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Exercise sheet 5 for Algebraic curves and the Weil conjectures

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Exercise 5.1. Let k be a perfect field and C/k be a smooth irreducible projective curve over k with function field $k(C)$. Take $t \in k(C)^\times$ a transcendental element over k such that $k(C)/k(t)$ is a finite separable field extension.

(1) Show that $\Omega^1_{k(C)/k} = k(C) \cdot dt$.

(2) Show that if $P \in (C/k)_0$ is a closed point and $t_P \in \mathcal{O}_{C/k,P}$ is a *local parameter* (i.e. it generates the maximal ideal), then $\Omega^1_{k(C)/k} = k(C) \cdot dt_P$. (*Hint:* First show that $\Omega^1_{C/k,P} = \mathcal{O}_{C,P} \cdot dt_P$. To this end observe that we know already that $\Omega^1_{C/k,P}$ is locally free, hence (by a lemma from the lecture) we know, that it suffices to see that $dt_P \otimes 1$ is a basis of $\Omega^1_{C/k,P} \otimes_{\mathcal{O}_{C,P}} k(P)$ which follows from a certain exact sequence for Ω^1.)

(3) Let $P \in (C/k)_0$ be a closed point and denote by $v_P : k(C)^\times \to \mathbb{Z}$ the associated discrete normalized valuation. Define $v_P(dt)$ as follows: take t_P a local parameter at P and write $dt = f_P dt_P$ with $f_P \in k(C)^\times$ (see (2)); then $v_P(dt) := v_P(f_P)$. Show that this definition is independent of the choice of the local parameter t_P.

(4) Define the Weil divisor $K_{C/k} := \sum_{P \in (C/k)_0} v_P(dt) \cdot [P] \in Z^1(C/k)$. Show that it is well-defined, i.e. the sum is finite.

(5) Show that there is a natural isomorphism
$$\omega_{C/k} \cong \mathcal{O}_{C/k}(K_{C/k}),$$
where $\omega_{C/k} = \Omega^1_{C/k}$ and for $U \subset C/k$ open $\Gamma(U, \mathcal{O}_{C/k}(K_{C/k})) = \{f \in k(C)^\times \mid \mathrm{Div}(f)_{|U} \geq -K_{C/k}\}$.

The divisor $K_{C/k}$ defined above is called the *canonical divisor of* C/k and it is well defined as an element in $\mathrm{CH}^1(C/k)$.

Exercise 5.2. Let $f : C' \to C$ be a dominant k-morphism between smooth connected curves/k.

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(1) Let $Q \in (C'/k)_0$ be a closed point with image $P = f(Q) \in (C/k)_0$. Define the natural number $e_Q = e(Q/P)$ as follows: Take a local parameter $t_P \in \mathcal{O}_{C,P}$, then set

$$e_Q := e(Q/P) := v_Q(t_P),$$

where $v_Q : k(C')^\times \to \mathbb{Z}$ is the normalized discrete valuation corresponding to Q and we view t_P as an element of $k(C')$ via the inclusion $k(C) \hookrightarrow k(C')$ defined by f. Show that e_Q is well-defined, i.e., independent of the choice of the parameter t_P.

(2) Let $D = \sum_i n_i[P_i]$, $P_i \in (C/k)_0$, be a Weil divisor on C. Set $f^*D := \sum_i n_i \cdot \sum_{Q\in(f^{-1}(P_i)/k)_0} e(Q/P_i)[Q]$. This defines a group homomorphism $f^* : Z^1(C/k) \to Z^1(C'/k) \to \mathrm{CH}^1(C'/k)$, where the second map is the quotient map. Show that f^* induces a well-defined map (again denoted by f^*) $\mathrm{CH}^1(C/k) \to \mathrm{CH}^1(C'/k)$.

Exercise 5.3. Let $f : C' \to C$ be a dominant k-morphism between smooth projective and connected curves over a perfect field k. Assume the corresponding function field extension $k(C')/k(C)$ is separable.

(1) For $Q \in (C'/k)_0$ define $r_Q \in \mathbb{N}_0$ as follows (notation as above): Take t_P a local parameter at $P = f(Q)$ and t_Q a local parameter at Q we can view dt_P as an element in $\omega_{C'/k,Q}$ and write $dt_P = g_P dt_Q$(see 5.1, (2)). Then we define

$$r_Q := v_Q(g_P).$$

Show that this definition is independent of the choices of t_P and t_Q.

(2) Let $p = \mathrm{char}(k) \geq 0$. Assume $p = 0$ or $e_Q = e(Q/P)$ (see 5.2, (1)) is prime to p. Show that $r_Q = e_Q - 1$.

(3) Show that in $\mathrm{CH}^1(C'/k)$

$$[K_{C'}] = f^*[K_C] + [R],$$

where $R = \sum_{Q\in(C'/k)_0} r_Q \cdot [Q]$ is the *ramification divisor.*

Exercise 5.4. Let $\mathbb{P}^2(\bar{k}) = U_0 \cup U_1 \cup U_2$ be the standard open cover over k, i.e., $U_i = \mathbb{P}^2(\bar{k}) \setminus Z(X_i)$, where X_0, X_1, X_2 are the coordinates on $\mathbb{P}^2$.

(1) Let $C = Z(F)$, where $F \in k[X_0, X_1, X_2]$ is an irreducible homogenous polynomial of degree n. We can view $C \subset \mathbb{P}^2/k$ as a prime Weil divisor and hence have the associated sheaf $\mathcal{O}_{\mathbb{P}^2}(-[C]) =: \mathcal{O}(-C)$. Show that $\mathcal{O}(-C)_{|U_i} = \mathcal{O}_{U_i} \cdot \frac{F}{X_i^n}$. (*Hint:* From commutative algebra we know that if $A = k[U]$ is the coordinate ring of a smooth affine k-variety with fraction field

K, then $A = \cap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$. Use this to show that $A = \{f \in K \mid \mathrm{div}_U(f) \geq 0\}$. Conclude.)

(2) Show that there is an exact sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_{\mathbb{P}^2_k} \to \mathcal{O}_C \to 0,$$

i.e. $\mathcal{O}(-C)$ is the ideal sheaf of the embedding $C \hookrightarrow \mathbb{P}^2/k$.

(3) Show that there is an isomorphism $\mathcal{O}(-C) \cong \mathcal{O}_{\mathbb{P}^2/k}(-n)$.

(4) Show that

$$\dim_k H^1(C, \mathcal{O}_C) = \frac{(n-2)(n-1)}{2}.$$

(*Hint:* Use the short exact sequence from (2), the associated long exact sequence in cohomology and Theorem 7.3 from the lecture.)