The principle of general local covariance and the quantization of Abelian gauge theories

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Outline of the Talk

- Motivations: The source of a problem
- Abelian gauge theories
- Quantizing and losing general local covariance
- Open problems

Based on

- M. Benini, C. D. and A. Schenkel, arXiv:1210.3457 [math-ph], to appear on Ann. Henri Poinc.
- M. Benini, C. D. and A. Schenkel, arXiv:1303.2515 [math-ph].
- M. Benini, C. D., H. Gottschalk, T.-P. Hack and A. Schenkel, in preparation

Which problem?

Starting from the seminal paper of Brunetti, Fredenhagen & Verch

- General local covariance has become the leading principle in AQFT,
- it works for bosonic and fermionic matter,
- It is a powerful concept to use in the study of structural properties of a QFT, e.g., renormalization....

What about gauge theories?

- First application: Maxwell's equations written in terms of the field strength tensor F[1],
- The theory is not generally locally covariant on account of topological obstructions.

[1] C.D., Benjamin Lang, Lett. Math. Phys. **101** (2012) 265

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What goes wrong with the vector potential? - I

One can construct the field algebra for the vector potential:

- $A \in \Omega^1(M)$ such that $\delta dA = 0$ where $\delta = *^{-1} d *$,
- A' is *gauge equivalent* to A if $\exists \chi \in C^\infty(M)$ such that $A' - A = d\chi$

Proposition

The space of solutions for Maxwell's equation $\delta dA = 0$ is

$$\mathcal{S}(M) = \{A \in \Omega^1(M) \mid \exists \omega \in \Omega^1_0(M) \text{ and } A = G(\omega) \text{ with } \delta\omega = 0\},$$

where $G = G^+ - G^-$ is built out of the fundamental solutions for $\Box \doteq d\delta + \delta d = \Box_g - R_{\mu\nu}$.

N.B. Since $\delta \circ G = G \circ \delta$, $\delta\omega = 0$ implies $\delta A = 0$ (Lorenz gauge)

What goes wrong with the vector potential? - II

One can associate to $\mathcal{S}(M)$ the field algebra $\mathcal{A}(M)$:

Proposition

The following statements hold true:

- The field algebra $\mathcal{A}(M)$ associated to the vector potential is not semisimple, that is it possesses an Abelian ideal generated by $\frac{\delta\Omega^2_{0,d}(M)}{\delta d\Omega^1_0(M)}$ whenever $H^2(M) \neq \{0\}$. Furthermore
- For any isometric embedding $\iota : M \to M'$ where $H^2(M) \neq \{0\}$ and $H^2(M') = \{0\}$ the corresponding $*$-homomorphism

$$\alpha_\iota : \mathcal{A}(M) \to \mathcal{A}(M') \text{ is not injective.}$$

Strategy

Why general local covariance fails?

The overall plan is the following:

- Consider **all** possible principal G-bundles with G connected and Abelian,
- Write Maxwell's equation as a theory on the bundle of connections,
- Characterize explicitly the **full** gauge group and analyze the classical dynamics,
- Construct the algebra of fields and study (the failure of) general local covariance.

(Un)expected connections with the Aharonov-Bohm effect appear!

Bundles for Dummies

Proposition:

Let M be a smooth manifold and G a Lie group (structure group). A **principal G-bundle** consists of a smooth manifold P together with a right, free G-action $r : P \times G \to G$, $r(p, g) = pg$ such that

1. M is the quotient P/G and the projection $\pi : P \to M$ is smooth,
2. P is locally trivial, that is, for every $x \in M$, there exists an open neighbourhood $U \subset M$ with $x \in U$ and a G-equivariant diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$.

To each P we can associate the *adjoint bundle*

$$ad(P) = P \times_{ad} \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G. $ad(P)$ is trivial, hence $M \times \mathfrak{g}$, if G is Abelian.

The gauge group

A smooth map $f : P \to P'$ where P, P' are principal G-bundles is

- a **bundle morphism** if $f(pg) = f(p)g$. This entails the existence of a map $\underline{f} : M \to M'$ such that $\underline{f} \circ \pi = \pi' \circ f$.
- a **bundle automorphism** if $P' = P$ and f is also a diffeomorphism. Hence we have a group $Aut(P)$.
- a **gauge transformation** if $f \in Aut(P)$ and $\underline{f} = id_M$. Hence we have a group $Gau(P) \subset Aut(P)$.

If G is Abelian and connected, than $G = \mathbb{R}^k \times T^n$, $n, k \in \mathbb{N}$ and

$$Gau(P) \simeq C^\infty(M; G)$$

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Connections

Goal: Write Maxwell's equations as a theory of connections.

Definition:

Let $\pi : P \to M$ be a principal G-bundle and let $\pi_* : TP \to TM$ be the induced map. Then

- we call **vertical bundle** the collection of all

$$V_p(P) = \{Y \in T_p(P) \mid \pi_*(Y) = 0\}, \quad p \in P,$$

- we call **connection** of P a smooth assignment to each $p \in P$ of a subvector space $H_p(P) \subset T_pP$ such that $T_pP = H_p(P) \oplus V_p(P)$ and $r_{g*}(H_p(P)) = H_{pg}(P)$ for all $g \in G$ and $p \in P$.

- A connection induces a notion of *horizontal lift*, *i.e.* $\forall (x, X) \in TM$ we associate a unique $X_p^\uparrow \in H_p(P)$ for any but fixed $p \in \pi^{-1}(x)$,

Connections: A second look

Essential point: The definition of connection is operatively almost useless.

Theorem

Let $\pi : P \to M$ be a principal G-bundle. Then the Atiyah sequence is exact:

$$0 \longrightarrow ad(P) \xrightarrow{\widetilde{\iota}} TP/G \xrightarrow{\widetilde{\pi}_*} TM \longrightarrow 0 \ .$$

Furthermore the choice of a connection for P is tantamount to $\widetilde{\lambda} : TM \to TP/G$ such that $\widetilde{\pi}_* \circ \widetilde{\lambda} = id_{TM}$. Hence the sequence splits: $TP/G = TM \oplus ad(P)$.

Notice:

- Assigning a connection is also equivalent to assigning $\omega \in \Omega^1(P; \mathfrak{g})$ such that $r_g^*(\omega) = ad_{g^{-1}}\omega$, for all $g \in G$ and $\omega(X^\xi) = \xi$ for all $\xi \in \mathfrak{g}$

The bundle of connections

Proposition:

Let $\pi : P \to M$ be a principal G-bundle and let $\pi_{Hom} : Hom(TM, TP/G) \to M$ be the homomorphism bundle. We call **bundle of connections** $\mathcal{C}(P)$, the sub-bundle $\pi_{\mathcal{C}} : \mathcal{C}(P) \to M$, of all linear maps $\widetilde{\lambda}_x : T_xM \to (TP/G)_x$ such that $\widetilde{\pi}_* \circ \widetilde{\lambda}_x = id_{T_xM}$.

Main consequence:

- The bundle of connections is an **affine bundle** modeled on the vector bundle $\pi'_{Hom} : Hom(TM, ad(P)) \to M$.

Affine spaces

Definition:

An **affine space** A modeled on a vector space V is a set endowed with an Abelian right group action $\Phi_A : A \times V \to A$

Notice that a map $f : A \to B$ between affine spaces is

- called **affine** if there exists a linear map $f_V : V_A \to V_B$ such that $\Phi_B \circ (f \times f_V) = f \circ \Phi_A$. f_V is called the linear part of f,
- compatible with the Abelian group action, if it is an affine map. We write

$$f(a) +_B f_V(v) = f(a +_A v), \quad \forall a \in A \text{ and } \forall v \in V_A.$$

The collection of all affine maps from A to $\mathbb{R}$ form $A^\dagger$, the **vector dual** of an affine space.

Affine bundles

An **affine bundle** is a triple $(M, \widetilde{A}, \widetilde{V})$ where M is a differentiable manifold and

1. $\widetilde{V} \equiv (M, \pi_E, E)$ is a vector bundle modeled on a vector space V,
2. $\widetilde{A} \equiv (M, \pi_F, F)$ is a fibre bundle such that, for all $x \in M$, $\pi_F^{-1}(x)$ is an affine space modeled on $\pi_E^{-1}(x)$,
3. The typical fiber of $\widetilde{A}$ is an affine space modeled on V,
4. For all $x \in M$, there exist a neighborhood U of x, a trivialization ψ of $\widetilde{A}$ on U and a trivialization ϕ of $\widetilde{V}$ on U such that, for all $y \in U$, the linear part of $\psi|_y$ coincides with $\phi|_y$, namely $\psi_V|_y = \phi|_y$

As with affine spaces, we can construct the **vector bundle dual** to any affine bundle.

The curvature of a connection

Notice: Henceforth we assume $G = U(1)$

Definition

Let $\pi : P \to M$ be a principal $U(1)$-bundle. Then we call **curvature** the assignment $\mathcal{F} : \Gamma^\infty(\mathcal{C}(P)) \to \Omega^2(P, \mathfrak{u}(1))$ such that

$$\mathcal{F}(\widetilde{\lambda}) = d_P \omega_{\widetilde{\lambda}},$$

where $\omega_{\widetilde{\lambda}} \in \Omega^1(P, \mathfrak{u}(1))$ is the connection 1-form associated to $\widetilde{\lambda}$.

Notice:

- $\mathcal{F}(\widetilde{\lambda})$ can be regarded as $F_{\widetilde{\lambda}} \in \Omega^2(M)$ via $F_{\widetilde{\lambda}}(X, Y) \doteq d_P \omega_{\widetilde{\lambda}}(X_p^\uparrow, Y_p^\uparrow)$,
- Let $\widetilde{\lambda}, \widetilde{\lambda}' \in \Gamma(\mathcal{C}(P))$, then there exists $\eta \in \Omega^1(M)$ such that

$$\widetilde{\lambda} = \widetilde{\lambda}' + \eta \Longrightarrow F_{\widetilde{\lambda}} = F_{\widetilde{\lambda}'} - d\eta.$$

Classification of a $U(1)$ bundle

Further properties of the curvature of a connection:

- For each $\widetilde{\lambda}$, $F_{\widetilde{\lambda}}$ is closed, hence $dF_{\widetilde{\lambda}} = 0$,
- The cohomology class $[F_{\widetilde{\lambda}}] \in H^2(M)$ does not depend on $\widetilde{\lambda} \in \Gamma^\infty(\mathcal{C}(P))$.

Theorem

Let $\pi : P \to M$ be a principal $U(1)$-bundle and let $\widetilde{\lambda} \in \Gamma^\infty(\mathcal{C}(P))$. Then $e_{\mathbb{R}}(P) \doteq -\frac{1}{2\pi}[F_{\widetilde{\lambda}}]$ is the **real Euler class** *of P. This is said to be* **natural***, that is, if $\pi' : P' \to M'$ is a second principal $U(1)$ bundle, any bundle morphism $f : P \to P'$ satisfies*

$$\underline{f}^* \left(e_{\mathbb{R}}(P')\right) = e_{\mathbb{R}}(P),$$

where $\underline{f} : M \to M'$ is such that $\underline{f} \circ \pi = \pi' \circ f$.

The Gauge group - I

It can be proven that:

- Given a principal G-bundle $\pi : P \to M$, let $\widetilde{\lambda} \in \Gamma^\infty(\mathcal{C}(P))$ and $f \in Gau(P)$. The gauge-transformed connection $\widetilde{\lambda}_f$ is

$$\widetilde{\lambda}_f(X) \doteq (\widetilde{f}_*^{-1})\lambda(X), \quad \forall X \in TM,$$

  where $\widetilde{f}_* : TP/G \to TP/G$ is induced by $f : P \to P$;
- If the structure group G is Abelian, then, for any $\chi \in C^\infty(M; \mathfrak{g})$, the application $\exp \circ \chi \in C^\infty(M; G)$ identifies a unique $f_\chi \in Gau(P)$. The set of all these f is called $Gau_0(P) \subseteq Gau(P)$, and
- For any $\widetilde{\lambda} \in \Gamma^\infty(\mathcal{C}(P))$ and for any $f_\chi \in Gau_0(P)$,

$$\widetilde{\lambda}_{f_\chi} = \widetilde{\lambda} - d\chi.$$

The Gauge group - II

What is the full structure of $Gau(P)$?

- Let $\mu_{U(1)} \in \Omega^1(U(1))$ be the Maurer-Cartan form for $U(1)$. Then, for every $f \in C^\infty(M; U(1))$, $f^*\mu_{U(1)} \in \Omega^1(M)$ and it is *closed*,
- It holds that $A_{U(1)} = \frac{\{f^*\mu_{U(1)} \mid f \in C^\infty(M;U(1))\}}{dC^\infty(M)} \subseteq H^1(M)$

Theorem:

The quotient $A_{U(1)}$ is isomorphic to $\check{\mathrm{H}}^1(M;\mathbb{Z})$, the first Čech cohomology group with integral coefficients.

$$\check{\mathrm{H}}^1(M;\mathbb{Z}) \hookrightarrow \check{\mathrm{H}}^1(M;\mathbb{R}) \simeq H^1(M).$$

The Phase Space - I

The equation of motion is given by setting to 0

$$MW = \delta \circ F : \Gamma^\infty(\mathcal{C}(P)) \to \Omega^1(M).$$

Notice that

- $\delta \circ F$ is an *affine differential operator* whose linear part is $\delta d : \Omega^1(M) \to \Omega^1(M)$.
- It admits a formal adjoint $MW^* : \Omega^1_0(M) \to \Gamma^\infty_0(\mathcal{C}(P)^\dagger)$ such that $\forall \lambda \in \Gamma^\infty(\mathcal{C}(P))$ and $\forall \eta \in \Omega^1_0(M)$

$$\langle MW^*(\eta), \lambda \rangle = \int_M d\mu(g)\, (MW^*(\eta))(\lambda) \doteq \int_M \eta \wedge *(MW(\lambda)).$$

- The formal adjoint is **unique** only if we single out from $\Gamma^\infty_0(\mathcal{C}(P)^\dagger)$

$$\mathrm{Triv} \doteq \{a\mathbb{I} \in \Gamma^\infty_0(\mathcal{C}(P)^\dagger) \mid a \in C^\infty_0(M) \text{ and } \int_M d\mu(g) a = 0\}.$$

The Phase Space - II

We have to implement gauge invariance

We start with the following set of *observables*:

$$\forall\varphi \in \Gamma_0^\infty(\mathcal{C}(P)^\dagger)/Triv \longrightarrow \mathcal{O}_\varphi : \Gamma^\infty(\mathcal{C}(P)) \to \mathbb{R},$$

such that $\mathcal{O}_\varphi(\lambda) = \int\limits_M d\mu(g)\ \varphi(\lambda)$.

Proposition

Invariance of an observable $\mathcal{O}_\varphi$ under gauge transformations implies that, if $\varphi_V \in \Omega_0^1(M)$ is the linear part of $\varphi \in \Gamma_0^\infty(\mathcal{C}(P)^\dagger)/Triv$

$$\langle \varphi_V, f^*(\mu_{U(1)})\rangle = 0 \qquad \forall f \in C^\infty(M; U(1)).$$

The Phase Space - III

We call phase space of a $U(1)$ gauge theory

$$\mathcal{E}^{inv} \doteq \{\varphi \in \Gamma_0^\infty(\mathcal{C}(P)^\dagger)/Triv \mid \langle \varphi_V, f^*(\mu_{U(1)})\rangle = 0 \quad \forall f \in C^\infty(M; U(1))\}.$$

Theorem

The following holds true:

1. for all $\varphi \in \mathcal{E}^{inv}$, $\delta\varphi_V = 0$,
2. The dynamics can be implemented via $MW(\lambda) = 0$, that is $\mathcal{E}^{inv} \to \mathcal{E} \doteq \mathcal{E}^{inv}/MW^*(\Omega_0^1(M))$.
3. The following bilinear form $\tau : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is presymplectic

$$\tau([\varphi], [\varphi']) \doteq \int_M \varphi_V \wedge *(G(\varphi'_V)),$$

where G is the causal propagator of $\Box = \delta d + d\delta$.

No Aharanov-Bohm observables

Let $\lambda, \lambda' \in \Gamma^\infty(\mathcal{C}(P))$ be two connections such that

$$F(\lambda) = F(\lambda') \Longrightarrow \lambda - \lambda' = \eta,$$

where $\eta \in \Omega^1(M)$ and $d\eta = 0$. Notice that

- η identifies $[\eta] \in H^1(M)$,
- $[\eta]$ is not necessarily in the image of $\check{\mathrm{H}}^1(M;\mathbb{Z})$ in $H^1(M)$,
- for all $\varphi \in \mathcal{E}^{inv}$, $\varphi_V = \delta\beta$, with $\beta \in \Omega^2_0(M)$ and

$$\mathcal{O}_\varphi(\lambda) = \mathcal{O}_\varphi(\lambda') + \langle \varphi_V, \eta \rangle = \mathcal{O}_\varphi(\lambda')$$

The algebra of observables does not separate all configurations!

The center of τ

The presymplectic form τ contains the following center

$$\mathcal{N} \doteq \{\varphi \in \mathcal{E}^{inv} \mid \varphi_V \in \delta\Omega^2_{0,d}\}/MW^*(\Omega^1_0(M)).$$

$\mathcal{N}$ is not trivial whenever $H^2_0(M) \simeq H^2(M) \neq \{0\}$.

The relevant categories

Two categories are playing a key role:

- The first is $\mathfrak{PrBu}$:
  1. Objects are principal $U(1)$-bundles P over a glob. hyp. spacetime M,
  2. Arrows are bundle morphisms $f : P \to P'$ such that $f(pg) = f(p)g$ for all $p \in P$ and $g \in U(1)$.
  3. For each arrow f the induced map $\underline{f} : M \to M'$ is an orientation, time orientation preserving, isometric embedding with causally compatible images.
- The second is $\mathfrak{PSymp}$:
  1. Objects are vector spaces V together with an antisym. bilinear map τ,
  2. Arrows are linear maps from two objects V and V' preserving τ and τ' (No injectivity).

The Phase Space Functor

Our construction entails the existence of a covariant functor

$$\mathfrak{PHSP} : \mathfrak{PrBu} \to \mathfrak{PSymp}$$

which assigns

- to every principal bundle P, the on-shell gauge invariant observables $(\mathcal{E}, \tau)$
- For each arrow $f : P \to P'$ a linear map $f_* : \mathcal{E} \to \mathcal{E}'$ induced by singling out the image of MW^* from the map $f_* : \mathcal{E}^{inv}/Triv \to \mathcal{E}'^{inv}/Triv'$ defined as follows
$$\int_{M'} d\mu(g')\,(f_*\varphi)(\lambda') = \int_M d\mu(g)\,\varphi(f^*\lambda'),$$
for each $\varphi \in \mathcal{E}^{inv}/Triv$ and each $\lambda' \in \Gamma^\infty(\mathcal{C}(P'))$.

What is working fine?

Essentially two aspects are still working as we would like:

1. Causality: observables spacelike separated and hence commuting in P so are in P'
2. The time slice axiom holds true.

The problem is:

What is not working fine

- The map between the space of observables is not injective in general!
  - Set P' as a/the principal $U(1)$ bundle over Minkowski spacetime.
  - Set P as the trivial principal $U(1)$-bundle on $M = \mathbb{R}^4 \setminus \left(J^+(0) \cup J^-(0)\right)$. Then $P'|_M = P$.
  - $H^2(\mathbb{R}^4) = \{0\}$ but $H^2(M) = \mathbb{R}$.
  - Let $\eta \in \Omega^2_0(M)$ such that $d\eta = 0$, but $\eta \neq d\alpha$. Let $F^*(\eta) \in \mathcal{E}^{inv}$ be

$$\int_M d\mu(g)\,(F^*(\eta))(\lambda) = \int_M \eta \wedge *F(\lambda) \quad \forall\lambda \in \Gamma^\infty(\mathcal{C}(P)).$$

  - Since $\eta \neq d\alpha$, then $F^*\eta \neq MW^*(\alpha)$. Yet $\underline{f}_*\eta = d\beta$, hence

$$f_*(F^*(\eta)) = F'^*(\underline{f}_*\eta) = MW'^*(\beta).$$

What we can measure...

- There is a rather interesting novel observable
  - Take any $\alpha \in \Omega^2_0(M)$ such that $\delta\alpha = 0$,
  - Take $F^*(\alpha) \in \mathcal{E}^{inv}$ defined by

$$\int_M d\mu(g)(F^*\alpha)(\lambda) = \int_M \alpha \wedge *F(\lambda) = \int_M F(\lambda) \wedge *\alpha \quad \forall \lambda \in \Gamma(\mathcal{C}(P)).$$

  - Notice that the right hand side is actually the pairing between $[*\alpha] \in H^2_0(M)$ and $[F(\lambda)] \in H^2(M)$
  - Observables similar to the one above can determine the cohomology class of the curvature of λ, namely the Euler class of the bundle.
  - The linear part of $F^*\alpha$ is $\delta\alpha = 0$. The observable is purely affine.

What else can we measure...

- There is a second kind of interesting observables
  - Take any $\beta \in \Omega^2_0(M)$ such that $d\alpha = 0$.
  - Take $F^*\beta \in \mathcal{E}^{inv}$ defined by

$$\int_M d\mu(g)(F^*\beta)(\lambda) = \int_M \beta \wedge *F(\lambda) = \int_M F(\lambda) \wedge *\beta \quad \forall \lambda \in \Gamma(\mathcal{C}(P)).$$

  - Notice that, if the connection is on-shell, the right hand side is actually the pairing between $[\beta] \in H^2_0(M)$ and $[*F(\lambda)] \in H^2(M)$.
  - These observables measures completely $[*F(\lambda)]$. It is a measure of the **electric charge**.
  - The linear part of $F^*\beta$ is $\delta\beta$. The observable is purely central.

A locally covariant quotient algebra

There is now a way to restore local covariance:

- The phase space $\mathcal{E}$ is replaced by $\mathcal{E}^{inv}/F^*(\Omega^2_{0,d})$.
- Change the definition of objects in $\mathfrak{PrBu}$.
- Keep the same covariant functor $\mathfrak{PHSP} : \mathfrak{PrBu} \to \mathfrak{PSymp}$.
- All maps are injective. Hence general local covariance is restored.

Yet, remember that our algebra is not separating

A sketch of the future

What about the sneaky configurations?

- Instead of observables $\lambda \mapsto \mathcal{O}_\varphi(\lambda)$ we consider those of exponential type:

$$\mathcal{W}_\varphi : \Gamma^\infty(M; \mathcal{C}(P)) \to \mathbb{C} \quad \lambda \mapsto \exp(2\pi i \mathcal{O}_\varphi(\lambda)),$$

  where $\varphi \in \Gamma_0^\infty(M, \mathcal{C}(P)^\dagger)$.

- We show that the collection of these new functionals forms a well-defined algebra,
- we select the sub-algebra of *gauge invariant* functionals (there are more now!),
- We prove that the new algebra is separating on gauge equivalence classes of configurations!

A sketch of the future - I

The good,

- The exponentiated algebra is "well-behaved" and separates the gauge equivalence classes (in a sense we account for AB observables)

The bad,

- General local covariance could not be implemented before, it cannot be now!

The ugly,

- The center of the new algebra does not coincide with that of the "linear" algebra,
- it cannot be consistently singled out for the algebra to recover the locality property.

Where are we?

We have proven that

- Maxwell's equations in their full glory as a $U(1)$ gauge theory can be quantized in the algebraic framework.
- The choice between F and A is no longer existent.
- The theory is not locally covariant. Amending the problem looks like playing tic-tac-toe. Alternatively work with 0 electric charge, but configurations are not fully separated.
- The Aharonov-Bohm observables are not present on account of the gauge group and of the linear structure of the dynamics.

Open issues:

- If we couple to P the Dirac bundle, can the construction get weirder? Probably not!
- Can we repeat our construction for non-Abelian gauge theories?