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Exercise sheet 4 for Algebraic curves and the Weil conjectures

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Exercise 4.1. Let k be a field with fixed algebraic closure $\bar{k}$. Show that there is an $\mathcal{O}_{\mathbb{P}^1/k}$-linear isomorphism

$$\mathcal{O}_{\mathbb{P}^1/k}(-2) \xrightarrow{\simeq} \omega_{\mathbb{P}^1/k}.$$

Conclude that $\Gamma(\mathbb{P}^1/k, \omega_{\mathbb{P}^1/k}) = 0$.

Exercise 4.2. Let k be a field of characteristic $\neq 2, 3$ with fixed algeraic closure $\bar{k}$. Let $a, b \in k$ and let $E \subset \mathbb{P}^2(\bar{k})$ be the projective variety $/k$ defined by $E = Z(X_2^2 X_0 - (X_1^3 + aX_1X_0^2 + bX_0^3))$.

(1) Set $U = Z(y^2 - (x^3 + ax + b))$, where $x = X_1/X_0, y = X_2/X_0$ and $W = Z(z - (u^3 + auz^2 + bz^3))$, where $u = X_1/X_2$, $z = X_0/X_2$. Show that $U, W \subset E/k$ are open and $E = U \cup W$.
(2) Show that E is an irreducible curve$/k$.
(3) Show that E is a smooth $/k$ if and only if $4a^3 + 27b^2 \neq 0$.

We assume $4a^3 + 27b^2 \neq 0$ in the following.

(4) Set $U_1 = U \setminus Z(y)$, $U_2 = U \setminus Z(3x^2 + a)$ and $U_3 = W \setminus Z(1 - 2auz - 3bz^2)$. Show that $E = U_1 \cup U_2 \cup U_3$ is an open covering.
(5) Define the differential forms

$$\alpha_1 := \frac{dx}{2y} \in \Gamma(U_1, \omega_E), \quad \alpha_2 := \frac{dy}{3x^2 + a} \in \Gamma(U_2, \omega_E),$$

$$\alpha_3 := -\frac{du}{1 - 2auz - 3bz^2} \in \Gamma(U_3, \omega_E),$$

where $\omega_E := \Omega^1_{E/k}$. Show that there is a differential $\alpha \in \Gamma(E, \omega_E)$ with $\alpha_{|U_i} = \alpha_i$, $i = 1, 2, 3$.
(6) Show that we have an isomorphism $\mathcal{O}_E \to \omega_E$, $f \mapsto f \cdot \alpha$.

Exercise 4.3. Let k be a field with fixed algebraic closure $\bar{k}$ and Y an affine k-variety with coordinate ring $k[Y] = A$. We write $\mathbb{P}^1_Y = \mathbb{P}^1 \times Y$ and $\mathcal{O}_{\mathbb{P}^1_Y}(r) = p_1^*\mathcal{O}_{\mathbb{P}^1/k}(r)$, where $p_1 : \mathbb{P}^1 \times Y \to \mathbb{P}^1$ is the projection.

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(1) Compute $H^1(\mathbb{P}^1_Y, \mathcal{O}_{\mathbb{P}^1_Y}(r))$ using Cech cohomology and the standard affine open cover of $\mathbb{P}^1_Y$.

(2) Show that there is a perfect pairing of finitely generated free A-modules

$$H^0(\mathbb{P}^1_Y, \mathcal{O}_{\mathbb{P}^1_Y}(-2-r)) \otimes_A H^1(\mathbb{P}^1_Y, \mathcal{O}_{\mathbb{P}^1_Y}(r)) \to H^1(\mathbb{P}^1_Y, \mathcal{O}_{\mathbb{P}^1_Y}(-2)) \cong A.$$

(Recall that a pairing $\phi : M \otimes_A N \to A$ is perfect if the induced maps $M \to \mathrm{Hom}_A(N, A)$, $m \mapsto \phi(m \otimes -)$, and $N \to \mathrm{Hom}_A(M, A)$ are isomorphisms.)

Exercise 4.4. Let k be a field with fixed algebraic closure $\bar{k}$ and X/k a smooth, irreducible, quasi-projective variety. Denote by $K = k(X)$ the function field of K.

(1) Let $V \subset X/k$ be a prime Weil divisor. For $U \subset X/k$ open define $\mathbb{Z}_V(U) = \mathbb{Z}$, if $U \cap V \neq \emptyset$, and $\mathbb{Z}_V(U) = 0$, else. Show that $\mathbb{Z}_V$ is a flasque sheaf on X. Deduce that $\oplus_V \mathbb{Z}_V$ is a flasque sheaf on X.

(2) Since X is smooth the local rings $\mathcal{O}_{X,V}$ are DVRs and hence define a normalized discrete valuation $\mathrm{ord}_V : K^\times \to Z$. Show that there is a surjective morphism of sheaves $K_X^\times \xrightarrow{\oplus_V \mathrm{ord}_V} \oplus_V \mathbb{Z}_V$, where $K_X^\times$ denotes the constant sheaf on X defined by $K^\times$.

(3) Conclude that we have a flasque resolution of $\mathcal{O}_X^\times$

$$0 \to \mathcal{O}_X^\times \to K_X^\times \to \oplus_V \mathbb{Z}_V \to 0.$$

(*Hint:* Use that $a \in \mathcal{O}_X^\times(U) \Leftrightarrow a \in K^\times$ and $\mathrm{ord}_V(a) = 0$, for all prime Weil divisors V with $V \cap U \neq \emptyset$.)

(4) Use the above resolution to compute

$$H^1(X, \mathcal{O}_X^\times) = \mathrm{CH}^1(X).$$

Remark 1. Without any assumptions on X one can show $H^1(X, \mathcal{O}_X^\times) \cong \check{H}^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$, see Exercise 3, for the second equality.