# A simplicial approach to the non-Abelian Chern-Simons path integral

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# Motivation: Why is the theory of 3-manifold quantum invariants interesting?

*1) It is beautiful:* surprising relations between many different areas of mathematics/physics like

- Algebra
- low-dimensional Topology
- Differential Geometry
- Functional Analysis and Stochastic Analysis
- Quantum field theory (in particular, Conformal field theory, Quantum Gravity, String theory)

*2) It is deep:* Fields Medals for Jones, Witten, Kontsevich

*3) It is useful:* Applications in Knot Theory and Quantum Gravity, ...

# List of approaches to 3-manifold quantum invariants

*Original heuristic approach*

0. Chern-Simons path integrals approach (Witten)

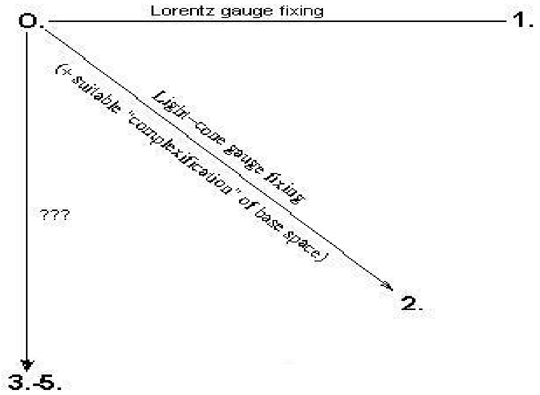
*Rigorous perturbative approaches*

1. Configuration space integrals
2. Kontsevich Integral

*Rigorous non-perturbative approaches*

3a. Quantum groups + Surgery (Reshetikhin/Turaev)
3b. Quantum groups + Shadow links (Turaev)
3c. Lattice gauge theories based on Quantum groups
4. Skein Modules
5. Geometric Quantization

# Some relations between the approaches



# Important open problems

(P1) Chern-Simons path integral $\overset{??}{\longleftrightarrow}$ rigorous non-perturbative approaches 3a, 3b, 3c, 4, 5.

(P2) Rigorous definition of original Chern-Simons path integral expressions?

Alternatively:

(P2') Rigorous definition of Chern-Simons path integral expressions after suitable gauge fixing?

# The shortterm goal

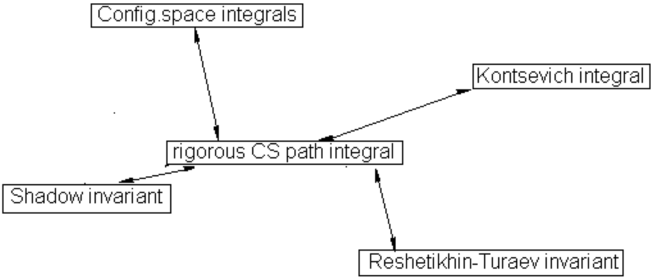
## Our Aim

Make progress regarding (P1) and (P2')

## Strategy

We consider special situation $M = \Sigma \times S^1$ and apply "torus gauge fixing" (cf. Blau/Thompson '93)

# The longterm "goal"/dream




**(all "arrows" being rigorous)**

(or something analogous for $BF_3$-theory with cosmological constant)

# The Chern-Simons path integral

Fix

- $M$: oriented connected 3-manifold (usually compact)
- $G$: simply-connected simple Lie subgroup of $U(N)$ ($N \in \mathbb{N}$ fixed)
- $k \in \mathbb{R} \backslash \{0\}$ (usually $k \in \mathbb{N}$)

Space of gauge fields:

$$\mathcal{A} = \{A \mid A \ \mathfrak{g}\text{-valued 1-form on } M\} \quad (\mathfrak{g} \subset u(N)\text{: Lie algebra of } G)$$

Action functional:

$$S_{CS} : \mathcal{A} \ni A \mapsto \frac{k}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \in \mathbb{R}$$

where $\mathrm{Tr} := c \, \mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}$ for suitable normalisation constant $c \in \mathbb{R}$

**Observation 1**

$S_{CS}$ is invariant under (orientation-preserving) diffeomorphisms

Fix

- "link" $L = (l_1, l_2, \ldots, l_n)$, $n \in \mathbb{N}$, in $M$
- n-tuple $(\rho_1, \rho_2, \ldots, \rho_n)$ of finite-dim. representations of $G$

**"Definition"**

$$Z(M, L) := \int \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA$$

where $DA$ is the "Lebesgue measure" on $\mathcal{A}$ and

$$\mathrm{Hol}_{l_i}(A) := \lim_{n\to\infty} \prod_{k=1}^{n} \exp(\tfrac{1}{n} A(l_i'(\tfrac{k}{n}))) \quad (\text{"holonomy of } A \text{ around } l_i\text{"})$$

**Observation 2**

$S_{CS}$ invariant under (orientation-preserving) diffeomorphisms $\Rightarrow$ $Z(M, L)$ only depends on diffeomorphism class of $M$ and isotopy class of $L$.

# The shadow invariant for $M = \Sigma \times S^1$

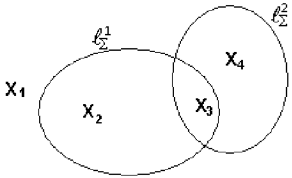
Special case: $M = \Sigma \times S^1$

Fix (framed) link $L = (l^1, l^2, \ldots, l^n)$

Loop projections onto $S^1$ and $\Sigma$:

$$l^1_{S^1}, l^2_{S^1}, ..., l^n_{S^1} \qquad \text{and} \qquad l^1_\Sigma, l^2_\Sigma, ..., l^n_\Sigma$$

$D(L)$: graph in $\Sigma$ generated by $l^1_\Sigma$, $l^2_\Sigma$, ..., $l^n_\Sigma$




$X_1, X_2, \ldots, X_m$: "faces" in $D(L)$

## Gleams

Each $X_t$ is equipped in a canonical way with a "gleam" $\mathrm{gl}_t \in \mathbb{Z}$
Gleams $(\mathrm{gl}_t)_t$ contain

- Information about crossings in $D(L)$,
- Information about winding numbers $\mathrm{wind}(l^j_{S^1})$

## "Shadow of $L$"

$sh(L) := (D(L), (\mathrm{gl}_t)_t)$

## Example 1: $D(L)$ has no crossing points

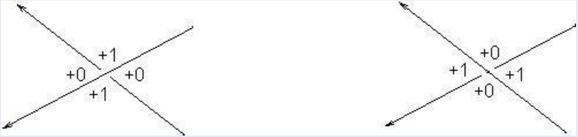
$$\mathrm{gl}_t = \sum_{\{j \mid l^j_\Sigma \text{ touches } X_t\}} \mathrm{wind}(l^j_{S^1}) \cdot \mathrm{sgn}(X_t; l^j_\Sigma)$$

where

$$\mathrm{sgn}(X_t; l^j_\Sigma) = \begin{cases} 1 & \text{if } X_t \text{ is ``inside'' of } l^j_\Sigma \\ -1 & \text{if } X_t \text{ is ``outside'' of } l^j_\Sigma \end{cases}$$

## Example 2: $D(L)$ has crossing points but $\mathrm{wind}(l^j_{S^1}) = 0$, $j \leq n$

Figure: Changes in the gleams at a given crossing point



Fix Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Let $k$ be as in Sec. 2 and additionally, $k > c_{\mathfrak{g}}$ ($c_{\mathfrak{g}}$ dual Coxeter number of $\mathfrak{g}$).

### Colors and Colorings

- "color": dominant weight of $\mathfrak{g}$ (w.r.t. $\mathfrak{t}$) which is "integrable at level" $k - c_{\mathfrak{g}}$.
- $\mathcal{C}$: set of colors
- "link coloring" : mapping $\gamma : \{l_1, l_2, \ldots, l_n\} \to \mathcal{C}$
- "area coloring": mapping $\mathrm{col} : \{X_1, \ldots, X_m\} \to \mathcal{C}$.
- *Col*: set of area colorings

Fix link coloring $\gamma : \{l_1, l_2, \ldots, l_n\} \to \mathcal{C}$.

### Example 3: $\mathfrak{g} = su(2)$, $\mathfrak{t}$ arbitrary

$\mathcal{C} \cong \{0, 1, 2, \ldots, k-2\}$

## "Fusion coefficients" $N_{\alpha\beta}^{\gamma} \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathcal{C}$

$$N_{\alpha\beta}^{\gamma} = \sum\nolimits_{\sigma \in W_k} (-1)^{\sigma} m_{\alpha}(\beta - \sigma(\gamma)),$$

where

- $m_{\alpha}(\beta)$: multiplicity of weight $\beta$ in character of $\alpha$.
- $W_k$: "quantum Weyl group" for $\mathfrak{g}$ and $k$.

## Remark

For our purposes the formula above is more useful than

$$N_{\alpha\beta}^{\gamma} = \sum\nolimits_{\delta} \frac{S_{\alpha\delta} S_{\beta\delta} S_{\gamma^*\delta}}{S_{0\delta}} \qquad \alpha, \beta, \gamma \in \mathcal{C},$$

where $(S_{\alpha\beta})_{\alpha\beta}$ is the $S$-matrix associated to $\mathfrak{g}$ and $k$.

## Some extra notation

- $R_+$: set of positive real roots (w.r.t. fixed Weyl chamber)
- $\rho := \frac{1}{2} \sum_{\alpha \in R_+} \alpha$
- $\langle \cdot, \cdot \rangle$: Killing metric normalized such that $\langle \alpha, \alpha \rangle = 2$ if $\alpha$ is a long root.

## "Shadow invariant" $|\cdot|$ for $\mathfrak{g}$ and $k$

$$|L| = \sum_{\mathrm{col} \in Col} \Big(\prod_{i=1}^{n} N_{\gamma(l_i)\, \mathrm{col}(Y_i^+)}^{\mathrm{col}(Y_i^-)}\Big) \Big(\prod_{t=1}^{m} (V_{\mathrm{col}(X_t)})^{\chi(X_t)} (U_{\mathrm{col}(X_t)})^{\mathrm{gl}_t}\Big)$$

$$\times \Big(\prod_{p \in \mathrm{CP}(L)} \mathrm{symb}_q(\mathrm{col}, p)\Big) \quad \text{where}$$

$\chi(X_t)$ : Euler characteristic of $X_t$

$Y_i^{+/-}$ : face touching $l_\Sigma^i$ from "inside" / "outside"

$$V_\lambda := \prod_{\alpha \in R_+} \frac{\sin \frac{\pi \langle \lambda + \rho, \alpha \rangle}{k}}{\sin \frac{\pi \langle \rho, \alpha \rangle}{k}}$$

$$U_\lambda := \exp\Big(\frac{\pi i}{k} \langle \lambda, \lambda + 2\rho \rangle\Big)$$

$\mathrm{symb}_q(\mathrm{col}, p)$ : associated q-6j-symbol for $q := \exp(\frac{2\pi i}{k})$

## Special case: $D(L)$ has no crossing points

$$|L| = \sum_{\text{col} \in Col} \Big(\prod_{i=1}^{n} N_{\gamma(l_i)\,\text{col}(Y_i^+)}^{\text{col}(Y_i^-)}\Big) \left( \prod_{t=1}^{m} (V_{\text{col}(X_t)})^{\chi(X_t)} (U_{\text{col}(X_t)})^{\text{gl}_t} \right)$$

## Aim of the rest of the Talk

Derive the formula above from the Chern-Simons path integral

# From the CS path integral to the shadow invariant

Gauge group: $\mathcal{G} = C^\infty(M, G)$

$\mathcal{G}$ operates on $\mathcal{A}$ from the right by

$$A \cdot \Omega = \Omega^{-1} A \Omega + \Omega^{-1} d\Omega \qquad \text{for} \quad \Omega \in \mathcal{G}, A \in \mathcal{A}$$

Gauge Fixing: Choice of system $\mathcal{A}_{\text{gf}}$ of representatives of $\mathcal{A}/\mathcal{G}$

**Example: "Axial gauge fixing" for $M = \mathbb{R}^3$**

In this case each $A \in \mathcal{A}$ can be written as $A = \sum_{i=0}^{2} A_i dx_i$.

$$\mathcal{A}_{\text{gf}} = \{A \mid A_0 = 0\}$$

is "essentially" a gauge-fixing.

## "Faddeev-Popov determinant"

If $\mathcal{A}_{\text{gf}}$ is "nice" enough there is a function $\triangle_{FadPop} : \mathcal{A}_{\text{gf}} \to \mathbb{R}$ such that (informally)

$$\int_{\mathcal{A}} \chi(A) DA = \int_{\mathcal{A}_{\text{gf}}} \chi(A) \triangle_{FadPop}(A) DA_{|\mathcal{A}_{\text{gf}}}$$

for every $\mathcal{G}$-invariant function $\chi : \mathcal{A} \to \mathbb{C}$

## Example

For $M = \mathbb{R}^3$ and $\mathcal{A}_{\text{gf}} := \mathcal{A}^{\perp} := \{A \in \mathcal{A} \mid A_0 = 0\}$ we have

$$\triangle_{FadPop}(A) = const. \qquad \Rightarrow$$

$$\int_{\mathcal{A}} \chi(A) DA \sim \int_{\mathcal{A}^{\perp}} \chi(A^{\perp}) DA^{\perp}, \qquad \text{with } DA^{\perp} := DA_{|\mathcal{A}^{\perp}}$$

## Example for usefulness of applying a gauge fixing

For $M = \mathbb{R}^3$ and $\mathcal{A}_{gf} := \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A_0 = 0\}$ we have

$$Z(M, L) = \int_{\mathcal{A}} \prod_i \mathrm{Tr}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA$$

$$\sim \int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA^\perp$$

$$\overset{(*)}{=} \int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}(\mathrm{Hol}_{l_i}(A^\perp)) \exp(i\tfrac{k}{4\pi} \int \mathrm{Tr}(dA^\perp \wedge A^\perp)) DA^\perp$$

$(*)$ holds because $A^\perp \wedge A^\perp \wedge A^\perp = 0$ for $A^\perp \in \mathcal{A}^\perp$.

The last integral involves a **Gauss-type measure!**

# Torus Gauge

## $M = \mathbb{R}^3$

$A = \sum_{i=0}^{2} A_i dx_i = A^\perp + A_0 dx_0$ where $A^\perp := A_1 dx_1 + A_2 dx_2$

Note that $A^\perp \in \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\frac{\partial}{\partial x_0}) = 0\}$

$\mathcal{A}_{\text{gf}} = \mathcal{A}^\perp = \{A \mid A_0 = 0\}$ **Axial gauge fixing**

## $M = \Sigma \times S^1$

Each $A \in \mathcal{A}$ we have $A = A^\perp + A_0 dt$ with $A_0 \in C^\infty(\Sigma \times S^1, \mathfrak{g})$ and $A^\perp \in \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\frac{\partial}{\partial t}) = 0\}$

$dt$ and $\frac{\partial}{\partial t}$ obtained by lifting obvious 1-form/vector field on $S^1$ to $\Sigma \times S^1$

**Problem:** $\mathcal{A}_{\text{gf}} = \mathcal{A}^\perp$ is not a gauge fixing! We need larger space

## 1. Option

$$\mathcal{A}_{\text{gf}} = \mathcal{A}^{\perp} \oplus \{Bdt \mid B \in C^{\infty}(\Sigma, \mathfrak{g})\}$$

## 2. Option: "torus gauge" (Blau/Thompson '93)

$$\mathcal{A}_{\text{gf}} = \mathcal{A}^{\perp} \oplus \{Bdt \mid B \in C^{\infty}(\Sigma, \mathfrak{t})\}$$

where $\mathfrak{t}$ is Lie algebra of fixed maximal torus $T \subset G$

**Example:** $T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \cong U(1)$ is a max. torus for $G = SU(2)$

## Observation

$\triangle_{FadPop}(A^{\perp} + Bdt)$ only depends on $B$ $\longrightarrow$ we set

$$\triangle_{FP}(B) := \triangle_{FadPop}(A^{\perp} + Bdt)$$

## Example

For $G = SU(2)$ and $T$ as above we have

$$\triangle_{FP}(B) \sim \prod_{\sigma} \sin^2(x(B(\sigma)))$$

for suitable isomorphism $x : \mathfrak{t} \to \mathbb{R}$

## Not the full story!

Topological obstructions

$\rightarrow$ strictly speaking torus gauge is not a gauge

$\rightarrow$ we must allow gauge transformations which have a singularity in a fixed point $\sigma_0 \in \Sigma$ (this causes also $A^\perp$ to have a singularity in $\sigma_0$)

$\rightarrow$ 1-1-correspondence

$$\{\text{relevant singularities of } A^\perp \text{ in } \sigma_0\} \quad \longleftrightarrow \quad [\Sigma, G/T] \cong \mathbb{Z}^{\dim(T)}$$

$\rightarrow$ extra summation $\sum_{h \in [\Sigma, G/T]} \cdots$ and factor depending on $h$ in our formulas

# Torus Gauge applied to CS theory on $M = \Sigma \times S^1$

"Definition" of $\triangle_{FP} \Rightarrow$

$$Z(M, L) = \int_{\mathcal{A}} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA$$

$$\sim \int_{C^\infty(\Sigma,\mathfrak{t})} \int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A^\perp + Bdt)) \exp(iS_{CS}(A^\perp + Bdt)) DA^\perp$$
$$\times \triangle_{FP}(B) DB \quad \text{where}$$

- $DA^\perp$: "Lebesgue measure" on $\mathcal{A}^\perp$
- $DB$: "Lebesgue measure" on $C^\infty(\Sigma, \mathfrak{t})$

**1. important Observation**

$S_{CS}(A^\perp + Bdt)$ quadratic in $A^\perp$ for fixed $B$

$\mathcal{A}_{\Sigma,V} :=$ Space of $V$-valued 1-forms on $\Sigma$ for $V \in \{\mathfrak{g}, \mathfrak{t}, \mathfrak{t}^\perp\}$

Identification $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}})$

$$\mathcal{A}^\perp_c := \{A^\perp \in \mathcal{A}^\perp \mid A^\perp \text{ constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}\text{-valued}\}$$
$$\check{\mathcal{A}}^\perp := \{A^\perp \in \mathcal{A}^\perp \mid \int_{S^1} A^\perp(t)dt \in \mathcal{A}_{\Sigma,\mathfrak{t}^\perp}\}$$

Decomposition $\mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}^\perp_c$

## 2. important Observation

$$S_{CS}(\check{A}^\perp + A^\perp_c + Bdt) = S_{CS}(\check{A}^\perp + Bdt) + \frac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c \cdot B)$$

## Final heuristic integral formula

$$Z(M,L) \sim \int_{C^\infty(\Sigma,\mathfrak{t})} \int_{\mathcal{A}_c^\perp} \int_{\check{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(\check{A}^\perp + A_c^\perp + Bdt)) d\check{\mu}_B^\perp(\check{A}^\perp)$$
$$\times \triangle_{FP}(B) Z(B) \exp(i\tfrac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp DB$$

where

$$d\check{\mu}_B^\perp(\check{A}^\perp) := \frac{1}{Z(B)} \exp(iS_{CS}(\check{A}^\perp + Bdt)) D\check{A}^\perp$$

with $Z(B) := \int \exp(iS_{CS}(\check{A}^\perp + Bdt)) D\check{A}^\perp$.

## 3. important Observation

Both $d\check{\mu}_B^\perp$ and $\exp(i\frac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp DB$ are of "Gauss-type"

## Comment

Useful to fix an auxiliary Riemannian metric $g_\Sigma$ on $\Sigma$: $\longrightarrow$

- Hodge star operator $\star : \mathcal{A}_{\Sigma,\mathfrak{g}} \to \mathcal{A}_{\Sigma,\mathfrak{g}}$
- scalar product $\ll \cdot, \cdot \gg$ on $\mathcal{A}_{\Sigma,\mathfrak{g}}$

$\longrightarrow$

- operator $\star : C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}}) \to C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}})$
- scalar product $\ll \cdot, \cdot \gg$ on $C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}})$

## Application: rewriting $d\check{\mu}_B^\perp = \exp(iS_{CS}(\check{A}^\perp + Bdt))D\check{A}^\perp$

$$d\check{\mu}_B^\perp = \exp(-i\tfrac{k}{4\pi} \ll \check{A}^\perp, \star(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))\check{A}^\perp \gg) D\check{A}^\perp$$

(Observe that $\star(\frac{\partial}{\partial t} + \mathrm{ad}(B))$ is symmetric w.r.t. $\ll \cdot, \cdot \gg$)

# Rigorous implementation

## The "discretization approach"

Fix $m \in \mathbb{N}$ and fix triangulation of $\Sigma$, $K$ being the underlying simplicial complex.

Let $C^p(K, V)$ denote the space of $V$-valued p-cochains for $K$, $p \in \mathbb{N}_0$

Discretization based on replacements

- $\mathcal{B} = C^\infty(\Sigma, \mathfrak{t}) = \Omega^0(\Sigma, \mathfrak{t}) \longrightarrow C^0(K, \mathfrak{t})$
- $\mathcal{A}_{\Sigma,\mathfrak{g}} = \Omega^1(\Sigma, \mathfrak{g}) \longrightarrow C^1(K, \mathfrak{g})$
- $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}}) \longrightarrow Maps(\mathbb{Z}_m, C^1(K, \mathfrak{g}))$

## Remark

Apart from $K$ we also use the dual $K'$ of $K$ in order to define a discrete Hodge star operator.

We need to use "field doubling" $\rightarrow$ we are now considering $BF_3$ theory !

(cf. Adams' work on Simplicial Abelian Gauge Theories).

# Evaluation of the path integral

Recall:

- We derived the heuristic formula

$$Z(M, L) \sim \int_{C^\infty(\Sigma,\mathfrak{t})} \int_{\mathcal{A}_c^\perp} \int_{\check{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(\check{A}^\perp + A_c^\perp + Bdt)) d\check{\mu}_B^\perp(\check{A}^\perp) \times \triangle_{FP}(B) Z(B) \exp(i\tfrac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp DB$$

- We are able to make rigorous sense of the r.h.s. of $Z(M, L)$

Let us now restrict ourselves to the special case where $L$ has no crossing points and evaluate $Z(M, L)$ explicitly.

For simplicity: heuristic treatment

## Some properties of rigorous analogue of $\check{\mu}_B^{\perp}$

- oscillatory complex measure of "Gauss-type"
- normalized
- zero mean
- non-definite covariance operator

## Toy model: complex "Gauss-type" measure $\mu$ on $\mathbb{R}^2$

$$\mu(x) = \frac{1}{2\pi} \exp(i\tfrac{1}{2}\langle x, Cx\rangle)dx \quad \text{where } C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Clearly, $\langle v, Cv\rangle = 0$ for $v = (1, 0)$

$\Rightarrow \quad \lim_{\epsilon\to 0} \int \langle x, v\rangle^n e^{-\epsilon|x|^2} d\mu(x) = 0 \qquad$ for all $n \in \mathbb{N}$

$\Rightarrow \quad \lim_{\epsilon\to 0} \int \Phi(\langle x, v\rangle) e^{-\epsilon|x|^2} d\mu(x) = \Phi(0)$ (for all "sufficiently nice" entire analytic functions $\Phi : \mathbb{R} \to \mathbb{R}$)

## 1. Step: Perform $\int \cdots d\check{\mu}^{\perp}_{B}(\check{A}^{\perp})$

$$\int_{\check{\mathcal{A}}^{\perp}} \prod_j \mathrm{Tr}_{\rho_j}(\mathrm{Hol}_{l_j}(\check{A}^{\perp} + A^{\perp}_c + Bdt))d\check{\mu}^{\perp}_B(\check{A}^{\perp})$$

$$= \prod_j \mathrm{Tr}_{\rho_j}(\mathrm{Hol}_{l_j}(0 + A^{\perp}_c + Bdt)) = \prod_j \mathrm{Tr}_{\rho_j}(\exp(\int_{l_j} A^{\perp}_c + \int_{l_j} Bdt)$$

$\longrightarrow$

$$Z(M, L) \sim \int_{C^{\infty}(\Sigma,\mathfrak{t})} \int_{\mathcal{A}^{\perp}_c} \prod_j \mathrm{Tr}_{\rho_j}(\exp(\int_{l_j} A^{\perp}_c + \int_{l_j} Bdt))$$

$$\times \triangle_{FP}(B)Z(B)\exp(i\tfrac{k}{2\pi}\int_{\Sigma} \mathrm{Tr}(dA^{\perp}_c \cdot B))DA^{\perp}_c DB$$

## 2. Step: Perform $\int \cdots DA_c^{\perp}$

Observe

1. $\mathrm{Tr}_{\rho_j}(e^b) = \sum_\alpha m_{\rho_j}(\alpha) e^{i\alpha(b)}$ if $b \in \mathfrak{t}$
2. $\int \mathrm{Tr}(dA_c^{\perp} \cdot B) = \ll B, \star dA_c^{\perp} \gg$
3. $\alpha(\int_{l_\Sigma^j} A_c^{\perp}) = \alpha(\int_{R_\Sigma^j} dA_c^{\perp}) = \ll \alpha \cdot 1_{R_\Sigma^j}, \star dA_c^{\perp} \gg$

$\longrightarrow$

$$\int \prod_j \mathrm{Tr}_{\rho_j}(\exp(\int_{l_j} A_c^{\perp} + \int_{l_j} Bdt) \exp(i \tfrac{k}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^{\perp} \cdot B)) DA_c^{\perp}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n} (\prod_j m_{\rho_j}(\alpha_j)) \quad [\ldots] \quad \int \exp(i \ll \tfrac{k}{2\pi} B - \sum_j \alpha_j 1_{R_\Sigma^j}, \ \star dA_c^{\perp} \gg) DA_c^{\perp}$$

$$\sim \sum_{\alpha_1,\ldots,\alpha_n} (\prod_j m_{\rho_j}(\alpha_j)) \quad [\ldots] \quad \delta(B - \tfrac{2\pi}{k} \sum_j \alpha_j 1_{R_\Sigma^j})$$

## 3. Step: Perform $\int \cdots DB$

$$\begin{aligned}
&Z(M, L) \\
&\sim \sum_{\alpha_1,\ldots,\alpha_n} \int (\prod_j m_{\rho_j}(\alpha_j))(\triangle_{FP}(B) Z(B))(\exp(\ldots))\delta(B - \tfrac{2\pi}{k} \sum_j \alpha_j 1_{R^j_\Sigma}) DB \\
&\overset{(*)}{=} \sum_{\{B = \frac{2\pi}{k} \sum_j 1_{R^j_\Sigma} \alpha_j\}} (\prod_j m_{\rho_j}(\alpha_j))(\triangle_{FP}(B) Z(B))(\exp(\ldots)) \\
&= \quad \ldots \\
&= \sum_{\mathrm{col} \in Col} (\prod_{j=1}^{n} N^{\mathrm{col}(Y_j^-)}_{\gamma(l_j)\, \mathrm{col}(Y_j^+)}) (\prod_{t=1}^{m} (V_{\mathrm{col}(X_t)})^{\chi(X_t)}) (\prod_{t=1}^{m} \exp(\mathrm{gl}_t\, U_{\mathrm{col}(X_t)})) \\
&= |L|
\end{aligned}$$

(step $(*)$ is not quite the full story

## Recall

Topological obstructions

$\rightarrow$ strictly speaking torus gauge is not a gauge
$\rightarrow$ we must allow $A_c^{\perp}$ to have a singularity in fixed point $\sigma_0$ of $\Sigma$
$\rightarrow$ 1-1-correspondence

$$\{\text{relevant singularities of } A_c^{\perp} \text{ in } \sigma_0\} \quad \longleftrightarrow \quad [\Sigma, G/T] \cong \mathbb{Z}^{\dim(T)}$$

$\rightarrow$ extra summation $\sum_{h \in [\Sigma, G/T]} \cdots$ (plus a term depending on the "winding number" of $h$) in some formulas
$\rightarrow$ this extra summation (combined with a suitable application of the Poisson summation formula) does indeed lead to the correct expressions at the end of last slide

# Open Questions

## Question 1

What about the case where $L$ does have crossing points:

Do we obtain quantum 6j-symbols?

## Question 2

Discretization approach possible for original (= non-gauge fixed) path integral?

# References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