Scale-Invariant Curvature Fluctuations from an Extended Semiclassical Gravity

Pinamonti & Siemssen, arXiv:1303.3241 [gr-qc]

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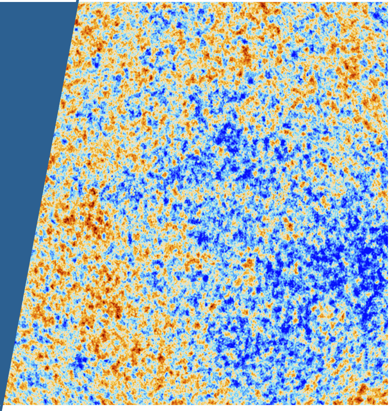


May 31, 2013 @ LQP32 in Wuppertal

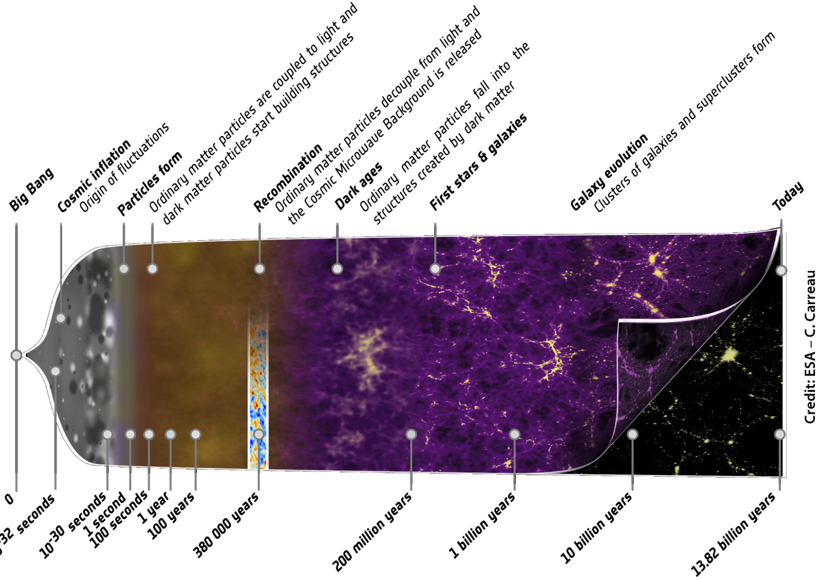
I

Introduction

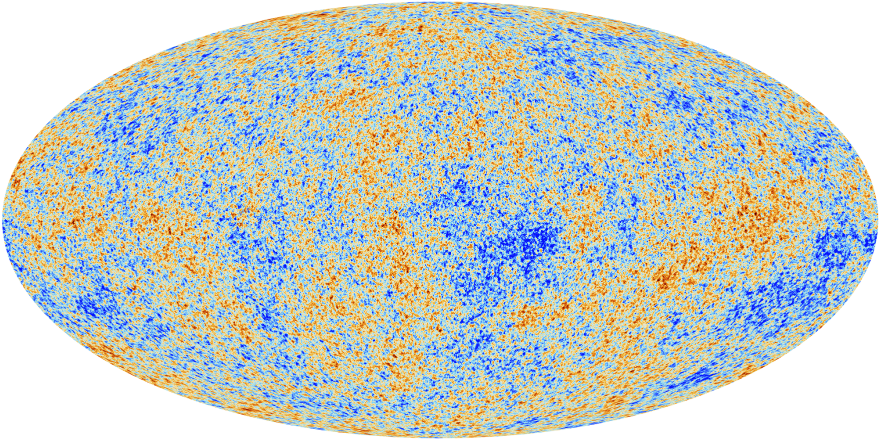
In the beginning the Universe was created. This has made a lot of people very angry and has been widely regarded as a bad move.

— Douglas Adams, The Restaurant at the End of the Universe

Big Bang, Inflation, ...



Cosmic microwave background



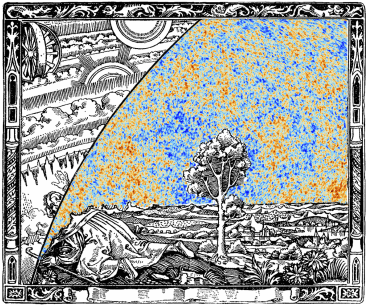
The anisotropies of the **CMB** as observed by the **Planck** space telescope
Credit: ESA and the Planck Collaboration

CMB fluctuations and cosmological parameters

The CMB temperature map is usually expanded in spherical harmonics:

$$T(\theta, \phi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi)$$

- $l = 0$ Mean temperature of $2.7255 \pm 0.0006\,\mathrm{K}$
- $l = 1$ Movement of the Earth relative to CMB ($\sim 10^{-3}$ K)
- $l \geq 2$ Density perturbations at last scattering ($\sim 10^{-5}$ K)



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Initial curvature fluctuations affect the density perturbations seen in the higher-order multipoles.
$\rightsquigarrow$ **Measurements constrain the initial curvature perturbations**:

$$P(k) = \frac{A_s}{k^3} \left(\frac{k}{k_0} \right)^{n_s - 1} \quad \text{with} \quad \left\langle \widehat{\Psi}(\vec{k})\, \widehat{\Psi}(\vec{k}') \right\rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}')\, P(k)$$

Planck measures $n_s = 0.9616 \pm 0.0094$.

II

Extending the semiclassical Einstein equation

Semiclassical Einstein equation

Motivation: We want to describe the interaction between **quantum** fields and a classical gravitational field.

$\rightsquigarrow$ **Einstein equation with quantum matter as source:**

$$G_{ab} = \omega(:T_{ab}:) \qquad (c = \hbar = 8\pi\, \mathrm{G} = 1)$$

G_{ab} Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab}$

$:T_{ab}:$ Normally-ordered stress-energy tensor

ω State of the quantum field

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Problems:

- Equating a classical quantity with a probabilistic quantity only meaningful if the fluctuations are small.
- Range of applicability unkown. Limiting case of a quantum gravity?

Extending the semiclassical Einstein equation

Think of the Einstein tensor G_{ab} as a random field: Equate moments of G_{ab} and T_{ab} for a state ω defined on the background spacetime $(M, \overline{g})$ specified by $\langle G_{ab} \rangle$:

$$\begin{aligned}
\big\langle G_{ab}(x_1) \big\rangle &= \omega\big(:T_{ab}(x_1):\big) \\
\big\langle \delta G_{ab}(x_1)\, \delta G_{c'd'}(x_2) \big\rangle &= \tfrac{1}{2}\, \omega\big(:\delta T_{ab}(x_1)::\delta T_{c'd'}(x_2): + :\delta T_{c'd'}(x_2)::\delta T_{ab}(x_1):\big) \\
&\vdots \\
\big\langle (\delta G)^{\boxtimes n}(x_1, \dots, x_n) \big\rangle &= \mathrm{Sym}\big[\omega\big(:\delta T:^{\boxtimes n}(x_1, \dots, x_n)\big)\big]
\end{aligned}$$

$$\text{with} \quad \delta G_{ab} = G_{ab} - \langle G_{ab} \rangle \quad \text{and} \quad :\delta T_{ab}: = :T_{ab}: - \,\omega(:T_{ab}:)$$

Similar to **stochastic gravity**, where a Gaussian stochastic source ξ_{ab} is added to the semiclassical Einstein equation *[Hu, Roura, Verdaguer, ...]*:

$$G_{ab} = \omega\big(:T_{ab}:\big) + \xi_{ab} \quad \text{with} \quad \langle \xi_{ab}\, \xi_{c'd'} \rangle = \mathrm{Sym}\big[\omega(:\delta T_{ab}::\delta T_{c'd'}:)\big]$$

III

Moments of the stress-energy tensor for φ

Conformally-coupled scalar field φ

The quantum matter: A **conformally-coupled massive scalar field** φ satisfying the Klein-Gordon equation

$$-\Box\,\varphi + \tfrac{1}{6}\,R\,\varphi + m^2\varphi = P\,\varphi = 0$$

in a **quasi-free Hadamard state** ω on $(M, \overline{g})$.

Remember: Hadamard states satisfy the **microlocal spectrum condition**

$$\mathrm{WF}(\omega_2) = \left\{(x, y, \xi, \eta) \in T^*(M \times M) \setminus 0 \mid (x, \xi) \sim (y, -\eta),\ \xi \triangleright 0\right\}$$

and are thus locally given by

$$\omega_2 = \lim_{\varepsilon \to 0^+} \left(\frac{U}{\sigma_\varepsilon} + V \ln \frac{\sigma_\varepsilon}{\lambda^2} \right) + W$$

- σ signed, squared, geodesic distance
- U, V depend on local geometry, $V = \sum_n V_n\, \sigma^n$
- W state-dependent part

Traced stress-energy tensor for φ (I)

A **quasi-free Hadamard state** ω for φ on $(M, \overline{g})$, *i.e.*,

$$\omega_2 = \lim_{\varepsilon \to 0^+} \left(\frac{U}{\sigma_\varepsilon} + \sum_n V_n \, \sigma^n \ln \frac{\sigma_\varepsilon}{\lambda^2} \right) + W,$$

results in

$$\omega(:T:) = -m^2 [W] + 2[V_1] + \alpha\, m^4 + \beta\, m^2 R + \gamma \,\Box\, R$$
$$\omega\big(:\delta T(x)::\delta T(y):\big) = 2 m^4 \omega_2^2(x, y)$$
$$\vdots$$

with $\quad T = \overline{g}^{ab}\, T_{ab} = -m^2 \varphi^2 + \tfrac{1}{3}\, \varphi\, P\, \varphi \quad$ and $\quad :\delta T_{ab}: = :T_{ab}: - \,\omega(:T_{ab}:)$

Note:

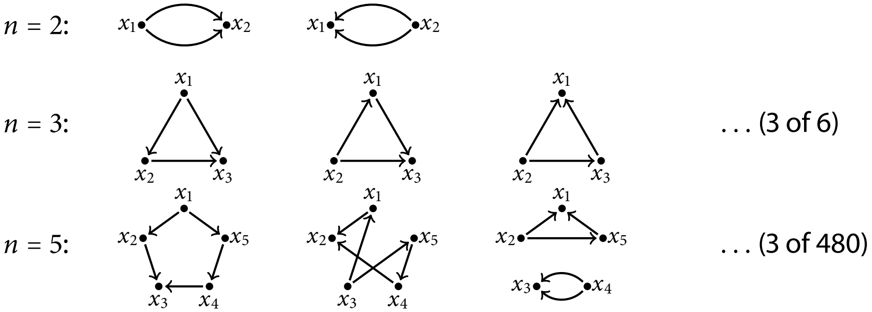
- No renormalization freedom beyond the first moment.
- ω_2^n is a well-defined distribution because $\mathrm{WF}(\omega_2)$ is convex.

Traced stress-energy tensor for φ (II)

Symmetrized higher moments:

$$\mathrm{Sym}\Big[\omega\big(:\delta T:^{\otimes n}(x_1,\dots,x_n)\big)\Big] = \frac{2^n}{n!}\, m^{2n} \left[\sum_G \prod_{i,j} \frac{\omega_2(x_i,x_j)^{\lambda^G_{ij}}}{\lambda^G_{ij}!}\right]$$

The sum is over all acyclical, directed graphs G with n vertices of degree 2:



IV

Fluctuations around a de Sitter universe

Background metric: exponentially expanding FLRW universe

$$\overline{g} = (H\tau)^{-2}\left(-\,\mathrm{d}\tau \otimes \mathrm{d}\tau + \delta_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j\right)$$

Task: Solve the (traced) first moment equation

$$\langle G \rangle = \overline{g}^{ab}\langle G_{ab}\rangle = -12H^2 = \omega(:T:)$$

Solution: Take as the state ω the Bunch–Davies vacuum

$$\omega_2(x_1, x_2) = \frac{m^2}{16\pi \sin(\pi \nu_+)}\, {}_2F_1\!\left(\nu_+, \nu_-; 2; \frac{\sigma_{\mathbb{M}}(x_1, x_2)}{4\tau_1\tau_2}\right)$$

where $2\nu_\pm = 3 \pm \sqrt{1 - 2m^2/H^2}$ and $\sigma_{\mathbb{M}}$ is the geodesic distance on Minkowski space.

Perturbed spacetime

Add **Newtownian perturbation** Ψ to the background metric $\bar{g}$:

$$g = (H\tau)^{-2}\big(-(1+2\Psi)\,\mathrm{d}\tau\otimes\mathrm{d}\tau + (1-2\Psi)\,\delta_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j\big).$$

Note:

- The perturbation Ψ is directly related to the Bardeen potentials.
- In single field inflation the same metric perturbation is used.
- The moments of Ψ can be constrained by observations.

The perturbed metric g yields the **linearized, traced, perturbed Einstein tensor**:

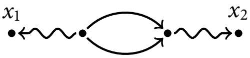
$$\delta G = \bar{g}^{ab}\big(G_{ab} - \langle G_{ab}\rangle\big) = 6H^2\tau^4\left(\frac{\partial^2}{\partial\tau^2} - \frac{1}{3}\vec{\nabla}^2\right)\tau^{-2}\Psi$$

NB: The trace is performed using the background metric.

Moments of Ψ

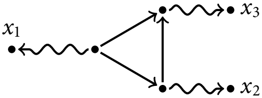
Inverting the differential equation for Ψ with the **retarded propagator** $\Psi = \Delta_R[\delta G]$, we can calculate the moments for Ψ ($\langle \Psi \rangle = 0$):

$$\langle \Psi(x_1)\, \Psi(x_2) \rangle = m^4 \left(\right.$$



$$\left. + \text{permutation} \right)$$

$$\langle \Psi(x_1)\, \Psi(x_2)\, \Psi(x_3) \rangle = \frac{2^3}{3!}\, m^6 \left(\right.$$



$$\left. + \text{permutations} \right)$$

... and analogously for higher moments.

Note:

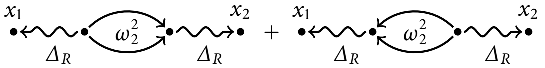
- Ψ is not a Gaussian random field.
- Only δT is taken as a source for Ψ.

V

Power spectrum and bispectrum of Ψ

Calculate the power spectrum $P(\tau, k)$, *i.e.*, the Fourier transform of $\langle \Psi \, \Psi \rangle$ at equal time τ:

$$\left\langle \Psi(x_1)\, \Psi(x_2) \right\rangle = m^4 \Big(\ldots \Big)$$



$$= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} P(\tau, k = |\vec{k}|)\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}\, \mathrm{d}^3\vec{k}$$

Power spectrum of Ψ: $P(\tau, k)$

Calculate the power spectrum $P(\tau, k)$, *i.e.*, the Fourier transform of $\langle \Psi\, \Psi \rangle$ at equal time τ:

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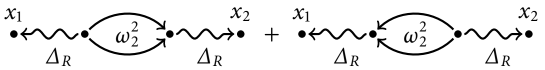


$$= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} P(\tau, k = |\vec{k}|)\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}\, \mathrm{d}^3\vec{k}$$

The square of the two-point function ω_2^2

We only use the **leading term** of the Bunch–Davies state

$$\omega_2(x_1, x_2) = H^2 \tau_1 \tau_2 \omega_{\mathbb{M}}(x_1, x_2) + \text{less singular terms}$$

and its Fourier-transformed square can be calculated as

$$\widehat{\omega^2_{\mathbb{M}}}(\tau_1, \tau_2, \vec{k}) = \frac{1}{16\pi^2} \int_k^\infty \mathrm{e}^{-\mathrm{i}p\,(\tau_1 - \tau_2)}\, \mathrm{d}p$$

NB: The less singular terms vanish for zero mass m.

Estimates of the power spectrum $P(\tau, k)$

Theorem (Useful estimate for large k)

The power spectrum P has the form

$$P(\tau, k) = \frac{\mathcal{P}(k\,\tau)}{k^3}$$

$$\text{with} \quad \mathcal{P}(k\,\tau) \leq \lim_{k\,\tau \to -\infty} \mathcal{P}(k\,\tau) = C = m^4\,\frac{3 - 2\sqrt{3}\,\operatorname{arccoth}\sqrt{3}}{192\pi^2}$$

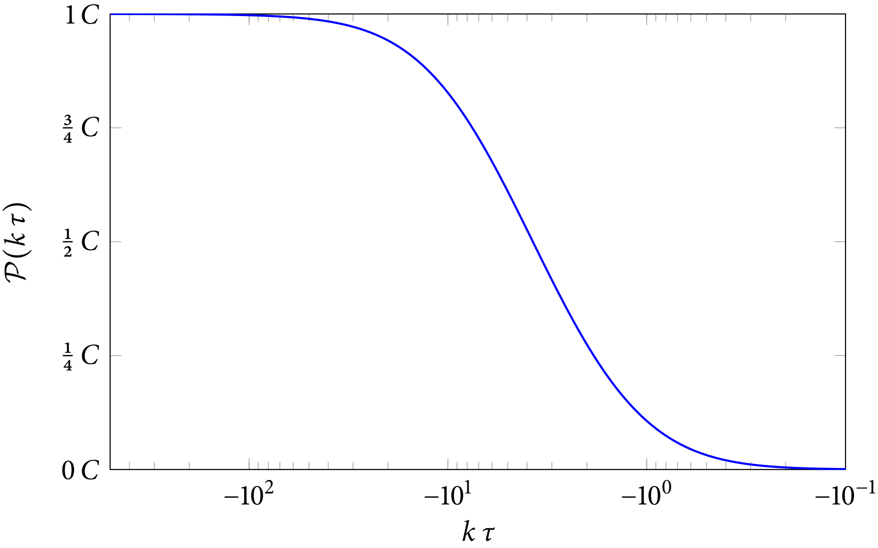
and is independent of the Hubble parameter H.

Theorem (Useful estimate for small k)

The power spectrum P satisfies

$$P(\tau, k) \leq \frac{m^4}{36\pi^2}\,\frac{\tau^2}{k} \quad \text{so that} \quad P(0, k) = 0.$$

Plot of the rescaled power spectrum $\mathcal{P}(k\,\tau)$



Bispectrum of Ψ: $B(\tau, \vec{k}_1, \vec{k}_2, \vec{k}_3)$

We want to calculate the bispectrum $B(\tau, \vec{k}_1, \vec{k}_2, \vec{k}_3)$, that is, the Fourier transform of $\langle \Psi\, \Psi\, \Psi \rangle$ at equal time τ:

$$\left\langle \Psi(\tau, \vec{x}_1)\, \Psi(\tau, \vec{x}_2)\, \Psi(\tau, \vec{x}_3) \right\rangle = \frac{1}{(2\pi)^9} \iiint_{\mathbb{R}^9} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\, B(\tau, \vec{k}_1, \vec{k}_2, \vec{k}_3) \times e^{i\,(\vec{k}_1\cdot\vec{x}_1 + \vec{k}_2\cdot\vec{x}_2 + \vec{k}_3\cdot\vec{x}_3)}\, d^3\vec{k}_1\, d^3\vec{k}_2\, d^3\vec{k}_3 .$$

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Theorem (Form of the bispectrum)

The bispectrum B has the form $(k_i = |\vec{k}_i|)$

$$B(\tau, \vec{k}_1, \vec{k}_2, \vec{k}_3) = \frac{\mathcal{B}(k_1\tau, k_2\tau, k_3\tau)}{k_1^2\, k_2^2\, k_3^2}$$

and is independent of the Hubble parameter H.

NB: Also in single field inflation a k^{-6} behavior is found *[Maldacena]*.

VI

Summary

Summary

- Proposed extension of semiclassical Einstein gravity

$$\Big\langle (\delta G)^{\boxtimes n}(x_1, \dots, x_n) \Big\rangle = \mathrm{Sym}\Big[\omega\Big({:}\delta T{:}^{\boxtimes n}(x_1, \dots, x_n)\Big)\Big]$$

- Applied extension to calculate curvature fluctuations in an exponentially expanding universe
- Results can be compared with observational cosmology: Almost scale-invariant curvature fluctuations are found
- Non-Gaussianities occur naturally

Fin.

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