AN EXPLICIT $\overline{\partial}$-INTEGRATION FORMULA FOR WEIGHTED HOMOGENEOUS VARIETIES II, FORMS OF HIGHER DEGREE

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Abstract. Let $\Sigma$ be a weighted homogeneous (singular) subvariety of $\mathbb{C}^n$. The main objective of this paper is to present a class of explicit integral formulae for solving the $\overline{\partial}$-equation $\omega = \overline{\partial} \lambda$ on the regular part of $\Sigma$, where $\omega$ is a $\overline{\partial}$-closed $(0,q)$-form with compact support and degree $q \geq 1$. Particular cases of these formulae yield $L^p$-bounded solution operators for $1 \leq p \leq \infty$ if $\Sigma$ is a homogeneous and pure dimensional subvariety of $\mathbb{C}^n$ with an arbitrary singular locus.

1. Introduction

As it is well known, solving the $\overline{\partial}$-equation forms one of the main pillars of complex analysis, but it also has deep consequences on algebraic geometry, partial differential equations and other areas. For example, the classical Dolbeault theorem implies that the $\overline{\partial}$-equation can be solved in all degrees on a Stein manifold, and it is known that an open subset of $\mathbb{C}^n$ is Stein if and only if the $\overline{\partial}$-equation can be solved in all degrees (on that set). Nevertheless, it is usually not easy to produce an explicit operator for solving the $\overline{\partial}$-equation on a given Stein manifold, even if we know that it can be solved. The construction of explicit operators depends strongly on the geometry of the manifold on which the equation is considered. There exists a vast literature about this problem on smooth manifolds, both in books and papers (see [10, 11], for example).

The respective Dolbeault theory on singular varieties has been developed only recently. Let $\Sigma$ be a singular subvariety of the space $\mathbb{C}^n$ and $\omega$ a bounded $\overline{\partial}$-closed differential form on the regular part of $\Sigma$. Fornæss,

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Gavosto and Ruppenthal have produced a general technique for solving the \(\overline{\partial}\)-equation \(\omega = \overline{\partial}\lambda\) on the regular part of \(\Sigma\), which they have successfully applied to varieties defined by the formula \(z^m = \prod_k w_k^b\) in \(\mathbb{C}^n\); see [9, 6] and [15]. Acosta, Solís and Zeron have developed an alternative technique for solving the \(\overline{\partial}\)-equation (if \(\omega\) is bounded) on the regular part of any singular quotient variety embedded in \(\mathbb{C}^n\) which is generated by a finite group of unitary matrices, like for instance hypersurfaces in \(\mathbb{C}^3\) with only a Rational Double Point singularity; see [1, 2] and [20].

Nevertheless, the research on calculating explicit operators for solving the \(\overline{\partial}\)-equation \(\omega = \overline{\partial}\lambda\) on the regular part of singular subvarieties \(\Sigma \subset \mathbb{C}^n\) is still at a very early state; the techniques mentioned in the previous paragraph do not produce useful explicit formulae. Ruppenthal and Zeron have proposed explicit operators for calculating solutions \(\lambda\) if \(\Sigma\) is a weighted homogeneous variety and \(\omega\) is a \(\overline{\partial}\)-closed \((0,1)\)-differential form with compact support; see [18]. The weighted homogeneous varieties are analysed, for they are a main model for classifying the singular subvarieties of \(\mathbb{C}^n\). A detailed analysis of the weighted homogeneous varieties is done in Chapter 2–§4 and Appendix B of [4]. The main objective of the present paper is to improve the explicit operators originally developed in [18] for calculating solutions \(\lambda\) to the \(\overline{\partial}\)-equation \(\omega = \overline{\partial}\lambda\) on the regular part of any weighted homogeneous variety \(\Sigma\) if \(\omega\) is a \(\overline{\partial}\)-closed \((0,q)\)-differential form with compact support and degree \(q \geq 1\). Furthermore, we produce \(\overline{\partial}\)-solution operators with \(L^p\)-estimates for \(1 \leq p \leq \infty\) if \(\Sigma\) is homogeneous with an arbitrary singular locus.

**Definition 1.** Let \(\beta \in \mathbb{Z}^n\) be a fixed integer vector with strictly positive entries \(\beta_k \geq 1\). A holomorphic polynomial \(Q(z)\) on \(\mathbb{C}^n\) is said to be **weighted homogeneous** of degree \(d \geq 1\) with respect to \(\beta\) if the following equality holds for all \(s \in \mathbb{C}\) and \(z \in \mathbb{C}^n\):

1. \(Q(s^\beta \ast z) = s^d Q(z),\) with the action:
2. \(s^\beta \ast (z_1, z_2, \ldots, z_n) := (s^{\beta_1} z_1, s^{\beta_2} z_2, \ldots, s^{\beta_n} z_n).\)

An algebraic subvariety \(\Sigma\) in \(\mathbb{C}^n\) is said to be **weighted homogeneous** with respect to \(\beta\) if \(\Sigma\) is the zero locus of a finite number of weighted homogeneous polynomials \(Q_k(z)\) of (maybe different) degrees \(d_k \geq 1\), but all of them with respect to the same fixed vector \(\beta\).
Let \( \Sigma \subset \mathbb{C}^n \) be any subvariety. We use the following notation along this paper. The regular part \( \Sigma^* = \Sigma_{reg} \) is the complex manifold consisting of the regular points of \( \Sigma \), and it is always endowed with the induced metric, so that \( \Sigma^* \) is a Hermitian submanifold in \( \mathbb{C}^n \) with corresponding volume element \( dV_\Sigma \) and induced norm \( | \cdot |_\Sigma \) on the Grassmannian \( \Lambda T^*\Sigma^* \). Thus, any Borel-measurable \((0,q)\)-form \( \omega \) on \( \Sigma^* \) admits a representation
\[
\omega = \sum_J f_J \, dz_J,
\]
where the coefficients \( f_J \) are Borel-measurable functions on \( \Sigma^* \) which satisfy the inequality \( |f_J(z)| \leq |\omega(z)|_\Sigma \) for all points \( z \in \Sigma^* \) and multi-indexes \( |J| = q \). Notice that such a representation is by no means unique. We refer to Lemma 2.2.1 in [15] for a more detailed treatment of that point. For \( 1 \leq p < \infty \), we also introduce the \( L^p \)-norm of a measurable \((0,q)\)-form \( \omega \) on an open set \( U \subset \Sigma^* \) via the formula:
\[
\|\omega\|_{L^p_{0,q}(U)} := \left( \int_U |\omega|^p_\Sigma \, dV_\Sigma \right)^{1/p}.
\]

We can now present the main result of this paper. We assume that the \( \overline{\partial} \)-differentials are calculated in the sense of distributions, for we work with Borel-measurable functions.

**Theorem 2 (Main).** Let \( \Sigma \) be a weighted homogeneous subvariety of \( \mathbb{C}^n \) with respect to a given vector \( \beta \in \mathbb{Z}^n \), where \( n \geq 2 \) and all entries \( \beta_k \geq 1 \). Consider the class of all \((0,q)\)-forms \( \omega \) given by \( \sum_J f_J \, dz_J \), where \( q \geq 1 \), the coefficients \( f_J \) are all Borel-measurable functions in \( \Sigma \), and \( z_1, \ldots, z_n \) are the Cartesian coordinates of \( \mathbb{C}^n \). Let \( \sigma \geq -q \) be any fixed integer. The operator \( S_\sigma^q \) below is well defined on \( \Sigma \) for all forms \( \omega \) which are essentially bounded and have compact support,
\[
S_\sigma^q \omega(z) := \sum_{|J|=q} \frac{N_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(u^\beta \ast z) \frac{w^\sigma(u^{J})}{\overline{u}(u - 1)} \, du \wedge \frac{d^{\sigma}u}{\overline{u}}
\]
with \( N_J = \sum_{j \in J, K = J \setminus \{j\}} \frac{\beta_j \overline{\sigma^K}}{\text{sgn}(j, K)} \) and \( \beta_J = \sum_{j \in J} \beta_j \).

Notice that the multi-indexes \( J \) and \( K \) are both ordered in an ascending way and that \( \text{sgn}(j, K) \) is the sign of the permutation used for ordering the elements of the \( q \)-tuple \( (j, K) \) into an ascending way. Finally, the form \( S_\sigma^q(\omega) \) is a solution of the \( \overline{\partial} \)-equation \( \omega = \overline{\partial} S_\sigma^q(\omega) \) on the regular part of \( \Sigma \setminus \{0\} \), whenever \( \omega \) is also \( \overline{\partial} \)-closed on the regular part of \( \Sigma \setminus \{0\} \).
The origin of $\mathbb{C}^n$ is in general a singular point of $\Sigma$ according to Definition 1, so that the regular parts of $\Sigma$ and $\Sigma \setminus \{0\}$ coincide. We will prove Theorem 2 in Section 2 of this paper. Similar techniques and a slight modification of equations (3) and (4) can also be used for producing a $\overline{\partial}$-solution operator with $L^p$-estimates on homogeneous subvarieties with arbitrary singular locus.

**Theorem 3 ($L^p$-Estimates).** Let $\Sigma$ be a pure $d$-dimensional homogeneous (cone) subvariety of $\mathbb{C}^n$, where $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Fix a real number $1 \leq p \leq \infty$ and an integer $1 \leq q \leq d$. Consider the class $L^p_{0,q}(\Sigma)$ of all $(0,q)$-forms $\omega$ given by $\sum_J f_J dz_J$, where the coefficients $f_J$ are all $L^p$-integrable functions in $\Sigma$, and $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. Choose $\sigma \in \mathbb{Z}$ to be the smallest integer such that

$$\sigma \geq \frac{2d - 2}{p} + 1 - q. \tag{5}$$

The operator $S^\sigma_q(\omega)$ below is well defined almost everywhere on $\Sigma$ for all forms $\omega$ which lie in $L^p_{0,q}(\Sigma)$ and have compact support on $\Sigma$:

$$S^\sigma_q(\omega)(z) := \sum_{|J| = q} \frac{R_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(uz) \frac{w^q \overline{u}^q d\overline{u}}{u(u-1)}, \tag{6}$$

where $R_J = \sum_{j \in J, K = J \setminus \{j\}} \frac{q \overline{z}_j d\overline{z}_K}{\text{sgn}(j,K)}$.

The form $S^\sigma_q(\omega)$ is a solution of the $\overline{\partial}$-equation $\omega = \overline{\partial} S^\sigma_q(\omega)$ on the regular part of $\Sigma \setminus \{0\}$, whenever $\omega$ is also $\overline{\partial}$-closed on the regular part of $\Sigma \setminus \{0\}$.

Finally, assuming that the support of $\omega$ is contained in an open ball $B_R$ of radius $R > 0$ and centre at the origin, there exists a strictly positive constant $C_{\Sigma}(R,\sigma)$ which does not depend on $\omega$ and such that:

$$\|S^\sigma_q(\omega)\|_{L^p_{0,-1}(\Sigma \cap B_R)} \leq C_{\Sigma}(R,\sigma) \cdot \|\omega\|_{L^p_{0,q}(\Sigma)}. \tag{7}$$

The case $p = \infty$ in the previous theorem is a corollary of Theorem 2 because the formulae (6) and (3) coincide in the homogeneous case (where all coefficients $\beta_J = q$). We will give the full proof of Theorem 3 in Section 3 of the present paper.
The obstructions to solving the \( \bar{\partial} \)-equation with \( L^p \)-estimates on subvarieties of \( \mathbb{C}^n \) are not completely understood in general. An \( L^2 \)-solution operator (for forms with non-compact support) is only known in the case where \( \Sigma \) is a complete intersection\(^1\) of pure dimension \( \geq 3 \) with only isolated singularities. This operator was constructed by Fornæss, Øvrelid and Vassiliadou in \([8]\) via an extension theorem for \( \bar{\partial} \)-cohomology groups originally presented by Scheja \([19]\). Usually, the \( L^p \)-results come with some obstructions to the solvability of the \( \bar{\partial} \)-equation. Different situations have been analysed in the works of Diederich, Fornæss, Øvrelid, Ruppenthal and Vassiliadou: It is shown that the \( \bar{\partial} \)-equation is solvable with \( L^p \)-estimates for forms lying in a closed subspace of finite codimension of the vector space of all the \( \bar{\partial} \)-closed \( L^p \)-forms if the variety has only isolated singularities \([3, 5, 8, 21, 16]\). Besides, in the paper \([7]\), the \( \bar{\partial} \)-equation is solved locally with some weighted \( L^2 \)-estimates for forms which vanish to a sufficiently high order on the (arbitrary) singular locus of the given varieties.

There is a second line of research about the \( \bar{\partial} \)-operator on complex projective varieties (see \([12, 13]\) for the state of the art and further references). Though that area has clearly a lot in common with the topic of \( \bar{\partial} \)-equations on analytic subvarieties of \( \mathbb{C}^n \), it is a somewhat different theory because of the strong global tools (like Serre duality) which cannot be used in the (local) situation of Stein spaces (due to the lack of compactness).

Since the estimates in Theorem 3 are given only for homogeneous varieties, we finally propose in Section 4 of this paper a useful technique for generalising the estimates in Theorem 3, so as to consider weighted homogeneous subvarieties instead of homogeneous ones.

\(^1\)More precisely: a Cohen-Macaulay space.
2. Proof of Main Theorem

We need the following result. The notation $L^1_{p,q}$ stands for the class of all the $(p,q)$-forms with $L^1$-integrable coefficients, so that the differentials are calculated in the sense of distributions.

**Theorem 4.** Let $U \subset \mathbb{C}^m$ be open, $2 \leq q \leq m$, and $\omega \in L^1_{0,q}(U)$ be a $\overline{\partial}$-closed form with compact support along the first coordinate $z_1$, that is, such that $\text{supp}(\omega) \cap F_y$ is compact in $U \cap F_y$ for all fibres $F_y = \mathbb{C} \times \{y\}$ with $y \in \mathbb{C}^{m-1}$. Assume that $\omega$ is given by:

$$\omega = \sum_{|J|=q, 1 \notin J} [a_J] d\overline{z}_J + \sum_{|K|=q-1, 1 \notin K} [a_{1,K}] d\overline{z}_1 \wedge d\overline{z}_K,$$

where the multi-indexes $J$ and $K$ are both ordered in an ascending way. The following operator

$$S_q(\omega) := \sum_{|K|=q-1, 1 \notin K} I[a_{1,K}] d\overline{z}_K,$$

with

$$I f(z_1, ..., z_m) := \frac{1}{2\pi i} \int_{t \in \mathbb{C}} f(t, z_2, ..., z_m) \frac{dt}{t - z_1},$$

is defined almost everywhere in $U$ and satisfies $\omega = \overline{\partial} S_q(\omega)$.

Notice that $S_q(\omega)$ is well defined in $U$ if $\omega$ is essentially bounded and has compact support along the first coordinate $z_1$.

**Proof.** It is clear that the restrictions $(a_{1,K})|_{F_y}$ are all $L^1$-integrable on the intersections $U \cap F_y$, for almost every fibre $F_y$, so that $\eta := S_q(\omega)$ is defined almost everywhere in $U$; see Appendix B of [14] or [11, 15]. We only need to show that $\overline{\partial} \eta = \omega$. The assumption $\overline{\partial} \omega = 0$ implies that the following equation holds for every multi-index $|J| = q$ with $1 \notin J$,

$$\frac{\partial [a_J]}{\partial \overline{z}_1} = \sum_{j \in J, K = J \setminus \{j\}} sgn(j, K) \frac{\partial [a_{1,K}]}{\partial \overline{z}_j}.$$

The function $sgn(j, K)$ is the sign of the permutation used for ordering the elements of the $q$-tuple $(j, K)$ into an ascending way. A direct application of the inhomogeneous Cauchy-Integral Formula in one complex variable and the fact that $\omega$ has compact support along the first coordinate yield the
following identity for every multi-index \(|K| = q-1\) with \(1 \notin K\):
\[
\bar{T}(a_{1,K}) = [a_{1,K}]d\bar{z}_1 \wedge d\bar{z}_K + \sum_{j \notin K, j \neq 1} I \left( \frac{\partial [a_{1,K}]}{\partial \bar{z}_j} \right) d\bar{z}_j \wedge d\bar{z}_K,
\]
and so we have that:
\[
\partial S_q(w) = \sum_{|K|=q-1, 1 \notin K} [a_{1,K}]d\bar{z}_1 \wedge d\bar{z}_K + \sum_{|J|=q, 1 \notin J} I(b_J) d\bar{z}_J
\]
with \(b_J := \sum_{j \in J, K=J \setminus \{j\}} \text{sgn}(j,K) \frac{\partial [a_{1,K}]}{\partial \bar{z}_j} \).

Recall that the multi-indexes \(J\) and \(K\) are both ordered in an ascending way and \(\text{sgn}(j,K)\) is the sign of the permutation used for ordering the elements of the \(q\)-tuple \((j,K)\) into an ascending way. Equation (8) implies that \(\partial S_q(w)\) is equal to \(\omega\), because \(a_K\) has compact support along the first coordinate, and so:
\[
I(b_J) = I \left( \frac{\partial [a_{J}]}{\partial \bar{z}_1} \right) = a_J.
\]

We may now proceed with the proof of the main theorem.

**Proof.** [Main Theorem 2]. We follow the proof originally presented in [18], so that we only point out the main points. Let \(\{Q_k\}\) be the set of polynomials which define the algebraic variety \(\Sigma\) as its zero locus. The definition of weighted homogeneous varieties implies that the polynomials \(Q_k(z)\) are all weighted homogeneous with respect to the same fixed vector \(\beta\). Equation (1) automatically yields that every point \(s^\beta * z\) lies in \(\Sigma\) for all \(s \in \mathbb{C}\) and \(z \in \Sigma\), and so each coefficient \(f_J(\cdot)\) in equation (3) is well evaluated in \(\Sigma\). Moreover, the coefficients \(\beta_k \geq 1\) and \(\beta_J \geq q\), for all index \(k\) and multi-index \(J\) of degree \(q\). Fixing any point \(z \in \Sigma\), the given hypotheses imply that the following Borel-measurable functions are all essentially bounded and have compact support in \(\mathbb{C}\),
\[
u \mapsto f_J(u^\beta * z).
\]

Hence, the operator \(S^\sigma_q(\omega)\) in (3)–(4) is well defined on \(\Sigma\) for each fixed integer \(\sigma \geq -q\) and all forms \(\omega\) which are essentially bounded and have compact support. We shall prove that \(S^\sigma_q(\omega)\) is also a solution of the equation \(\omega = \bar{T} S^\sigma_q(\omega)\) if the \((0,q)\)-form \(\omega\) is \(\bar{T}\)-closed. We may suppose, without loss
of generality and because of the given hypotheses, that the regular part of \( \Sigma \) does not contain the origin. Let \( \xi \neq 0 \) be any fixed point in the regular part of \( \Sigma \). We may suppose by simplicity that the first entry \( \xi_1 \neq 0 \), and so we define the following mapping and subvariety:

\[
(9) \quad \eta(y) := (y_1/\xi_1)^\beta \ast (\xi_1, y_2, \ldots, y_n), \quad \text{for } y \in \mathbb{C}^n, \\
Y := \{ \hat{y} \in \mathbb{C}^{n-1} : Q_k(\xi_1, \hat{y}) = 0 \forall k \}.
\]

The action \( s^\beta \ast z \) was given in (2). We have that \( \eta(\xi) = \xi \), and that the following identities hold for all \( s \in \mathbb{C} \) and \( \hat{y} \in \mathbb{C}^{n-1} \), recall equation (1) and the fact that \( \Sigma \) is the zero locus of the polynomials \( \{Q_k\} \):

\[
(10) \quad Q_k(\eta(s, \hat{y})) = (s/\xi_1)^{d_k} Q_k(\xi_1, \hat{y}) \quad \text{and} \\
\eta(C^* \times Y) = \{ z \in \Sigma : z_1 \neq 0 \}.
\]

The symbol \( C^* \) stands for \( \mathbb{C} \setminus \{0\} \). The mapping \( \eta(y) \) is locally a biholomorphism whenever the first entry \( y_1 \neq 0 \). Whence, the point \( \xi \) lies in the regular part of the variety \( C \times Y \), because \( \xi = \eta(\xi) \) also lies in the regular part of \( \Sigma \) and \( \xi_1 \neq 0 \). Thus, we can find a biholomorphism

\[
\pi = (\pi_2, \ldots, \pi_n) : U \rightarrow Y \subset \mathbb{C}^{n-1}
\]

defined from an open and bounded domain \( U \) in \( \mathbb{C}^m \) onto an open set in the regular part of \( Y \), such that \( \pi(\zeta) \) is equal to \( (\xi_2, \ldots, \xi_n) \) for some \( \zeta \in U \).

Consider the following holomorphic mapping defined for all points \( s \in \mathbb{C} \) and \( x \in U \),

\[
(11) \quad \Pi(s, x) := s^\beta \ast (\xi_1, \pi(x)) = \eta(s\xi_1, \pi(x)) \in \Sigma.
\]

The image \( \Pi(C \times U) \) will be known as a **generalised cone** from now on. Notice that \( \Pi(C^* \times U) \) lies in the regular part of \( \Sigma \), for \( \pi(U) \) is contained in the regular part of \( Y \). The mapping \( \Pi(s, x) \) is locally a biholomorphism whenever \( s \neq 0 \), because \( \eta \) is also a local biholomorphism for \( y_1 \neq 0 \); and the image \( \Pi(1, \zeta) \) is equal to \( \xi \). Hence, recalling the form \( \omega \) and the operator \( S_q^\nu(\omega) \) defined in (3)–(4), we only need to prove that the pull-back \( \Pi^* \omega \) is equal to the differential \( \overline{\partial} \Pi^* S_q^\nu(\omega) \) inside \( C^* \times U \), in order to conclude that the \( \overline{\partial} \)-equation \( \omega = \overline{\partial} S_q^\nu(\omega) \) holds in a neighbourhood of \( \xi \) in \( \Sigma \). We can use equations (2) and (11) in order to calculate the pull-back \( \Pi^* \omega \) when \( \omega \) is given by \( \sum_j f_j d\overline{e_j} \). To simplify the notation, let \( \pi_1(x) := \xi_1 \) for all \( x \in U \),
so that \(d\pi_1 = 0\).

\[
\begin{align*}
[\Pi^* \omega](s, x) &= \sum_{|J|=q} f_J(\Pi(s, x)) s^{\beta_J} \bigwedge_{j \in J} d\pi_j(x) + \\
&+ \sum_{|J|=q, j \in J} \frac{f_J(\Pi) \beta_j s^{\beta_J-1} \pi_j(x)}{\text{sgn}(j, J \setminus \{j\})} d\pi_s(x) \bigwedge_{k \in J \setminus \{j\}} d\pi_k(x).
\end{align*}
\]

Recall that \(\beta_J = \sum_{j \in J} \beta_j \geq q \geq -\sigma\), the multi-index \(J\) is ordered in an ascending way, and \(\text{sgn}(\alpha_1, ..., \alpha_q)\) is the sign of the permutation used for ordering the elements of the \(q\)-tuple \((\alpha_1, ..., \alpha_q)\) into an ascending way. The given hypotheses on \(\omega\) yield that the pull-back \(\Pi^* \omega\) is \(\partial\)-closed and bounded in \(C^* \times U\), and so it is also \(\overline{\partial}\)-closed in \(C \times U\); see Lemma 4.3.2 in [15] or Lemma (2.2) in [20]. The same argument applies to the \(\overline{\partial}\)-closed and essentially bounded form

\[
s^{\sigma+1} [\Pi^* \omega](s, x) \in L^\infty_{0,q}(C \times U).
\]

The open set \(U\) is bounded in \(C^m\). Thus, it follows from (12) and (13), by the use of Lemma 7.2.2 in [15, p. 186] or Lemma 3.6 in [17], that \(s^\sigma \Pi^* \omega\) is both \(L^1_{0,q}(C \times U)\) and \(\overline{\partial}\)-closed in \(C \times U\). It is easy to see that each coefficient \(f_J(\Pi(s, x))\) has compact support with respect to the first coordinate \(s\), so that we can apply Theorem 4 to \(t^\sigma \Pi^* \omega(t, x)\) and calculate the form:

\[
S_q(t^\sigma \Pi^* \omega) = \sum_{|J|=q} \Theta_J \frac{1}{2\pi i} \int_{t \in C} f_J(\Pi(t, x)) \frac{t^\sigma(t^{\beta_J}) dt \wedge \overline{dt}}{t(t-s)}
\]

with \(\Theta_J = \sum_{j \in J} \frac{\beta_j \pi_j(x)}{\text{sgn}(j, J \setminus \{j\})} \bigwedge_{k \in J \setminus \{j\}} d\pi_k(x)\).

Theorem 4 implies that

\[
s^\sigma [\Pi^* \omega](s, x) = \overline{\partial} S_q(t^\sigma [\Pi^* \omega](t, x)).
\]

Hence, we only need to verify that the form \(S_q(t^\sigma \Pi^* \omega)/s^\sigma\) is equal to the pull-back \(\Pi^* S_q^\sigma(\omega)\) of the form defined in (3), in order to conclude that \(\omega = \overline{\partial} S_q^\sigma(\omega)\), as desired. We begin by calculating the pull-back \(\Pi^* \mathcal{N}\) of the differential form \(\mathcal{N}_J\) given in (4). Notice that \(\pi_1(x) \equiv \xi_1\), so that \(d\pi_1 = 0\)
and recall equations (2) and (11).

\[ \Pi^* \mathcal{R}_J = \sum_{j \in J} \frac{\beta_j s^{\beta_j} \pi_j(x)}{\text{sgn}(j, J \setminus \{j\})} \wedge \int_{\mathbb{C}} f_j(\Pi(us, x)) \frac{u^\sigma(\overline{u}^{\beta_j})d\overline{u}}{\overline{u}(u - 1)}. \]

Suppose that \( J = (\alpha_1, ..., \alpha_a, j, \alpha_1, ..., \alpha_b, k, \alpha_1, ..., \alpha_c) \), then:

\[ \text{sgn}(j, J \setminus \{j\}) \text{sgn}(k, J \setminus \{j, k\}) = (-1)^a(-1)^{a+b}, \]

\[ \text{sgn}(k, J \setminus \{k\}) \text{sgn}(j, J \setminus \{j, k\}) = (-1)^{a+b+1}(-1)^a. \]

So that the last sum in equation (15) vanishes, and so the pull-back \( \Pi^* \mathcal{R}_J \) is identically equal to \( \overline{s^{\beta}} \Theta_J \), with \( \Theta_J \) defined in (14). Finally, we calculate the pull-back of the form \( S_q^\sigma(\omega) \) given in (3), notice that \( \Pi(us, x) \) is equal to \( u^\beta \cdot \Pi(s, x) \),

\[ \Pi^* S_q^\sigma(w) = \sum_{|J|=q} \frac{\overline{s^{\beta}} \Theta_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(\Pi(us, x)) \frac{u^\sigma(\overline{u}^{\beta})d\overline{u}}{u(u - 1)}. \]

The change of variables \( t = us \) yields that the form \( S_q^\sigma(t^\sigma \Pi^* \omega)/s^\sigma \) in (14) is equal to the identity above, and so \( \omega = \overline{\partial} S_q^\sigma(\omega) \), as desired. \( \square \)

3. \( L^p \)-Estimates

We prove Theorem 3 in this section. Recall that \( \Sigma \) is a pure \( d \)-dimensional homogeneous (cone) subvariety of \( \mathbb{C}^n \) with arbitrary singular locus, so that \( n \geq 2 \) and each entry \( \beta_k = 1 \) in Definition 1. Moreover, given a fixed real number \( 1 \leq p \leq \infty \) and an integer \( 1 \leq q \leq d \), we consider the class \( L^p_{0,q} \) of all \((0,q)\)-forms \( \omega \) expressed as \( \sum_J f_J d\overline{z}_J \), where the coefficients \( f_J \) are all \( L^p \)-integrable functions in \( \Sigma \), and \( z_1, ..., z_n \) are the Cartesian coordinates of \( \mathbb{C}^n \). Assume that the support of each form \( \omega \in L^p_{0,q} \) is contained in the open ball \( B_R \) of radius \( R > 0 \) and centre at the origin. Fix \( \sigma \in \mathbb{Z} \) to be the smallest integer such that

\[ \sigma \geq \frac{2d - 2}{p} + 1 - q. \]

We begin by showing that \( S_q^\sigma \) in (6) defines a bounded operator

\[ S_q^\sigma : L^p_{0,q}(\Sigma \cap B_R) \rightarrow L^p_{0,q-1}(\Sigma \cap B_R), \]
where:

\[
S^q_\sigma \omega(z) = \sum_{|J|=q} \frac{\mathcal{N}_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(uz) \frac{u^q \overline{u^q}}{\pi (u - 1)} du
\]

and \(\mathcal{N}_J = \sum_{j \in J, K = J \setminus \{j\}} q \frac{\overline{z}_j}{\text{sgn}(j, K)} \).

Recall that the multi-indexes \(J\) and \(K\) are both ordered in an ascending way and that \(\text{sgn}(j, K)\) is the sign of the permutation used for ordering the elements of the \(q\)-tuple \((j, K)\) into an ascending way. Notice that the case \(p = \infty\) in (18) is a corollary of Theorem 2, because the formulae (3) and (19) coincide when the variety \(\Sigma\) is homogeneous (so that all \(\beta_J = q\)). Hence, we can suppose from now on that \(p < \infty\), and we only need to prove that the following inequality holds for every multi-indexes \(|J| = q\) and \(j \in J\) in order to conclude that (18) and (7) holds,

\[
\int_{z \in \Sigma \cap B_R} \left| \sum_{|J|=q} \mathcal{N}_J \int_{|u| < \|z\|} f_J(uz) \frac{u^q \overline{u^q}}{(u - 1) u} du \right|^p dV_{\Sigma} \lesssim \|\omega\|_{L_{p,q}^p(\Sigma)}^p.
\]

Notice that the support of \(f_J(z)\) is contained in the open ball \(B_R\) of radius \(R\), and that \(dV_{\Sigma}\) are the respective volume forms on \(\Sigma\). Further, we may use the variable \(u\) instead of its complex conjugate \(\overline{u}\) because we work under an absolute value sign. Let \(\delta < 1\) be any fixed real number. It is easy to deduce the existence of a finite real constant \(M_1\) such that the following inequalities hold for all complex numbers \(\hat{t}\) and \(\hat{w}\):

\[
\int_{|w| < R} \left| \overline{z}_j \int_{|u| < \|z\|} f_J(uz) \frac{u^q \overline{u^q}}{(u - 1) u} du \right|^p dV_{\Sigma} \lesssim \|\omega\|_{L_{p,q}^p(\Sigma)}^p.
\]

Moreover, let \(\tilde{\Sigma}\) be the projective variety associated to \(\Sigma\) in the space \(\mathbb{C}P^{n-1}\), for \(\Sigma\) is a pure \(d\)-dimensional homogeneous subvariety of \(\mathbb{C}^n\). We also use the fact that any integral on \(\Sigma\) can be decompose as a pair of nested integrals on \(\mathbb{C}\) and \(\tilde{\Sigma}\), that is:

\[
\int_{z \in \Sigma} \Phi(z) dV_{\Sigma}(z) = \int_{[z] \in \tilde{\Sigma}} \int_{t \in \mathbb{C}} \Phi(zt) |t|^{2d-2} dV_{\mathbb{C}}(t) dV_{\tilde{\Sigma}}([z]);
\]
where on the right hand side, \( \hat{z} \in \Sigma \) is any representative of \( [z] \in \tilde{\Sigma} \) with \( \| \hat{z} \| = 1 \). Finally, since \( \sigma \in \mathbb{Z} \) is the smallest integer which satisfies (17), we have that the following constant

\[
(22) \quad \delta := \sigma + q - 1 + \frac{2 - 2d}{p} \quad \text{satisfies} \quad 0 \leq \delta < 1.
\]

We can now use the results presented in the paragraphs above in order to calculate (20) and (7), with the change of variables \( w = ut \),

\[
\int_{z \in \Sigma \cap B_R} \int_{|u| < R/\|z\|} f_J(uz) u^{\sigma+q} \frac{dV_C}{(u-1)u} dV_{\Sigma}^{p} \leq \int_{|z| < \Sigma} \int_{|t| < R} \frac{f_J(uz) u^{\sigma+q}}{|u-1|u} dV_C dV_{\Sigma}^{p} |t|^{2d-2} dV_{C} dV_{\Sigma}^{e} = \int_{|z| < \Sigma} \int_{|t| < R} \frac{f_J(wz) w^{\sigma+q-1}}{(w-t) |t|^2} dV_{C}^{p} \leq \int_{|z| < \Sigma} \int_{|t| < R} f_J(tz) w^{\sigma+q-1} |p| dV_{C} dV_{\Sigma} \leq \int_{z \in \Sigma \cap B_R} |f_J(z)|^{p} R^{\delta} dV_{\Sigma} \lesssim \|f_J\|_{L^p(\Sigma)}^{p} \leq \|\omega\|_{L^p_{\lambda,q}(\Sigma)}^{p} < \infty.
\]

We have used (22) and (21) with \( h(w) = f_J(wz) w^{\sigma+q-1} \). That completes the proof of equations (20) and (7).

Finally, notice that the operators \( S^\gamma_q(\omega) \) given in (3), (6) and (19) are all the same, because the coefficients \( \beta_j = q \) for every multi-index \( |J| = q \). Therefore, we can show that the operator \( S^\gamma_q(\omega) \) satisfies the differential equation \( \omega = \bar{\partial} S^\gamma_q(\omega) \) following step by step the proof presented in Section 2. We only need to rewrite the pull-back given in (12), which is \( \bar{\partial} \)-closed in the product \( \mathbb{C}^* \times U \),

\[
(23) \quad [\Pi^* \omega](u, x) = \sum_{|J|=q} f_J(\Pi(u, x)) \bar{u}^J \bigwedge_{j \in J} \bar{d}\pi_j(x) + \sum_{|J|=q, j \in J} \frac{f_J(\Pi)}{\text{sgn}(j, J \setminus \{j\})} \bar{u}^J \bigwedge_{k \in J \setminus \{j\}} \bar{d}\pi_k(x).
\]
And we must show that $u^\sigma \Pi^* \omega (u, x)$ lies in $L^1_{0,q}(\mathbb{C} \times U)$, where $U$ is a bounded domain in $\mathbb{C}^m$. Thus, we have that $u^\sigma \Pi^* \omega$ is also $\overline{\partial}$-closed in $\mathbb{C} \times U$, because of Lemma 7.2.2 in [15, p. 186] or Lemma 3.6 in [17]. We can then apply Theorem 4 and follow step by step the proof presented in Section 2 from equation (14) to the end of that section.

Recall that the integer $\sigma \geq 2d - 2 - 2p + q - 1$. We begin showing that the form $u^\sigma \Pi^* \omega$ lies in $L^p_{0,q}(\mathbb{C} \times U)$. Notice that $\Pi(u, x)$ is equal to $u(\xi_1, \pi(x))$ because each entry $\beta_k = 1$ in (2) and (11). It is easy to calculate the pull-back of the volume form $dV_\Sigma$:

$$\Pi^* dV_\Sigma = \sum_{|I|=|J|=d} \beta_{I,J}(z) dz_I \wedge d\overline{z}_J \bigg|_{z=u(\xi_1, \pi(x))}$$

$$= \Theta(x) |u|^{2d-2} |du \wedge d\overline{u}| \wedge \bigwedge_{k=1}^{d-1} [dx_k \wedge d\overline{x}_k].$$

Recall that $x$ lies in the bounded open set $U \subset \mathbb{C}^{d-1}$. Since $\Sigma$ is a pure $d$-dimensional homogeneous (cone) subvariety of $\mathbb{C}^n$, the coefficients $\beta_{I,J}(z)$ are all invariant under the transformations $z \mapsto uz$, and so $\Theta(x)$ only depends on the values of $\pi(x)$ and all its partial derivatives (it is constant with respect to $u$). The fact that $\Pi$ is a biholomorphism from $\mathbb{C}^* \times U$ onto its image also implies that $\Theta$ cannot vanish. Hence, choosing a smaller set $U$ if it is necessary, we can suppose that $|\Theta|$ is bounded from below by a constant $M_2 > 0$.

On the other hand, since $\Pi(u, x) = u(\xi_1, \pi)$ and the support of each $f_J(z)$ is contained in a ball of radius $R > 0$ and centre at the origin, we have that every $f_J(\Pi(u, x))$ vanishes if $|u\xi_1| > R$. Thus, equation (23) and the analysis done in the paragraphs above imply that the form $u^\sigma \Pi^* \omega$ lies in $L^p_{0,q}(\mathbb{C} \times U)$, because the following inequalities hold for every multi-index $J$ and exponent $b = 0, 1$:

$$\int_{\mathbb{C} \times U} |u|^{\sigma+q-b} f_J(\Pi)|^p dV_{\mathbb{C} \times U} \lesssim \int_{\mathbb{C} \times U} |f_J(\Pi)|^p |u|^{2d-2} dV_{\mathbb{C} \times U}$$

$$= \int_{\Pi(\mathbb{C} \times U)} |f_J|^p dV_\Sigma \lesssim \|\lambda\|^p_{L^2_{0,1}(\Sigma)} < \infty.$$
the form $u^q \Pi^\omega$ is $L^1_{0,q}$ and $\overline{\partial}$-closed in $\mathbb{C} \times U$; see for example Lemma 7.2.2 in [15, p. 186] or Lemma 3.6 in [17]. We can then apply Theorem 4 and follow step by step the proof presented in Section 2 from equation (14) to the end of that section, in order to conclude that the operator $S^q_\sigma(\omega)$ satisfies the differential equation $\omega = \overline{\partial}S^q_\sigma(\omega)$, as we want.

4. Weighted Homogeneous Estimates

We want to close this paper presenting a useful technique for generalising the estimates given in Theorem 3, so as to consider weighted homogeneous subvarieties instead of cones. Let $\Sigma \subset \mathbb{C}^n$ be a weighted homogeneous subvariety defined as the zero locus of a finite set of polynomials $\{Q_k\}$. Thus, the polynomials $Q_k(z)$ are all weighted homogeneous with respect to the same vector $\beta \in \mathbb{Z}^n$, and each entry $\beta_k \geq 1$. Define the following holomorphic mapping:

$$\Phi : \mathbb{C}^n \to \mathbb{C}^n, \quad \text{with} \quad \Phi(x) = (x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_n^{\beta_n}).$$

It is easy to see that each polynomial $Q_k(\Phi)$ is homogeneous, and so the subvariety $X \subset \mathbb{C}^n$ defined as the zero locus of $\{Q_k(\Phi)\}$ is a cone. Consider a $(0,q)$-form $\omega$ given by the sum $\sum_J f_J d\overline{\sigma}_J$, where the coefficients $f_J$ are all Borel-measurable functions with compact support in $\Sigma$, and $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. We may follow two different paths in order to solve the equation $\overline{\partial}\lambda = \omega$. We may calculate the pull-back:

$$\Phi^*\omega = \sum_{|J|=q} f_J(\Phi(x)) \left( \prod_{j \in J} \frac{x_j^{\beta_j} - 1}{\beta_j x_j^{\beta_j}} \right) d\overline{\sigma}_J;$$

and then apply Theorems 2 and 3 on the cone $X$, so as to get the following operators:

$$S^q_\sigma(\Phi^*\omega) := \sum_{|J|=q} \frac{\widehat{N}_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(\Phi(ux)) \frac{u^q(\overline{u}^\beta) d\overline{u} \wedge du}{\overline{u}(u - 1)}$$

with $\widehat{N}_J = \sum_{j \in J, K = J \setminus \{j\}} \frac{\beta_j x_j^{\beta_j}}{\text{sgn}(j,K)} \left( \prod_{k \in K} \beta_k x_k^{\beta_k - 1} \right) d\overline{x}_K$. 

On the other hand, we may use the main Theorem 2 on the weighted homogeneous variety $\Sigma$, so as to get:

$$S^\sigma_q(\omega) := \sum_{|J|=q} \mathbb{N}_J \int_{u \in \mathbb{C}} f_J(u^\beta \ast z) \frac{u^\sigma(u^{\bar{J}})}{\pi (u - 1)} \quad \text{with}$$

$$\mathbb{N}_J = \sum_{j \in J, K=J \backslash \{j\}} \frac{\beta_j \bar{z}_j d\bar{z}_K}{\text{sgn}(j, K)} \quad \text{and} \quad \beta_J = \sum_{j \in J} \beta_j.$$

We can easily verify that $\Phi^* S^\sigma_q(\omega)$ is equal to $S^\sigma_q(\Phi^* \omega)$. The main problem is that $\Phi^* \omega$ may not lie necessarily in $L^p_{0, q}(X)$ for $p < \infty$.

**References**


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