ON HARTOGS' EXTENSION THEOREM ON $(n-1)$-COMPLETE COMPLEX SPACES

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Abstract. Let X be a connected normal complex space of dimension $n \geq 2$ which is cohomologically $(n-1)$-complete, and let $\pi : M \to X$ be a resolution of singularities. By use of Takegoshi's generalization of the Grauert-Riemenschneider vanishing theorem, we deduce $H^1_{cpt}(M, \mathcal{O}) = 0$, which in turn implies Hartogs' extension theorem on X by the $\overline{\partial}$-technique of Ehrenpreis.

1. Introduction

The well-known Hartogs' extension theorem states that for every open subset $D \subset \mathbb{C}^n$, $n \geq 2$, and $K \subset D$ compact such that $D \setminus K$ is connected, the holomorphic functions on $D \setminus K$ extend to holomorphic functions on D. Whereas first versions of Hartogs' extension theorem were obtained by filling Hartogs' figures with analytic discs (Hartogs' original idea [Ha]), no such geometrical proof was known for the general theorem in complex number space $\mathbb{C}^n$ for a long time. Proofs of the general theorem in $\mathbb{C}^n$ usually depend on the Bochner-Martinelli-Koppelman kernel or on the solution of the $\overline{\partial}$-equation with compact support (the famous idea due to Ehrenpreis [E], see also [Hö]).

Only recently, Merker and Porten were able to fill the gap by giving an involved geometrical proof of Hartogs' extension theorem in $\mathbb{C}^n$ in the spirit of Hartogs' original idea by using a finite number of parameterized families of holomorphic discs and Morse-theoretical tools for the global topological control of monodromy, but no $\overline{\partial}$-theory or intergal kernels except the Cauchy kernel (see [MP1]).

Since the key ingredient of this strategy is the existence of a strongly $(n-1)$-convex exhaustion function, it is natural to ask wether the result remains true for $(n-1)$-complete complex spaces. In fact, Hartogs' theorem was generalized to $(n-1)$-complete manifolds by Andreotti and Hill [AH] using cohomological results (the $\overline{\partial}$-method), but no proof was known until now for the more general case of $(n-1)$-complete normal complex spaces. Merker and Porten were able to carry over their strategy and to prove Hartogs' extension theorem in this general situation (see [MP2]).

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The present paper is an answer to the question whether it could be possible to use $\overline{\partial}$-theoretical considerations for reproducing the result of Merker and Porten on a $(n-1)$-complete space X by the simple and striking strategy of Ehrenpreis. More precisely, we solve a $\overline{\partial}$-equation with compact support on a desingularization of X in order to derive the following statement by the technique of Ehrenpreis. For our strategy, it is enough to assume that X is cohomologically $(n-1)$-complete. Note that $(n-1)$-complete spaces are cohomologically $(n-1)$-complete by the work of Andreotti and Grauert [AG], but the converse is not known.

Theorem 1.1. *Let X be a connected normal complex space of dimension $n \geq 2$ which is cohomologically $(n-1)$-complete. Furthermore, let D be a domain in X and $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then each holomorphic function $f \in \mathcal{O}(D \setminus K)$ has a unique holomorphic extension to the whole set D.*

In the special case of a Stein space X with only isolated singularities, a $\overline{\partial}$-theoretical proof of Hartogs' extension theorem was already given in [R].

The main points of the proof of Theorem 1.1 are as follows: Let M be a complex manifold of dimension n, X a complex space and $\pi : M \to X$ a proper modification. Then it follows by Takegoshi's generalization [T] of the Grauert-Riemenschneider vanishing theorem [GRie] that

$$R^q \pi_* \Omega^n_M = 0,\ q > 0, \tag{1}$$

where $R^q \pi_* \Omega^n_M$ are the higher direct images of the canonical sheaf Ω^n_M on M (see Theorem 2.4). By use of the Leray spectral sequence, (1) yields

$$H^q(M, \Omega^n_M) \cong H^q(X, \pi_* \Omega^n_M),\ q > 0,$$

and this implies by Serre duality that

$$H^1_{cpt}(M, \mathcal{O}) \cong H^{n-1}(M, \Omega^n_M) = 0$$

if X is cohomologically $(n-1)$-complete (see Theorem 2.6). This is elaborated in section 2, while we will show in section 3 that vanishing of $H^1_{cpt}(M, \mathcal{O})$ gives Hartogs' extension theorem on X by the $\overline{\partial}$-technique of Ehrenpreis, because the extension problem on X can be reduced to an extension problem on the desingularization M. In the last section, we give a few remarks on Takegoshi's vanishing theorem for convenience of the reader. We remark that Takegoshi's result was used in a similar fashion in an earlier paper of Colţoiu and Silva [CS].

For a more detailed introduction to Hartogs' theorem with a full historical record, remarks and references, we refer to [MP1] and [MP2]. Though the method of Merker and Porten is technically more involved and harder to reproduce than the $\overline{\partial}$-method, it has the advantage that it works as well for meromorphic functions which is out of scope of the $\overline{\partial}$-method. In fact, Merker and Porten proved the extension theorem even for the extension of meromorphic functions (previously considered in the smooth case by Koziarz and Sarkis [KS]).

2. Dolbeault Cohomology of Proper Modifications

In its general form, the Grauert-Riemenschneider vanishing theorem (see [GRie], Satz 2.1) states:

Theorem 2.1. *Let X be an n-dimensional compact irreducible reduced complex space with n independent meromorphic functions (Moishezon), and let $\mathcal{S}$ be a quasi-positive coherent analytic sheaf without torsion on X. Then:*

$$H^q(X, \mathcal{S} \otimes \Omega^n_X) = 0, \quad q > 0,$$

where Ω^n_X is the sheaf of holomorphic n-forms on X (the canonical sheaf), defined in the sense of Grauert and Riemenschneider.

This generalization of Kodaira's famous vanishing theorem is also proved by means of harmonic theory. The main point in the proof is ([GRie], Satz 2.3):

Theorem 2.2. *Let X be a projective complex space, $\mathcal{S}$ a quasi-positive coherent analytic sheaf on X without torsion, and let $\pi : M \to X$ be a resolution of singularities, such that $\hat{\mathcal{S}} = \mathcal{S} \circ \pi$ is locally free on M. Then:*

$$R^q\pi_*(\hat{\mathcal{S}} \otimes \Omega^n_M) = 0, \quad q > 0.$$

Here, $\hat{\mathcal{S}} = \mathcal{S} \circ \pi$ denotes the torsion-free preimage sheaf:

$$\mathcal{S} \circ \pi := \pi^*\mathcal{S}/T(\pi^*\mathcal{S}),$$

where $T(\pi^*\mathcal{S})$ is the coherent torsion sheaf of the preimage $\pi^*\mathcal{S}$ (see [G], p. 61). As a simple consequence of Theorem 2.2, one can deduce:

Corollary 2.3. *Let M be a Moishezon manifold of dimension n, and X a projective variety such that $\pi : M \to X$ is a resolution of singularities. Then:*

$$R^q\pi_*\Omega^n_M = 0, \quad q > 0,$$

where $R^q\pi_*\Omega^n_M$, $q > 0$, are the higher direct image sheaves of Ω^n_M.

As Grauert and Riemenschneider mention already in their original paper [GRie], this statement is of local nature and doesn't depend on the projective embedding (whereas their proof does). And in fact, the result was generalized later by K. Takegoshi (see [T], Corollary I; and also [O]):

Theorem 2.4. *Let M be a complex manifold of dimension n, and X a complex space such that $\pi : M \to X$ is a proper modification. Then:*

$$R^q\pi_*\Omega^n_M = 0, \quad q > 0.$$

The nice proof consists mainly of a vanishing theorem on weakly 1-complete Kähler manifolds which is based on L^2-estimates for the $\overline{\partial}$-operator. For convenience of the reader, we will give some remarks on the proof in section 4. As an easy consequence, we obtain:

Theorem 2.5. *Let M be a complex manifold of dimension n, and X a complex space such that $\pi : M \to X$ is a proper modification. Then:*

$$H^{n,q}(M) \cong H^q(M, \Omega^n_M) \cong H^q(X, \pi_*\Omega^n_M). \tag{2}$$

Proof. The proof follows directly by the Leray spectral sequence. □

Now, if the space X has nice properties, we can deduce consequences for the Dolbeault cohomology on M. In this paper, we are particularly interested in q-complete spaces. Recall that a complex space X is q-complete in the sense of Andreotti and Grauert [AG] if it has a strongly q-convex exhaustion function. X is called cohomologically q-complete if

$$H^k(X, \mathcal{F}) = 0$$

for any coherent analytic sheaf $\mathcal{F}$ and all $k \geq q$. Note that q-complete spaces are cohomologically q-complete by the work of Andreotti and Grauert [AG], but the converse is not known.

Theorem 2.6. *Let M be a complex manifold of dimension n, and X a cohomologically q-complete complex space such that $\pi : M \to X$ is a proper modification. Then:*

$$H^{n-k}_{cpt}(M, \mathcal{O}) \cong H^k(M, \Omega^n_M) = 0, \quad k \geq q. \tag{3}$$

Proof. Since X is q-complete, it follows from Theorem 2.5 that

$$H^k(M, \Omega^n_M) \cong H^k(X, \pi_*\Omega^n_M) = 0, \quad k \geq q, \tag{4}$$

because $\pi_*\Omega^n_M$ is coherent by Grauert's direct image theorem (see [G]). Serre's criterion ([S], Proposition 6) tells us that we can apply Serre duality ([S], Théorème 2) to the cohomology groups in (4), and we get the duality

$$H^{n-k}_{cpt}(M, \mathcal{O}) \cong H^k(M, \Omega^n_M), \quad k \geq q.$$

□

3. Proof of Theorem 1.1

The assumption about normality implies that X is reduced. Let

$$\pi : M \to X$$

be a resolution of singularities, where M is a complex connected manifold of dimension n, and π is a proper holomorphic surjection. Let $E := \pi^{-1}(\operatorname{Sing} X)$ be the exceptional set of the desingularization. Note that

$$\pi|_{M \setminus E} : M \setminus E \to X \setminus \operatorname{Sing} X \tag{5}$$

is a biholomorphic map. For the topic of desingularization we refer to [AHL], [BM] and [Hau]. M is non-compact because

$$H^0_{cpt}(M, \mathcal{O}) = 0$$

by Theorem 2.6. Keep in mind that $H^1_{cpt}(M, \mathcal{O}) = 0$, as well.

First, we observe that the extension problem on X can be reduced to an analogous extension problem on M. Let

$$D' := \pi^{-1}(D),\ K' := \pi^{-1}(K),\ F := f \circ \pi \in \mathcal{O}(D' \setminus K').$$

Well-known properties of normal complex spaces (see [GRe] for a reference) imply that K' is a compact subset of the domain $D' \subset M$ such that $D' \setminus K'$ is connected. That means that the assumptions on D and K behave well under desingularization, and it is enough to construct an extension of F to D', because $\pi_* \mathcal{O}_M = \mathcal{O}_X$ by normality of X. The existence of such an extension follows easily from $H^1_{cpt}(M, \mathcal{O}) = 0$ by Ehrenpreis' $\overline{\partial}$-technique (see [Hö]) as we will describe in the remainder of this section for convenience of the reader.

Let

$$\chi \in C^\infty_{cpt}(M)$$

be a smooth cut-off function that is identically one in a neighborhood of K' and has compact support

$$C := \operatorname{supp} \chi \subset\subset D'.$$

Consider

$$G := (1 - \chi) F \in C^\infty(D'),$$

which is an extension of F to D', but unfortunately not holomorphic. We have to fix it by the idea of Ehrenpreis. So, let

$$\omega := \overline{\partial} G \in C^\infty_{(0,1),cpt}(D'),$$

which is a $\overline{\partial}$-closed $(0,1)$-form with compact support in D'. We may consider ω as a form on M with compact support. But $H^1_{cpt}(M, \mathcal{O}) = 0$ by Theorem 2.6. So, there exists $g \in C^\infty_{cpt}(M)$ such that

$$\overline{\partial} g = \omega,$$

and g is holomorphic on $M \setminus C$ (where $\omega = \overline{\partial} G = \overline{\partial} F = 0$). Let

$$\widetilde{F} := (1 - \chi) F - g \in \mathcal{O}(D'). \tag{6}$$

Since M is non-compact, it follows by standard arguments that $g \equiv 0$ on an open subset of $D' \setminus C \subset D' \setminus K'$, and so $\widetilde{F}$ is the desired extension by the identity theorem.

4. Remarks on Takegoshi's Vanishing Theorem 2.4

Let N be a complex manifold of dimension n and $\Phi \in C^\infty(N)$ a real valued function on N. We denote by $H(\Phi)_p$ the complex Hessian of Φ at $p \in N$, and set

$$\sigma(\Phi) = \max_{p \in N} \operatorname{rank} H(\Phi)_p.$$

For Φ and Ω^n_N, the canonical sheaf on N, we denote by $A(p, q)$ the following assertion: $\sigma(\Phi) = n - p + 1$ and $H^q(N, \Omega^n_N)$, $H^{q+1}(N, \Omega^n_N)$ are Hausdorff.

N is called weakly 1-complete if it possesses a C^∞ plurisubharmonic exhaustion function. The key point in Takegoshi's vanishing theorem is the following result which is proved by use of Kähler identities and a priori L^2 estimates for the $\overline{\partial}$-operator ([T], Theorem 2.1):

Theorem 4.1. *Let N be a weakly 1-complete Kähler manifold with a C^∞-plurisubharmonic exhaustion function Φ. Suppose with respect to Ω^n_N and Φ that $A(p,q)$ is true for $q \geq p \geq 1$. Then: $H^q(N, \Omega^n_N) \cong H^{n-q}_{cpt}(N, \mathcal{O}) = 0$.*

Now, for the proof of Theorem 2.4, a crucial question is how to get a Kähler metric involved if $\pi : M \to X$ is a proper modification, but M not necessary of Kähler type. This problem can be settled by Hironaka's Chow lemma (see [Hi]):

Theorem 4.2. *Let $\pi : M \to X$ be a proper modification of complex spaces, X reduced. Then there exists a proper modification $\pi' : M' \to X$ which is a locally finite (with respect to X) sequence of blow-ups and a holomorphic map $h : M' \to M$ such that $\pi' = \pi \circ h$.*

Note that $h : M' \to M$ is a proper modification, too. Now, let $x \in X$. Then x has a Stein neighborhood V with a C^∞ strictly plurisubharmonic exhaustion function φ such that $V' := \pi'^{-1}(V)$ is of Kähler type and has a C^∞ plurisubharmonic exhaustion function $\varphi \circ \pi'$. Note that the $H^q(V', \Omega^n_{M'})$ are Hausdorff for all $q \geq 1$ because V' is holomorphically convex (see [L], Theorem 2.1, and [P], Lemma II.1, and use the Remmert reduction). So, $H^q(V', \Omega^n_{M'}) = 0$ for $q \geq 1$ by Theorem 4.1. A similar reasoning shows that $R^q h_* \Omega^n_{M'} = 0$ (cf. [T], paragraph 3) and this yields

$$H^q(\pi^{-1}(V), \Omega^n_M) \cong H^q(V', h_*\Omega^n_M) = H^q(V', \Omega^n_{M'}) = 0 \tag{7}$$

because $h_*\Omega^n_{M'} = \Omega^n_M$. (7) gives Takegoshi's Theorem 2.4 for

$$\left(R^q \pi_* \Omega^n_M\right)_x = \varinjlim_{x \in V} H^q(\pi^{-1}(V), \Omega^n_M).$$

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