Precanonical quantization: from foundations to quantum gravity

Igor Kanatchikov

National Center of Quantum Information in Gdańsk (KCIK)

Sopot, Poland

E-mail: kanattsi@gmail.com, kai@fuw.edu.pl

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Different strategies towards quantum gravity:

- Apply QFT to GR (e.g. WDW, path integral)
- Adapt GR to QFT (e.g. Ashtekar variables, Shape dynamics)
- Change the fundamental microscopic dynamics, GR as an effective emergent theory (e.g. string theory, GFT, induced gravity, quantum/non-commutative space-times)

- **To try: Adapt quantum theory to GR?**
  - **Take the relativistic space-time seriously,**
  - **Avoid the distinguished role of time dimension in the formalism of quantum theory.**

$\longrightarrow$ The distinguished role of time is rooted already in the classical canonical Hamiltonian formalism underlying canonical quantization.

$\longrightarrow$ Fields as infinite-dimensional Hamiltonian systems evolving in time.

How to circumvent it?

$\rightarrow$ **De Donder-Weyl (”precanonical”) Hamiltonian formalism.**

Precanonical means that the mathematical structures of the canonical Hamiltonian formalism can be derived from those of the DW (or multisymplectic, or polysymplectic) formalism. Precanonical and canonical coincide at $n = 1$.

$\longrightarrow$ ”Precanonical quantization” is based on the structures of DW Hamiltonian formalism.

- De Donder-Weyl Hamiltonian formulation.
- Mathematical structures of DW theory.
  Poisson-Gerstenhaber brackets on forms.
- Applications of P-G brackets:
  field equations, geometric prequantization.
- Precanonical quantization of scalar field theory.
- Precanonical quantization vs. functional Schrödinger representation.
- Precanonical quantization of gravity.
  A. in metric variables,
  B. in vielbein variables.
- Discussion.

De Donder-Weyl (precanonical) Hamiltonian formalism

- Lagrangian density: $L = L(y^a, y^a_\mu, x^\nu)$.
- polymomenta: $p^\mu_a := \partial L / \partial y^a_\mu$.
- DW (covariant) Hamiltonian function: $H := y^a_\mu p^\mu_a - L$,
  $\hookrightarrow \; H = H(y^a, p^\mu_a, x^\mu)$.
- DW covariant Hamiltonian form of field equations:

$$\partial_\mu y^a(x) = \partial H / \partial p^\mu_a, \;\; \partial_\mu p^\mu_a(x) = -\partial H / \partial y^a.$$

- New regularity condition: $\det \left|\left| \partial^2 L / \partial y^\mu_a \partial y^\nu_b \right|\right| \neq 0$.
  $\hookrightarrow$ No usual constraints,
  $\hookrightarrow$ No space-time decomposition,
  $\hookrightarrow$ Finite-dimensional covariant analogue of the configuration space: (y^a, x^μ).

De Donder-Weyl (precanonical) Hamiltonian formalism
2. DWHJ

- DW Hamilton-Jacobi equation on n functions $S^\mu = S^\mu(y^a, x^\nu)$:

$$\partial_\mu S^\mu + H\left(y^a, p_a^\mu = \frac{\partial S^\mu}{\partial y^a}, x^\nu\right) = 0.$$

- Can DWHJ be a quasiclassical limit of some Schrödinger like formulation of QFT?
- How to quantize fields using the DW analogue of the Hamiltonian formalism?

  The potential advantages would be:

  – Explicit compliance with the relativistic covariance principles,

  – Finite dimensional covariant analogue of the configuration space: (y^a, x^μ) instead of $(y(\mathbf{x}), t)$.

- What are the Poisson brackets in DW theory? What is the analogue of canonically conjugate variables, the starting point of quantization?

DW Hamiltonian formulation: Examples

Nonlinear scalar field theory

$$L = \frac{1}{2}\partial_\mu y \partial^\mu y - V(y)$$

DW Legendre transformation:

$$p^\mu = \frac{\partial L}{\partial \partial_\mu y} = \partial^\mu y$$

$$H = \partial_\mu p^\mu - L = \frac{1}{2} p_\mu p^\mu + V(y)$$

DW Hamiltonian equations:

$$\partial_\mu y(x) = \partial H / \partial p^\mu = p_\mu,$$

$$\partial_\mu p^\mu(x) = -\partial H/\partial y = -\partial V/\partial y$$

are equivalent to : $\Box y + \partial V/\partial y = 0.$

DW Hamiltonian formulation: Einstein's gravity

Einstein's gravity in metric variables

- $\Gamma\Gamma$ action: $\mathfrak{L}(g^{\alpha\beta}, \partial_\nu g^{\alpha\beta})$;
- Field variables: $\mathfrak{h}^{\alpha\beta} := \sqrt{\mathfrak{g}} g^{\alpha\beta}$, with $\mathfrak{g} := |\mathrm{det}(g_{\mu\nu})|$,
- Polymomenta: $Q^\alpha_{\beta\gamma} := \frac{1}{8\pi G}(\delta^\alpha_{(\beta}\Gamma^\delta_{\gamma)\delta} - \Gamma^\alpha_{\beta\gamma})$;
- DW Hamiltonian density: $\mathfrak{H}(\mathfrak{h}^{\alpha\beta}, Q^\alpha_{\beta\gamma})$,

$$\mathfrak{H} = \sqrt{\mathfrak{g}} H = 8\pi G\, \mathfrak{h}^{\alpha\gamma} \left(Q^\delta_{\alpha\beta} Q^\beta_{\gamma\delta} + \frac{1}{1-n} Q^\beta_{\alpha\beta} Q^\delta_{\gamma\delta} \right);$$

- Einstein field equations in DW Hamiltonian form

$$\begin{aligned} \partial_\alpha \mathfrak{h}^{\beta\gamma} &= \partial\mathfrak{H}/\partial Q^\alpha_{\beta\gamma}, \\ \partial_\alpha Q^\alpha_{\beta\gamma} &= -\partial\mathfrak{H}/\partial\mathfrak{h}^{\beta\gamma}. \end{aligned}$$

- No constraints analysis!
  - Gauge fixing is still necessary to single out physical modes.

Mathematical structures of the DW formalism. Brief outline.

1. Finite dimensional ”polymomentum phase space” (y^a, p_a^μ, x^ν)
2. Polysymplectic $(n+1)$-form: $\Omega = dy^a \wedge dp_a^\mu \wedge \omega_\mu$,
   with $\omega_\mu = \partial_\mu \lrcorner (dx^1 \wedge dx^2 \wedge ... \wedge dx^n)$.
3. Horizontal differential forms $F_{\nu_1 \dots \nu_p}(y, p, x) dx^{\nu_1} \wedge ... \wedge dx^{\nu_p}$
   as dynamical variables.
4. Poisson brackets on differential forms follow from $X_F \lrcorner\, \Omega = dF$.

$\Rightarrow$ Hamiltonian forms F,

$\Rightarrow$ **Co-exterior** product of Hamiltonian forms:

$\overset{p}{F} \bullet \overset{q}{F} := *^{-1}(*\overset{p}{F} \wedge *\overset{q}{F})$.

$\Rightarrow$ Graded Lie (Nijenhuis) bracket.

$\Rightarrow$ **Gerstenhaber algebra.**

5. The bracket with H generates $d\bullet$ on forms.
6. Canonically conjugate variables from the analogue of the Heisenberg subalgebra:

$$\{[p_a^\mu \omega_\mu, y^b]\} = \delta_a^b, \quad \{[p_a^\mu \omega_\mu, y^b \omega_\nu]\} = \delta_a^b \omega_\nu, \quad \{[p_a^\mu, y^b \omega_\nu]\} = \delta_a^b \delta_\nu^\mu.$$

Geometric setting 1

- **Classical fields $y^a = y^a(x)$ are sections in the *covariant configuration bundle* $Y \to X$ over an oriented n-dimensional space-time manifold X with the volume form ω.**
- **local coordinates in $Y \to X$: (y^a, x^μ).**
- **$\bigwedge^p_q(Y)$ denotes the space of p-forms on Y which are annihilated by $(q+1)$ arbitrary vertical vectors of Y.**
- **$\bigwedge^n_1(Y) \to Y$:**
  - generalizes the cotangent bundle,
  - models the *multisymplectic phase space*.
- **Multisymplectic structure:**

$$\Theta_{MS} = p^\mu_a dy^a \wedge \omega_\mu + p\,\omega, \quad \omega_\mu := \partial_\mu \lrcorner \omega.$$

- **A section $p = -H(y^a, p^\mu_a, x^\nu)$ yields the *Hamiltonian Poincaré-Cartan form* Ω_{PC}:**

$$\Omega_{PC} = dp^\mu_a \wedge dy^a \wedge \omega_\mu + dH \wedge \omega$$

- ***Extended polymomentum phase space*:**

  $(y^a, p^\nu_a, x^\nu) =: (z^v, x^\mu) = z^M$

  Z**:** $\bigwedge^n_1(Y)/\bigwedge^n_0(Y) \to Y.$

- **Canonical structure on Z:**

  $\Theta := [p^\mu_a dy^a \wedge \omega_\mu \mod \bigwedge{}^n_0(Y)]$

- **Polysymplectic form**

$$\Omega := [d\Theta \mod \bigwedge{}^{n+1}_1(Y)]$$

$$\Omega = -dy^a \wedge dp^\mu_a \wedge \omega_\mu$$

- **DW equations in geometric formulation:**

$$\overset{n}{X} \lrcorner\, \Omega = dH$$

Hamiltonian multivector fields and Hamiltonian forms

- **A multivector field of degree p, $\overset{p}{X} \in \bigwedge^p TZ$, is called *vertical* if $\overset{p}{X} \lrcorner F = 0$ for any form $F \in \bigwedge_0^*(Z)$.**
- **The polysymplectic form establishes a map of horizontal $p-$form $\overset{p}{F} \in \bigwedge_0^p(Z)$ to vertical multivector fields of degree $(n-p)$, $\overset{n-p}{X}{}_F$, called *Hamiltonian*:**

$$\overset{n-p}{X}{}_F \lrcorner \Omega = d\overset{p}{F}.$$

- **The forms for which the map (2) exists are called *Hamiltonian*.**
- **The natural product operation of Hamiltonian forms is the *co-exterior* product**

$$\overset{p}{F} \bullet \overset{q}{F} := *^{-1}(*\overset{p}{F} \wedge *\overset{q}{F}) \in \bigwedge{}_0^{p+q-n}(Z)$$

- **co-exterior product is graded commutative and associative.**

Poisson-Gerstenhaber brackets

- **P-G brackets:**

$$\{[\overset{p}{F}_1, \overset{q}{F}_2]\} = (-1)^{(n-p)} \overset{n-p}{X}_1 \lrcorner d\overset{q}{F}_2 = (-1)^{(n-p)} \overset{n-p}{X}_1 \lrcorner \overset{n-q}{X}_2 \lrcorner \Omega$$
$$\in \bigwedge\nolimits_0^{p+q-n+1}(Z).$$

- **The space of Hamiltonian forms with the operations $\{[\ ,\ ]\}$ and**
  - **is a (Poisson-)Gerstenhaber algebra, viz.**

$$\{[\overset{p}{F}, \overset{q}{F}]\} = -(-1)^{g_1 g_2}\{[\overset{q}{F}, \overset{p}{F}]\},$$

$$(-1)^{g_1 g_3}\{[\overset{p}{F}, \{[\overset{q}{F}, \overset{r}{F}]\}]\} + (-1)^{g_1 g_2}\{[\overset{q}{F}, \{[\overset{r}{F}, \overset{p}{F}]\}]\}$$
$$+ \ (-1)^{g_2 g_3}\{[\overset{r}{F}, \{[\overset{p}{F}, \overset{q}{F}]\}]\} = 0,$$

$$\{[\overset{p}{F}, \overset{q}{F} \bullet \overset{r}{F}]\} = \{[\overset{p}{F}, \overset{q}{F}]\} \bullet \overset{r}{F} + (-1)^{g_1(g_2+1)} \overset{q}{F} \bullet \{[\overset{p}{F}, \overset{r}{F}]\},$$

$g_1 = n - p - 1$, $g_2 = n - q - 1$, $g_3 = n - r - 1$.

Applications of P-G brackets

- **The pairs of "canonically conjugate variables":**

$$\{[p_a^\mu \omega_\mu, y^b]\} = \delta_a^b, \quad \{[p_a^\mu \omega_\mu, y^b \omega_\nu]\} = \delta_a^b \omega_\nu, \quad \{[p_a^\mu, y^b \omega_\nu]\} = \delta_a^b \delta_\nu^\mu.$$

- **DW Hamiltonian equation in the bracket form:**

$$\mathbf{d}\bullet F = -\sigma(-1)^n \{[H, F]\} + d^h \bullet F,$$

for Hamiltonian $(n-1)-$**form** $F := F^\mu \omega_\mu$**;**

$$\mathbf{d}\bullet \overset{p}{F} := \frac{1}{(n-p)!} \partial_M F^{\mu_1 \dots \mu_{n-p}} \partial_\mu z^M dx^\mu \bullet \partial_{\mu_1 \dots \mu_{n-p}} \lrcorner\, \omega,$$

$$d^h \bullet \overset{p}{F} := \frac{1}{(n-p)!} \partial_\mu F^{\mu_1 \dots \mu_{n-p}} dx^\mu \bullet \partial_{\mu_1 \dots \mu_{n-p}} \lrcorner\, \omega,$$

$\sigma = \pm 1$ **for the Euclidean/Minkowskian signature of** X**.**

- **More general:** $\mathbf{d} F = \{[H\omega, \overset{p}{F}]\} + d^h F.$

Application to quantization of fields

- **Geometric prequantization of P-G brackets.**

  Prequantization map $F \to O_F$ **acting on (prequantum) Hilbert space fulfills three prorties:**

  (Q1) the map $F \to O_F$ **is linear;**
  (Q2) if F **is constant, then** O_F **is the corresponding multiplication operator;**
  (Q3) the Poisson bracket of dynamical variables is related to the commutator of the corresponding operators:

$$[O_{F_1}, O_{F_2}] = -i\hbar O_{\{F_1,F_2\}},$$
$$[A, B] := A \circ B - (-1)^{\deg A \deg B} B \circ A.$$

- **Explicit construction of prequantum operator of form F:**

$$\boxed{O_F = i\hbar[X_F, d] + (X_F \lrcorner \Theta) \bullet + F \bullet}$$

is a inhomogeneous operator, acts on prequantum wave functions $\Psi(y,p,x)$ – inhomogeneous forms on the polymomentum phase space.

- **"Prequantum Schrödinger equation"**

$$X_0 \lrcorner \Omega_{MS} = 0 \rightarrow O_0 \Psi = 0 \Rightarrow \boxed{i\sigma\hbar\, d \bullet \Psi = O_H(\Psi)}$$

- **Polarization:** $\Psi(y,p,x) \rightarrow \Psi(y,x)$.

- Normalization of prequantum wave functions leads to the metric structure on the space-time!

  $\Rightarrow$ **Co-exterior algebra $\rightarrow$ Clifford algebra.**

Geometric prequantization $\rightarrow$ precanonical quantization

- **The metric structure $\Rightarrow$ Clifford algebra**

"quantization map" $\boxed{q : \omega_\mu \bullet \rightarrow \frac{1}{\varkappa}\gamma_\mu}$

- **Precanonical analogue of the Schrödinger equation:**

$$\boxed{i\hbar\varkappa\gamma^\mu\partial_\mu\Psi = \hat{H}\Psi}$$

$\Psi(y, x)$ **- Clifford-valued wave function on $Y \rightarrow X$.**
$\hookrightarrow$ **Reproduces DW Hamiltonian equations on the average! (Ehrenfest theorem).**
$\hookrightarrow$ **Conserved probability current $\int dy \overline{\Psi}\gamma^\mu\Psi$.**
$\hookrightarrow$ **Reproduces DWHJ in the classical limit.**

- **For free scalar field theory:**

$$\boxed{\widehat{p}^\mu = -i\hbar\varkappa\gamma^\mu\frac{\partial}{\partial y},} \quad \boxed{\widehat{H} = -\frac{1}{2}\hbar^2\varkappa^2\frac{\partial^2}{\partial y^2} + \frac{1}{2}m^2y^2.}$$

- **The spectrum of** $\widehat{H}$**:** $(N+\frac{1}{2})\varkappa m$**.**
- $\langle M|y|M\pm 1\rangle \neq 0 \Rightarrow$ **quantum particles as transitions?**
- **The ground state** $(N=0)$ **solution (up to a normalisation factor)**

$$\Psi_0(y,\mathbf{q}) = e^{-\frac{1}{2\varkappa}q_\mu\gamma^\mu y^2}, \tag{3}$$

  which corresponds to the eigenvalues $k_0^t = \frac{1}{2}\omega_{\mathbf{q}}$**,** $k_0^i = \frac{1}{2}q^i$**.**
- **Higher excited states can be easily found to correspond to** $k_N^\mu = (N+\frac{1}{2})q^\mu$**.**
- **Define** $\hat{y}(x) = e^{-i\hat{P}_\mu x^\mu} y e^{-i\hat{P}_\mu x^\mu}, \quad i\partial_\mu\Psi = \hat{P}_\mu\Psi$ **(precanonical SE).**
  $\Rightarrow \langle 0|\hat{y}(x)\hat{y}(x')|0\rangle = \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} e^{-ik_\mu(x-x')^\mu}$.
  $\hookrightarrow y$ **in "ultra-Schrödinger representation" is well-defined, unlike** $y(0)$ **in Källén-Lehmann "spectral representation" calculation.**

Canonical vs. precanonical

A: Schrödinger functional rep.: $\mathbf{\Psi}([y(\mathbf{x})],t)$

$$i\partial_t \mathbf{\Psi} = \widehat{\mathbf{H}}\mathbf{\Psi}$$

$$\widehat{\mathbf{H}} = \int d\mathbf{x} \left\{ -\frac{1}{2}\frac{\delta^2}{\delta y(\mathbf{x})^2} + \frac{1}{2}(\nabla y(\mathbf{x}))^2 + V(y(\mathbf{x})) \right\}$$

B: Precanonical quantization: $\Psi(y,x)$

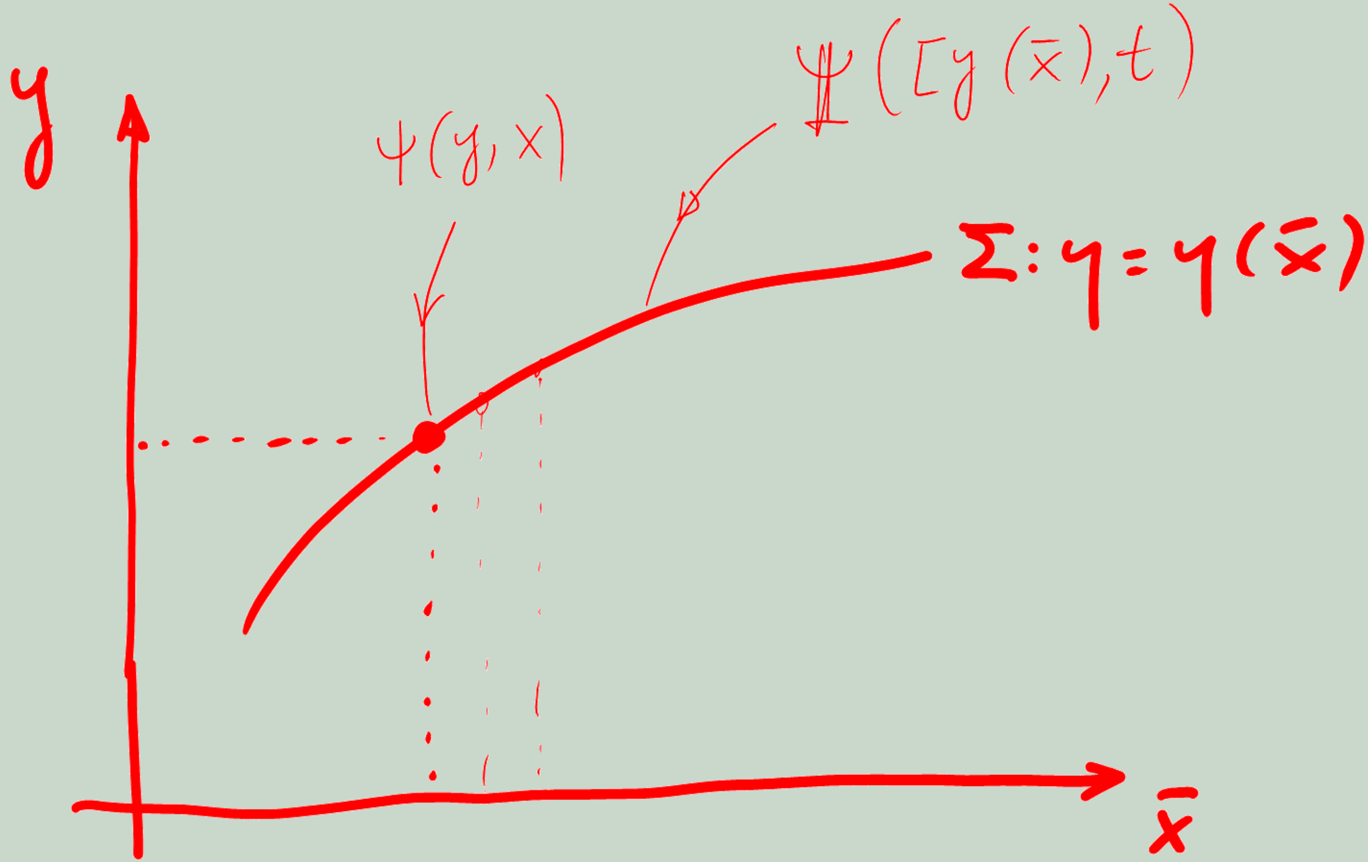
$$i\varkappa\gamma^\mu\partial_\mu\Psi = \widehat{H}\Psi$$

$\varkappa$ **is a "very large" constant of dimension** $\mathsf{L}^{-(n-1)}$**,**

$$\widehat{H} = -\frac{1}{2}\varkappa^2\partial_{yy} + V(y)$$

How those two descriptions can be related?

Canonical vs. precanonical



$$\longrightarrow \quad \Psi \sim \prod_{x} \psi\big(y = y(\bar{x}), \bar{x}, t\big)$$

Canonical vs. precanonical: HJ theory

- **Canonical Hamilton-Jacobi equation,** $\mathbf{S}([y(\mathbf{x})], t)$

$$\partial_t \mathbf{S} + \mathbf{H}\left(y^a(\mathbf{x}), p_a^0(\mathbf{x}) = \frac{\delta \mathbf{S}}{\delta y(\mathbf{x})}, t \right) = 0$$

- **Canonical HJ can be derived from DWHJ equation**

$$\partial_\mu S^\mu + H\left(y^a, p_a^\mu = \frac{\partial S^\mu}{\partial y^a}, x^\mu \right) = 0$$

- **Canonical HJ eikonal functional vs. DWHJ eikonal functions:**

$$\mathbf{S} = \int_\Sigma (S^\mu \omega_\mu)|_\Sigma \rightarrow \int d\mathbf{x}\, S^0(y = y(\mathbf{x}), \mathbf{x}, t)$$

$\Sigma := (y = y(\mathbf{x}), t)$ **("the Cauchy surface")**

Canonical vs. precanonical: Schrödinger functional

- **Denote** $\Psi(y,x)|_\Sigma := \Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)$**. Let**

$$\boldsymbol{\Psi}([y^a(\mathbf{x})],t) = \boldsymbol{\Psi}([\Psi_\Sigma(t)],[y^a(\mathbf{x})]).$$

- **The time evolution of the Schrödinger wave functional is determined by the time evolution of precanonical wave function:**

$$i\partial_t \boldsymbol{\Psi} = \int d\mathbf{x}\ \mathsf{Tr}\left\{ \frac{\delta \boldsymbol{\Psi}}{\delta \Psi^T_\Sigma(y^a(\mathbf{x}),\mathbf{x},t)} i\partial_t \Psi_\Sigma(y^a(\mathbf{x}),\mathbf{x},t) \right\}$$

- **The time evolution of** Ψ_Σ **is given by the precanonical Schrödinger equation restricted to** Σ**:**

$$i\partial_t \Psi_\Sigma(\mathbf{x}) = -i\beta\gamma^i \frac{d}{dx^i}\Psi_\Sigma(\mathbf{x}) + i\beta\gamma^i\partial_i y(\mathbf{x})\partial_y \Psi_\Sigma(\mathbf{x}) + \frac{1}{\varkappa}\beta(\widehat{H}\Psi)_\Sigma(\mathbf{x})$$

Hence (for scalar field theory):

$$i\partial_t \mathbf{\Psi} = \int d\mathbf{x} \mathsf{Tr} \left\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x}, t)} \left[-i\beta\gamma^i \frac{d}{dx^i} \Psi_\Sigma(\mathbf{x}) + i\beta\gamma^i \partial_i y(\mathbf{x}) \partial_y \Psi_\Sigma(\mathbf{x}) - \frac{1}{2} \varkappa \beta \partial_{yy} \Psi_\Sigma + \frac{1}{\varkappa} \beta V(y(\mathbf{x})) \Psi_\Sigma \right] \right\}$$

$$\text{c.f. :} \quad \frac{\delta \mathbf{\Psi}}{\delta y(\mathbf{x})} = \mathsf{Tr} \left\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x}, t)} \partial_y \Psi_\Sigma(\mathbf{x}) \right\} + \frac{\bar{\delta} \mathbf{\Psi}}{\bar{\delta} y(\mathbf{x})} \,,$$

$$\begin{aligned} \frac{\delta^2 \mathbf{\Psi}}{\delta y(\mathbf{x})^2} = {} & \mathsf{Tr} \left\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x}, t)} \, \boxed{\delta(\mathbf{0}) \partial_{yy} \Psi_\Sigma(\mathbf{x})} \right\} \\ & + \mathsf{Tr}\,\mathsf{Tr} \left\{ \frac{\delta^2 \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x}) \otimes \delta \Psi_\Sigma^T(\mathbf{x})} \, \partial_y \Psi_\Sigma(\mathbf{x}) \otimes \partial_y \Psi_\Sigma(\mathbf{x}) \right\} \\ & + 2\,\mathsf{Tr} \left\{ \frac{\delta \bar{\delta} \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x}) \, \bar{\delta} y(\mathbf{x})} \, \partial_y \Psi_\Sigma(\mathbf{x}) \right\} + \frac{\bar{\delta}^2 \mathbf{\Psi}}{\bar{\delta} y(\mathbf{x})^2} \,. \end{aligned}$$

Canonical vs. precanonical: Schrödinger functional 2

$$\int d\mathbf{x}\,\mathsf{Tr}\left\{\frac{\delta\boldsymbol{\Psi}}{\delta\Psi_\Sigma^T(\mathbf{x})}\,\frac{1}{\varkappa}\beta V(y(\mathbf{x}))\Psi_\Sigma(\mathbf{x}))\right\}\rightarrow\int d\mathbf{x}V(y(\mathbf{x}))\;\boldsymbol{\Psi}\;,$$

$$\Rightarrow\quad \mathsf{Tr}\left\{\frac{\delta\boldsymbol{\Psi}}{\delta\Psi_\Sigma^T(\mathbf{x})}\,\beta\Psi_\Sigma(\mathbf{x})\right\}=\varkappa\boldsymbol{\Psi}\quad\forall\mathbf{x}$$

$$\Rightarrow\mathsf{Tr}\left\{\frac{\delta^2\boldsymbol{\Psi}}{\delta\Psi_\Sigma^T(\mathbf{x})\otimes\delta\Psi_\Sigma^T(\mathbf{x})}\beta\Psi_\Sigma(\mathbf{x})\right\}=\frac{\delta\boldsymbol{\Psi}}{\delta\Psi_\Sigma^T(\mathbf{x})}\,(\varkappa-\beta\delta(\mathbf{0}))$$

- $\Rightarrow\;\beta\varkappa\rightarrow\delta(\mathbf{0})$ **i.e. the "inverse quantization map" at** $1/\varkappa\rightarrow 0.$
- **The term** $\varkappa\beta\partial_{yy}\Psi_\Sigma$ **reproduces the first term in** $\delta^2\boldsymbol{\Psi}/\delta y(\mathbf{x})^2.$
- **The terms proportional to** $\partial_y\Psi_\Sigma(\mathbf{x})$ **should cancel**

$$\Rightarrow\qquad\frac{\delta\boldsymbol{\Psi}}{\delta\Psi_\Sigma^T(\mathbf{x})}i\beta\gamma^i\partial_i y(\mathbf{x})+\frac{\delta\overline{\delta}\boldsymbol{\Psi}}{\delta\Psi_\Sigma^T(\mathbf{x})\overline{\delta}y(\mathbf{x})}=0. \qquad (4)$$

- **Using the condition $\beta\varkappa \to \delta(\mathbf{0})$ and**

$$\mathbf{\Phi}(\mathbf{x}) := \frac{\delta \mathbf{\Psi}}{\delta \Psi_{\Sigma}^{T}(\mathbf{x})} \tag{5}$$

$$\Rightarrow \qquad \varkappa \frac{\bar{\delta} \mathbf{\Phi}(\mathbf{x})}{\bar{\delta} y(\mathbf{x})} + \mathbf{\Phi}(\mathbf{x}) i \delta(\mathbf{0}) \gamma^{i} \partial_{i} y(\mathbf{x}) = 0 \ , \tag{6}$$

$$\Rightarrow \qquad \mathbf{\Phi}(\mathbf{x}) = \mathbf{\Xi}([\Psi_{\Sigma}]; \breve{\mathbf{x}}) \ e^{-iy(\mathbf{x})\gamma^{i}\partial_{i} y(\mathbf{x})/\varkappa}, \tag{7}$$

where $\Xi([\Psi_{\Sigma}]; \breve{\mathbf{x}})$ is a functional of $\Psi_{\Sigma}(\mathbf{x}')$ at $\mathbf{x}' \neq \mathbf{x}$, so that

$$\delta \mathbf{\Phi}(\mathbf{x}) / \delta \Psi_{\Sigma}^{T}(\mathbf{x}) = 0 \quad \Leftrightarrow \quad \frac{\delta^{2} \mathbf{\Psi}}{\delta \Psi_{\Sigma}(\mathbf{x}) \otimes \delta \Psi_{\Sigma}(\mathbf{x})} = 0. \tag{8}$$

- **Eqs. (5,7) lead to the solution:**

$$\mathbf{\Psi} = \mathsf{Tr} \left\{ \mathbf{\Xi}([\Psi_{\Sigma}]; \breve{\mathbf{x}}) \ e^{-iy(\mathbf{x})\gamma^{i}\partial_{i} y(\mathbf{x})/\varkappa} \ \Psi_{\Sigma}(\mathbf{x}) \right\} . \tag{9}$$

- **The total derivative term in $i\partial_t\Psi$ integrated by parts:**

$$\int d\mathbf{x}\ \mathsf{Tr}\left\{\left(i\frac{d}{dx^i}\mathbf{\Phi}\right)\gamma^i\Psi_\Sigma(\mathbf{x})\right\}, \tag{10}$$

taking the total derivative $\frac{d}{dx^i}$ of Φ in (7):

$$\frac{d}{dx^i}\mathbf{\Phi}(\mathbf{x}) = -\frac{i}{\varkappa}\mathbf{\Xi}(\mathbf{x})e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}\Big(\gamma^k\partial_k y(\mathbf{x})\partial_i y(\mathbf{x})+y(\mathbf{x})\gamma^k\partial_{ik}y(\mathbf{x})\Big)$$

and using the expression of Ψ in (9):

$$\mathbf{Eq.}(\mathbf{10}) \;\Rightarrow -i\mathbf{\Psi}\int d\mathbf{x}\,(\gamma^k\partial_k y(\mathbf{x})\partial_i y(\mathbf{x}) + y(\mathbf{x})\gamma^k\partial_{ik}y(\mathbf{x}))\gamma^i \tag{11}$$

$\Rightarrow$ **vanishes upon integrating by parts.**

- **The functional $\Xi([\Psi_\Sigma(\mathbf{x})])$ in (9) is specified by noticing that the formula (9) is valid for *any* x. It can be achieved only if the functional Ψ has the continuous product structure, viz.**

$$\boldsymbol{\Psi} = \mathsf{Tr}\left\{\prod_{\mathbf{x}} e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)\right\}$$

- **Expresses the Schrödinger wave functional $\boldsymbol{\Psi}([y(\mathbf{x})],t)$ in terms of precanonical wave functions $\Psi(y,x)$ restricted to Σ.**
- **Implies the inverse of the ″quantization map″ $\beta\varkappa \rightarrow \delta(\mathbf{0})$ in the limit of infinitesimal ″elementary volume″ $1/\varkappa \rightarrow 0$.**
- **$\Rightarrow$ QFT based on canonical quantization is a singular limit of QFT based on precanonical quantization.**

Precanonical quantization of metric gravity

- **A guess:**

$$i\hbar\kappa\mathfrak{e}\widehat{\not\nabla}\Psi = \widehat{\mathfrak{H}}\Psi, \tag{12}$$

- **with** $\widehat{\not\nabla} := \gamma^\mu(\partial_\mu + \hat{\theta}_\mu)$**, the quantized covariant Dirac operator,**
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu := 2g^{\mu\nu}$**,** $\hat{\theta}_\mu$ **the spin-connection operator.**
-

$$\widehat{Q}^\alpha_{\beta\gamma} = -i\hbar\kappa\gamma^\alpha\left\{\sqrt{\mathfrak{g}}\frac{\partial}{\partial\mathfrak{h}^{\beta\gamma}}\right\}_{ord}, \tag{13}$$

$$\widehat{\mathfrak{H}} = -\frac{16\pi}{3}G\hbar^2\kappa^2\left\{\sqrt{\mathfrak{g}}\mathfrak{h}^{\alpha\gamma}\mathfrak{h}^{\beta\delta}\frac{\partial}{\partial\mathfrak{h}^{\alpha\beta}}\frac{\partial}{\partial\mathfrak{h}^{\gamma\delta}}\right\}_{ord} \tag{14}$$

- **Problems:**
  - Classical Q **transforms as connection vs.** $Q\hat{\sigma}_{\alpha\beta} \sim \gamma^\sigma \otimes \frac{\partial}{\partial\partial h^{\alpha\beta}}$
  - $e\partial e$ **part of spin-connection can't be expressed in terms of** Q**:**

$$\theta_\mu = e \otimes e\Gamma_\mu + e\partial_\mu e$$

- **$\Rightarrow$ Assume the hybrid approach, viz. the remaining (not quantizable) objects needed to formulate the covariant Schrödinger equation are introduced in a self-consistent with the underlying quantum dynamics of Ψ way as averaged notions.**
- **Diffeomorphism covariant wave equation for "hybrid" quantum gravity:**

$$i\hbar\kappa\widetilde{\mathfrak{e}\nabla\!\!\!/}\Psi + i\hbar\kappa(\mathfrak{e}\gamma^{\mu}\theta_{\mu})^{op}\Psi = \widehat{\mathfrak{H}}\Psi \tag{15}$$

- $\widetilde{\nabla\!\!\!/} = \tilde{e}^{\mu}_{A}(x)\gamma^{A}(\partial_{\mu} + \tilde{\theta}_{\mu}(x))$ **is the Dirac operator constructed using the self-consistent field $\tilde{e}^{\mu}_{A}(x)$:**

$$\tilde{e}^{\mu}_{A}(x)\tilde{e}^{\nu}_{B}(x)\eta^{AB} := \langle g^{\mu\nu}\rangle(x),$$

$$\langle g^{\mu\nu}\rangle(x) = \int \overline{\Psi}(g,x)g^{\mu\nu}\Psi(g,x)[\mathfrak{g}^{(n+1)/2}\prod_{\alpha\leq\beta} dg^{\alpha\beta}]; \tag{16}$$

- **Quantum superposition principle is effectively valid on the self-consistent space-time.**

- **The operator part of the spin-connection:**

$$(\sqrt{\mathfrak{g}}\gamma^\mu\theta_\mu)^{op} = -4\pi i G\hbar\kappa \left\{ \sqrt{\mathfrak{g}}\mathfrak{g}^{\mu\nu} \frac{\partial}{\partial \mathfrak{h}^{\mu\nu}} \right\}_{ord} \tag{17}$$

- **To complete the description, impose the De Donder-Fock harmonic gauge:** $\partial_\mu \left\langle \sqrt{\mathfrak{g}} g^{\mu\nu} \right\rangle (x) = 0$. **In the present context this is the gauge condition on the wave function** $\Psi(g^{\mu\nu}, x^\nu)$ **rather than on the metric field.**

- **Can the hybrid description be circumvented in vielbein/spin-connection variables?**

DW formulation of first order e-θ gravity.

- **EH Lagrangean density** $\mathfrak{L} = \frac{1}{2\kappa_E}(R + 2\Lambda)\sqrt{-\mathfrak{g}}$**:**

$$\mathfrak{L} = \frac{1}{\kappa_E}\mathfrak{e} e_I^{[\alpha} e_J^{\beta]}(\partial_\alpha \theta_\beta^{IJ} + \theta_\alpha{}^{IK}\theta_{\beta K}{}^J) + \frac{1}{\kappa_E}\Lambda\mathfrak{e}$$

- **Polymomenta** $\rightarrow$ **primary constraints:**

$$\mathfrak{p}^\alpha_{\theta^{IJ}_\beta} = \frac{\partial \mathfrak{L}}{\partial_\alpha \theta^{IJ}_\beta} \approx \frac{1}{\kappa_E}\mathfrak{e} e_I^{[\alpha} e_J^{\beta]}, \quad \mathfrak{p}^\alpha_{e^I_\beta} = \frac{\partial \mathfrak{L}}{\partial_\alpha e^I_\beta} \approx 0.$$

- **DW Hamiltonian density:**

$$\mathfrak{H} = \mathfrak{p}_\theta \partial\theta + \mathfrak{p}_e \partial e - \mathfrak{L} + \lambda(\mathfrak{p}_\theta - \mathfrak{e} e \wedge e) + \mu \mathfrak{p}_e$$

- **On the constraints surface:**

$$\mathfrak{H}|_C \approx -\mathfrak{p}^\alpha_{\theta^{IJ}_\beta}\theta^{\alpha IK}\theta_{\beta K}{}^J - \frac{1}{\kappa_E}\Lambda\mathfrak{e}$$

DW formulation of first order e-θ gravity – Constraints.

- **Preservation of constraints $\Leftrightarrow$ DW equations or vanishing PG brackets of $(n-1)$-forms $C^\alpha \omega_\alpha$ constructed from the constraints $C^\alpha \approx 0$ with $\mathfrak{H}$.**
- **From ∂e and $\partial \theta \Rightarrow \mu = 0, \lambda = 0$**
-
$$\partial_\alpha \mathfrak{p}^\alpha_{e^I_\beta} = -\frac{\partial \mathfrak{H}}{\partial e^I_\beta}$$
  $\Rightarrow$ **Einstein equations.**
-
$$\partial_\alpha \mathfrak{p}^\alpha_{\theta^{IJ}_\beta} = -\frac{\partial \mathfrak{H}}{\partial \theta^{IJ}_\beta}$$
  $\Rightarrow$ **expression of θ^{IJ}_β i.t.o. ∂e.**

Quantization of e-θ gravity

$$\mathfrak{p}^{\alpha}_{e^I_\beta} \approx 0 \Rightarrow \frac{\partial \Psi}{\partial e^I_\beta} = 0 \Rightarrow \Psi(\theta, e, x) \to \Psi(\theta, x)$$

$$\mathfrak{e} e^{[\alpha}_I e^{\beta]}_J \gamma^{IJ} = \mathfrak{e}\gamma^{\alpha\beta} \approx \kappa_E\, \mathfrak{p}^{\alpha}_{\theta^{IJ}_\beta} \gamma^{IJ}$$

$$\widehat{\mathfrak{p}}^{\alpha}_{\theta^{IJ}_\beta} = -i\hbar\varkappa\mathfrak{e}\gamma^{[\alpha} \frac{\partial}{\partial \theta^{IJ}_{\beta]}}$$

$$\Rightarrow \quad \boxed{\widehat{\gamma}^{\beta} = -i\hbar\varkappa\kappa_E \gamma^{IJ} \frac{\partial}{\partial \theta^{IJ}_\beta}} \quad \to \hat{e}^{\beta}_I$$

DW Hamiltonian operator, $\widehat{\mathfrak{H}} =: \widehat{\mathfrak{e}H}$:

$$\boxed{\widehat{H} = \hbar^2\varkappa^2\kappa_E \frac{\partial}{\partial \theta^{IJ}_\alpha} \gamma^{IJ} \frac{\partial}{\partial \theta^{KL}_\beta} \theta_\alpha{}^{KM} \theta_{\beta M}{}^{L} - \frac{1}{\kappa_E}\Lambda}$$

Covariant Schrödinger equation for quantum gravity

$$\boxed{i\hbar\varkappa\widehat{\not\nabla}\Psi = \widehat{H}\Psi}$$

with the "quantized Dirac operator":

$$\widehat{\not\nabla} := (\gamma^\mu(\partial_\mu + \theta_\mu))^{op}, \qquad \theta_\mu := \frac{1}{4}\theta_{\mu IJ}\gamma^{IJ}$$

$$\Rightarrow \qquad \boxed{\widehat{\not\nabla} = -i\hbar\varkappa\kappa_E\gamma^{IJ}\frac{\partial}{\partial\theta^{IJ}_\mu}(\partial_\mu + \tfrac{1}{4}\theta_{\mu KL}\gamma^{KL})}$$

Hence, **precanonical counterpart of WDW:**

$$\gamma^{IJ}\frac{\partial}{\partial\theta^{IJ}_\mu}\left(\partial_\mu + \frac{1}{4}\theta_{\mu KL}\gamma^{KL} - \frac{\partial}{\partial\theta^{KL}_\beta}\theta_\mu{}^{KM}\theta_{\beta M}{}^L\right)\Psi(\theta, x) + \frac{\Lambda}{\hbar^2\varkappa^2\kappa_E^2}\Psi(\theta, x) = 0.$$

$\hookrightarrow$ **Ordering ambiguities!**

Defining the Hilbert space

- **The scalar product:** $\langle\Phi|\Psi\rangle := \int[d\theta]\overline{\Phi}\Psi.$

  $\hookrightarrow$ **Misner-like covariant measure on the space of θ-s:**

$$[d\theta] = \mathfrak{e}^{-n(n-1)}\prod_{\mu IJ} d\theta^{IJ}_{\mu}.$$

  $\hookrightarrow$ $[d\theta]$ **is operator-valued, because**

$$\mathfrak{e} := \det(e^I_\alpha), \quad \widehat{e^\alpha_I} \sim \gamma^J \frac{\partial}{\partial\theta^{IJ}_\alpha}.$$

  $\hookrightarrow$ **Weyl ordering in $\widehat{[d\theta]}$:**

$$\langle\Phi|\Psi\rangle := \int \overline{\Phi}\,\widehat{[d\theta]}_W \Psi.$$

Further definition of the Hilbert space

- **Boundary condition** $\Psi(\theta \to \infty) \to 0$.

  $\to$ **Excludes (almost) infinite curvatures** $R = d\theta + \theta \wedge \theta$**.**

  $\to$ **To be explored, how it will play together with the OVM in the singularity avoidance.**

- **Huge gauge freedom in spin-connection coefficients is removed by fixing the De Donder-Fock gauge condition: the choice of harmonic coordinates on the average:**

$$\partial_\mu \left\langle \Psi(\theta, x) \middle| \widehat{\gamma}^\mu \middle| \Psi(\theta, x) \right\rangle = 0.$$

  $\to$ **Gauge fixing on the level of states** Ψ**, not spin-connections or vielbeins.**

  $\to$ **To be explored if this gauge fixing is sufficient and should not be complemented by further conditions.**

Precanonical quantum cosmology, a toy model. 1

$n = 4$, $k = 0$ **FLRW metric with a harmonic time coordinate** τ

$$ds^2 = a(\tau)^6 d\tau^2 - a(\tau)^2 d\mathbf{x}^2 = \eta_{IJ} e^I_\mu e^J_\nu dx^\mu dx^\nu .$$

$$e^0_\nu = a^3 \delta^0_\nu, e^J_\nu = a\delta^J_\nu, \quad J = 1, 2, 3$$

$$\omega_i^{0I} = -\omega_i^{I0} = \dot{a}/2a^3 =: \omega, \quad i = I = 1, 2, 3$$

Our analogue of WDW:

$$\Big(2 \sum_{i=I=1}^{3} \alpha^I \partial_\omega \partial_i + 3\omega\partial_\omega + \lambda\Big)\Psi = 0,$$

$\alpha^I := \gamma^{0I}$, $\lambda := \frac{3}{2} + \Lambda/(\hbar\varkappa\kappa_E)^2$, **Weyl ordering.**

$\hookrightarrow$ **The correct value of** Λ **can be obtained from the constant of order unity which results from the operator ordering, if** $\varkappa \sim 10^{-3} GeV^3$.

Precanonical quantum cosmology, a toy model. 2

By separating variables $\Psi := u(x)f(\omega)$**:**

$$2\sum_{i=I}\alpha^I\partial_i u = iqu,$$

the imaginary unit comes from the anti-hermicity of ∂_i**,**

$$(iq\partial_\omega + 3\omega\partial_\omega + \lambda)f = 0.$$

Solution $f \sim (iq+3\omega)^{-\lambda}$ **yields the probability density**

$$\boxed{\rho(\omega) := \bar{f}f \sim (9\omega^2 + q^2)^{-\lambda}.}$$

(similar to t**-distribution).**

$\hookrightarrow$ **At** $\lambda > 1/2$ **(required by** $\mathsf{L}^2[(-\infty,\infty),[d\omega] = d\omega]$ **normalizability in** ω**-space)** $\rho(\omega)$ **has a bell-like shape centered at the zero universe's expansion rate** $\dot{a} = 0$**.**

$\hookrightarrow$ **The most probable expansion rate can be shifted by accepting complex values of** q**, and the inclusion of minimally coupled matter fields changes** λ**.**

Precanonical quantum cosmology, a toy model. 3

$\hookrightarrow$ **Although our toy model bears some similarity with the minisuperspace models, its origin and the content are different:**

- **It is obtained from the full quantum Schrödinger equation when ω is one-component, NOT via quantization of a reduced mechanical model deduced under the assumption of spatial homogeneity.**
- **Naive assumption of spatial homogeneity of the wave function: $\partial_i \Psi = 0$, or $q = 0$, would not be compatible with normalizability of Ψ in ω-space!**
- **Instead, our model implies a quantum gravitational structure of space at the scales $\sim \mathsf{Re}\frac{1}{q}$ and $\sim \mathsf{Im}\frac{1}{q}$ given by the configuration of "weyleon" $u(x)$.**

Concluding remarks 1.

- **Standard QFT in the functional Schrödinger represenation is a**

$$\beta\varkappa \rightarrow \delta^{n-1}(0)$$

"limit" of QFT based on precanonical quantization.
$\hookrightarrow$ The latter regularizes some of the singularities of the former? The details are to be explored!

Concluding remarks 2.

- **How to extract physics of quantum gravity from the above pre-canonical counterpart of WDW equation?**

$$\gamma^{IJ}\frac{\partial}{\partial\theta^{IJ}_{\mu}}\left(\partial_\mu + \frac{1}{4}\theta_{\mu KL}\gamma^{KL} - \frac{\partial}{\partial\theta^{KL}_{\beta}}\theta_{\mu}{}^{KM}\theta_{\beta M}{}^{L}\right)\ \Psi(\theta,x)$$
$$+\ \frac{\Lambda}{\hbar^2\varkappa^2\kappa_E^2}\Psi(\theta,x) = 0.$$

$\hookrightarrow$ **Multidimensional generalized hypergeometric equation.**
$\hookrightarrow$ **Quantum geometry in terms of** $\langle\theta', x'|\theta, x\rangle$**.**

- **Precanonical formulation is**
  - **inherently non-perturbative,**
  - **manifestly covariant,**
  - **background-independent,**
  - **mathematically well-defined,**
  - **works in any number of dimensions and metric signature.**

Concluding remarks 3.

$\hookrightarrow$ **Metric structure is emergent:**

$$\langle g^{\mu\nu}\rangle(x) = \int \overline{\Psi}(\theta, x)\widehat{[d\theta]}\,\widehat{g^{\mu\nu}}\Psi(\theta, x),$$

$$\widehat{g^{\mu\nu}} = -\hbar^2\varkappa^2\kappa_E^2\frac{\partial^2}{\partial\theta_\mu^{IA}\theta_\nu^{JB}}\eta^{IJ}\eta^{AB}$$

$\hookrightarrow$ **Ehrenfest theorem vs. the ordering of operators and OVM (work in progress).**

Concluding remarks 4.

- Λ **or couplings with matter fields are crucial to determine the characteristic scales.**

  $\hookrightarrow$ **"Naturality"** $\frac{\Lambda}{\hbar^2 \varkappa^2 \kappa_E^2} \sim n^6 \Rightarrow \varkappa$ **at roughly** $\sim 10^2 MeV$ **scale!**

  $\hookrightarrow$ **If** $\varkappa$ **is Planckian, then** Λ **is estimated to be** $\sim 10^{120}$ **higher than observed (as usual), i.e.** $\varkappa$ **is consistent with the UV cutoff scale in standard QFT.**

  $\hookrightarrow$ **Include matter fields to see their impact on the estimation? E.g. the conformal coupling term with the scalar field leads to** $\frac{\xi}{2\varkappa^2} R\phi^2$ **term.**

- **Misner-like OVM in the definition of the scalar product as a specifics of quantum gravity and its probabilistic interpretation?**

THE END

Many thanks for your attention.