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Exercise sheet 7 for Algebraic curves and the Weil conjectures

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Exercise 7.1. Let k be a perfect field and C a smooth projective and *geometrically connected* curve over k. Show that the duality trace map which we defined in the lecture $\mathrm{Tr}_{C/k} : H^1(C, \omega_C) \to k$ is an isomorphism. (*Hint:* We saw in the course that if C is geometrically connected, then $H^0(C, \mathcal{O}_C) = k$.)

Exercise 7.2. Let k be a perfect field of characteristic $\neq 2, 3$. Let $a, b \in k$ with $4a^3 + 27b^2 \neq 0$ and let $E \subset \mathbb{P}^2(\bar{k})$ be the smooth projective curve $E = Z(X_2^2 X_0 - (X_1^3 + aX_1X_0^2 + bX_0^3))$ over k, see Exercise 4.2. With the notation from Exercise 4.2 (and the lecture), show that $\sum_P \mathrm{Tr}_{P/k}(\mathrm{Res}_P(ydx)) = 0$ by a direct calculation and without using the residue formula.

Exercise 7.3. Let k be a perfect field of characteristic $\neq 2, 3$ with an algebraic closure $\bar{k}$. Denote by $C \subset \mathbb{P}^2(\bar{k})$ the closure of the affine curve given by $Z(y^2 - (x^3 + 4)) \subset \mathbb{A}^2(\bar{k})$.

(1) Show that C is smooth projective and geometrically connected.
(2) Show that the points $P_\pm := (x, y) = (0, \pm 2)$ lie in C and that $x \in \mathcal{O}_{C,P_\pm}$ is a local parameter for both.
(3) Denote by $K := k(C)$ the function field of C. Show that there exists a differential form $\alpha \in \Omega^1_{K/k}$ such that $\alpha \in \omega_{C,P}$, for all $P \neq P_\pm$, and $\alpha \pm \frac{dx}{x} \in \omega_{C,P_\pm}$. (*Hint:* Use Exercise 7.1 and the explicit description of $\mathrm{Tr}_{C/k}$ from the lecture.)

Exercise 7.4. Let k be a perfect field. Let E be an *elliptic curve* over k, i.e. it is a smooth projective and connected curve with a rational point and of genus $g = \dim_k H^1(E, \mathcal{O}_E) = 1$. Show that $\omega_E \cong \mathcal{O}_E$. (*Hint*: Use Serre Duality to see that ω_E has a global nowhere vanishing section and conclude with Nakayama's Lemma.)

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