A $\partial$-THEORETICAL PROOF OF HARTOGS’ EXTENSION
THEOREM ON STEIN SPACES WITH ISOLATED
SINGULARITIES

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Abstract. Let $X$ be a connected normal Stein space of pure dimension $d \geq 2$
with isolated singularities only. By solving a weighted $\partial$-equation with compact
support on a desingularization of $X$, we derive Hartogs’ Extension Theorem
on $X$ by the $\partial$-idea due to Ehrenpreis.

1. Introduction

Whereas first versions of Hartogs’ Extension Theorem (e.g. on polydiscs) were
usually obtained by filling Hartogs’ Figures with holomorphic discs, no such ge-
ometrical proof was known for the general Theorem in complex number space
for a long time. Proofs of the general Theorem in $\mathbb{C}^n$ usually depend on the
Bochner-Martinelli-Koppelman kernel or on the solution of the $\partial$-equation with
compact support (the famous idea due to Ehrenpreis [Eh]).

Only recently, J. Merker and E. Porten were able to fill the gap by giving a ge-
ometrical proof of Hartogs’ Extension Theorem in $\mathbb{C}^n$ (see [MePo1]) by using a fi-
nite number of parameterized families of holomorphic discs and Morse-theoretical
tools for the global topological control of monodromy, but no $\partial$-theory or intergal
kernels (except the Cauchy kernel).

They also extended their result to the general case of $(n−1)$-complete normal
complex spaces (see [MePo2]), where no proof was known until now at all. One
reason is the lack of global integral kernels or an appropriate $\partial$-theory for singular
complex spaces. The present paper is a partial answer to the question of Merker
and Porten wether it could be possible to use some $\partial$-theoretical considerations
for reproducing their result on $(n−1)$-complex spaces. More precisely, we solve a
weighted $\partial$-equation with compact support on a desingularization of $X$, in order
to derive the following statement by the principle of Ehrenpreis:

Theorem 1.1. Let $X$ be a connected normal Stein space of pure dimension $d \geq 2$
with isolated singularities only. Furthermore, let $\Omega$ be a domain in $X$ and $K \subset \Omega$
a compact subset such that $\Omega \setminus K$ is connected. Then each holomorphic function
$f \in O(\Omega \setminus K)$ has a unique holomorphic extension to the whole set $\Omega$.

For a more detailed introduction to the topic, we refer to [MePo1] and [MePo2].

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2. Proof of Theorem 1.1

The assumption about normality implies that $X$ is reduced. For convenience, we may assume that $X$ is embedded properly into a complex number space.$^1$ Let

$$\pi : M \to X$$

be a resolution of singularities, where $M$ is a complex connected manifold of dimension $d$, and $\pi$ is a proper holomorphic surjection. Let $E := \pi^{-1}(\text{Sing} X)$ be the exceptional set of the desingularization. Note that

$$\pi|_{M \setminus E} : M \setminus E \to X \setminus \text{Sing} X$$

is a biholomorphic map. For the topic of desingularization we refer to [AHL], [BiMi] and [Ha]. It follows that $M$ is a 1-convex complex manifold, and that there exists a strictly plurisubharmonic exhaustion function

$$\rho : M \to [-\infty, \infty),$$

such that $\rho$ takes the value $-\infty$ exactly on $E$ (see [CoMi]). We can assume that $\rho$ is real-analytic on $M \setminus E$. Let

$$\Omega' := \pi^{-1}(\Omega), \ K' := \pi^{-1}(K), \ F := f \circ \pi \in \mathcal{O}(\Omega' \setminus K').$$

Note that $K'$ is compact because $\pi$ is a proper map. $\Omega \setminus K$ is a connected normal complex space. Hence, $\Omega \setminus K \setminus \text{Sing} X$ is still connected. So, the same is true for $\Omega' \setminus K' \setminus E$ because of (1). But then $\Omega' \setminus K'$ and $\Omega'$ are connected, too. That means that the assumptions on $\Omega$ and $K$ behave well under desingularization. Choose $\delta > 0$ such that

$$K' \subset D := \{z \in M : \rho(z) < \delta\},$$

which is possible since $\rho$ is an exhaustion function. But $\rho$ is also strictly plurisubharmonic outside $E$, and it follows that $D$ is strongly pseudoconvex. We will use the fact that

$$\dim H^q(D, \mathcal{S}) < \infty$$

for all coherent analytic sheaves $\mathcal{S}$ and $q \geq 1$. This result goes back to Grauert who originally proved it in case $\mathcal{S} = \mathcal{O}$ (see [Gr1]). Let

$$\chi \in C^\infty_{\text{cpt}}(M)$$

be a smooth cut-off function that is identically one in a neighborhood of $K'$ and has compact support in $D \cap \Omega'$ such that

$$\Omega' \setminus C \quad (\text{with } C := \text{supp } \chi)$$

is connected. That is possible if we choose the neighborhood of $K'$ small enough.

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$^1$See [Na1]. The result is due to R. Remmert
Consider
\[ G := (1 - \chi)F \in C^\infty(\Omega'), \]
which is an extension of \( F \) to \( \Omega' \), but unfortunately not holomorphic. We have to fix it by the idea of Ehrenpreis. So, let
\[ \omega := \overline{\partial}G \in C^\infty_{0,1}(\Omega'), \]
which is in fact a \( \overline{\partial} \)-closed \((0,1)\)-form with compact support in \( C \subset D \cap \Omega' \). We are now searching a solution of the \( \overline{\partial} \)-equation
\[ \overline{\partial}g = \omega. \] (4)
But we only know that \( D \) is a strongly pseudoconvex subset of a 1-convex complex manifold. This is now the place to introduce some ideas from the \( \overline{\partial} \)-theory on singular complex spaces. We will use a result of Fornæss, Øvrelid and Vassiliadou presented in [FOV1], Lemma 2.1. We must verify their assumptions. So, let
\[ \widetilde{D} := \pi(D). \]
That is a strongly pseudoconvex neighborhood of the isolated singularities in \( X \). Hence, it is a holomorphically convex subset of a Stein space by the results of Narasimhan (see [Na2]), and therefore Stein itself.

Let \( \mathcal{J} \) be the sheaf of ideals of the exceptional set \( E \) in \( M \). Now, the result of Fornæss, Øvrelid and Vassiliadou reads as:

**Theorem 2.1.** Let \( \mathcal{S} \) be a torsion-free coherent analytic sheaf on \( D \), and \( q > 0 \). Then there exists a natural number \( T \in \mathbb{N} \) such that
\[ i_q : H^q(D, \mathcal{J}^T \mathcal{S}) \to H^q(D, \mathcal{S}) \]
is the zero map, where \( i_q \) is the map induced by the natural inclusion \( \mathcal{J}^T \mathcal{S} \hookrightarrow \mathcal{S} \).

This statement reflects the fact that the cohomology of \( M \) is concentrated along the exceptional set \( E \), and can be killed by putting enough pressure on \( E \). We will now use Theorem 2.1 with the choices \( q = n - 1 \) and \( \mathcal{S} = \Omega^0_D \) the canonical sheaf on \( D \). So, there exists a natural number \( \mu > 0 \) such that
\[ i_{n-1} : H^{n-1}(D, \mathcal{J}^\mu \Omega^0_D) \to H^{n-1}(D, \Omega^0_D) \] (5)
is the zero map. We will use Serre Duality (cf. [Se]) to change to the dual statement. But, can we apply Serre-Duality to the non-compact manifold \( D \)? The answer is yes, because higher cohomology groups are finite-dimensional on \( D \) by the result of Grauert (2), and we can use Serre’s criterion ([Se], Proposition 6). So, we deduce:
\[ i_c : H^1_{\text{cpt}}(D, \mathcal{O}_D) \to H^1_{\text{cpt}}(D, \mathcal{J}^{-\mu} \mathcal{O}_D) \] (6)
is the zero map, where \( i_c \) is induced by the natural inclusion \( \mathcal{O}_D \hookrightarrow \mathcal{J}^{-\mu} \mathcal{O}_D \). This statement means that we can have a solution for the \( \overline{\partial} \)-equation (4) with compact support in \( D \) that has a pole of order \( \mu \) (at most) along \( E \). Let us make that precise.
$J^{-\mu}\mathcal{O}_D$ is a subsheaf of the sheaf of germs of meromorphic functions $\mathcal{M}_D$. We will now construct a fine resolution for $J^{-\mu}\mathcal{O}_D$. Let $\mathcal{C}^\infty_{0,q}$ denote the sheaf of germs of smooth $(0,q)$-forms on $D$. We consider $J^{-\mu}\mathcal{C}^\infty_{0,q}$ as subsheaves of the sheaf of germs of differential forms with measurable coefficients. Now, we define a weighted $\overline{\partial}$-operator on $J^{-\mu}\mathcal{C}^\infty_{0,q}$. Let $f \in (J^{-\mu}\mathcal{C}^\infty_{0,q})_z$ for a point $z \in M$. Then $f$ can be written as $f = h^{-\mu} f_0$, where $h \in (\mathcal{O}_D)_z$ generates $J_z$ and $f_0 \in (\mathcal{C}^\infty_{0,q})_z$. Let

$$
\overline{\partial}_{-\mu} f := h^{-\mu} \overline{\partial} f_0 = h^{-\mu} \overline{\partial}(h^\mu f).
$$

We obtain the sequence

$$
0 \to J^{-\mu}\mathcal{O}_D \to J^{-\mu}\mathcal{C}^\infty_{0,0} \xrightarrow{\overline{\partial}_{-\mu}} J^{-\mu}\mathcal{C}^\infty_{0,1} \xrightarrow{\overline{\partial}_{-\mu}} \cdots \xrightarrow{\overline{\partial}_{-\mu}} J^{-\mu}\mathcal{C}^\infty_{0,d} \to 0,
$$

which is exact by the Grothendieck-Dolbeault Lemma and well-known regularity results. It is a fine resolution of $J^{-\mu}\mathcal{O}_D$ since the $J^{-\mu}\mathcal{C}^\infty_{0,q}$ are closed under multiplication by smooth cut-off functions. Therefore, the abstract Theorem of de Rham implies that:

$$
H^q(D, J^{-\mu}\mathcal{O}_D) \cong \frac{\ker (\overline{\partial}_{-\mu} : J^{-\mu}\mathcal{C}^\infty_{0,q}(D) \to J^{-\mu}\mathcal{C}^\infty_{0,q+1}(D))}{\text{Im} (\overline{\partial}_{-\mu} : J^{-\mu}\mathcal{C}^\infty_{0,q-1}(D) \to J^{-\mu}\mathcal{C}^\infty_{0,q}(D))},
$$

and we have the analogous statement for forms and cohomology with compact support. Recall that $\omega \in \mathcal{C}^\infty_{0,1}(D)$ is $\overline{\partial}$-closed with compact support in $D$. By the natural inclusion, we have that $\omega \in J^{-\mu}\mathcal{C}^\infty_{0,1}(D)$, too, and it is in fact $\overline{\partial}_{-\mu}$-closed. But then (6) tells us that there exists a solution $g \in J^{-\mu}\mathcal{C}^\infty(D)$ such that $\overline{\partial}_{-\mu} g = \omega$, and $g$ has compact support in $D$. So, $g \in \mathcal{C}^\infty(M \setminus E)$ with support in $D$, and

$$
\overline{\partial} g = \omega = \overline{\partial} G = \overline{\partial}((1 - \chi)F) \quad \text{on } \Omega' \setminus E.
$$

Recall that

$$
\supp \omega \subset C = \supp \chi,
$$

and that $\Omega' \setminus C$ is connected (3). But then $M \setminus C$ and $M \setminus C \setminus E$ are connected, and $g$ is a holomorphic function on $M \setminus C \setminus E$ with support in $D$. Hence

$$
\supp g \subset C.
$$

So,

$$
\tilde{F} := (1 - \chi)F - g \in \mathcal{O}(\Omega' \setminus E)
$$

equals $F$ on $\Omega' \setminus E \setminus C$. 

But then
\[ \tilde{f} := ((1 - \chi)F - g) \circ \pi^{-1} \in \mathcal{O}(\Omega \setminus \text{Sing } X) \]
equals \( f \) on some open set, and it has an extension to the whole domain \( \Omega \) by
Riemann’s Extension Theorem for normal spaces (see [GrRe], for example). The
extension \( \tilde{f} \) is unique because \( \Omega \setminus K \) is connected and \( X \) is globally and locally
irreducible (see again [GrRe]). That completes the proof of Theorem 1.1. With
a little more effort, one can remove the assumption that \( X \) should contain only
finitely many singularities, because \( K \) has a neighborhood in \( \Omega \) that contains only
a finite number of singular points.

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