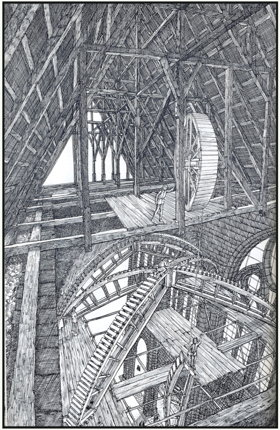

Construction of Hadamard states by pseudo-differential calculus

Michał Wrochna

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joint work with Christian Gérard

31.05.13

Physical states for QFT on curved space-times

- On general curved space-times, no notion of vacuum state.
- Substitute for vacuum state: Hadamard states, characterized by the singularity structure of their two-point functions [Kay, Wald, etc. '70-'80];
- Most important property: quantum stress-energy tensor can be renormalized w.r.t. a Hadamard state;
- Since [Radzikowski '96], Hadamard condition formulated in terms of wave front set.
- Problem: few examples of Hadamard states have been constructed.

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Overview

- We reconsider the construction of Hadamard states on space-times with metric well-behaved at spatial infinity;
- Working on a fixed Cauchy surface, we can use rather standard pseudo-differential analysis.
- We construct a large class of Hadamard states with ΨDO two-point functions, in particular all pure Hadamard states.
- We give a new construction of Hadamard states on general globally hyperbolic space-times.

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Quasi-free states: neutral case

Let $(\mathcal{X}, \sigma)$ be a symplectic space and $\mathcal{A}$ its Weyl CCR C^*-algebra, generated by elements $W(f)$, $f \in \mathcal{X}$, with

$$W(f)^* = W(-f), \quad W(f)W(g) = \mathrm{e}^{-\mathrm{i}\sigma(f,g)/2}W(f+g), \quad f, g \in \mathcal{X}.$$

A state ω on $\mathcal{A}$ is quasi-free if there is a symmetric form η s.t.

$$\omega(W(f)) = \mathrm{e}^{-\frac{1}{2}\eta(f,f)}, \qquad f \in \mathcal{X}.$$

- A symmetric form η on $\mathcal{X}$ defines a quasi-free state iff the two-point function $\lambda = \eta_{\mathbb{C}} + \frac{\mathrm{i}}{2}\sigma_{\mathbb{C}}$ satisfies $\lambda \geq 0$.
- A symmetric form λ on $\mathbb{C}\mathcal{X}$ is the two-point function of a quasi-free state iff $\lambda \geq 0$ and $\lambda \geq \mathrm{i}\sigma_{\mathbb{C}}$.

The field operators $\phi(f)$ in the GNS rep. of ω satisfy

$$[\phi(f), \phi(g)] = \mathrm{i}\sigma(f,g), \quad \omega(\phi(f)\phi(g)) = \lambda_{\mathbb{R}}(f,g).$$

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Quasi-free states: charged case

Let $(\mathcal{Y}, \sigma)$ be a complex symplectic space (with some complex structure j) and $\mathcal{A}$ the Weyl CCR C^*-algebra of $(\mathcal{Y}_{\mathbb{R}}, \mathrm{Re}\sigma)$.

A quasi-free state ω on $\mathcal{A}$ is gauge-invariant if

$$\omega(W(y)) = \omega(W(\mathrm{e}^{\mathrm{j}\theta} y)), \quad 0 \leq \theta < 2\pi, \quad y \in \mathcal{Y}.$$

Let $\phi(y)$ be the ('neutral') field operators in the GNS rep. of ω. The charged fields:

$$\psi(y) := \frac{1}{\sqrt{2}}(\phi(y) + \mathrm{i}\phi(\mathrm{j}y)), \quad \psi^*(y) := \frac{1}{\sqrt{2}}(\phi(y) - \mathrm{i}\phi(\mathrm{j}y))$$

$$[\psi(y_1), \psi^*(y_2)] = \mathrm{i}\sigma(y_1, y_2), \quad \omega(\psi(y_1)\psi^*(y_2)) =: \lambda(y_1, y_2)$$
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Quasi-free states: unitary group

Let $(\mathcal{Y}, \sigma)$ be a complex symplectic space and $\mathcal{A}$ the Weyl CCR C^*-algebra. The unitary group:

$$U(\mathcal{Y}, \mathrm{i}\sigma) = \{u : \ u^* \sigma u = \sigma\}.$$

Recall that λ is the two-point function of a gauge-invariant quasi-free state on $\mathcal{A}$ iff

$$\text{(Pos)} \quad \boxed{\lambda \geq 0 \text{ and } \lambda \geq \mathrm{i}\sigma}.$$

- If λ satisfies (Pos) then so does $u^* \lambda u$ for any $u \in U(\mathcal{Y}, \mathrm{i}\sigma)$.
- If λ_1 and λ_2 are two-point functions of pure states, then there exists $u \in U(\mathcal{Y}, \mathrm{i}\sigma)$ such that $\lambda_2 = u^* \lambda_1 u$.

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Klein-Gordon equations

Consider a globally hyperbolic space-time $(M, g_{\mu\nu}dx^{\mu}dx^{\nu})$.
We fix a smooth vector potential $A_{\mu}(x)dx^{\mu}$ and a mass term.

- Klein-Gordon operator:

$$P(x, D_x) = |g|^{-\frac{1}{2}}(\partial_{\mu} + \mathrm{i}A_{\mu})|g|^{\frac{1}{2}}g^{\mu\nu}(\partial_{\nu} + \mathrm{i}A_{\nu}) + m^2,$$

  where $|g| = \det[g_{\mu\nu}]$, $[g^{\mu\nu}] := [g_{\mu\nu}]^{-1}$.

- $P(x, D_x)$ admits unique advanced/retarded fundamental solutions $E_{\pm}$ solving:

$$P(x, D_x) \circ E_{\pm} = 1\!\!1,$$

$$\mathrm{supp} E_{\pm} f \subset J^{\pm}(\mathrm{supp} f), \ f \in C_0^{\infty}(M),$$

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Symplectic space of solutions

Let $\mathrm{Sol}_{\mathrm{sc}}(P) \subset C^\infty(M)$ be the space of smooth space-compact solutions of

$$\text{(KG)} \quad P(x, D_x)\phi = 0.$$

- $E = E_+ - E_-$, called the commutator function.
- One has $\mathrm{Sol}_{\mathrm{sc}}(P) = EC_0^\infty(M)$.

$(\mathrm{Sol}_{\mathrm{sc}}(P), E)$ is our complex symplectic space. We look for Λ s.t.

$$\Lambda \geq 0 \text{ and } \Lambda \geq \mathrm{i}E \quad +\text{Hadamard condition}.$$

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Symplectic space of Cauchy data

Fix a Cauchy hypersurface Σ and set

$$\begin{aligned}\rho: \quad \mathrm{Sol}_{\mathrm{sc}}(P) &\to C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma) \\ \phi &\mapsto (\phi_{|\Sigma}, \mathrm{i}^{-1} n^\mu(\nabla_\mu + \mathrm{i}A_\mu)\phi_{|\Sigma}) =: (\rho_0\phi, \rho_1\phi).\end{aligned}$$

Denote by σ the canonical symplectic form on $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$:

$$\sigma(f,g) := -\mathrm{i}\int_\Sigma (\overline{f_0} g_1 + \overline{f_1} g_0) ds, \ f, g \in C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma),$$

- One has $E(u_1, u_2) = \sigma(\rho \circ Eu_1, \rho \circ Eu_2)$ for $u_1, u_2 \in C_0^\infty(M)$.
- Hence $(C_0^\infty(M), E)$ is isomorphic to $(C_0^\infty(\Sigma) \otimes \mathbb{C}^2, \sigma)$.

We can thus work with $(C_0^\infty(\Sigma) \otimes \mathbb{C}^2, \sigma)$ and look for λ s.t.

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Pseudo-differential operators

If $u \in \mathcal{D}'(\mathbb{R}^n)$, $WF(u)$ can be defined using pseudo-differential operators. Denote by $S^m(\mathbb{R}^{2d})$, $m \in \mathbb{R}$ the symbol class

$$a \in S^m(\mathbb{R}^{2d}) \text{ if } \partial_x^\alpha \partial_k^\beta a(x,k) \in O\big((1+|k|^2)^{\frac{m-|\beta|}{2}}\big),\ \alpha, \beta \in \mathbb{N}^d.$$

The Weyl quantization of a is the operator

$$a(x, D_x)u(x) := (2\pi)^{-d} \iint \mathrm{e}^{\mathrm{i}(x-y)k} a(\tfrac{x+y}{2}, k) u(y) dy dk.$$

$\Psi^m(\mathbb{R}^d) := \mathrm{Op}^{\mathrm{w}}(S^m(\mathbb{R}^{2d})) =$ pseudo-differential operators.

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Basic property: $a(x, D_x) \in \Psi^m(\mathbb{R}^d)$ maps $H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d)$. In particular $a(x, D_x) \in \Psi^{-\infty}(\mathbb{R}^d)$ maps to smooth functions. The characteristic set of a

$$\mathrm{Char}(a) := \{(x, k) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) : \ a_m(x, k) = 0\}.$$

$\to$ a is elliptic iff $\mathrm{Char}(a) = \emptyset$. Then there exists $a^{(-1)} \in \Psi^{-m}(\mathbb{R}^d)$ such that $a^{(-1)}a = 1\!\!1 \bmod \Psi^{-\infty}$.

Definition
$(x, k) \notin WF(u)$ (the wave front set) iff there exists $\chi \in C_0^\infty$ and $a \in S^0(\mathbb{R}^{2d})$ with $\chi(x) \neq 0$, $(x, k) \notin \mathrm{Char}(a)$ and

$$a(x, D_x)\chi u \in \mathcal{S}(\mathbb{R}^d).$$

$\to$ provides criteria for existence of $u \cdot v$, $u|_X$.

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$(x, k) \notin WF(u)$ (the wave front set) iff there exists $\chi \in C_0^\infty$ and $a \in S^0(\mathbb{R}^{2d})$ with $\chi(x) \neq 0$, $(x, k) \notin \mathrm{Char}(a)$ and

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Hadamard states

- Denote $p(x,\xi) = g^{\mu\nu}(x)\xi_\mu\xi_\nu$ the principal symbol of $P(x, D_x)$,
- $\mathcal{N} = p^{-1}(\{0\})$ *energy surface*, $\mathcal{N}_\pm = \{(x,\xi) \in \mathcal{N} \ : \ \xi \in V^*_\pm(x)\}$, *positive/negative energy surfaces*, $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_-$,
- For $X_i = (x_i, \xi_i)$ write $X_1 \sim X_2$ if $X_1, X_2 \in \mathcal{N}$, X_1, X_2 on the same Hamiltonian curve of p.

Definition ([Radzikowski '96])

Λ satisfies the Hadamard condition iff

$$WF(\Lambda)' \subset \{(X_1, X_2),\ X_1 \sim X_2 :\ X_1 \in \mathcal{N}^+\}.$$

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Examples of Hadamard states

- On general space-times + arbitrary smooth potentials, Hadamard states exist [Fulling, Narcowich, Wald '80].
- If (M, g) is asymptotically flat at null infinity, distinguished Hadamard states [Dappiaggi, Moretti, Pinamonti '09]
- The 'Unruh state' on Schwarzschild space-time [Dappiaggi, Moretti, Pinamonti '11].
- If (M, g) has a compact Cauchy surface, construction by pseudo-differential methods [Junker '97].
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Can one use a ΨDO-based construction for non-compact Cauchy surfaces? Can one construct *all* Hadamard states?

Model Klein-Gordon equation

We consider first the following Model case:

- $M = \mathbb{R}^{1+d}$, $x = (t, \mathrm{x}) \in \mathbb{R}^{1+d}$

$$a(t, \mathrm{x}, D_{\mathrm{x}}) = -\sum_{j,k=1}^{d} \partial_{\mathrm{x}^j} a^{jk}(x)\partial_{\mathrm{x}^k} + \sum_{j=1}^{d} b^j(x)\partial_{\mathrm{x}^j} - \partial_{\mathrm{x}^j}\overline{b}^j(x) + m(x),$$

- $[a^{jk}]$ uniformly elliptic, a^{jk}, b^j, m uniformly bounded with all derivatives in x, in bounded time intervals.
- We consider $P(x, D_x) = \partial_t^2 + a(t, \mathrm{x}, D_{\mathrm{x}})$.
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Parametrix for the Cauchy problem

Consider the Cauchy problem for P:

$$(C)\quad \begin{cases} \partial_t^2\phi(t) + a(t, \mathrm{x}, D_{\mathrm{x}})\phi(t) = 0, \\ \phi(0) = f_0, \\ \mathrm{i}^{-1}\partial_t\phi(0) = f_1, \end{cases}$$

- essential step to construct Hadamard states for P: characterize solutions with wavefront set in $\mathcal{N}^{\pm}$ in terms of their Cauchy data.
- method: construct a sufficiently explicit *parametrix* for the Cauchy problem (C).
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In the static case, (C) is solved by:

$$U(t)f = \frac{1}{2}\mathrm{e}^{\mathrm{i}ta^{1/2}}\left(f_0 + a^{-1/2}f_1\right) + \frac{1}{2}\mathrm{e}^{-\mathrm{i}ta^{1/2}}\left(f_0 - a^{-1/2}f_1\right).$$

Theorem

There exist $b(t) \in C^\infty(\mathbb{R}, \Psi^1(\mathbb{R}^d))$, $d \in \Psi^0(\mathbb{R}^d)$, $r \in \Psi^{-1}(\mathbb{R}^d)$ *(unique mod* $\Psi^{-\infty}(\mathbb{R}^d)$*), such that if*

$$U_+(t)f = \mathrm{Texp}(\mathrm{i}\textstyle\int_0^t b(s)\mathrm{d}s)d\left(f_0 + rf_1\right),$$
$$U_-(t)f = \mathrm{Texp}(-\mathrm{i}\textstyle\int_0^t b^*(s)\mathrm{d}s)d^*(f_0 - r^*f_1)$$

then $U(t)f := \left(U_+(t) + U_-(t)\right)f$ *solves the Cauchy problem* (C) *up to* C^∞.
Above, $b(t) = a^{1/2}(t) + \frac{1}{\sqrt{2}}(a^{-1/2}(t))\mathrm{i}\partial_t a^{1/2}(t)$ *mod* $\Psi^{-\infty}$.
Moreover, $WF(U_\pm(t)f) \subset \mathcal{N}_\pm$.

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In the proofs, we use:

- If $m \geq 0$, $a \in \Psi^m(\mathbb{R}^d)$ is elliptic in $\Psi^m(\mathbb{R}^d)$ and symmetric:
  $\rightarrow$ a is selfadjoint on $H^m(\mathbb{R}^d)$;
  $\rightarrow$ if $f \in S^p(\mathbb{R})$, $p \in \mathbb{R}$, then $f(a) \in \Psi^{mp}(\mathbb{R}^d)$ [Bony '96].
- Pseudo-differential operators act $a : \mathcal{E}' \rightarrow \mathcal{D}'$ — problems with compositions!
  $\rightarrow$ Instead consider $\mathcal{H} := \bigcap_m H^m$ and $a : \mathcal{H}' \rightarrow \mathcal{H}'$.
- Transport equations:
  $\rightarrow$ Fix $a \in \Psi^0(\mathbb{R}^d)$. Equations of the form

  $$b = a + F(b) \bmod \Psi^{-\infty}$$

  where $F : \Psi^m(\mathbb{R}^d) \rightarrow \Psi^{m-1}(\mathbb{R}^d)$, can be solved uniquely mod $\Psi^{-\infty}$.
- Egorov's theorem:
  $\rightarrow$ Gives the wave front set of $\mathrm{Texp}(\mathrm{i}\int_0^t b(s)\mathrm{d}s)u$, $u \in \mathcal{H}'(\mathbb{R}^d)$ for $b \in \Psi^1(\mathbb{R}^d)$.

We obtained $WF(U_{\pm}(t)f) \subset \mathcal{N}_{\pm}$. First consequence:
Define finite energy solutions

$$\mathrm{Sol}_E(P) := \{\phi \in C^0(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d)) : \ P\phi = 0\},$$

and positive/negative wavefront set solutions

$$\mathrm{Sol}_E^+(P, r) := \{\phi \in \mathrm{Sol}_E(P) : \ \phi(0) = -\mathrm{i}\, r\partial_t\phi(0))\},$$
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Theorem

One has $\pm\mathrm{i}\sigma > 0$ on $\mathrm{Sol}_E^{\pm}(P, r)$, and the spaces $\mathrm{Sol}_E^{\pm}(P, r)$ are symplectically orthogonal.

This decomposition depends on the choice of r. There is no distinguished one, but we can restrict to the set:

$$\mathcal{R} := \{r \in \Psi^{-1} : \ r = b^{*(-1)} + \Psi^{-\infty}, \ c\, a^{-1/2} \le r + r^* \le Ca^{-1/2}\}.$$

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Construction of Hadamard states

Once having fixed $r \in \mathcal{R}$ (not unique in the construction!), set

$$T(r) := (r + r^*)^{-\frac{1}{2}} \begin{pmatrix} \mathbb{1} & r \\ \mathbb{1} & -r^* \end{pmatrix}.$$

It diagonalizes the symplectic form:

$$\tilde{\sigma} := (T(r)^{-1})^* \circ \sigma \circ T(r)^{-1} = \begin{pmatrix} -\mathrm{i}\mathbb{1} & 0 \\ 0 & \mathrm{i}\mathbb{1} \end{pmatrix}.$$

If λ is a form on $C_0^\infty(\mathbb{R}^d) \otimes \mathbb{C}^2$ (Cauchy data), set

$$\tilde{\lambda} := (T(r)^{-1})^* \circ \lambda \circ T(r)^{-1} =: \begin{pmatrix} \tilde{\lambda}_{++} & \tilde{\lambda}_{+-} \\ \tilde{\lambda}_{-+} & \tilde{\lambda}_{--} \end{pmatrix}$$

Hadamard states with ΨDO two-point function

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Theorem

Let λ be a form with $\Psi\mathrm{DO}$ entries. Then Λ satisfies the Hadamard condition iff:

$$\tilde{\lambda}_{+-}, \tilde{\lambda}_{-+}, \tilde{\lambda}_{--} \in \Psi^{-\infty}(\mathbb{R}^d).$$

To get states, we need additionally $\tilde{\lambda} \geq 0$, $\tilde{\lambda} \geq \mathrm{i}\tilde{\sigma}$.

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Pure Hadamard states

Theorem

Let λ be a form with $\Psi\mathrm{DO}$ entries. It defines a Hadamard and pure state iff there exists $c_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^d)$ s.t.

$$\begin{aligned}\tilde{\lambda}_{++} &= \mathbb{1} + c_{-\infty} c_{-\infty}^*, \\ \tilde{\lambda}_{--} &= c_{-\infty}^* c_{-\infty}, \\ \tilde{\lambda}_{+-} &= \tilde{\lambda}_{-+}^* = c_{-\infty}(\mathbb{1} + c_{-\infty}^* c_{-\infty})^{1/2}\end{aligned}$$

Choose $c_{-\infty} = 0$ above. The corresponding two-point function is:

$$\lambda(r) = \begin{pmatrix} (r + r^*)^{-1} & -(r + r^*)^{-1} r^* \\ -r(r + r^*)^{-1} & r(r + r^*)^{-1} r^* \end{pmatrix}$$

and defines the canonical Hadamard state (associated to r).

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Let $\mathcal{Y} = \mathcal{H}(\mathbb{R}^d) \otimes \mathbb{C}^2$ and recall $U(\mathcal{Y}, \mathrm{i}\sigma)$ consists of transformations which preserve σ.

One can characterize the elements $U(\mathcal{Y}, \mathrm{i}\sigma)$ which preserve the Hadamard condition. There is a remarkable large subgroup:

$$U_{-\infty}(\mathcal{Y}, \mathrm{i}\sigma) := \{u \in U(\mathcal{Y}, \mathrm{i}\sigma) : \ u - \mathbb{1} \in \Psi^{-\infty}(\mathbb{R}^d) \otimes M_2(\mathbb{C})\}.$$

Theorem

Define a group $\mathcal{G}$ by

$$\mathcal{G} = \{(g, f) : \ g - \mathbb{1}, \ f \in \Psi^{-\infty}, \ g, g^* \in GL(L^2(\mathbb{R}^d)), f = -f^*\},$$

$$\mathrm{Id} = (\mathbb{1}, 0), \quad G_2 G_1 = (g_2 g_1, (g_2^*)^{-1} f_1 g_2^{-1} + f_2) \text{ for } G_i = (g_i, f_i).$$

There is a group homomorphism $\mathcal{G} \ni G \mapsto u_G \in U_{-\infty}(\mathcal{Y}, \mathrm{i}\sigma)$ and a transitive group action $\mathcal{G} \ni G \mapsto \alpha_G(r) \in \mathcal{R}$ such that

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One can characterize the elements $U(\mathcal{Y}, \mathrm{i}\sigma)$ which preserve the Hadamard condition. There is a remarkable large subgroup:

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Theorem

Define a group $\mathcal{G}$ by

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Static case

Consider the *static* case $a(t, \mathrm{x}, D_{\mathrm{x}}) = a(\mathrm{x}, D_{\mathrm{x}})$ independent on t. Then one can define the ground state and thermal state. In this case — preferred choice $r = a^{-1/2} \in \mathcal{R}$.

- ground state:

$$\tilde{\lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

- thermal state:

$$\tilde{\lambda}_{\beta} = \begin{pmatrix} \mathrm{e}^{-\beta a^{1/2}}(\mathbb{1} - \mathrm{e}^{-\beta a^{1/2}})^{-1} & 0 \\ 0 & (\mathbb{1} - \mathrm{e}^{-\beta a^{1/2}})^{-1} \end{pmatrix}.$$

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Arbitrary globally hyperbolic space-times

- $M = \mathbb{R} \times \Sigma$.
- Choose an open set Ω in M, and open, pre-compact sets U_n, $\tilde{U}_n$ in Σ such that:

  (i) $U_n \Subset \tilde{U}_n$, $\bigcup_n U_n = \Sigma$,

  (ii) $\tilde{U}_n$ are coordinate charts for Σ,

  (iii) $y \in \Omega,\ J(y) \cap U_n \neq \emptyset \ \Rightarrow\ y \in]-\delta_n, \delta_n[\times \tilde{U}_n =: \tilde{\Omega}_n$,

  (iv) Ω is a neighborhood of Σ in M.

- Fix a partition of unity $1 = \sum_n \chi_n^2$ of Σ, with $\chi_n \in C_0^\infty(U_n)$.

We have $\sigma = \sum_n \chi_n^* \sigma \chi_n$. Now we set $\lambda := \sum_{n \in \mathbb{N}} \chi_n^* \lambda_n \chi_n$, where λ_n are obtained by transporting Hadamard states constructed on $]-\delta_n, \delta_n[\times V_n \subset \mathbb{R} \times \mathbb{R}^d$ along coordinate maps $\varphi_n : \tilde{U}_n \to V_n$.

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Outlook

- Construction of Hadamard states on arbitrary globally-hyperbolic space-times.
- Transparent results if metric components well-behaved at spatial infinity — explicit description of pure quasi-free states $\leftrightarrow$ useful to investigate their local properties;
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  - replace space-like Cauchy surface by characteristic one;
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