Locally C* Algebras and C* Bundles

Daniel Paulino

II. Institüt für Theoretishe Physik, Universität Hamburg
&
Instituto de Matemática e Estatística, Universidade de São Paulo

Wuppertal, 01 June 2013

The Gelfand Theorem

A well know result from the theory of C^* algebras, the following theorem establishes a correspondence between locally compact topological spaces and commutative C^* algebras.

The Gelfand Theorem

A well know result from the theory of C^* algebras, the following theorem establishes a correspondence between locally compact topological spaces and commutative C^* algebras.

Gelfand Theorem

Given a commutative C^* algebra A there is a locally compact topological space X such that $A \approx C_0(X)$. Moreover, if the algebra A is unital, then the space X is compact.

X is constructed from the algebra A as a certain space of states or equivalently classes of irreducible representations.

X is constructed from the algebra A as a certain space of states or equivalently classes of irreducible representations.

A natural question then is:

- For a noncommutative algebra, can we construct an underling “space” for which it can be seen as an “algebra of functions”?

Different Spectra and the Gelfand Theorem

Equivalent ways to define the space X in the statement of Gelfand Theorem.

Different Spectra and the Gelfand Theorem

Equivalent ways to define the space X in the statement of Gelfand Theorem.

- The usual notion of *Spectrum* as the space of pure states, or of irreducible representations $\hat{A}$.

Different Spectra and the Gelfand Theorem

Equivalent ways to define the space X in the statement of Gelfand Theorem.

- The usual notion of *Spectrum* as the space of pure states, or of irreducible representations $\hat{A}$.
- The *Maximal Spectrum*, Max$A$, i. e. the space of maximal ideals.

Different Spectra and the Gelfand Theorem

Equivalent ways to define the space X in the statement of Gelfand Theorem.

- The usual notion of *Spectrum* as the space of pure states, or of irreducible representations $\hat{A}$.
- The *Maximal Spectrum*, $\mathrm{Max}A$, i. e. the space of maximal ideals.
- The *Primitive Spectrum*, $\mathrm{Prim}A$, composed by the ideals of the from $\ker \pi$ for π an irreducible representation of A.

Different Spectra and the Gelfand Theorem

Equivalent ways to define the space X in the statement of Gelfand Theorem.

- The usual notion of *Spectrum* as the space of pure states, or of irreducible representations $\hat{A}$.
- The *Maximal Spectrum*, $\mathrm{Max}A$, i. e. the space of maximal ideals.
- The *Primitive Spectrum*, $\mathrm{Prim}A$, composed by the ideals of the from $\ker \pi$ for π an irreducible representation of A.

The topology in the space is derived from the *Hull-Kernel Topology* in the ideal space of the algebra.

The Dauns-Hofmann Theorem

An important result about the primitive spectrum is the following:

The Dauns-Hofmann Theorem

An important result about the primitive spectrum is the following:

Dauns-Hofmann Theorem

Given a C^* algebra A, a function in $f \in C_b(\mathrm{Prim} A)$ and an element $a \in A$ there is a unique element, $f \cdot a$ in A defined by:

$$f \cdot a - f(\ker \pi)a \in \ker \pi \qquad \forall \ker \pi \in \mathrm{Prim} A$$

The Dauns-Hofmann Theorem

An important result about the primitive spectrum is the following:

Dauns-Hofmann Theorem

Given a C^* algebra A, a function in $f \in C_b(\mathrm{Prim}A)$ and an element $a \in A$ there is a unique element, $f \cdot a$ in A defined by:

$$f \cdot a - f(\ker \pi)a \in \ker \pi \qquad \forall \ker \pi \in \mathrm{Prim}A$$

This in turn defines an homomorphism $\Psi : C_b(\mathrm{Prim}A) \to \mathcal{Z}M(A)$, which is obviously unital.

Stone-Čech Compactification

For any topological space X, $C_b(X)$ is a commutative unital C^* algebra.

Stone-Čech Compactification

For any topological space X, $C_b(X)$ is a commutative unital C^* algebra.

The original Gelfand theorem allows us to construct a new compact space βX, such that

Stone-Čech Compactification

For any topological space X, $C_b(X)$ is a commutative unital C^* algebra.

The original Gelfand theorem allows us to construct a new compact space βX, such that

$$C_b(X) \approx C(\beta X)$$

Stone-Čech Compactification

For any topological space X, $C_b(X)$ is a commutative unital C^* algebra.

The original Gelfand theorem allows us to construct a new compact space βX, such that

$$C_b(X) \approx C(\beta X)$$

The space βX is called the *Stone-Čech Compactification* of X.

Stone-Čech Compactification

For any topological space X, $C_b(X)$ is a commutative unital C^* algebra.

The original Gelfand theorem allows us to construct a new compact space βX, such that

$$C_b(X) \approx C(\beta X)$$

The space βX is called the *Stone-Čech Compactification* of X.

The canonical map $\beta : X \to \beta X$, is such that, for every compact Y, every continuous map $F : X \to Y$ lifts to a map $\beta F : \beta X \to Y$ such that $F = \beta F \circ \beta$

For the primitive spectrum, Prim$A$, the Stone-Čech compactification is just the primitive spectrum, Prim$ZM(A)$, of the center of the multiplier algebra of A.

For the primitive spectrum, Prim$A$, the Stone-Čech compactification is just the primitive spectrum, Prim$ZM(A)$, of the center of the multiplier algebra of A.

The homomorphism Ψ above is lifted from the canonical isomorphism $\Phi : C(\mathrm{Prim} ZM(A)) \to ZM(A)$.

For the primitive spectrum, Prim$A$, the Stone-Čech compactification is just the primitive spectrum, Prim$ZM(A)$, of the center of the multiplier algebra of A.

The homomorphism Ψ above is lifted from the canonical isomorphism $\Phi : C(\text{Prim}ZM(A)) \to ZM(A)$.

We call the compact space Prim$ZM(A)$ the space of points of the algebra A and denote it by pt$A$.

C(X)-Algebras

For compact space X a $C(X)$-algebra is a C^* algebra A which is a module over the ring of functions $C(X)$ and for which the C^*-norm satisfies:

$$\|f \cdot a\| \leqslant \|f\|_\infty \|a\|$$

C(X)-Algebras

For compact space X a $C(X)$-algebra is a C^* algebra A which is a module over the ring of functions $C(X)$ and for which the C^*-norm satisfies:

$$\|f \cdot a\| \leqslant \|f\|_\infty \|a\|$$

Equivalently a $C(X)$-algebra can be defined as a C^* algebra with a homomorphism between $C(X)$ and the center of its multiplier algebra.

C(X)-Algebras

For compact space X a $C(X)$-algebra is a C^* algebra A which is a module over the ring of functions $C(X)$ and for which the C^*-norm satisfies:

$$\|f \cdot a\| \leqslant \|f\|_\infty \|a\|$$

Equivalently a $C(X)$-algebra can be defined as a C^* algebra with a homomorphism between $C(X)$ and the center of its multiplier algebra.

The Dauns-Hofmann theorem show that every C^* algebra A is a $C(\text{pt}A)$-algebra.

C^* Bundles

A C^* *Bundle* over a locally compact space X is a topological space $\mathcal{A}$ with a continuous and open surjection $\rho : \mathcal{A} \longrightarrow X$ such that:

C^* Bundles

A C^* *Bundle* over a locally compact space X is a topological space $\mathcal{A}$ with a continuous and open surjection $\rho : \mathcal{A} \longrightarrow X$ such that:

- Each fiber $\mathcal{A}_x = \rho^{-1}(x)$ is a C^*-algebra.

C^* Bundles

A C^* *Bundle* over a locally compact space X is a topological space $\mathcal{A}$ with a continuous and open surjection $\rho : \mathcal{A} \longrightarrow X$ such that:

- Each fiber $\mathcal{A}_x = \rho^{-1}(x)$ is a C^*-algebra.
- The maps $+, \cdot : \mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\cdot : \mathbb{C} \times \mathcal{A} \to \mathcal{A}$ and $* : \mathcal{A} \longrightarrow \mathcal{A}$ induced by the algebraic structure are all continuous.

C^* Bundles

A C^* *Bundle* over a locally compact space X is a topological space $\mathcal{A}$ with a continuous and open surjection $\rho : \mathcal{A} \longrightarrow X$ such that:

- Each fiber $\mathcal{A}_x = \rho^{-1}(x)$ is a C^*-algebra.
- The maps $+, \cdot : \mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\cdot : \mathbb{C} \times \mathcal{A} \to \mathcal{A}$ and $* : \mathcal{A} \longrightarrow \mathcal{A}$ induced by the algebraic structure are all continuous.
- The map $\| \cdot \| : \mathcal{A} \to \mathbb{R}$ defined by the norm is upper-semicontinuous.

For X compact, the space of continuous sections $\Gamma(\mathcal{A})$ of a C^* bundle, with the algebraic structure induced by the fibers and the norm

For X compact, the space of continuous sections $\Gamma(\mathcal{A})$ of a C^* bundle, with the algebraic structure induced by the fibers and the norm

$$\|\varphi\| = \sup_{x \in X} \|\varphi(x)\|$$

For X compact, the space of continuous sections $\Gamma(\mathcal{A})$ of a C^* bundle, with the algebraic structure induced by the fibers and the norm

$$\|\varphi\| = \sup_{x \in X} \|\varphi(x)\|$$

is a C^*-algebra.

For X compact, the space of continuous sections $\Gamma(\mathcal{A})$ of a C^* bundle, with the algebraic structure induced by the fibers and the norm

$$\|\varphi\| = \sup_{x \in X} \|\varphi(x)\|$$

is a C^*-algebra.

Moreover, it is clear that $\Gamma(\mathcal{A})$ is also a $C(X)$-algebra.

The Sectional Representation Theorem

In fact, section algebras are the only examples of $C(X)$-algebras.

The Sectional Representation Theorem

In fact, section algebras are the only examples of $C(X)$-algebras.

Sectional Representation Theorem

Given a C^* algebra A and a compact topological space X, the following statements are equivalent:

- The algebra A is a $C(X)$-algebra.
- There is an C^* bundle $\mathcal{A}$ over X such that $A \approx \Gamma(\mathcal{A})$.

The Generalized Gelfand Theorem

Combining both the results, we prove that every C^*-algebra is isomorphic to the section algebra of a C^* bundle over its space of points.

The Generalized Gelfand Theorem

Combining both the results, we prove that every C^*-algebra is isomorphic to the section algebra of a C^* bundle over its space of points.

The Generalized Gelfand Theorem

Given a C^* algebra A there is a C^* bundle $\mathcal{A} \to \mathrm{pt}A$, such that

$$A \approx \Gamma(\mathcal{A}).$$

Thus justifying the notation $\mathrm{pt}A$.

Commutative Case

What the is the meaning of this theorem for a commutative C^* algebra?

Commutative Case

What the is the meaning of this theorem for a commutative C^* algebra?

When A is unital, $\mathrm{Prim} A = \mathrm{pt} A$ is compact and $\mathrm{pt} A \times \mathbb{C}$ is the C^* bundle given by the generalized Gelfand theorem, so that:

Commutative Case

What the is the meaning of this theorem for a commutative C^* algebra?

When A is unital, $\mathrm{Prim}A = \mathrm{pt}A$ is compact and $\mathrm{pt}A \times \mathbb{C}$ is the C^* bundle given by the generalized Gelfand theorem, so that:

$$A \approx \Gamma(\mathrm{pt}A \times \mathbb{C}) \approx C(\mathrm{pt}A)$$

Commutative Case

What the is the meaning of this theorem for a commutative C^* algebra?

When A is unital, $\mathrm{Prim}A = \mathrm{pt}A$ is compact and $\mathrm{pt}A \times \mathbb{C}$ is the C^* bundle given by the generalized Gelfand theorem, so that:

$$A \approx \Gamma(\mathrm{pt}A \times \mathbb{C}) \approx C(\mathrm{pt}A)$$

recovering precisely the original commutative Gelfand theorem.

When A is nonunital one uses the canonical map β between $\mathrm{Prim}A$ and $\mathrm{pt}A = \beta\mathrm{Prim}A$.

When A is nonunital one uses the canonical map β between $\mathrm{Prim}A$ and $\mathrm{pt}A = \beta\mathrm{Prim}A$.

$$C_0(\mathrm{Prim}A) \approx \{f \in C(\mathrm{pt}A) \mid \forall x \notin \beta(\mathrm{Prim}A),\ f(x) = 0\}$$

When A is nonunital one uses the canonical map β between $\mathrm{Prim}A$ and $\mathrm{pt}A = \beta\mathrm{Prim}A$.

$$C_0(\mathrm{Prim}A) \approx \{f \in C(\mathrm{pt}A) \mid \forall x \notin \beta(\mathrm{Prim}A),\ f(x) = 0\}$$

Define a sub-bundle $\mathcal{A} \subset \mathrm{pt}A \times \mathbb{C}$ by

When A is nonunital one uses the canonical map β between $\mathrm{Prim}A$ and $\mathrm{pt}A = \beta\mathrm{Prim}A$.

$$C_0(\mathrm{Prim}A) \approx \{f \in C(\mathrm{pt}A) \,|\, \forall x \notin \beta(\mathrm{Prim}A),\ f(x) = 0\}$$

Define a sub-bundle $\mathcal{A} \subset \mathrm{pt}A \times \mathbb{C}$ by

$$\mathcal{A}_x = \begin{cases} \mathbb{C} & x \in \beta(\mathrm{Prim}A) \\ \{\emptyset\} & x \notin \beta(\mathrm{Prim}A) \end{cases}$$

When A is nonunital one uses the canonical map β between $\mathrm{Prim}A$ and $\mathrm{pt}A = \beta\mathrm{Prim}A$.

$$C_0(\mathrm{Prim}A) \approx \{f \in C(\mathrm{pt}A) \mid \forall x \notin \beta(\mathrm{Prim}A),\ f(x) = 0\}$$

Define a sub-bundle $\mathcal{A} \subset \mathrm{pt}A \times \mathbb{C}$ by

$$\mathcal{A}_x = \begin{cases} \mathbb{C} & x \in \beta(\mathrm{Prim}A) \\ \{\emptyset\} & x \notin \beta(\mathrm{Prim}A) \end{cases}$$

$C_0(\mathrm{Prim}A) \approx \Gamma(\mathcal{A})$ and our result reduces to the original Gelfand theorem for nonunital commutative C^*-algebras.

The previous remarks point to an interesting feature of the generalized Gelfand theorem.

The previous remarks point to an interesting feature of the generalized Gelfand theorem.

Due to that identification one can not distinguish between the algebra of functions vanishing at infinity and some subalgebra of the functions over the Stone-Čech compactification.

The previous remarks point to an interesting feature of the generalized Gelfand theorem.

Due to that identification one can not distinguish between the algebra of functions vanishing at infinity and some subalgebra of the functions over the Stone-Čech compactification.

This hints C^*-algebras are ill suited to deal with noncompact spaces.

Noncompactness through the algebra of functions

How does one deal then with noncompact spaces?

Noncompactness through the algebra of functions

How does one deal then with noncompact spaces?

The first hint comes from the consequences of noncompactness for the algebra of continuous functions.

Noncompactness through the algebra of functions

How does one deal then with noncompact spaces?

The first hint comes from the consequences of noncompactness for the algebra of continuous functions.

A locally compact space X is noncompact if and only if there is a continuous unbounded function $f \in C(X)$.

Noncompactness through the algebra of functions

How does one deal then with noncompact spaces?

The first hint comes from the consequences of noncompactness for the algebra of continuous functions.

A locally compact space X is noncompact if and only if there is a continuous unbounded function $f \in C(X)$.

So instead of imposing conditions over the behavior at infinity we should loosen our assumpions over the algebra of functions.

Sheaves and Conditions at Infinity

A different hint of this comes from sheaf theory.

Sheaves and Conditions at Infinity

A different hint of this comes from sheaf theory.

The functors C_0 or C_b do not define sheaves, the natural functor to work with is just C, the algebra of all continuous functions.

Sheaves and Conditions at Infinity

A different hint of this comes from sheaf theory.

The functors C_0 or C_b do not define sheaves, the natural functor to work with is just C, the algebra of all continuous functions.

To reproduce the sheaf structure of the commutative case, we need to allow for unbounded functions in our formalism.

Locally C^* Algebras

A *Locally C^*-algebra* is a complete locally convex $*$-algebra for which the topology can be generated by a system of C^*-seminorms.

Locally C^* Algebras

A *Locally C^*-algebra* is a complete locally convex $*$-algebra for which the topology can be generated by a system of C^*-seminorms.

Examples of such algebras are the continuous functions, or continous sections of a C^* bundle, over a locally compact space X equipped with the topology generated by the seminorms

Locally C^* Algebras

A *Locally C^*-algebra* is a complete locally convex $*$-algebra for which the topology can be generated by a system of C^*-seminorms.

Examples of such algebras are the continuous functions, or continous sections of a C^* bundle, over a locally compact space X equipped with the topology generated by the seminorms

$$\|a\|_K = \sup_{x \in K} \|a(x)\|$$

For $K \subset X$ compact.

Denote by $S(A)$ the set of all continuous C^* seminorms in a locally C^* algebra A with the usual order relation, and define, for every $s \in S(A)$ the algebra

Denote by $S(A)$ the set of all continuous C^* seminorms in a locally C^* algebra A with the usual order relation, and define, for every $s \in S(A)$ the algebra

$$A_s = A/\ker s.$$

Denote by $S(A)$ the set of all continuous C^* seminorms in a locally C^* algebra A with the usual order relation, and define, for every $s \in S(A)$ the algebra

$$A_s = A/\ker s.$$

These algebras are C^* algebras thenselves and

Denote by $S(A)$ the set of all continuous C^* seminorms in a locally C^* algebra A with the usual order relation, and define, for every $s \in S(A)$ the algebra

$$A_s = A/\ker s.$$

These algebras are C^* algebras thenselves and

$$A = \varprojlim A_s,$$

Denote by $S(A)$ the set of all continuous C^* seminorms in a locally C^* algebra A with the usual order relation, and define, for every $s \in S(A)$ the algebra

$$A_s = A/\ker s.$$

These algebras are C^* algebras thenselves and

$$A = \varprojlim A_s,$$

so that every locally C^* algebra is the inverse limit a family of C^* algebras. The family $(A_s)_{s \in S(A)}$ is called the *Michael-Arens* decomposition of A.

The Primitive Spectrum

Given a locally C^* algebra A define it's *primitive spectrum*, $\text{Prim}A$, as the space of ideals of the from $\ker \pi$ where π is an irreducible representation of A.

The Primitive Spectrum

Given a locally C^* algebra A define it's *primitive spectrum*, $\mathrm{Prim}A$, as the space of ideals of the from $\ker \pi$ where π is an irreducible representation of A.

For a closed ideal $I \subset A$ define its *hull*, $h(I)$, by

The Primitive Spectrum

Given a locally C^* algebra A define it's *primitive spectrum*, $\mathrm{Prim}A$, as the space of ideals of the from $\ker\pi$ where π is an irreducible representation of A.

For a closed ideal $I \subset A$ define its *hull*, $h(I)$, by

$$h(I) = \{P \in \mathrm{Prim}A,\ I \subset P\},$$

The Primitive Spectrum

Given a locally C^* algebra A define it's *primitive spectrum*, $\mathrm{Prim}A$, as the space of ideals of the from $\ker \pi$ where π is an irreducible representation of A.

For a closed ideal $I \subset A$ define its *hull*, $h(I)$, by

$$h(I) = \{P \in \mathrm{Prim}A,\ I \subset P\},$$

similarly, given an subset $F \subset \mathrm{Prim}A$ define it's kernel by

The Primitive Spectrum

Given a locally C^* algebra A define it's *primitive spectrum*, $\mathrm{Prim}A$, as the space of ideals of the from $\ker\pi$ where π is an irreducible representation of A.

For a closed ideal $I \subset A$ define its *hull*, $h(I)$, by

$$h(I) = \{P \in \mathrm{Prim}A,\ I \subset P\},$$

similarly, given an subset $F \subset \mathrm{Prim}A$ define it's kernel by

$$k(F) = \bigcap_{P \in F} P$$

The Primitive Spectrum

Given a locally C^* algebra A define it's *primitive spectrum*, $\mathrm{Prim}A$, as the space of ideals of the from $\ker \pi$ where π is an irreducible representation of A.

For a closed ideal $I \subset A$ define its *hull*, $h(I)$, by

$$h(I) = \{P \in \mathrm{Prim}A,\ I \subset P\},$$

similarly, given an subset $F \subset \mathrm{Prim}A$ define it's kernel by

$$k(F) = \bigcap_{P \in F} P$$

The hulls of all closed ideals in a locally C^* algebras are the closed sets of a topology, denoted by *hull-kernel topology*.

For $s \in S(A)$, there is an isomorphism between $h(\ker s)$ and $\mathrm{Prim} A_s$.

For $s \in S(A)$, there is an isomorphism between $h(\ker s)$ and $\mathrm{Prim} A_s$.

Since $(A_s)_{s \in S(A)}$ is a inverse system of C^* algebras, by the contravariant nature of Prim, $(\mathrm{Prim} A_s)_{s \in S(A)}$ is a direct system of topological spaces and we have the set theoretical isomorphism:

For $s \in S(A)$, there is an isomorphism between $h(\ker s)$ and $\mathrm{Prim} A_s$.

Since $(A_s)_{s \in S(A)}$ is a inverse system of C^* algebras, by the contravariant nature of Prim, $(\mathrm{Prim} A_s)_{s \in S(A)}$ is a direct system of topological spaces and we have the set theoretical isomorphism:

$$\mathrm{Prim} A \approx \varinjlim \mathrm{Prim} A_s.$$

For $s \in S(A)$, there is an isomorphism between $h(\ker s)$ and $\mathrm{Prim} A_s$.

Since $(A_s)_{s \in S(A)}$ is a inverse system of C^* algebras, by the contravariant nature of Prim, $(\mathrm{Prim} A_s)_{s \in S(A)}$ is a direct system of topological spaces and we have the set theoretical isomorphism:

$$\mathrm{Prim} A \approx \varinjlim \mathrm{Prim} A_s.$$

We define then the *direct limit topology* on $\mathrm{Prim} A$ as the one provided by the identification above.

The Commutative Gelfand Theorem

We arrive then at our first important theorem about locally C^* algebras:

The Commutative Gelfand Theorem

We arrive then at our first important theorem about locally C^* algebras:

Commutative Gelfand Theorem

Given a commutative, unital locally C^* algebra A, the direct-limit topology turns $\mathrm{Prim} A$ into a compactly generated space such that

$$C(\mathrm{Prim} A) \approx A,$$

When the former is equipped with the topology of uniform convergence over compact sets of the form $\mathrm{Prim} A_s$ for $s \in S(A)$

The Multiplier Algebra and the Space of Points

For a locally C^* algebra A, define it's multiplier algebra, $M(A)$, in the same way as for C^* algebras. It is a locally C^* algebra such that

The Multiplier Algebra and the Space of Points

For a locally C^* algebra A, define it's multiplier algebra, $M(A)$, in the same way as for C^* algebras. It is a locally C^* algebra such that

$$S(M(A)) \approx S(A)$$

The Multiplier Algebra and the Space of Points

For a locally C^* algebra A, define it's multiplier algebra, $M(A)$, in the same way as for C^* algebras. It is a locally C^* algebra such that

$$S(M(A)) \approx S(A)$$

and

$$M(A) = \varprojlim M(A_s)$$

The Multiplier Algebra and the Space of Points

For a locally C^* algebra A, define it's multiplier algebra, $M(A)$, in the same way as for C^* algebras. It is a locally C^* algebra such that

$$S(M(A)) \approx S(A)$$

and

$$M(A) = \varprojlim M(A_s)$$

Define the space of points for a locally C^* algebra as

$$\mathrm{pt}A = \mathrm{Prim}ZM(A) \approx \varinjlim \mathrm{Prim}ZM(A_s) = \varinjlim \mathrm{pt}A_s$$

The Generalized Gelfand Theorem

We are now ready to state our main result:

The Generalized Gelfand Theorem

We are now ready to state our main result:

Generalized Gelfand Theorem

For a locally C^* algebra A, $\mathrm{pt}A$ is a compactly generated space and there is a C^* bundle $\mathcal{A} \to \mathrm{pt}A$ such that

$$A \approx \Gamma(\mathcal{A})$$

when the latter is equipped with the topology of uniform convergence over the compacts $\mathrm{pt}A_s$ for $s \in S(A)$.

The Sheaf defined by a Locally C^* Algebra

The last theorem shows that to every locally C^* algebra we can associate a C^* bundle and so, using local sections, a sheaf of algebras.

The Sheaf defined by a Locally C^* Algebra

The last theorem shows that to every locally C^* algebra we can associate a C^* bundle and so, using local sections, a sheaf of algebras.

Unfortunately this is not always sheaf of locally C^* algebras.

The Sheaf defined by a Locally C^* Algebra

The last theorem shows that to every locally C^* algebra we can associate a C^* bundle and so, using local sections, a sheaf of algebras.

Unfortunately this is not always sheaf of locally C^* algebras.

There may be open sets in $\mathrm{pt}A$ which are not compactly generated, and so that the algebra of local sections admits no locally C^* topology.

Perfect Algebras

A result proven by Apostol gives a sufficient condition to associate a sheaf of locally C^* algebras to a given one.

Perfect Algebras

A result proven by Apostol gives a sufficient condition to associate a sheaf of locally C^* algebras to a given one.

Given a seminorm $s \in S(A)$ we say an element $a \in A$ is *supported* in s if

Perfect Algebras

A result proven by Apostol gives a sufficient condition to associate a sheaf of locally C^* algebras to a given one.

Given a seminorm $s \in S(A)$ we say an element $a \in A$ is *supported* in s if

$$a \cdot \ker s = 0.$$

Perfect Algebras

A result proven by Apostol gives a sufficient condition to associate a sheaf of locally C^* algebras to a given one.

Given a seminorm $s \in S(A)$ we say an element $a \in A$ is *supported* in s if

$$a \cdot \ker s = 0.$$

The ideal of elements supported on s is denoted by $\operatorname{Sppd} s$.

We say then that a locally C^* algebra is *perfect* if

We say then that a locally C^* algebra is *perfect* if

$$A = \overline{\sum_{s \in S(A)} \operatorname{Sppd} s}.$$

We say then that a locally C^* algebra is *perfect* if

$$A = \overline{\sum_{s \in S(A)} \operatorname{Sppd} s}.$$

A unital, commutative locally C^* algebra is perfect if, and only if, it is isomorphic to the algebra of continuous functions over a locally compact space with the compact-open topology.

We say then that a locally C^* algebra is *perfect* if

$$A = \overline{\sum_{s \in S(A)} \operatorname{Sppd} s}.$$

A unital, commutative locally C^* algebra is perfect if, and only if, it is isomorphic to the algebra of continuous functions over a locally compact space with the compact-open topology.

Equivalently, a compactly generated space is locally compact if, and only if, every function over it can be approximated by functions with compact support.

C^* Algebras and Sheaves

Generaly, if $ZM(A)$ is perfect, pt$A$ is locally compact, a hereditary propriety, so that $U \mapsto \Gamma(\mathcal{A}, U)$ is a sheaf of locally C^* algebras.

C^* Algebras and Sheaves

Generaly, if $ZM(A)$ is perfect, pt$A$ is locally compact, a hereditary propriety, so that $U \mapsto \Gamma(\mathcal{A}, U)$ is a sheaf of locally C^* algebras.

Clearly, this hold when the topology of your original algebra is normable, i.e. for C^* algebras.

C^* Algebras and Sheaves

Generaly, if $ZM(A)$ is perfect, pt$A$ is locally compact, a hereditary propriety, so that $U \mapsto \Gamma(\mathcal{A}, U)$ is a sheaf of locally C^* algebras.

Clearly, this hold when the topology of your original algebra is normable, i.e. for C^* algebras.

Theorem

Every C^* algebra is the algebra of global sections of a sheaf of locally C^* algebras over a compact space.

Outlook

This results suggest a different approach to noncommutative geometry.

Outlook

This results suggest a different approach to noncommutative geometry.

- It provides a strict extension of the Gelfand theorem which was the original motivation for noncommutative geometry in the first place.

Outlook

This results suggest a different approach to noncommutative geometry.

- It provides a strict extension of the Gelfand theorem which was the original motivation for noncommutative geometry in the first place.
- The sheaf structure, crucial for every local construction in geometry, is naturally encoded in the ideal structure of the algebra.

Outlook

This results suggest a different approach to noncommutative geometry.

- It provides a strict extension of the Gelfand theorem which was the original motivation for noncommutative geometry in the first place.
- The sheaf structure, crucial for every local construction in geometry, is naturally encoded in the ideal structure of the algebra.

Outlook

- The next natural step is to try to generalize the notion of differential structure to this framework.

Outlook

- The next natural step is to try to generalize the notion of differential structure to this framework.
  - Unbounded derivations.

Outlook

- The next natural step is to try to generalize the notion of differential structure to this framework.
  - Unbounded derivations.
  - Differential Structures (Blackadar, Cuntz, Bhatt).

Outlook

- The next natural step is to try to generalize the notion of differential structure to this framework.
  - Unbounded derivations.
  - Differential Structures (Blackadar, Cuntz, Bhatt).
- Investigate also the relation with usual $AQFT$ provided by a resent by Ruzzi and Vasseli, which, given a net of C^* algebras over a topological space, allows the construction of a C^* bundle over the same space.