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Exercise sheet 6 for Algebraic curves and the Weil conjectures

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Let C be a smooth geometrically connected curve over a perfect field k. Then we will prove in the course that there is a canonical isomorphism of k-vector spaces

$$(*)\quad H^0(C/k, \omega_C \otimes_{\mathcal{O}_C} L^\vee) \xrightarrow{\simeq} H^1(C/k, L)^\vee := \mathrm{Hom}_k(H^1(C/k, L), k),$$

where $\omega_C = \Omega^1_{C/k}$, L is any invertible sheaf on C/k and $L^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_C}(L, \mathcal{O}_C)$ is its dual. In the following you can use this statement.

Exercise 6.1. In the situation above, show:

(1) $\dim_k H^0(C/k, \omega_C) = \dim_k H^1(C/k, \mathcal{O}_C) =: g$ (It is the *genus of* C).
(2) Assume $C = Z(F) \subset \mathbb{P}^2(\bar{k})$, where $F \in k[X_0, X_1, X_2]$ is a homogenous polynomial of degree n. Then $g = \frac{(n-2)(n-1)}{2}$.
(3) $\dim_k H^0(C/k, L) - \dim_k H^0(C/k, \omega_C \otimes_{\mathcal{O}_C} L^\vee) = 1 - g + \deg L$, where L is an invertible sheaf on C/k.
(4) Let $K_{C/k} \in CH^1(C/k)$ be the canonical divisor of C/k, i.e. $\omega_C \cong \mathcal{O}_C(K_{C/k})$. Show $\deg(K_{C/k}) = 2g - 2$.
(5) If $\deg L \geq 2g - 1$, then $\dim_k H^0(C/k, L) = 1 - g + \deg L$.
(6) Assume $\deg L = 0$. Show that $\dim_k H^0(C/k, L) = 1$, if $L \cong \mathcal{O}_C$, and $= 0$, else. (*Hint:* Here you don't need (*) or Riemann-Roch, just write $L \cong \mathcal{O}_C(D)$ for some divisor D and look what you get.)

Exercise 6.2. Let C be a smooth projective geometrically connected curve over k of genus g.

(1) Show that any non-constant function $t \in k(C)$ (i.e. $t \in k(C) \setminus k$) induces a dominant k-morphism $C \to \mathbb{P}^1_k$ such that the corresponding function field inclusion is given by $k(t) \subset k(C)$.
(2) Assume $g = 0$. Then C is k-isomorphic to $\mathbb{P}^1_k$ if and only if C has a k-rational point $P_0 \in C(k)$. (*Hint:* If there exists $P_0 \in C(k)$ use Exercise 6.1, (5), to show that there exists a

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non-constant function $f \in k(C)^\times$ with a simple pole only at P_0. Then consider the k-morphism $\pi : C \to \mathbb{P}^1_k$ induced by f as in (1) and show that $[P_0] = \pi^*[\infty]$. Now apply deg and conclude.)

Exercise 6.3. Let $\pi : C' \to C$ be a dominant k-morphism between smooth projective and geometrically connected curves over k of respective genus $g(C')$ and $g(C)$. Denote by $K' = k(C')$ and $K = k(C)$ the function fields and assume that that the field extension K'/K induced by π is separable and of degree $[K' : K] = n$. Show the Hurwitz genus formula

$$2g(C') - 2 = n \cdot (2g(C) - 2) + \deg R,$$

where R is the ramification divisor, see Exercise 5.3 (it is an effective divisor on C'.) (*Hint:* Exercise 5.3.)

Exercise 6.4. Let $\pi : C' \to C$ be as in Exercise 6.2 above. We say π is *étale*, if for all closed points Q the ramification index is one, i.e. $e_Q = e(Q/P) = 1$, with $P = \pi(Q)$; in this case we say that π or C' is a *connected finite étale cover of* C.

(1) Show: π is étale $\Longleftrightarrow \chi(C', \mathcal{O}'_C) = n \cdot \chi(C, \mathcal{O}_C)$, where $\chi(C, \mathcal{O}_C) = \dim_k H^0(C/k, \mathcal{O}_C) - \dim_k H^1(C/k, \mathcal{O}_C)$ is the Euler characteristic of C.
(2) Show that the only connected finite étale cover of $\mathbb{P}^1_k$ is $\mathbb{P}^1_k$ itself. (This can be seen as a geometric version of Minkowski's Theorem from Number Theory: The field of rational numbers $\mathbb{Q}$ does not admit a non-trivial unramified extension.)