Massless Wigner particles in conformal field theory are free

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Introduction

Is there any nontrivial CFT in 4 dimensions?

- Baumann, 1982: asymptotically complete, dilation-covariant scalar field is free.
- Weinberg, 2012: any conformal field which creates massless particle is free.

cf. complementary approach by Nikolov, Rehren, Todorov...

Main result

In any **conformal** net, the massless particle spectrum is generated by a free field subnet. The free field net decouples from the rest as a tensor product component if it is scalar (**No asymptotic completeness, no field**).

cf. in 2 dimensions, massless wave spectrum in CFT is generated by a tensor product subnet (T. 2012). A tensor product net may have a nontrivial extension (Rehren 2000, Kawahigashi-Longo 2004).

Modular theory in CFT

The conformal group $\mathscr{C}$ is generated by Poincaré transformations, dilations and special conformal transformations. They contain:

$$\Lambda_t a_\pm = \frac{(1+a_\pm) - e^{-2\pi t}(1-a_\pm)}{(1+a_\pm) - e^{-2\pi t}(1+a_\pm)}, \quad a\pm = a_0 \pm a_1.$$

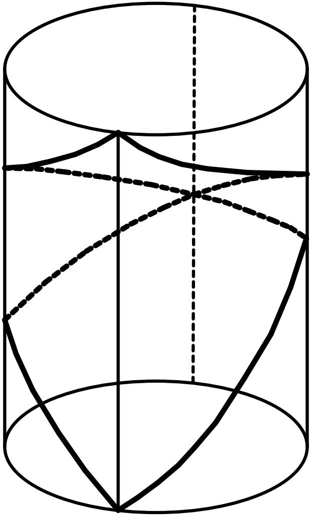
A **conformal net** is a net of von Neumann algebras $\{\mathcal{A}(O)\}$ parametrized by O, subject to the standard requirements and local covariance by a representation $U(g)$ of $\widetilde{\mathscr{C}}$, the universal covering of $\mathscr{C}$.

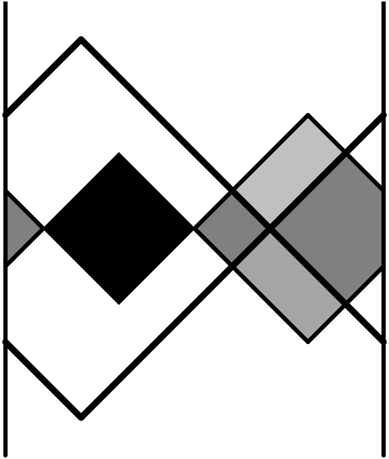
Theorem (Brunetti-Guido-Longo, 1993)

Any conformal net extends to the cylinder $\widetilde{M} = S^3 \times \mathbb{R}$ and Haag duality holds on $\widetilde{M}$. The modular group for the following regions are:

- *The future lightcone V_+: dilations.*
- *The standard wedge W: the boosts which preserve W.*
- *The standard double cone $|a_0| + |a| < 1$: Λ_t above.*

The spacetime cylinder





Unitary projective representations of the conformal group

The conformal group $\mathscr{C}$ is locally isomorphic to $\mathrm{SU}(2,2)$, whose maximally compact subgroup is $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$. Accordingly, the representations are classified by (d, j_1, j_2).

Theorem (Mack 1977)

Irreducible unitary projective representations of $\mathscr{C}$ with positive energy are:

- *trivial representation* $d = j_1 = j_2 = 0$
- $j_1 \neq 0 \neq j_2$, $d > j_1 + j_2 + 2$. $m > 0$ *and* $s = |j_1 - j_2|, \cdots j_1 + j_2$
- $j_1 \neq 0 \neq j_2$, $d = j_1 + j_2 + 2$. $m > 0$ *and* $s = j_1 + j_2$
- $j_1 j_2 = 0$, $d > j_1 + j_2 + 1$. $m > 0$ *and* $s = j_1 + j_2$
- $j_1 j_2 = 0$, $d = j_1 + j_2 + 1$. $m = 0$, *helicity* $j_1 - j_2$.

The **only massless representations** are the last ones.
Main result: if the representation U of a CFT contains one of them, then it is generated by the free field subnet.

Scattering theory for massless particles

- $\mathcal{A}$: a Poincaré covariant net.
- $U(\tau)$: representation of translations, whose spectral projection of the boundary of the future lightcone is nontrivial.

For $x \in \mathcal{A}(O)$ which is smooth under translation,

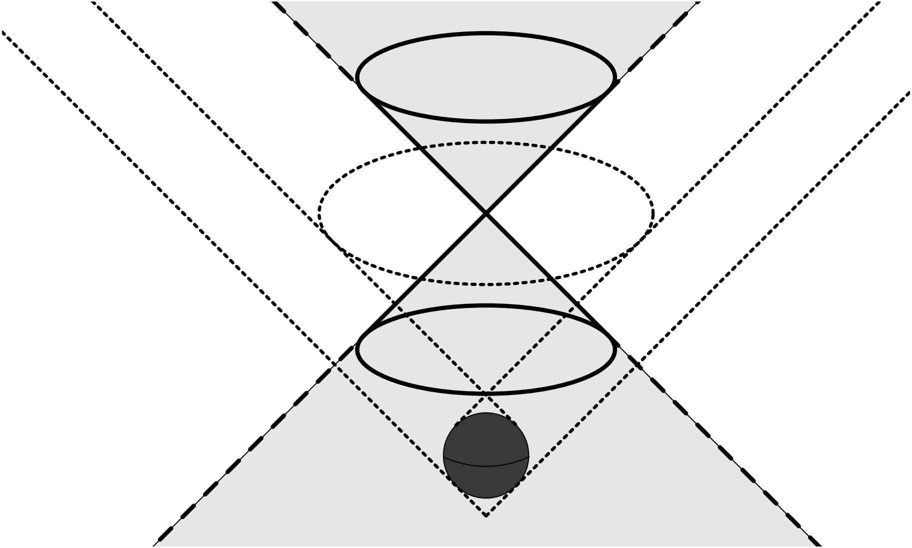
$$\Phi^{\mathrm{out}}(x) = \lim_{T\to\infty} \int dt\, h_T(t) \int d\omega(n)\, t\mathrm{Ad}\, U(\tau(t, tn))(\partial_0 x),$$

where h_T is a certain regularizing function, $d\omega$ is the rotation-invariant measure on S^3, ∂_0 is the time-derivative.

Theorem (Buchholz 1977, T. in preparation)

- $\Phi^{\mathrm{out}}(x)$ *is self-adjoint and* $\mathcal{A}(V_{O,+})\Omega$ *is a core, where* $V_{O,+}$ *is the future tangent of* O.
- $\mathrm{Ad}\, U(g)(\Phi^{\mathrm{out}}(x)) = \Phi^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))$. *This holds also for a conformal transformation* g *if* $\mathcal{A}$ *is a conformal net.*

Approximating asymptotic field



Proof under Global Conformal Invariance

Simpler proof under stronger assumption.

Global Conformal Invariance: the action of $\widetilde{\mathscr{C}}$ factors through $\mathscr{C}$ and the net is defined on the compactified Minkowski space $\bar{M}$.
E.g. the free scalar field.

Consequence: $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute when O_1 and O_2 are timelike separated.
Moreover, $\mathcal{A}(V_+)' = \mathcal{A}(V_-)$ because of Takesaki's theorem: by GCI $\mathcal{A}(V_-) \subset \mathcal{A}(V_+)'$ and the dilations are the modular group for both, the vacuum Ω is cyclic for both, then they must coincide.

Proof under Global Conformal Invariance

We define the free subnet $\mathcal{A}^{\mathrm{out}}(O) := \{e^{i\Phi^{\mathrm{out}}(x)} : x \in \mathcal{A}(O)\}''$. This is **conformally covariant with respect to the same** $U(g)$.

By the scattering theory, $\mathcal{A}^{\mathrm{out}}(V_-) \subset \mathcal{A}(V_+)' = \mathcal{A}(V_-)$. By conformal covariance of the both nets, $\mathcal{A}^{\mathrm{out}}$ is a **subnet** of $\mathcal{A}$ and generate all the massless particle spectrum by definition.

What is the structure of the full net $\mathcal{A}$?

Decoupling of the free subnet

If the particle spectrum is only scalar, then the free subnet $\mathcal{A}^{\rm out}$ has no nontrivial DHR sector (Araki 1963, Driessler 1979) and has split property (Buchholz-Jacobi 1978, Buchholz-Wichmann 1977).

Theorem

Let $\mathcal{F} \subset \mathcal{A}$ be an inclusion of conformal nets. If $\mathcal{F}$ has split property and has no nontrivial DHR sector, then $\mathcal{A}(O) = \mathcal{F}(O) \vee \mathcal{C}_0(O)$, where $\mathcal{C}_0(O) = \mathcal{A}(O) \cap \mathcal{F}(O)'$ is the coset net.

Proof (cf. Carpi-Conti 2001): by assumption, $\mathcal{F}(O) = \pi_0(\mathcal{F}(O)) \otimes \mathbb{C}\mathbb{1}$, $\mathcal{F}(O) \subset \mathcal{A}(O) = \pi_0(\mathcal{F}(O)) \otimes \mathcal{B}(\mathcal{K})$.
Since $\mathcal{F}(O)$ is a factor, $\mathcal{A}(O) = \pi_0(\mathcal{F}(O)) \otimes \mathcal{C}_0(O)$ (Ge-Kadison 1993).

Corollary: $\mathcal{A}(O) = \mathcal{A}^{\rm out}(O) \otimes \mathcal{C}(O)$

General case (work in progress)

Without GCI, not necessarily $\mathcal{A}(V_-) = \mathcal{A}(V_+)'$, so it is not clear whether the asymptotic algebra $\mathcal{A}^{\mathrm{out}}(V_-)$ is a subalgebra of $\mathcal{A}(V_-)$.

Use **directed** asymptotic field (suggested by Buchholz 1977).

For a smooth function f on S^3,

$$\Phi_f^{\mathrm{out}}(x) = \lim_{T\to\infty} \int dt\, h_T(t) \int d\omega(n)\, f(n) t \mathrm{Ad}\, U(\tau(t, tn))(\partial_0 x),$$

then it holds that

$$\Phi_f^{\mathrm{out}}(x)\Omega = P_1 f\left(\frac{\mathbf{P}}{|\mathbf{P}|}\right)\Omega.$$

The resolvents $R_\pm(\Phi_f^{\mathrm{out}}(x))$ is contained in a spacelike cone. Especially, one can obtain an asymptotic field **contained in the spacelike complement of a double cone** O: $R_\pm(\Phi_f^{\mathrm{out}}(x)) \in \mathcal{A}(O')$.

General case (work in progress)

For a fixed double cone O_1, take all such $R_\pm(\Phi_f^{\mathrm{out}}(x)) \in \mathcal{A}(O_1')$. This set is invariant under rotations, but may be not invariant under the modular group.

Take a subalgebra $\mathcal{N} = \{\mathrm{Ad}\,\Delta^{it}(R_\pm(\Phi_f^{\mathrm{out}}(x)))\}'' \subset \mathcal{A}(O_1)' = \mathcal{A}(O_1^{\mathrm{d}})$, where the latter is the spacelike complement on $\widetilde{M}$.

Define a net $\mathcal{A}^{\mathrm{out}}(O) = \mathrm{Ad}\,U(g)(\mathcal{N})$ with g such that $gO_1^{\mathrm{d}} = O$. This is a **well-defined covariant subnet** of $\mathcal{A}$ because of rotation invariance of $\mathcal{N}$.

By Haag duality and Reeh-Schlieder property of $\mathcal{A}^{\mathrm{out}}$ defined before, one has $\mathcal{A}^{\mathrm{out}} = \mathcal{A}^{\mathrm{dir}}$. In particular, $\mathcal{A}^{\mathrm{out}} \subset \mathcal{A}$. The rest is as before.

Conclusion

In CFT, **massless particles are free** and **they decouple if scalar**.

Open problems:

Can one say anything about $m > 0$ spectrum?

In dilation-covariant net? Does conformal covariance follow from dilation covariance?

Non free CFT? **Supersymmetric Yang-Mills**? Interacting massless nets by twisting? (cf. Tanimoto 2012, 2013 (**local**), Bischoff-Tanimoto 2012)

Technical appendix

Theorem (Buchholz 1977, T. in preparation)

- $\Phi^{\mathrm{out}}(x)$ *is self-adjoint and* $\mathcal{A}(V_{O,+})\Omega$ *is a core, where* $V_{O,+}$ *is the future tangent of* O.
- $\mathrm{Ad}\, U(g)(\Phi^{\mathrm{out}}(x)) = \Phi^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))$ *for* $g \in \widetilde{\mathscr{C}}$.

- if $x \in \mathcal{A}_{N_0}$, then the first part is ok.
- find $\{x_m\} \subset \mathcal{A}_{N_0}(V_{O,-})$ s.t. $P_1 x_m \Omega \to P_1 x \Omega = \xi$ (a la Buchholz).
- $\Phi^{\mathrm{out}}(x_m)$ is convergent in the strong resolvent sense to a self-adjoint operator $\Phi^{\mathrm{out}}(\xi)$, which one can easily calculate on $\mathcal{A}(V_{O,+})\Omega$.
- $\mathcal{A}(V_{O,+})\Omega$ is a core. first, $\{y \cdot \xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n\}$ is a core by Nelson. $\xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ can be reached by $\mathcal{A}(V_{O,+})\Omega$ since $\|\Phi^{\mathrm{out}}(x)^2\Omega\| < \infty$.
- $\Phi^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))$ is an extension of $\mathrm{Ad}\, U(g)(\Phi^{\mathrm{out}}(x))$ on $\mathcal{A}(gV_{O,+})\Omega$, where g preserves O and $V_{O,g}$. Covariance is OK also for $g \in \mathcal{P}_+^\uparrow$ and dilations, so for $\widetilde{\mathscr{C}}$.