Characterization of rational conformal QFTs and their boundary conditions[1]

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[1] work in progress with Roberto Longo and Yasuyuki Kawahigashi

- Algebraic quantum field theory: A family of algebras containing all local observables associated to space-time regions.
- Many structural results, recently also construction of interesting models
- Conformal field theory (CFT) in 1 and 2 dimension described by AQFT quite successful, e.g. partial classification results (e.g. $c < 1$) (Kawahigashi and Longo, 2004)
- Boundary Conformal Quantum Field Theory (BCFT) on Minkowski half-plane: (Longo and Rehren, 2004)

$\mathcal{H}$ Hilbert space, $\mathcal{I}$ = family of proper intervals on $S^1 \cong \overline{\mathbb{R}}$

$$\mathcal{I} \ni I \longmapsto \mathcal{A}(I) = \mathcal{A}(I)'' \subset \mathrm{B}(\mathcal{H})$$

A. Isotony. $I_1 \subset I_2 \Longrightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

B. Locality. $I_1 \cap I_2 = \varnothing \Longrightarrow [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

C. Möbius covariance. There is a unitary representation U of the Möbius group ($\cong \mathrm{PSL}(2,\mathbb{R})$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

D. Positivity of energy. U is a positive-energy representation, i.e. generator L_0 of the rotation subgroup (conformal Hamiltonian) has positive spectrum.

E. Vacuum. $\ker L_0 = \mathbb{C}\Omega$ and Ω (vacuum vector) is a unit vector cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

▸ Consequences ▸ Complete Rationality

- Irreducibility. $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I) = B(\mathcal{H})$
- Reeh-Schlieder theorem. Ω is cyclic and separating for each $\mathcal{A}(I)$.
- Bisognano-Wichmann property. The Tomita-Takesaki modular operator Δ_I and and conjugation J_I of the pair $(\mathcal{A}(I),\Omega)$ are

$$U(\Lambda(-2\pi t)) = \Delta^{\mathrm{i}t},\ t\in\mathbb{R} \qquad \text{dilation}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

  (Gabbiani and Fröhlich, 1993), (Guido and Longo, 1995)
- Haag duality. $\mathcal{A}(I') = \mathcal{A}(I)'$.
- Factoriality. $\mathcal{A}(I)$ is III_1-factor (in (Connes, 1973) classification)
- Additivity. $I\subset\bigcup_i I_i \Longrightarrow \mathcal{A}(I)\subset\bigvee_i\mathcal{A}(I_i)$ (Fredenhagen and Jörß, 1996).

example complete rationality

A representation of $\mathcal{A}$ is a family of representations $\pi = \{\pi_I : \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}$ on a Hilbert space $\mathcal{H}_\pi$ such that

$$\pi_J \restriction \mathcal{A}(I) = \pi_I \qquad I \subset J.$$

Fact: Let π be a non-degenerated represenatation on a seperable space $\mathcal{H}_\pi$ and let $I \in \mathcal{I}$ then there is a unitary equivalent representation ρ on $\mathcal{H}$, such that:

1. $\rho_{I'} = \mathrm{id}_{\mathcal{A}(I')}$, i.e. ρ is **localized** in I.
2. $\rho_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for all $J \supset I$, i.e. ρ_J is an **endomorphism** of $\mathcal{A}(J)$.

We call ρ_J an DHR endomorphism. It is enough to look into representation localized in I. $\mathrm{Rep}^I(\mathcal{A})$ is a full subcategory of $\mathrm{End}(\mathcal{N})$ with $\mathcal{N} = \mathcal{A}(I)$. A **sector** is a unitary equivalence class $[\pi]$. We can define the **fusion** by composition of DHR endomorphisms.

$$[\pi^1] \times [\pi^2] := [\rho^1 \circ \rho^2] \qquad \rho^i \in [\pi^i] \text{ localized in } I$$

$\mathrm{End}(\mathcal{N})$

Let $\mathcal{N}$ be a **type III factor** and $\mathrm{End}(\mathcal{N})$ the **C*-tensor category** with:

- *objects:* **endomorphisms** $\rho \in \mathrm{End}(\mathcal{N})$
- *arrows:* **intertwiner** $t : \rho \to \sigma$ with

$$t \in \mathrm{Hom}(\rho, \sigma) = \{s \in \mathcal{N} : s\rho(n) = \sigma(n)s \text{ for all } n \in \mathcal{N}\}$$

- $\otimes$*-product:* $\rho \otimes \sigma = \rho \circ \sigma$ (composition), $s : \sigma \to \sigma'$ and $t : \tau \to \tau'$ then $s \otimes t : \sigma : \tau \circ \sigma \to \tau' \circ \sigma'$ given by $s \otimes t = s\sigma(t) = \sigma'(t)s$.

A **sector** $[\rho]$ is the unitary equivalence class ($\rho \sim \rho' \Leftrightarrow \rho(\,\cdot\,) = U\rho'(\,\cdot\,)$ for some $U \in \mathcal{N}$ unitary). Direct sums :

$$[\rho] \oplus [\sigma] = [\mathrm{Ad}w_1 \circ \rho + \mathrm{Ad}w_2 \circ \sigma] \qquad w_1 w_1^* + w_2 w_2^* = 1, w_i^* w_j = \delta_{ij}$$

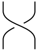
Proposition (Longo)

Irreducible finite depth subfactors $\iota(\mathcal{N}) \subset \mathcal{M} \longleftrightarrow$ Q-systems (θ, w, x) in $\mathrm{End}(\mathcal{N})$, where $\theta = \bar{\iota} \circ \iota \in \mathrm{End}(\mathcal{N})$ is the **dual canonical endomorphism**, an algebra object with unit $w^* : \theta \to \mathrm{id}$ and counit $x : \theta \to \theta \circ \theta$.

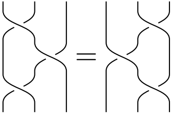
Fusion coefficients:

$$[\rho] \times [\sigma] = \bigoplus_{[\tau]} N^{\tau}_{\rho\sigma} [\tau]$$

with fusion coefficients $N^{\tau}_{\rho\sigma} = \dim \mathrm{Hom}(\rho\sigma, \tau)$.
The fusion is commutative $[\pi^1] \times [\pi^2] = [\pi^2] \times [\pi^1]$ and there is a natural choice of unitaries, the **braiding**:

$\varepsilon(\rho, \sigma) =$  $: \rho \circ \sigma \to \sigma \circ \rho \qquad \rho, \sigma \in \mathrm{Rep}^I(\mathcal{A})$

the braiding. Fulfills naturality (braiding fusion equations) and Yang-Baxter identity:

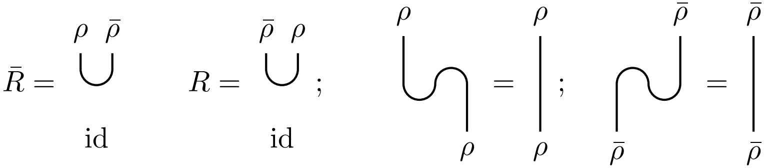


Conjugates

Let us consider $\mathrm{Rep}^I_\mathrm{f}(\mathcal{A})$, i.e. only representations with finite statistical dimension $d\rho < \infty$, where $[\mathcal{M} : \mathcal{N}]$ denotes the minimal (Jones) index:

$$(d\rho)^2 = \left[\rho_{J'}(\mathcal{A}(J'))' : \rho_J(\mathcal{A}(J))\right] \equiv [\mathcal{A}(I) : \rho_I(\mathcal{A}(I))]$$

For $[\rho]$ one can define a conjugate DHR sector $[\bar\rho]$ by $\bar\rho_{I'} = j \circ \rho_I \circ j$ where j is the anti-automorphism of $\mathcal{A}(I)$ given by Bisognano–Wichmann property. Then there exist $\bar R \in \mathrm{Hom}(\mathrm{id}, \rho \circ \bar\rho)$ and $R \in \mathrm{Hom}(\mathrm{id}, \bar\rho \circ \rho)$ fulfilling the zig-zag identity:

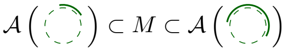


Unitary ribbon category

Complete rationality

Completely rational conformal net (Kawahigashi, Longo, Müger (2001))

- Split property. For every relatively compact inclusion of intervals $\exists$ intermediate **type** I **factor** M



- Finite μ-index: finite Jones index of subfactor

where the intervals are splitting the circle.

Consequences

- Strong additivity. (Longo and Xu, 2004) Additivity for touching intervals:

- Only finite sectors, each sector has finite statistical dimension
- **Modularity:** The category of DHR sectors is modular, i.e. non degenerated braiding.

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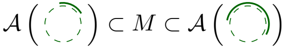

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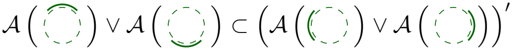

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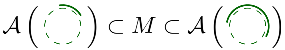

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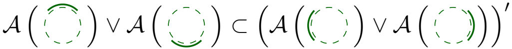

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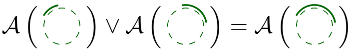

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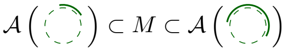

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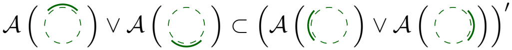

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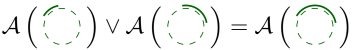

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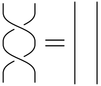
- **Modularity:** The category of DHR sectors is modular, i.e. non degenerated braiding.

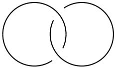
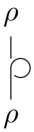
Modularity

If the net $\mathcal{A}$ is completely rational then $\mathrm{Rep_f}(\mathcal{A})$ is a modular C*-tensor category (unitary MTC):

1. **Finite #** of **sectors**.
2. The **braiding** is **non-degenerated**, i.e.

$$\varepsilon(\rho,\sigma)\varepsilon(\sigma,\rho) \equiv$$  $$= 1 \text{ for all } \rho \quad \Longrightarrow \quad [\sigma] = N[\mathrm{id}]$$

   identity is the only *transparent* object, with respect to the braiding or equivalently S-matrix (Rehren) is unitary:

$S_{\rho\sigma} \sim \rho$  σ ; $T_{\rho\rho} \sim$  = conformal spin

$$SS^* = TT^* = 1, \qquad (ST)^3 = S^2, \qquad S^4 = 1$$

   Unitary representation of the "modular group" $\mathrm{SL}(2,\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Example

G compact Lie group
Loop group: $\mathrm{L}G = \mathrm{C}^\infty(S^1, G)$ (point wise multiplication)
Projective representations $\longleftrightarrow$ representations of a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{\mathrm{L}G} \longrightarrow \mathrm{L}G \longrightarrow 1$$

$\pi_{0,k}$ projective **positive-energy** and **vacuum** representation (classified by the level k)

$$I \longmapsto \mathcal{A}_{G,k}(I) = \pi_{0,k}(\mathrm{L}_I G)''$$

is a **conformal net**; $\mathrm{L}_I G$ loops supported in I.

Example

$G = \mathrm{SU}(n)$ gives completely rational conformal net (Xu, 2000)

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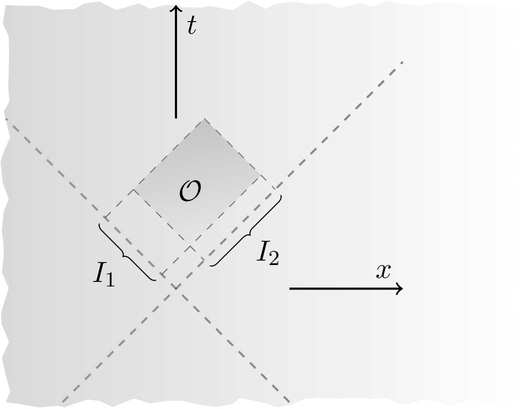
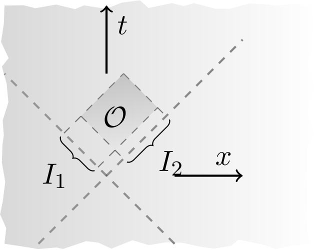
- **Minkowski space** $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$
- **Double cone** $\mathcal{O} = I_1 \times I_2$ where I_1, I_2 disjoint intervals





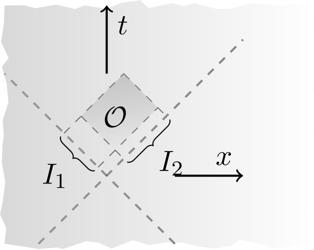
Let us fix a **completely rational conformal net** $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$.
One can define a **chiral conformal net** on Minkowski space by

$$\mathcal{A}_2(\mathcal{O}) = \mathcal{A}(I_1) \otimes \mathcal{A}(I_2) \subset \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$$

Non-chiral nets are given by irreducible local extensions

$$\mathcal{B}_2(O) \supset \mathcal{A}_2(O) \equiv \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

on a bigger Hilbert space $\mathcal{H}_{\mathcal{B}_2}$.



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Classification Problem I

Given **completely rational conformal net** $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$. Find (up to unitary equivalence) **all local irreducible extensions**

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Only **finitely** many extensions (Izumi,Popa,Longo,...).

Outline

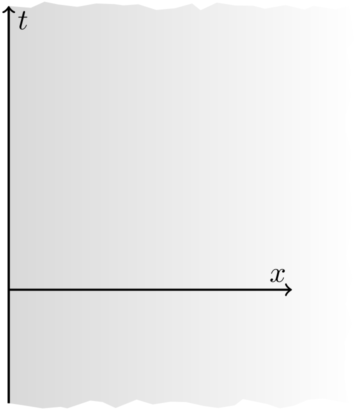
Conformal Nets

Nets on Minkowski space

Nets on Minkowski half-plane

Boundary conditions

- **Minkowski half-plane** $x > 0$, $ds^2 = dt^2 - dx^2$
- **Double cone** $\mathcal{O} = I_1 \times I_2$ where I_1, I_2 disjoint intervals



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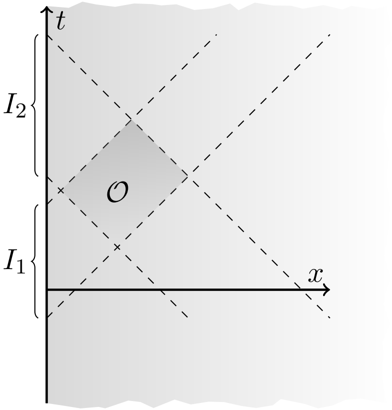

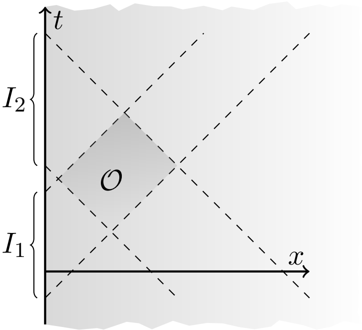

Given **completely rational conformal net** $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$.
Trivial (=chiral) **boundary conformal net** (Longo and Rehren, 2004) is given by

$$\mathcal{A}_+(\mathcal{O}) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathrm{B}(\mathcal{H}).$$

General **boundary conformal nets** given by (Möbius covariant) irreducible local extensions

$$\mathcal{B}_+(O) \subset \mathcal{A}_+(O) \equiv \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$$

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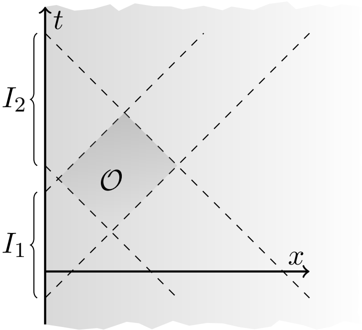

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Classification Problem II

Given **completely rational conformal net** $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$. Find (up to unitary equivalence) all **boundary conformal nets** $\mathcal{B}_+ \supset \mathcal{A}_+$, i.e. **all local irreducible extensions**

$$\mathcal{B}_+(O) \supset \mathcal{A}_+(O) \equiv \mathcal{A}(I_1) \vee \mathcal{A}(I_2).$$

Two covariant nets $\mathcal{R}_1, \mathcal{R}_2$ are called **locally isomorphic** if there is an family of isomorphism $\Phi_O : \mathcal{R}_1(O) \to \mathcal{R}_2(O)$ such that:

$$\Phi_{\tilde{O}} \restriction \mathcal{R}_1(O) = \Phi_O \qquad\qquad O \subset \tilde{O}$$
$$U_2(g)\Phi_O(x)U_2(g)^* = \Phi_{gO}(U_1(g)xU_1(g)^*) \qquad x \in \mathcal{R}_1(O)\,.$$

The chiral conformal net $\mathcal{A}_2$ on $\mathbb{M}$ and the chiral boundary conformal net $\mathcal{A}_+$ on $\mathbb{M}_+$ are locally equivalent (due to split property), i.e.

$$\mathcal{A}_2 \restriction \mathbb{M}_+ \cong \mathcal{A}_+$$

Given $\mathcal{B}_2 \supset \mathcal{A}_2$ we say the net $\mathcal{B}_+ \supset \mathcal{A}_+$ is a (conformal) **boundary condition** (with chiral symmetry $\mathcal{A}$) if the above local equivalence extends to:

$$\mathcal{B}_2 \restriction \mathbb{M}_+ \cong \mathcal{B}_+.$$

Classification Problem III

Given **completely rational conformal net** $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$ and a **local irreducible extensions**

$$\mathcal{B}_2(O) \supset \mathcal{A}_2(O) \equiv \mathcal{A}(I_1) \otimes \mathcal{A}(I_2).$$

Find **all boundary conditions** (with chiral symmetry $\mathcal{A}$), i.e. all $\mathcal{B}_+ \supset \mathcal{A}_+$

$$\mathcal{B}_+ \cong \mathcal{B}_2 \restriction \mathbb{M}_+$$

Only **finitely** many extensions (Izumi,Popa,Longo,...).
All these boundary conditions can be obtained by an operator algebraic construction (Carpi, Kawahigashi and Longo 2012).

Main motivation of this talk

How do this boundary conditions look like?

Partial answer was given already in (Longo and Rehren, 2004).

We got an immediate answer by the following "curious identity" due to (Evans, 2002) for non-degenerate braided subfactors.

$$\bigoplus_{\mu\nu \in {}_\mathcal{N}\Delta_\mathcal{N}} Z_{\mu\nu}[\mu \circ \bar{\nu}] = \bigoplus_{a \in {}_\mathcal{N}\Delta_\mathcal{M}} [a \circ \bar{a}]$$

but unfortunately this is not the whole story.

Let us first go back to:

Classification Problem I

Given **completely rational conformal net** $\mathcal{A}$ on $S^1 \cong \overline{\mathbb{R}}$. Find (up to unitary equivalence) **all local irreducible extensions**

$$\mathcal{B}_2(O) \supset \mathcal{A}_2(O) \equiv \mathcal{A}(I_1) \otimes \mathcal{A}(I_2).$$

With a given conformal net $\mathcal{B}_2$ on $\mathbb{M}$ we can associate a maximal net $\mathcal{B}_2^{\max}(O) \supset \mathcal{B}_2(O)$ by

$$\mathcal{B}_2(O) \to \mathcal{B}_+(O) \to \mathcal{B}_+^d(O) = \mathcal{B}_+(O')' \to \mathcal{B}_2^{\max}(O)$$

Then $\mathcal{B}_2$ is characterized by the intermediate subfactor

$$\mathcal{A}_2(O) \subset \mathcal{B}_2(O) \subset \mathcal{B}_2^{\max}(O)$$

Properties of $\mathcal{B}_2^{\max}$:

1. $\mathcal{B}_2$ has no DHR superselection sectors ($\mu = 1$).
2. $\mathcal{B}_2$ is modular invariant.

$\mathcal{B}_2^{\max}$ is **modular invariant**: let $\iota : \mathcal{A}_2(O) \hookrightarrow \mathcal{B}_2^{\max}(O)$ then $\theta_2 = \bar{\iota} \circ \iota \in \mathrm{End}(\mathcal{A}_2(O))$ is the dual canonical endomorphism which decomposes as:

$$[\theta_2] = \bigoplus_{\mu,\nu \in \mathrm{Sect}(\mathcal{A})} Z_{\mu\nu} [\mu] \otimes [\bar{\nu}]$$

with multiplicities $Z_{\mu\nu} \in \mathbb{N}_0$ and the matrix Z commutes with the S and T matrices.
Let $\mathcal{A} \subset \mathcal{B}$ be a chiral (non-local) extension. There are two canonical tensor functors

$$\alpha^{\pm} : \mathrm{Rep}^I_{\mathrm{f}}(\mathcal{A}) \to \mathrm{End}(\mathcal{B}(I))$$

(Longo and Rehren, 1995; Böckenhauer and Evans, 1998) called α-induction defined using the braiding and opposite braiding, respectively.
Using this (Rehren, 2000) constructed a local extension $\mathcal{A}_2 \subset \mathcal{B}_2$, where $\mathcal{B}_2$ is maximal and $Z_{\mu\nu} = \dim \mathrm{Hom}(\alpha^+_\mu, \alpha^-_\nu)$ a **modular invariant** (Böckenhauer, Evans, Kawahigashi 1999).

Two extensions are $\mathcal{A} \subset \mathcal{B}_a, \mathcal{B}_b$ are called **Morita** equivalent if the two module categories ${}_{\mathcal{N}}\mathcal{X}_{\mathcal{M}_\bullet}$ generated by sectors

$$ {}_{\mathcal{A}}\Delta_{\mathcal{B}_\bullet} = \{[a] \subset [\rho\bar{\iota}_\bullet] : \rho \in \mathrm{Rep_f}(\mathcal{A}), a \text{ irred.}\}, \qquad \bullet = a, b $$

respectively, are isomorphic as module categories. ($\iota_\bullet : \mathcal{A} \hookrightarrow \mathcal{B}_\bullet$ is the inclusion and $\bar{\iota}_\bullet$ its conjugate.)
Give rise to NIMreps (non-negative integer representations) of the fusion rules in $\mathrm{Rep_f}(\mathcal{A})$.
To $a \in {}_{\mathcal{A}}\Delta_{\mathcal{B}}$ one can relate an extension $\mathcal{A} \subset \mathcal{B}_a$ which is Morita equivalent to $\mathcal{A} \subset \mathcal{B}$. All Morita equivalent extensions are given this way (Ostrik, 2003).
α-induction construction applied to Morita equivalent extensions $\mathcal{A} \subset \mathcal{B}$ give the same two-dimensional extensions $\mathcal{A}_2 \subset \mathcal{B}_2$ (Longo and Rehren, 2004).

Proposition (conjectured by (Kong and Runkel, 2010))

The α-induction construction coincides with the **full centre** construction in the categorical framework.

This enables us to use all the results about the full centre. The full centre maps of two "extensions" is equivalent iff the extensions are Morita equivalent (Kong and Runkel, 2008).

Classification

Let $\mathcal{A}$ be a completely rational net. There is a one-to-one correspondence (up to unitary equivalence) between:

- Maximal two-dimensional extensions $\mathcal{A}_2 \subset \mathcal{B}_2$.
- Morita equivalence classes of extensions $\mathcal{A} \subset \mathcal{B}$.
- Morita equivalence classes of Q-systems in $\mathrm{Rep_f}(\mathcal{A})$.

Given now a $\mathcal{B}_2 \supset \mathcal{A}_2$ maximal extension. What are its boundary conditions?
For two extensions: $\mathcal{A} \subset \mathcal{B}$ we define ${}_{\mathcal{B}}\mathcal{X}_{\mathcal{B}}$ the category defined by:

$$ {}_{\mathcal{B}}\Delta_{\mathcal{B}} = \{[\beta] \subset [\iota \circ \rho \circ \bar{\iota}] : \rho \in \mathrm{Rep_f}(\mathcal{A}), \beta \text{ irred.}\}, \qquad \bullet = a, b, $$

Proposition (Grossman and Snyder, 2012)

Given $\mathcal{A} \subset \mathcal{B}$ and $a, b \in {}_{\mathcal{A}}\Delta_{\mathcal{B}}$ giving equivalent extensions $\mathcal{A} \subset \mathcal{B}_a, \mathcal{B}_b$. Then there exists an invertible object $\beta \in {}_{\mathcal{B}}\mathcal{X}_{\mathcal{B}}$, such that $a = b\beta$.

Proposition

The simple elements in the Morita equivalence classe of $\mathcal{A} \subset \mathcal{B}$ are in one-to-one correspondence with

$$ {}_{\mathcal{A}}\Delta_{\mathcal{B}} / {}_{\mathcal{B}}\Delta_{\mathcal{B}}^{\times} \equiv \mathrm{Skeleton}\big[{}_{\mathcal{A}}\mathcal{X}_{\mathcal{B}} / \mathrm{Pic}({}_{\mathcal{B}}\mathcal{X}_{\mathcal{B}})\big], $$

where ${}_{\mathcal{B}}\Delta_{\mathcal{B}}^{\times} = \{\beta \in {}_{\mathcal{B}}\Delta_{\mathcal{B}} : \beta \text{ invertible}\}$ and $\mathrm{Pic}(\mathcal{C}) = \{a \in \mathcal{C} : \exists b \in \mathcal{C} : a \otimes b = \mathrm{id}\}$ is the **Picard group** of a tensor category $\mathcal{C}$.

Given $\mathcal{B}_2 \supset \mathcal{A}_2$ a maximal two-dimensional extension. Then it comes from the Morita equivalence class $\{\mathcal{A} \subset \mathcal{B}_a\}_a$.

Proposition Holography (Longo and Rehren, 2004)

There is a one-to-one correspondence between Haag-dual local extensions $\mathcal{A}_+ \subset \mathcal{B}_+$ on Minkowski half-plane and non-local extensions $\mathcal{A} \subset \mathcal{B}$.

Let us assume that $\mathcal{B}_+ \supset \mathcal{A}_+$ is a boundary condition for $\mathcal{B}_2$ and us the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{B} \supset \mathcal{A} & & \\ \sim \updownarrow & \searrow^{\alpha\text{-induction}} & \\ \mathcal{B}_+ \subset \mathcal{A}_+ & \xrightarrow[\text{removing bndry}]{} & \mathcal{B}_2 \supset \mathcal{A}_2 \equiv \mathcal{A} \otimes \mathcal{A} \end{array}$$

Proposition

Let $\mathcal{A}$ be completely rational and $\mathcal{B}_2 \supset \mathcal{A}_2 = \mathcal{A} \otimes \mathcal{A}$ maximal local conformal net on Minkowski space.
Then there exist a (up to Morita equivalents unique) $\mathcal{A} \subset \mathcal{B}$ which gives $\mathcal{B}_2$.
There is a one-to-one correspondence between

1. Boundary conditions $\mathcal{B}_+^a$ of $\mathcal{B}_2$.
2. Extensions $\mathcal{B}_a \supset \mathcal{A}$ Morita equivalent to $\mathcal{B} \supset \mathcal{A}$.
3. Elements in $[a] \in {}_{\mathcal{A}}\Delta_{\mathcal{B}}/{}_{\mathcal{B}}\Delta_{\mathcal{B}}^{\times}$.

- Given chiral observables/symmetry in terms of a conformal net $\mathcal{A}$ we can characterize all conformal theories on Minkowski space and all its boundary condition having this chiral symmetry
- Non-conformal boundary non-chiral boundary conditions (chiral examples are given by (Longo, Witten (2011)), (B. (2012))).
- Defects (Longo,Rehren).
- Deform CQFTs using defects to obtain integrable QFTs (Runkel,...)
- Operator algebraic constuction on riemann surfaces, extended CFTs, etc. (Henriques)
- Minkowski CFT $\leftrightarrow$ euclidean CFT (Wick rotation)
- Relation to 3D TFTs, for example "duality" of
  - 1D Positive energy reps of $\mathrm{L}G$.
  - 2D Wess Zumino Witten model
  - 3D Chern-Simons theory.

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Characterization of rational conformal QFTs and their boundary conditions[2]

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[2] work in progress with Roberto Longo and Yasuyuki Kawahigashi

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