

Noncommutative gravity à la Wess et al.

- ◇ Basic idea: [Aschieri,Blohmann,Dimitrijević,Meyer,Schupp,Wess]

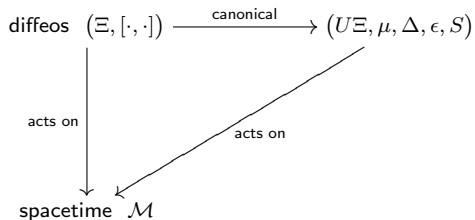
diffeos $(\Xi, [\cdot, \cdot])$

acts on

spacetime \mathcal{M}

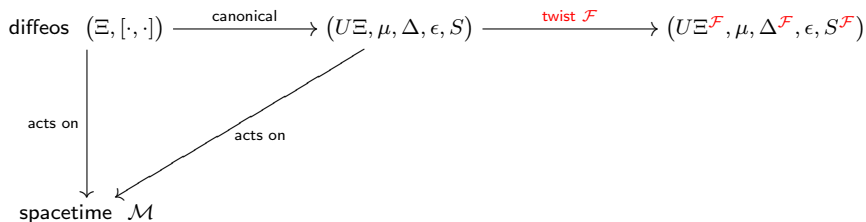
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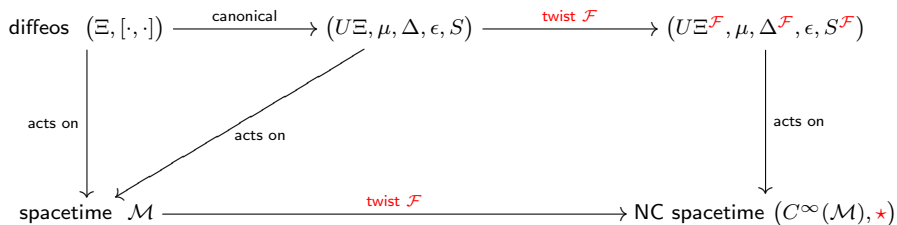
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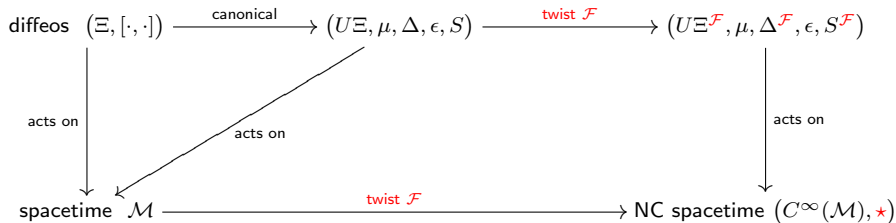
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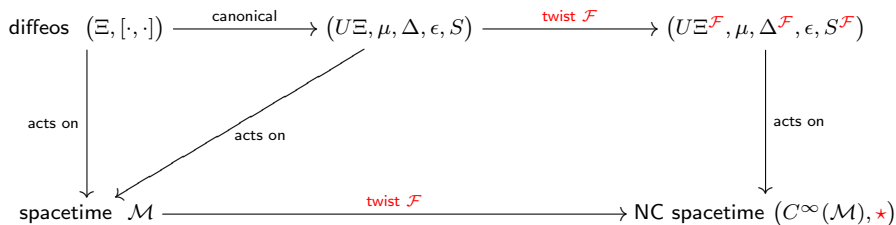


- NC geometry via imposing covariance under $U\Xi^{\mathcal{F}}$

$$- h \star k = \bar{f}^\alpha(h) \bar{f}_\alpha(k) \quad " = h e^{\frac{i\lambda}{2} \overleftarrow{\partial}_\mu \Theta^{\mu\nu} \overrightarrow{\partial}_\nu} k "$$

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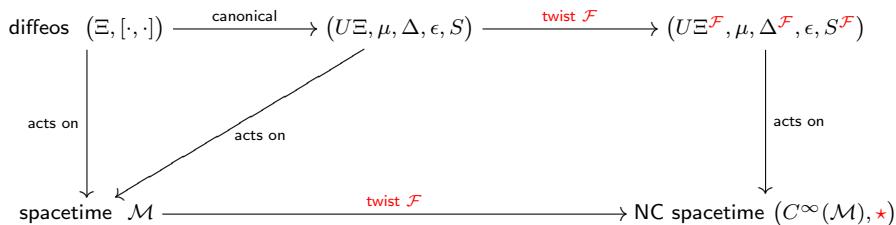


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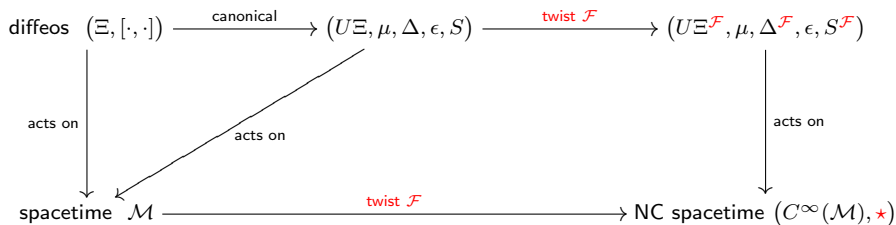


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- NC Einstein equation

$$R^\star_{ab} - \frac{1}{2} g_{ab} \star \mathfrak{R}^\star = 8\pi G_N T^\star_{ab}$$

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Mathematical structure

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Twist deformation of homomorphisms and connections

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Theorem (textbook)

- Given a twist, there is a new Hopf algebra $H^{\mathcal{F}}$ with coproduct $\Delta^{\mathcal{F}}(\cdot) = \mathcal{F} \Delta(\cdot) \mathcal{F}^{-1}$ and antipode $S^{\mathcal{F}}(\cdot) = \chi S(\cdot) \chi^{-1}$, where $\chi = f^\alpha S(f_\alpha)$.

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- Given also an H -module A -bimodule V , there is an $H^{\mathcal{F}}$ -module A_\star -bimodule V_\star with $a \star v = (\bar{f}^\alpha \triangleright a) \cdot (\bar{f}_\alpha \triangleright v)$ and $v \star a = (\bar{f}^\alpha \triangleright v) \cdot (\bar{f}_\alpha \triangleright a)$.

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NB: $D_{\mathcal{F}}$ is a **quantization isomorphism**, mapping one-to-one classical endomorphisms $P(v \cdot a) = P(v) \cdot a$ to deformed ones $P_\star(v \star a) = P_\star(v) \star a$.

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Ex: Consider the dual module $V' := \text{Hom}_A(V, A)$. Then $D_{\mathcal{F}}$ ensures that $(V_\star)' \simeq (V')_\star$.

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$$\begin{array}{ccc}
 V_\star & \xrightarrow{\tilde{D}_{\mathcal{F}}(\nabla)} & V_\star \otimes_{A_\star} \Omega_\star^1 \\
 & \searrow^{D_{\mathcal{F}}(\nabla)} & \uparrow \iota^{-1} \\
 & & (V \otimes_A \Omega^1)_\star
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With isomorphism

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Theorem

The map $\tilde{D}_{\mathcal{F}} : \text{Con}_A(V) \rightarrow \text{Con}_{A_\star}(V_\star)$ is an isomorphism between connections on V and connections on V_\star .

Product module homomorphisms and connections

Lifting homomorphisms

- ? Given $P \in \text{Hom}_A(V, \tilde{V})$ is there a lift to $\text{Hom}_A(V \otimes_A W, \tilde{V} \otimes_A W)$ and $\text{Hom}_A(W \otimes_A V, W \otimes_A \tilde{V})$?

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 - ◇ Second lift i.g. not: $w \otimes_A v \mapsto w \otimes_A P(v)$ is incompatible with A -linearity!

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- ◇ For *central* connections $\text{id} \otimes_R \nabla_W = \text{id} \otimes \nabla_W$, but for noncentral connections the \otimes_R is important!

Noncommutative gravity solutions revisited

Existing results

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Global approach to NC gravity solutions

We use the methods developed above, in particular the isomorphisms

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 - $D_{\mathcal{F}} : (\text{End}_A(V), \circ_*) \rightarrow (\text{End}_{A_*}(V_*), \circ)$
 - $D_{\mathcal{F}} : (\text{Hom}_A(V, W))_* \rightarrow \text{Hom}_{A_*}(V_*, W_*)$
 - $\tilde{D}_{\mathcal{F}} : \text{Con}_A(V) \rightarrow \text{Con}_{A_*}(V_*)$
- ◇ Lift $\text{Hom}_A(V, \tilde{V}) \rightarrow \text{Hom}_A(W \otimes_A V, W \otimes_A \tilde{V})$ for quasitriangular Hopf algebras and quasi-commutative algebras and modules
- ◇ Sum $\oplus_R : \text{Con}_A(V) \times \text{Con}_A(W) \rightarrow \text{Con}_A(V \otimes_A W)$ for triangular Hopf algebras and quasi-commutative algebras and modules

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- ◇ Open issues:
 - existence and uniqueness of NC Levi-Civita connection
 - introducing $*$ -structures and reality conditions